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Lacunary series in $\mathcal{Q}_K$ spaces

by

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Abstract. Under mild conditions on the weight function K we characterize lacunary series in the so-called $\mathcal{Q}_K$ spaces.

1. Introduction. Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Green's function for $\mathbb{D}$ is given by

$$g(z,w) = \log\frac{1}{|\sigma_w(z)|} = \log\left|\frac{1-\overline{w}z}{w-z}\right|,$$

where

$$\sigma_w(z) = \frac{w-z}{1-\overline{w}z}$$

is a Möbius transformation of $\mathbb{D}$.

Given a function $K:(0,\infty)\to[0,\infty)$, we consider the space $\mathcal{Q}_K$ of all functions $f\in H(\mathbb{D})$ for which

$$\|f\|^2_{\mathcal{Q}_K} = \sup_{w\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2 K(g(z,w))\,dA(z) < \infty,$$

where $H(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$ and dA is the Euclidean area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D})=1$. It is easy to check that $\|\ \|_{\mathcal{Q}_K}$ is a complete seminorm on $\mathcal{Q}_K$ and it is Möbius invariant, that is,

$$\|f\circ\sigma\|_{\mathcal{Q}_K} = \|f\|_{\mathcal{Q}_K}, \qquad \sigma\in\mathrm{Aut}(\mathbb{D}),$$

where $\mathrm{Aut}(\mathbb{D})$ is the group of all Möbius maps of the unit disk. Earlier studies on $\mathcal{Q}_K$ spaces can be found in [8], [9], [15]–[18].

It is clear that each $\mathcal{Q}_K$ contains all constant functions. If $\mathcal{Q}_K$ consists of just the constant functions, we say that it is *trivial*. It follows from the general theory of Möbius invariant function spaces (see [2] for example) that

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$\mathcal{Q}_K$ is nontrivial if and only if it contains the coordinate function z, and in this case, $\mathcal{Q}_K$ contains all polynomials.

From a change of variables we see that the coordinate function z belongs to $\mathcal{Q}_K$ if and only if

$$\sup_{w\in\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\,K\left(\log\frac{1}{|z|}\right)dA(z)<\infty.$$

Simplifying the above integral in polar coordinates, we conclude that $\mathcal{Q}_K$ is nontrivial if and only if

$$\sup_{t\in(0,1)}\int_0^1\frac{(1-t)^2}{(1-tr^2)^3}\,K\left(\log\frac{1}{r}\right)r\,dr<\infty. \tag{1}$$

Throughout the paper we always assume that condition (1) above is satisfied, so that the space $\mathcal{Q}_K$ we study is nontrivial. Another standing assumption we make for the rest of the paper is that the weight function K is nondecreasing.

An important tool in the study of $\mathcal{Q}_K$ spaces is the auxiliary function φ_K defined by

$$\varphi_K(s)=\sup_{0<t\le 1}\frac{K(st)}{K(t)},\qquad 0<s<\infty.$$

The following condition has played a crucial role in the study of $\mathcal{Q}_K$ spaces during the last few years:

$$\int_1^\infty\varphi_K(s)\,\frac{ds}{s^2}<\infty. \tag{2}$$

See [9], [17], [18] for example. This condition will be crucial for us here as well. The main result of the paper is the following.

Main Theorem. *If K satisfies condition* (2), *then a lacunary series*

$$f(z)=\sum_{k=1}^\infty a_k z^{n_k}$$

belongs to $\mathcal{Q}_K$ if and only if

$$\sum_{k=1}^\infty n_k|a_k|^2\,K\left(\frac{1}{n_k}\right)<\infty.$$

Recall that a function

$$f(z)=\sum_{k=1}^\infty a_k z^{n_k}$$

is called a *lacunary series* if

$$\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1.$$

Such series are often used to construct examples of analytic functions in various function spaces.

A special case is worth mentioning. When $K(t) = t^p$, $0 \le p < \infty$, the resulting $\mathcal{Q}_K$ space is usually denoted by $\mathcal{Q}_p$. It is well known that $\mathcal{Q}_p$ coincides with BMOA if $p = 1$, and $\mathcal{Q}_p$ is the Bloch space $\mathcal{B}$ if $p > 1$. We remind the reader that $\mathcal{B}$ consists of analytic functions f in $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

The most interesting case is when $0 < p < 1$; such $\mathcal{Q}_p$ spaces are distinct Möbius invariant Banach spaces that are strictly contained in BMOA. See [19] for the relatively new theory of $\mathcal{Q}_p$ spaces.

