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Hadamard Gaps and $\mathcal{N}_K$-type Spaces in the Unit Ball

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Abstract

In this paper, we introduce a class of holomorphic Banach spaces $\mathcal{N}_K$ of functions on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$. We develop the necessary and sufficient condition for $\mathcal{N}_K(\mathbb{B})$ spaces to be non-trivial and we discuss the nesting property of $\mathcal{N}_K(\mathbb{B})$ spaces. Also, we obtain some characterizations of functions with Hadamard gaps in $\mathcal{N}_K(\mathbb{B})$ spaces. As a consequence, we prove a necessary and sufficient condition for that $\mathcal{N}_K(\mathbb{B})$ spaces coincides with the Beurling-type space.

Keywords

$\mathcal{N}_K$-type Spaces, Beurling-Type Space, Hadamard Gaps

1. Introduction

Through this paper, $\mathbb{B}$ is the unit ball of the n-dimensional complex Euclidean space $\mathbb{C}^n$, $\mathbb{S}$ is the boundary of $\mathbb{B}$. We denote the class of all holomorphic functions, with the compact-open topology on the unit ball $\mathbb{B}$ by $\mathcal{H}(\mathbb{B})$.

For any $z=(z_1,z_2,\cdots,z_n)$, $w=(w_1,w_2,\cdots,w_n)\in\mathbb{C}^n$, the inner product is defined by $\langle z,w\rangle=(z_1\overline{w_1},z_2\overline{w_2},\cdots,z_n\overline{w_n})$, and write $|z|=\sqrt{\langle z,w\rangle}$.

Let $\mathrm{d}v$ be the Lebesgue volume measure on $\mathbb{C}^n$, normalized so that $v(\mathbb{B})\equiv 1$ and $\mathrm{d}\sigma$ be the surface measure on $\mathbb{S}$. Once again, we normalize σ so that $\sigma(\mathbb{B})\equiv 1$. For $z\in\mathbb{B}$ and $r>0$ let $\mathbb{B}_r=\{z\in\mathbb{B}:|z|\le r\}$.

For $\zeta\in\mathbb{B}$ the measures v and σ are related by the following formula:

$$\int_{\mathbb{B}} f\mathrm{d}v = 2n\int_0^1 r^{2n-1}\mathrm{d}r\int_{\mathbb{S}} f(r\zeta)\mathrm{d}\sigma(\zeta). \tag{1}$$

The identity

$$\int_{\mathbb{S}} f\mathrm{d}\sigma = \int_{\mathbb{S}} \mathrm{d}\sigma(\zeta)\frac{1}{2\pi}\int_0^{2\pi} f\left(\mathrm{e}^{\mathrm{i}\theta}\zeta\right)\mathrm{d}\theta, \tag{2}$$

is called integration by slices, for all $0 \le \theta \le 2\pi$ (see [1]).

For every point $a \in \mathbb{B}$ the Möbius transformation $\varphi_a : \mathbb{B} \to \mathbb{B}$ is defined by

$$\varphi_a(z) = \frac{a - P_a(z) - S_a Q_a(z)}{1 - \langle z, a\rangle}, \tag{3}$$

where $S_a = \sqrt{1-|z|^2}, P_a(z) = \frac{a\langle z, a\rangle}{|a|^2}, P_0 = 0$ and $Q_a = I - P_a(z)$ (see [1] or [2]).

The map φ_a has the following properties that $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_a = \varphi_a^{-1}$ and

$$1 - \langle \varphi_a(z), \varphi_a(w)\rangle = \frac{\left(1-|a|^2\right)\left(1-\langle z, w\rangle\right)}{\left(1-\langle z, a\rangle\right)\left(1-\langle a, w\rangle\right)},$$

where z and w are arbitrary points in $\mathbb{B}$. In particular,

$$1 - |\varphi_a(z)|^2 = \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\langle z, a\rangle|^2}, \tag{4}$$

For $a \in \mathbb{B}$ the Möbius invariant Green function in the unit ball $\mathbb{B}$ denoted by $G(z,a) = g(\varphi_a(z))$ where $g(z)$ is defined by:

$$g(z) = \frac{n+1}{2n}\int_{|z|}^{1}\left(1-t^2\right)^{n-1} t^{1-2n}\mathrm{d}t. \tag{5}$$

For $n > 1$, we have

$$\frac{1}{C_n}\left(1-r^2\right)^n t^{-2(n-1)} \le C_n\left(1-r^2\right)^n t^{-2(n-1)}, \tag{6}$$

where C_n is a constant depending on n only.