It is well known that a lacunary series belongs to BMOA if and only if it is in the Hardy space H^2; see [5] for example. It is also well known that a lacunary series is in the Bloch space if and only if its Taylor coefficients are bounded; see [20] for example. Lacunary series in $\mathcal{Q}_p$ are characterized in [4]. More specifically, if $0 \le p \le 1$, then a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

is in $\mathcal{Q}_p$ if and only if

$$\sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty.$$

Since the function $K(t) = t^p$ satisfies condition (2) if and only if $p < 1$, our main result covers $\mathcal{Q}_p$ spaces for $0 \le p < 1$, but it misses the classical case of BMOA (corresponding to $p = 1$). Nevertheless, it should be clear from these remarks that condition (2) is very sharp.

2. Preliminaries on weight functions. The function theory of $\mathcal{Q}_K$ obviously depends on the properties of K. Given two weight functions K_1 and K_2, we write $K_1 \lesssim K_2$ if there exists a constant $C > 0$, independent of t, such that $K_1(t) \le CK_2(t)$ for all t. The notation $K_1 \gtrsim K_2$ is used in a similar fashion. When $K_1 \lesssim K_2 \lesssim K_1$, we write $K_1 \approx K_2$.

It is clear that $K_1 \lesssim K_2$ implies $\mathcal{Q}_{K_2} \subset \mathcal{Q}_{K_1}$. In particular, K_1 and K_2 give rise to the same $\mathcal{Q}_K$ space whenever $K_1 \approx K_2$. The converse is false in general, as is demonstrated by the fact that $\mathcal{Q}_p$ equals the Bloch space for all $p > 1$.

In this section we collect several results about the weight functions that are needed for subsequent sections and are of some independent interest.

Although a few of the results in this section are buried in [8] and [9], we include proofs here for the sake of completeness and ease of reference.

LEMMA 1. *If*
$$K_1(t) = \begin{cases} K(t), & 0 < t \le 1, \\ K(1), & 1 \le t < \infty, \end{cases}$$
then $\mathcal{Q}_K = \mathcal{Q}_{K_1}$.

Proof. Since K is nondecreasing, we have $K_1 \le K$, so $\mathcal{Q}_K \subset \mathcal{Q}_{K_1}$. In particular, both spaces are nontrivial Möbius invariant spaces.

Since $K(\log(1/|z|))$ is a radial function, integration in polar coordinates shows that $f \mapsto f'(0)$ is a bounded linear functional on any nontrivial $\mathcal{Q}_K$ space. By [12], each such space $\mathcal{Q}_K$ is contained in the Bloch space.

Fix a function $f \in \mathcal{Q}_{K_1}$ and consider the integrals
$$I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, dA(z).$$
We must show that $I(a)$ is bounded for $a \in \mathbb{D}$. To this end, we write $I(a) = I_1(a) + I_2(a)$, where
$$I_1(a) = \int_{|\varphi_a(z)|>e^{-1}} |f'(z)|^2 K(g(z,a))\, dA(z),$$
$$I_2(a) = \int_{|\varphi_a(z)|\le e^{-1}} |f'(z)|^2 K(g(z,a))\, dA(z).$$

It is clear that
$$I_1(a) \le \int_{\mathbb{D}} |f'(z)|^2 K_1(g(z,a))\, dA(z),$$
so there exists a positive constant C_1 such that $I_1(a) \le C_1$ for all $a \in \mathbb{D}$.

By a change of variables, we have
$$\begin{aligned} I_2(a) &= \int_{|\varphi_a(z)|\le e^{-1}} |f'(z)|^2 K\left(\log \frac{1}{|\varphi_a(z)|}\right) dA(z) \\ &= \int_{|z|\le e^{-1}} |f'(\varphi_a(z))|^2 K\left(\log\frac{1}{|z|}\right) \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}\, dA(z) \\ &= \int_{|z|\le e^{-1}} \frac{|f'(\varphi_a(z))|^2(1-|\varphi_a(z)|^2)^2}{(1-|z|^2)^2} K\left(\log\frac{1}{|z|}\right) dA(z). \end{aligned}$$
Since f is in the Bloch space, we can find a constant $C_2 > 0$ such that
$$I_2(a) \le C_2 \int_{|z|\le e^{-1}} K\left(\log\frac{1}{|z|}\right) dA(z) \le C_2 \int_{\mathbb{D}} K\left(\log\frac{1}{|z|}\right) dA(z).$$

By condition (1), the last integral above is convergent, so there exists a constant $C_3 > 0$ such that $I_2(a) \le C_3$ for all $a \in \mathbb{D}$. This shows that $I(a)$ is bounded in a, or equivalently, f belongs to $\mathcal{Q}_K$. ∎

The significance of Lemma 1 is that the space $\mathcal{Q}_K$ only depends on the behavior of $K(t)$ for t close to 0. In particular, when studying $\mathcal{Q}_K$ spaces, we can always assume that $K(t) = K(1)$ for $t \ge 1$. However, we do not make this assumption in our main theorems.

LEMMA 2. *If K satisfies condition* (2), *then the function*

$$K^*(t) = t \int_t^\infty K(s)\,\frac{ds}{s^2}, \qquad 0 < t < \infty,$$

has the following properties:

(i) *K^* is nondecreasing on $(0, \infty)$.*
(ii) *$K^*(t)/t$ is nonincreasing on $(0, \infty)$.*
(iii) *$K^*(t) \ge K(t)$ for all $t \in (0, \infty)$.*
(iv) *$K^* \lesssim K$ on $(0, 1]$.*

If $K(t) = K(1)$ for $t \ge 1$, then we also have

(v) *$K^*(t) = K^*(1) = K(1)$ for $t \ge 1$, so $K^* \approx K$ on $(0, \infty)$.*

Proof. If $t \in (0, 1]$, then a change of variables gives

$$\begin{aligned} K^*(t) &= t \int_t^\infty K(s)\,\frac{ds}{s^2} = \int_1^\infty K(ts)\,\frac{ds}{s^2} = K(t) \int_1^\infty \frac{K(ts)}{K(t)}\,\frac{ds}{s^2} \\ &\le K(t) \int_1^\infty \varphi_K(s)\,\frac{ds}{s^2}. \end{aligned}$$

So condition (2) implies that $K^*(t) \lesssim K(t)$ for $t \in (0, 1]$. This yields property (iv) and shows that $K^*(t)$ is well defined for all $t > 0$.

Since

$$\frac{K^*(t)}{t} = \int_t^\infty K(s)\,\frac{ds}{s^2}$$

and K is nonnegative, we see that the function $K^*(t)/t$ is decreasing. This proves (ii). Property (v) follows from a direct calculation.

Using the assumption that K is nondecreasing again, we obtain

$$K^*(t) = t \int_t^\infty K(s)\,\frac{ds}{s^2} \ge tK(t) \int_t^\infty \frac{ds}{s^2} = K(t)$$

for all $0 < t < \infty$. This proves property (iii).

It remains for us to show that K^* is nondecreasing. To this end, we fix $0 < t < T < \infty$ and consider the difference

$$D = K^*(T) - K^*(t) = T \int_T^\infty \frac{K(s)\,ds}{s^2} - t \int_t^\infty \frac{K(s)\,ds}{s^2}$$
$$= (T-t) \int_T^\infty \frac{K(s)\,ds}{s^2} - t \int_t^T \frac{K(s)\,ds}{s^2}.$$

Since K is nondecreasing and nonnegative, we have

$$D \ge (T-t)K(T) \int_T^\infty \frac{ds}{s^2} - tK(T) \int_t^T \frac{ds}{s^2} = 0.$$

This proves property (i) and completes the proof of the lemma. ∎

Note that condition (2) is critically needed only in the proof of (iv). Without condition (2), properties (i), (ii), and (iii) remain valid, provided that K^* is allowed to be identically infinite.

Corollary 3. *If K satisfies condition* (2), *then there exists a constant $C > 0$ such that $K(2t) \le CK(t)$ for all $0 \le 2t \le 1$.*

Proof. For any $t > 0$, we have

$$\frac{K^*(2t)}{K^*(t)} = 2\,\frac{\int_{2t}^\infty \frac{K(s)\,ds}{s^2}}{\int_t^\infty \frac{K(s)\,ds}{s^2}} \le 2.$$

The desired estimate now follows from parts (iii) and (iv) of Lemma 2. ∎

If we started out with a weight function K with the property that $K(t) = K(1)$ for $t \ge 1$, then the conclusion of Corollary 3 could be strengthened to $K(2t) \approx K(t)$ for $t > 0$.