Let $H^\infty(\mathbb{B})$ denote the Banach space of bounded functions in $\mathcal{H}(\mathbb{B})$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$.

For $\alpha > 0$, the Beurling-type space (sometimes also called the Bers-type space) $H_\alpha^\infty(\mathbb{B})$ in the unit ball $\mathbb{B}$ consists of those functions $f \in \mathcal{H}(\mathbb{B})$ for which

$$\|f\|_{H_\alpha^\infty(\mathbb{B})} = \sup_{z\in\mathbb{B}} |f(z)|\left(1-|z|^2\right)^\alpha < \infty. \tag{7}$$

Let $K : (0,\infty) \to [0,\infty)$ is a right-continuous, non-decreasing function and is not equal to zero identically. The $\mathcal{N}_K(\mathbb{B})$ space consists of all functions $f \in \mathcal{H}(\mathbb{B})$ such that

$$\|f\|_K^2 = \sup_{z\in\mathbb{B}} \int_{\mathbb{B}} |f(z)|^2 K(G(z,a))\mathrm{d}v(z) < \infty. \tag{8}$$

Clearly, if $K(t) = t^p$, then $\mathcal{N}_K(\mathbb{B}) = \mathcal{N}_p(\mathbb{B})$. For $K(t) = 1$ it gives the Bergman space $\mathcal{A}^2(\mathbb{B})$. If $\mathcal{N}_K(\mathbb{B})$ consists of just the constant functions, we say that it is trivial.

We assume from now that all $K:(0,\infty)\to[0,\infty)$ to appear in this paper are right-continuous and nondecreasing function, which is not equal to 0 identically.

In [3], several basic properties of $\mathcal{N}_K(\mathbb{B})$ are proved, in connection with the Beurling-type space $H_\alpha^\infty(\mathbb{B})$. In particular, an embedding theorem for $\mathcal{N}_K(\mathbb{B})$ and $H_\alpha^\infty(\mathbb{B})$ is obtained, together with other useful properties. Hadamard gaps series and Hadamard product on $\mathcal{N}_K$ spaces of holomorphic function in the case of the unit disk has been studied quite well in [4] and [5].

Through this, paper, given two quantities A_f and B_f both depending on a function $f\in\mathcal{H}(\mathbb{B})$, we are going to write $A_f\lesssim B_f$ if there exists a constant $C>0$, independent of f, such that $A_f\le CB_f$ for all f. When $A_f\lesssim B_f\lesssim A_f$, we write $A_f\approx B_f$. If the quantities A_f and B_f are equivalent, then in particular we have $A_f<\infty$ if and only if $B_f<\infty$. As usual, the letter C will denote a positive constant, possibly different on each occurrence.

In this paper, we introduce $\mathcal{N}_K(\mathbb{B})$ spaces, in terms of the right continuous and non-decreasing function $K:(0,\infty)\to[0,\infty)$ on the unit ball $\mathbb{B}$. We discuss the nesting property of $\mathcal{N}_K(\mathbb{B})$. We prove a sufficient condition for $\mathcal{N}_K(\mathbb{B})=H_\alpha^\infty(\mathbb{B})$, $\alpha=\frac{n+1}{2}$ (the Beurling-type space). Also we generalize the necessary condetion to $\mathcal{N}_K(\mathbb{B})$ for a kind of lacunary series. As aplplication, we show that the sufficient condition is also a necessary to $\mathcal{N}_K(\mathbb{B})=H_{\frac{n+1}{2}}^\infty(\mathbb{B})$.

2. $\mathcal{N}_K$ Spaces in the Unit Ball

In this section we prove some basic Banach space properties of $\mathcal{N}_K(\mathbb{B})$ space. A sufficient and necessary condition for $\mathcal{N}_K(\mathbb{B})$ to be non-trivial is given. We discuss the nesting property of $\mathcal{N}_K(\mathbb{B})$ spaces and prove a sufficient condition for $\mathcal{N}_K(\mathbb{B})=H_{\frac{n+1}{2}}^\infty(\mathbb{B})$.