Proposition 4. *If K satisfies condition* (2), *then we can find another nonnegative weight function K^* such that $\mathcal{Q}_K = \mathcal{Q}_{K^*}$ and that the new weight function K^* has the following properties*:

(a) *K^* is nondecreasing on $(0, \infty)$.*
(b) *K^* satisfies condition* (1).
(c) *K^* satisfies condition* (2).
(d) *$K^*(2t) \approx K^*(t)$ on $(0, \infty)$.*
(e) *K^* is differentiable* (*up to any given order*) *on $(0, \infty)$.*
(f) *K^* is concave on $(0, \infty)$.*
(g) *$K^*(t) = K^*(1)$ for $t \ge 1$.*

(h) $K^*(t)/t$ *is nonincreasing on* $(0,\infty)$.
(i) $K^*(t) \approx K(t)$ *on* $(0,1]$.

Proof. By Lemma 1, we may assume that $K(t) = K(1)$ for all $t \ge 1$. Under this assumption, the function K^* from Lemma 2 then satisfies $K^* \approx K$ on $(0,\infty)$. Moreover, properties (a), (b), (c), (g), (h), and (i) all hold.

Property (d) follows from the proof of Corollary 3.

If we repeat the construction $K \mapsto K^*$, then we can make the new weight function differentiable up to any desired order. So property (e) holds.

If the function K is differentiable, which we may assume by property (e), then

$$\frac{d}{dt}K^*(t) = \int_t^\infty \frac{K(s)\,ds}{s^2} - \frac{K(t)}{t} \quad \text{and} \quad \frac{d^2}{dt^2}K^*(t) = -\frac{K'(t)}{t} \le 0.$$

This shows that K^* is concave on $(0,\infty)$ and completes the proof of the proposition. ∎

Theorem 5. *If K satisfies condition* (2), *then for any $\alpha > 0$ and $0 \le \beta < 1$ we have*

$$\int_0^1 r^{\alpha-1}\left(\log\frac{1}{r}\right)^{-\beta} K\left(\log\frac{1}{r}\right) dr \approx C(\beta)\left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K\left(\frac{1-\beta}{\alpha}\right),$$

where $C(\beta)$ is a constant depending on β alone.

Proof. Let

$$I = \int_0^1 r^{\alpha-1}\left(\log\frac{1}{r}\right)^{-\beta} K\left(\log\frac{1}{r}\right) dr.$$

By a change of variables,

$$I = \int_0^\infty e^{-\alpha t} t^{-\beta} K(t)\,dt.$$

We write $I = I_1 + I_2$, where

$$I_1 = \int_0^{(1-\beta)/\alpha} e^{-\alpha t} t^{-\beta} K(t)\,dt, \qquad I_2 = \int_{(1-\beta)/\alpha}^\infty e^{-\alpha t} t^{-\beta} K(t)\,dt.$$

Since K is nondecreasing, we have

$$I_1 \le K\left(\frac{1-\beta}{\alpha}\right) \int_0^{(1-\beta)/\alpha} e^{-\alpha t} t^{-\beta}\,dt.$$

Making the change of variables $t = (1-\beta)s/\alpha$, we obtain

$$I_1 \le \left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K\left(\frac{1-\beta}{\alpha}\right) \int_0^1 e^{-(1-\beta)s} s^{-\beta}\, ds$$

$$= C(\beta)\left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K\left(\frac{1-\beta}{\alpha}\right).$$

By part (iii) of Lemma 2, we have

$$I_2 \le \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t} t^{1-\beta}\, \frac{K^*(t)}{t}\, dt.$$

According to part (ii) of Lemma 2, the function $K^*(t)/t$ is decreasing on $(0,\infty)$, so

$$I_2 \le \frac{K^*((1-\beta)/\alpha)}{(1-\beta)/\alpha} \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t} t^{1-\beta}\, dt.$$

A change of variables ($t = (1-\beta)s/\alpha$) in the integral above leads to

$$I_2 \le \left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K^*\left(\frac{1-\beta}{\alpha}\right) \int_1^{\infty} e^{-(1-\beta)s} s^{1-\beta}\, ds.$$

This together with part (iv) of Lemma 2 shows that

$$I_2 \lesssim C(\beta)\left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K\left(\frac{1-\beta}{\alpha}\right).$$

Combining this with what was proved in the previous paragraph, we have

$$I \lesssim C(\beta)\left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K\left(\frac{1-\beta}{\alpha}\right).$$

On the other hand,

$$I \ge \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t} t^{-\beta} K(t)\, dt.$$

The assumption that K is nondecreasing gives

$$I \ge K\left(\frac{1-\beta}{\alpha}\right) \int_{(1-\beta)/\alpha}^{\infty} e^{-\alpha t} t^{-\beta}\, dt.$$

Make a change of variables according to $t = (1-\beta)s/\alpha$. Then

$$I \ge C(\beta)\left(\frac{1-\beta}{\alpha}\right)^{1-\beta} K\left(\frac{1-\beta}{\alpha}\right).$$

This completes the proof of the theorem. ■

3. Lacunary series in $\mathcal{Q}_K$. We begin with an estimate of the weighted Dirichlet integral in terms of Taylor coefficients.

THEOREM 6. *If K satisfies condition* (2) *and*

$$f(z)=\sum_{n=0}^{\infty} a_n z^n,$$

then

$$\int_{\mathbb{D}} |f'(z)|^2 K\left(\log\frac{1}{|z|}\right) dA(z) \approx \sum_{n=1}^{\infty} n\,|a_n|^2 K\left(\frac{1}{n}\right).$$

Proof. Write

$$I(f)=\int_{\mathbb{D}} |f'(z)|^2 K\left(\log\frac{1}{|z|}\right) dA(z).$$

Integrating in polar coordinates leads to

$$I(f)=2\sum_{n=1}^{\infty} n^2|a_n|^2\int_0^1 r^{2n-1}K\left(\log\frac{1}{r}\right) dr.$$

We apply Theorem 5 with $\beta=0$ and $\alpha=2n$ to obtain

$$I(f)\approx\sum_{n=1}^{\infty} n|a_n|^2\,K\left(\frac{1}{2n}\right).$$

The desired result then follows from Corollary 3. ∎

We are now ready to prove the main result of the paper.

THEOREM 7. *If K satisfies condition* (2), *then a lacunary series*

$$f(z)=\sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $\mathcal{Q}_K$ if and only if

$$\sum_{k=1}^{\infty} n_k|a_k|^2\,K\left(\frac{1}{n_k}\right)<\infty. \tag{3}$$

Proof. First assume that

$$f(z)=\sum_{k=1}^{\infty} a_k z^{n_k}$$

is a lacunary series in $\mathcal{Q}_K$. Then

$$\int_{\mathbb{D}} |f'(z)|^2 K\left(\log\frac{1}{|z|}\right) dA(z)=\int_{\mathbb{D}} |f'(z)|^2 K(g(z,0))\,dA(z)<\infty,$$

which, according to Theorem 6, implies condition (3).

Next assume that condition (3) holds. We proceed to estimate the integral

$$I(a)=\int_{\mathbb{D}}|f'(z)|^2K(g(z,a))\,dA(z),\qquad a\in\mathbb{D}.$$

As the first step, we show that for any $a\in\mathbb{D}$,

$$I(a)\le 2\int_0^1 r\Big[\sum_{k=1}^{\infty}n_k|a_k|r^{n_k-1}\Big]^2K\Big(\log\frac{1}{r}\Big)\,dr. \tag{4}$$

To this end, we write $z=re^{i\theta}$ in polar form and observe that

$$|f'(z)|\le\sum_{k=1}^{\infty}n_k|a_k|r^{n_k-1}.$$

It follows that

$$I(a)\le 2\int_0^1\Big[\sum_{k=1}^{\infty}n_k|a_k|r^{n_k-1}\Big]^2r\,dr\,\frac{1}{2\pi}\int_0^{2\pi}K(g(re^{i\theta},a))\,d\theta.$$

By Proposition 4, we may as well assume that K is concave. Then

$$\frac{1}{2\pi}\int_0^{2\pi}K(g(re^{i\theta},a))\,d\theta\le K\bigg(\frac{1}{2\pi}\int_0^{2\pi}g(re^{i\theta},a)\,d\theta\bigg).$$

By Jensen's formula, the integral

$$\frac{1}{2\pi}\int_0^{2\pi}g(re^{i\theta},a)\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log\bigg|\frac{1-\bar{a}re^{i\theta}}{re^{i\theta}-a}\bigg|\,d\theta$$

is equal to $\log(1/|a|)$ for $0<r\le|a|$ and $\log(1/r)$ for $|a|<r<1$. In particular,

$$\frac{1}{2\pi}\int_0^{2\pi}g(re^{i\theta},a)\,d\theta\le\log\frac{1}{r}.$$

From this we deduce inequality (4).