Lemma 2.1

Let $f(z)=\sum_{k=1}^{\infty}a_k z^k$ be a non-constant function, where $k=(k_1,k_2,\cdots,k_n)$ is an n-tuple of non-negative integers and $z^k=\left(z_1^{k_1},z_2^{k_2},\cdots,z_n^{k_n}\right)$.

Then, $z^k\in\mathcal{N}_K(\mathbb{B})$ if $a_k\ne 0$.

Proof:

Let k be such that Let k be such that $a_k\ne 0$ and let $F_k(z)=a_k z^k$. Suppose that

$$U_\theta f(z)=f\left(z_1 e^{i\theta_1},z_2 e^{i\theta_2},\cdots,z_n e^{i\theta_n}\right)=f\circ U_\theta(z),$$

where $U_\theta(z)=\left(z_1 e^{i\theta_1},z_2 e^{i\theta_2},\cdots,z_n e^{i\theta_n}\right)$. Then, we have

$$\begin{aligned}F_k(z)&=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f\left(z_1 e^{i\theta_1},\cdots,z_n e^{i\theta_n}\right)e^{-ik_1\theta_1}\cdots e^{-ik_n\theta_n}\,d\theta_n\\&=\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}(U_\theta f)(z)e^{-ik_1\theta_1}\cdots e^{-ik_n\theta_n}\,d\theta_n.\end{aligned}\tag{9}$$

By Jensen's inequality on convexity,

$$\left|F_k(z)\right|^2 \le \frac{1}{(2\pi)^{2n}}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|U_\theta f(z)\right|^2 \mathrm{d}\theta_1\cdots\mathrm{d}\theta_n. \tag{10}$$

Consequently,

$$\begin{aligned}&\int_{\mathbb{B}}\left|F_k(z)\right|^2 K(G(z,a))\mathrm{d}\lambda(z)\\ &\le \left\|U_\theta f\right\|_K^2 \frac{1}{(2\pi)^{2n}}\int_0^{2\pi}\cdots\int_0^{2\pi}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n \le \left\|U_\theta f\right\|_K^2.\end{aligned} \tag{11}$$

Because $U_\theta(z)\in Aut(\mathbb{B})$ we have $\left\|U_\theta f\right\|_K = \|f\|_K$. Therefore,

$$\left\|F_k f\right\|_K = \left\|a_k z^k\right\|_K \le \|f\|_K$$

and $z^k \in \mathcal{N}_K(\mathbb{B})$. The lemma is proved.

Theorem 2.1 The Holomorphic function spaces $\mathcal{N}_K(\mathbb{B})$, contains all polynomials if

$$\int_0^1 r^{2n-1}K(g(r))\mathrm{d}r < \infty. \tag{12}$$

Otherwise, $\mathcal{N}_K(\mathbb{B})$ contains only constant functions.

Proof:

First assume that (12) holds. Let $f(z)$ be a polynomial i.e. (there exists a $M>0$ such that $\left|f(z)\right|^2 \le M, \forall z\in\overline{\mathbb{B}} = \mathbb{B}\cup\mathbb{S}$). Then,

$$\begin{aligned}&\int_{\mathbb{B}}\left|f(z)\right|^2 K(G(z,a))\mathrm{d}v(z)\\ &= 2n\int_0^1 r^{2n-1}K(g(r))\mathrm{d}r\int_{\mathbb{S}}\left|f(\phi_a(r\zeta))\right|^2\mathrm{d}\sigma(\zeta)\\ &\le 2nM\int_0^1 r^{2n-1}K(g(r))\mathrm{d}r.\end{aligned} \tag{13}$$

Since a is arbitrary, it follows that

$$\|f\|_K^2 \le 2nM\int_0^1 r^{2n-1}K(g(r))\mathrm{d}r < \infty. \tag{14}$$

Thus, $f\in\mathcal{N}_K(\mathbb{B})$ and the first half of the theorem is proved.

Now, we assume that the integral in (12) is divergent. Let $\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_n)$ is an n-tuple of non-negative integers $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n \ge 1$, $f(z) = z^\alpha$.