Our second step is to prove that inequality (4) implies

$$I(a)\lesssim\sum_{n=0}^{\infty}\Big[\sum_{n_k\in I_n}n_k|a_k|\Big]^2\frac{1}{2^n}K\Big(\frac{1}{2^n}\Big), \tag{5}$$

where

$$I_n=\{k:2^n\le k<2^{n+1},\ k\in\mathbb{N}\}.$$

To this end, we combine the elementary estimates

$$\begin{aligned}\sum_{n=0}^{\infty}2^{n/2}r^{2^n}&\le\sqrt{2}\sum_{n=0}^{\infty}\int_{2^n}^{2^{n+1}}t^{-1/2}r^{t/2}\,dt\\&\le\sqrt{2}\int_0^{\infty}t^{-1/2}r^{t/2}\,dt=2\,\Gamma\Big(\frac{1}{2}\Big)\Big(\log\frac{1}{r}\Big)^{-1/2}\end{aligned}$$

with the Cauchy–Schwarz inequality to produce

$$\begin{aligned}\Big[\sum_{k=1}^{\infty} n_k|a_k|r^{n_k}\Big]^2 &= \Big[\sum_{n=0}^{\infty}\sum_{n_k\in I_n} n_k|a_k|r^{n_k}\Big]^2 \le \Big[\sum_{n=0}^{\infty}\sum_{n_k\in I_n} n_k|a_k|r^{2^n}\Big]^2\\ &\le \Big[\sum_{n=0}^{\infty} 2^{n/2}r^{2^n}\Big]\Big[\sum_{n=0}^{\infty} 2^{-n/2}r^{2^n}\Big(\sum_{n_k\in I_n} n_k|a_k|\Big)^2\Big]\\ &\le \frac{2\,\Gamma(1/2)}{(\log(1/r))^{1/2}}\sum_{n=0}^{\infty} 2^{-n/2}r^{2^n}\Big[\sum_{n_k\in I_n} n_k|a_k|\Big]^2.\end{aligned}$$

This together with (4) and Theorem 5 and Corollary 3 gives

$$\begin{aligned}I(a) &\le 2\int_0^1 r^{-1}\Big[\sum_{k=1}^{\infty} n_k|a_k|r^{n_k}\Big]^2 K\Big(\log\frac{1}{r}\Big)\,dr\\ &\lesssim \sum_{n=0}^{\infty} 2^{-n/2}\Big[\sum_{n_k\in I_n} n_k|a_k|\Big]^2\int_0^1 r^{2^n-1}\Big(\log\frac{1}{r}\Big)^{-1/2}K\Big(\log\frac{1}{r}\Big)\,dr\\ &\lesssim \sum_{n=0}^{\infty}\Big[\sum_{n_k\in I_n} n_k|a_k|\Big]^2\frac{1}{2^n}\,K\Big(\frac{1}{2^n}\Big).\end{aligned}$$

Thus, inequality (5) holds.

If $n_k\in I_n$, then $n_k<2^{n+1}$. It follows from the monotonicity of K and Corollary 3 that

$$\frac{1}{n_k}\,K\Big(\frac{1}{n_k}\Big)\ge\frac{1}{2^{n+1}}\,K\Big(\frac{1}{2^{n+1}}\Big)\gtrsim\frac{1}{2^n}\,K\Big(\frac{1}{2^n}\Big).$$

Combining this with (5), we obtain

$$I(a)\lesssim\sum_{n=0}^{\infty}\Big[\sum_{n_k\in I_n} n_k|a_k|\sqrt{\frac{1}{n_k}\,K\Big(\frac{1}{n_k}\Big)}\,\Big]^2. \tag{6}$$

Note that everything so far in the proof works for an arbitrary analytic function, not just for a lacunary series. Our final step, though, does make use of the fact that f is a lacunary series. More specifically, if

$$\frac{n_{k+1}}{n_k}\ge\lambda>1$$

for all k, then the Taylor series of $f(z)$ has at most $[\log_\lambda 2]+1$ terms $a_kz^{n_k}$ such that $n_k\in I_n$ for $n\in\mathbb{N}$. By (6) and Hölder's inequality,

$$\begin{aligned}I(a)&\lesssim([\log_\lambda 2]+1)\sum_{n=0}^{\infty}\sum_{n_k\in I_n} n_k|a_k|^2\,K\Big(\frac{1}{n_k}\Big)\\ &=([\log_\lambda 2]+1)\sum_{k=1}^{\infty} n_k|a_k|^2K\Big(\frac{1}{n_k}\Big).\end{aligned}$$

This shows that condition (3) implies $f \in \mathcal{Q}_K$. The proof of the theorem is now complete. ∎

4. Lacunary series in $\mathcal{Q}_{K,0}$. Let $\mathcal{Q}_{K,0}$ denote the subspace of $\mathcal{Q}_K$ consisting of functions f with

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, dA(z) = 0.$$

The following result together with Theorem 7 characterizes lacunary series in $\mathcal{Q}_{K,0}$.

THEOREM 8. *Let*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

be a lacunary series. If K satisfies condition (2), *then $f \in \mathcal{Q}_K$ if and only if $f \in \mathcal{Q}_{K,0}$.*

Proof. Suppose the lacunary series f belongs to $\mathcal{Q}_K$. We must show that $I(a) \to 0$ as $|a| \to 1^-$, where

$$I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, dA(z), \qquad a \in \mathbb{D}.$$

From the proof of Theorem 7, we know that $f \in \mathcal{Q}_K$ implies

$$\int_0^1 r\Big[\sum_{k=1}^{\infty} n_k |a_k| r^{n_k - 1}\Big]^2 K\bigg(\log\frac{1}{r}\bigg)\, dr < \infty.$$

Thus for any given $\varepsilon > 0$ there exists some $\sigma \in (0,1)$ such that

$$2\int_{\sigma}^1 r\Big[\sum_{k=1}^{\infty} n_k |a_k| r^{n_k - 1}\Big]^2 K\bigg(\log\frac{1}{r}\bigg)\, dr < \varepsilon.$$

We may assume that

$$\lim_{|a|\to 1^-} K\bigg(\log\frac{1}{|a|}\bigg) = 0.$$

Otherwise, $\mathcal{Q}_K$ coincides with the Dirichlet space $\mathcal{D}$ (see [8]), and the desired result is obvious.

We write $I(a) = I_1(a) + I_2(a)$, where

$$I_1(a) = \int_{|z|<\sigma} |f'(z)|^2 K(g(z,a))\, dA(z),$$

$$I_2(a) = \int_{\sigma \le |z| < 1} |f'(z)|^2 K(g(z,a))\, dA(z).$$

By arguments used in the second paragraph of the proof of Theorem 7, we have

$$I_1(a) \le 2K\left(\log\frac{1}{|a|}\right)\int_0^\sigma \Big[\sum_{k=1}^\infty n_k|a_k|r^{n_k-1}\Big]^2 r\,dr$$

whenever $\sigma < |a| < 1$, because in this case

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta}, a)\,d\theta = \log\frac{1}{|a|}.$$

In particular, $I_1(a) \to 0$ as $|a| \to 1^-$. Similarly, we have

$$I_2(a) \le 2\int_\sigma^1 \Big[\sum_{k=1}^\infty n_k|a_k|r^{n_k-1}\Big]^2 K\left(\log\frac{1}{r}\right) r\,dr < \varepsilon.$$

It follows that

$$\limsup_{|a|\to 1^-} I(a) \le \varepsilon.$$

Since ε is arbitrary, we conclude that $I(a) \to 0$ as $|a| \to 1^-$. So $f \in \mathcal{Q}_{K,0}$ and the proof is complete. ∎

Carefully checking the proof of Theorems 7 and 8, we also obtain the following sufficient condition for a function to be in $\mathcal{Q}_{K,0}$ (and hence in $\mathcal{Q}_K$) in terms of Taylor coefficients.

THEOREM 9. *If K satisfies condition* (2), *and if*

$$f(z) = \sum_{n=0}^\infty a_n z^n$$

satisfies the condition

$$\sum_{n=0}^\infty \Big[\sum_{k\in I_n} k|a_k|\Big]^2 \frac{1}{2^n} K\left(\frac{1}{2^n}\right) < \infty,$$

then $f \in \mathcal{Q}_{K,0}$.

Proof. We leave the details to the interested reader. ∎

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