Then, we have $\left|f(r\xi)\right|^2 = r^{2|\alpha|}\left|\xi^\alpha\right|^2$ and

$$\int_{\mathbb{S}}\left|(r\zeta)^\alpha\right|^2\mathrm{d}\sigma(r\zeta) \ge \frac{r^{2|\alpha|}(n-1)!\alpha!}{(n-1+|\alpha|)!} \ge Cr^{2|\alpha|}. \tag{15}$$

Thus,

$$\|f\|_K \ge \frac{nC}{2^{2|\alpha|-1}}\int_{1/2}^1 r^{2n-1}K(g(r))\mathrm{d}r. \tag{16}$$

There exists $a \in \mathbb{B}$ such that $f(a) \neq 0$, by the subharmonicity of $\left|f \circ \varphi_a(r\xi)\right|$,

$$\|f\|_K \geq \frac{3n}{2}|f(a)|^2 \int_0^{1/2} \frac{r^{2n-1}}{\left(1-r^2\right)^{n+1}} K(g(r)) \mathrm{d}r. \tag{17}$$

Combining (17) and (18), we see that (12) implies that $\|f\|_K = \infty$.

It is proved that $f \notin \mathcal{N}_K(\mathbb{B})$ and, since α is arbitrary, any non-constant polynomial is not contained in $\mathcal{N}_K(\mathbb{B})$. Using Lemma 2.1, we conclude that $\mathcal{N}_K(\mathbb{B})$ contains only constant functions. The theorem is proved.

Theorem 2.2

Let K_1 and K_2 satisfy (12). If there exist a constant $t_0 > 0$ such that $K_2(t) \lesssim K_1(t)$ for $t \in (0, t_0)$, then $\mathcal{N}_{K_1}(\mathbb{B}) \subseteq \mathcal{N}_{K_2}(\mathbb{B})$. As a consequence, $\mathcal{N}_{K_1}(\mathbb{B}) = \mathcal{N}_{K_2}(\mathbb{B})$. if $K_2(t) \approx K_1(t)$ for $t \in (0, t_0)$.

Proof: Let $f \in \mathcal{N}_{K_1}(\mathbb{B})$. We note that from the property of $g(z)$, there exists a constant $\delta > 0$, such that $g(z) < t_0$ if $|z| > \delta$. Then, we have

$$\int_{\mathbb{B}} |f(z)|^2 K_2(G(z,a)) \mathrm{d}v(z) = \int_{\mathbb{B}_\delta} + \int_{|z| \geq \delta} \left|f(\phi_a(z))\right|^2 K_2(g(z)) \mathrm{d}v(z) \tag{18}$$

where

$$\int_{\mathbb{B}_\delta} \left|f(\phi_a(z))\right|^2 K_2(g(z)) \mathrm{d}v(z) \leq \|f\|_\infty^2 \int_{\mathbb{B}_\delta} \left(1-|z|^2\right)^{-n} K_2(g(z)) \mathrm{d}v(z)$$

$$\leq 2n\|f\|_\infty^2 \int_0^\delta r^{2n-1} K_2(g(r)) \mathrm{d}r < \infty,$$

and

$$\int_{|z| \geq \delta} \left|f(\phi_a(z))\right|^2 K_2(g(z)) \mathrm{d}v(z)$$

$$\leq \int_{|z| \geq \delta} \left|f(\phi_a(z))\right|^2 K_1(g(z)) \mathrm{d}v(z) \leq \|f\|_{K_1}^2 < \infty.$$

This show that $\|f\|_{K_2} < \infty$ and, consequently, $f \in \mathcal{N}_{K_2}(\mathbb{B})$.

Theorem 2.3

Let $K : (0, \infty) \to [0, \infty)$ be nondecreasing function, then $\mathcal{N}_K(\mathbb{B}) \subset H_{\frac{n+1}{2}}^\infty(\mathbb{B})$.

Proof: The theorem proved in [3].

Theorem 2.4

$\mathcal{N}_K(\mathbb{B}) = H_{\frac{n+1}{2}}^\infty(\mathbb{B})$ if

$$\int_0^1 \frac{r^{2n-1}}{\left(1-r^2\right)^{n+1}} K(g(r)) \mathrm{d}r < \infty. \tag{19}$$

Proof: Let $f \in H_{\frac{n+1}{2}}^\infty(\mathbb{B})$. Then,

$$
\begin{aligned}
&\int_{\mathbb{B}} |f(z)|^2 K(G(z,a))\,\mathrm{d}v(z) \\
&\le \|f\|^2_{H^\infty_{\frac{n+1}{2}}(\mathbb{B})} \int_{\mathbb{B}} \left(1-|z|^2\right)^{-n} K(g(z)) \frac{\mathrm{d}v(z)}{\left(1-|z|^2\right)^{n+1}} \\
&\le 2n\|f\|^2_{H^\infty_{\frac{n+1}{2}}(\mathbb{B})} \int_0^1 \frac{r^{2n-1}}{\left(1-r^2\right)^{n+1}} K(g(r))\,\mathrm{d}r.
\end{aligned} \tag{20}
$$

Thus, $\|f\|_K < \infty$ and $f \in \mathcal{N}_K(\mathbb{B})$. This shows that $H^\infty_{\frac{n+1}{2}}(\mathbb{B}) \subset \mathcal{N}_K(\mathbb{B})$. By Theorem 2.3, we have $\mathcal{N}_K(\mathbb{B}) \subset H^\infty_{\frac{n+1}{2}}(\mathbb{B})$. The proof of theorem is complete.

3. Hadamard Gaps in $\mathcal{N}_K$ Spaces in the Unit Ball

In this section we prove a necessary condition for a lacunary series defined by a normal sequence to belong to $\mathcal{N}_K(\mathbb{B})$ space. As an implication of Theorem 2.4, we prove that (19) is also necessary for $\mathcal{N}_K(\mathbb{B}) = H^\infty_{\frac{n+1}{2}}(\mathbb{B})$.

Recall that an $f \in \mathcal{H}(\mathbb{B})$ written in the form $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$ where P_{n_k} is a homogeneous polynomial of degree n_k, is said to have Hadamard gaps (also known as lacunary series) if there exists a constant $c > 1$ such that (see e.g. [6])

$$
\frac{n_{k+1}}{n_k} \ge c, \ \forall k \ge 0. \tag{21}
$$

Let $\Lambda_n \subset \mathbb{S}$ for $n = n_0, n_0+1, \cdots$. The sequence of homogeneous polynomials

$$
P_n(z) = \sum_{\zeta \in \Lambda_n} \langle z, \zeta \rangle^n, \tag{22}
$$

is called a normal sequence if it possesses the following property (see [7]):

- $|P_n(z)| \le C|z|^n$ for $z \in \mathbb{B}$;
- $\sum_{\xi,\zeta \in \Lambda_n} \xi, \zeta^n \ge \frac{n^{k+1}}{C}$.

In what following, we will consider all lacunary series defined by normal sequences of homogeneous polynomials. To formulate our main result, we denote

$$
L_j = \int_{\mathbb{S}} \left|P_{n_j}(\zeta)\right|^2 \mathrm{d}\sigma(\zeta). \tag{23}
$$

Theorem 3.1

Let $P_n(z)$ be a normal sequence and let $I_K = \left\{n \in \mathbb{N} : 2^k \le n \le 2^{k+1}\right\}$. Then a lacunary series $f(z) = \sum_{k=0}^{\infty} P_{n_k}(z)$, belongs to $\mathcal{N}_K(\mathbb{B})$ if

$$
\sum_{k=0}^{\infty} \frac{n_k^m}{2^k} K\left(n_k^{-m}\right) \sum_{n_j \in I_k} L_j < \infty. \tag{24}
$$

Proof: Let $f \in \mathcal{N}_K(\mathbb{B})$. Then, we have

$$\int_{\mathbb{B}} |f(z)|^2 K(G(z,a))\,\mathrm{d}v(z) \ge \int_{\mathbb{B}} \left|\sum_{k=0}^{\infty} P_{n_k}(z)\right|^2 K(g(|z|))\,\mathrm{d}v(z) \\ \ge \sum_{k=0}^{\infty} \frac{n_k}{2^k} \sum_{n_j \in I_k} L_j \int_0^1 r^{2m-1} K(g(r))\,\mathrm{d}r, \tag{25}$$

where

$$\left|\sum_{k=0}^{\infty} P_{n_k}(z)\right|^2 = \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{n_j \in I_k} \left|P_{n_k}(\zeta)\right|^2. \tag{26}$$

By (6) for $\frac{1}{2} \le r \le 1$, we have

$$K(g(r)) \ge K\left(c^{-1}(1-r)^m\right). \tag{27}$$

Consequently,

$$\int_0^1 r^{2m-1} K(g(r))\,\mathrm{d}r \ge \int_{\frac{1}{2}}^1 r^{2m-1} K\left(c^{-1}(1-r)^m\right)\mathrm{d}r \ge \int_0^{\log 2} \mathrm{e}^{-2mt} K\left(c_1^{-1} t^m\right)\mathrm{d}t \\ \ge K\left(n_k^{-m}\right) \int_{c_1 n_k^{-1}}^{\log 2} \mathrm{e}^{-2mt}\,\mathrm{d}t \ge n_k^{m-1} K\left(n_k^{-m}\right) \int_{c_1}^{n_k \log 2} \mathrm{e}^{-2t}\,\mathrm{d}t. \tag{28}$$

Let k' be sufficiently large such that $n_{k'} \log 2 \ge c_1 + 1$. Then, for $k \ge k'$,

$$\int_0^1 r^{2m-1} K(g(r))\,\mathrm{d}r \ge n_k^{m-1} K\left(n_k^{-m}\right). \tag{29}$$

And

$$\int_{\mathbb{B}} |f(z)|^2 K(G(z,a))\,\mathrm{d}v(z) \ge C \sum_{k=k'}^{\infty} \frac{n_k^m}{2^k} K\left(n_k^{-m}\right) \sum_{n_j \in I_k} L_j. \tag{30}$$

This shows (24) and the theorem is proved.

Theorem 3.2

$\mathcal{N}_K(\mathbb{B}) = H^{\infty}_{\frac{n+1}{2}}(\mathbb{B})$ if and only if (18) holds.

Proof: The sufficient condition was proved by Theorem 2.4. Now we prove the necessary condition, assume that $\mathcal{N}_K(\mathbb{B}) = H^{\infty}_{\frac{n+1}{2}}(\mathbb{B})$. Among lacunary series defined by normal sequences, we consider

$$f(z) = \sum_{k=k_0}^{\infty} P_{2^k}(z), \tag{31}$$

where $P_{2^k} = \sum_{\zeta \in \Lambda_n} \langle z, \zeta \rangle^{2^k}$ and $\left|P_{2^k}\right| = C|z|^{2^k}$ for $k \ge k_0, 2^{k_0} \ge n_0$ and $z \in \mathbb{B}$.

Thus

$$|f(z)|\left(1-|z|^2\right)^{n+1} \le \left(1-|z|^2\right)^{n+1} \sum_{k=k_0}^{\infty} \left|P_{2^k}(z)\right| \le C \sum_{n=1}^{\infty} |z|^n \le C. \tag{32}$$

This shows that $f \in H^{\infty}_{\frac{n+1}{2}}(\mathbb{B})$ and, consequently, $f \in \mathcal{N}_K(\mathbb{B})$. By Theorem 3.1, we have

$$\sum_{k=1}^{\infty} 2^{k(m-1)} K\left(2^{-mk}\right) < \infty. \tag{33}$$

By (6), we have

$$\int_{1/2}^{1} \frac{r^{2m-1}}{\left(1-r^2\right)^{m+1}} K\left(g\left(r\right)\right) \mathrm{d}r \le \int_0^{c^{1/m}\log 2} t^{-m-1} K\left(t^m\right) \mathrm{d}t. \tag{34}$$

On the other hand,

$$\begin{aligned} \int_0^{1/2} t^{-m-1} K\left(t^m\right) \mathrm{d}t &= \sum_{k=1}^{\infty} \int_{2^{-k-1}}^{2^{-k}} t^{-m-1} K\left(t^m\right) \mathrm{d}t \\ &= \sum_{k=1}^{\infty} 2^{-(k+1)} 2^{-m-1} K\left(2^{-mk}\right), \end{aligned} \tag{35}$$

since K is non-decreasing. Thus,

$$\int_{1/2}^{1} \frac{r^{2m-1}}{\left(1-r^2\right)^{m+1}} K\left(g\left(r\right)\right) \mathrm{d}r < \infty. \tag{36}$$

Combining this, we obtain (18). The theorem is proved.

4. Conclusion

Our aim of the present paper is to characterize the holomorphic functions with Hadamard gaps in $\mathcal{N}_K$-type spaces on the unit ball, where K is the right continuous and non-decreasing function. Our main results will be of important uses in the study of operator theory of holomorphic function spaces.

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