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SENSITIVITY OF OPTIMUM SOLUTIONS TO PROBLEM PARAMETERS

JAROSLAW SOBIESZCZANSKI-SOBIESKI,
JEAN-FRANCOIS BARTHELEMY, AND
KATHLEEN M. RILEY

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National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23665

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Jaroslaw Sobieszczanski-Sobieski*
NASA Langley Research Center
Hampton, Virginia

Jean-Francois Barthelemy**
Virginia Polytechnic Institute and State University
Blacksburg, Virginia

and

Kathleen M. Riley†
Kentron International, Inc.
Hampton, Virginia

Abstract

In the optimum sensitivity problem, one seeks to determine the values of derivatives of the optimal objective function and design variables with respect to those physical quantities which were kept constant as problem parameters during optimization. Examples of these sensitivity derivatives might include derivatives of cross-sectional area and structural mass with respect to allowable stress and derivatives of fuel consumed and wing aspect ratio with respect to aircraft range. Derivation of the sensitivity equations that yield the sensitivity derivatives directly, which avoids the costly and inaccurate "perturb-and-reoptimize" approach, is discussed and solvability of the equations is examined. The equations apply to optimum solutions obtained by direct search methods as well as those generated by procedures of the sequential unconstrained minimization technique (SUMT) class. Applications are discussed for the use of the sensitivity derivatives in extrapolation of the optimal objective function and design variable values for incremented parameters, optimization with multiple objectives, and decomposition of large optimization problems. Several aspects of these applications and verification of the sensitivity equation are presented through numerical examples.

Nomenclature

a	weighting factor; allowable value when in subscript
A	cross-section area
b	behavior variable
C	a convergence tolerance governing termination of optimization
f	function in general mathematical sense
F	objective function
g	vector of m constraints
g_j	j-th constraint, assumed to be active unless noted otherwise; constraint is violated when $g_j > 0$
m	number of constraints active at constrained minimum
n	number of design variables in optimization problem
P	penalty term in penalty function ϕ, and force
p_k	k-th parameter; in differentiation with respect to p_k, k is omitted
r	draw-down factor in SUMT
W	weight
X, x_i	respectively: vector of design variables, and i-th design variable
ε	constraint violation tolerance (typically a small number, e.g., 0.025)
λ	vector of m Lagrange multipliers
λ_j	j-th multiplier corresponding to j-th active constraint
ϕ	penalty function

Subscripts and Superscripts:

a	allowable value
e	extrapolated value
h	number of objectives in a multiobjective optimization
i	i-th design variable, i-th weighting factor
j	j-th constraint, j-th weighting factor
k	k-th parameter
l	lower bound
o	initial value in extrapolation
p	prescribed quantity
q	refers to q-th design variable
u	upper bound

Overbar denotes quantities at optimum.

Differential notation:

$$\dot{f}^{(i)} \equiv \partial f/\partial x_i; \quad \ddot{f}^{(i,q)} \equiv \partial^2 f/\partial x_i\, \partial x_q$$

$f' \equiv \partial f/\partial p$ assuming constant x_i-values

$$\dot{f}'^{(i)} \equiv \partial^2 f/\partial x_i\, \partial p$$

in this notation:

$$\dot{g} \equiv \left[\dot{g}_1, \dot{g}_2, \ldots \dot{g}_j \ldots \dot{g}_m\right], \quad n \times m \text{ matrix}$$

$$\dot{g}_j \equiv \left\{\dot{g}_j^{(1)}, \dot{g}_j^{(2)}, \ldots \dot{g}_j^{(i)} \ldots \dot{g}_j^{(n)}\right\}, \text{ column-vector of length } n$$

*Head, Multidisciplinary Analysis and Optimization Branch, Loads and Aeroelasticity Division
**Graduate Student, Department of Aerospace and Ocean Engineering
†Scientific Programmer, Hampton Technical Center

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$$\dot{g}^{(i)} \equiv \left\{\dot{g}_1^{(i)}, \dot{g}_2^{(i)}, \ldots, \dot{g}_j^{(i)}, \ldots \dot{g}_m^{(i)}\right\}$$ row-vector of length m

$$\ddot{g}^{(i,q)} \equiv \left\{\ddot{g}_1^{(i,q)}, \ddot{g}_2^{(i,q)}, \ldots \ddot{g}_m^{(i,q)}\right\},$$ row vector of length m

Introduction

Nonlinear mathematical programing has become well established as a tool for defining optimal engineering designs as local constrained minima. A typical constrained minimization problem entails a group of physical quantities which are used as design variables and a group of constant quantities termed parameters of the problem. It is of obvious interest to know, when the optimization is completed, the sensitivity of the constrained minimum to the parameters of the problem. Mathematically, this requires the determination of the partial derivatives of the objective function and design variables with respect to the parameters of interest. These derivatives are referred to as sensitivity derivatives. For example, in structural optimization, it would be useful to determine the effect on optimal structural mass and cross-sectional dimensions of changes in allowable stress or displacements. In an aircraft configuration optimization the information of interest would be the sensitivity of optimal block fuel consumption and wing aspect ratio and area, to variations of required range and payload.

Generation of sensitivity derivatives by finite difference approximations requires reoptimization of the problem with incremented values of the parameters. This is a costly procedure burdened with the difficulty of assessing numerical errors. A preferable approach is to obtain the sensitivity derivatives directly from an appropriate set of equations. This approach, known as optimum sensitivity analysis, became well-established as a routine tool in linear programing.[1] In contrast, sensitivity analysis in nonlinear mathematical programing is still at an early stage of development, and its incorporation into engineering practice is as yet to be accomplished. The relatively limited literature available on the subject, primarily in the discipline of operations research, is well represented by Refs. 2, 3, 4, and 5. Specifically, Ref. 2 gives a solution to the problem of finding the increments of the optimum objective function and variables caused by simultaneous small changes of the problem parameters, and contains references to other relevant early works. Reference 3 develops a solution for sensitivity in the context of a sequential unconstrained minimization technique (SUMT) and a particular form of a penalty function. Some computational experience with that solution including an application in resource management is reported in Ref. 4. The authors are aware of no reported research on optimum sensitivity analysis applied to optimization of structures and other engineering systems with the exception of a simple torsion bar example in Ref. 5.

The objectives of this paper are to show how the equations capable of yielding the sensitivity derivatives (the sensitivity equations) can be obtained for a constrained optimum regardless of the type of optimization algorithm that was used to arrive at the optimum point; to review the solvability of the sensitivity equations; and to report on applications in structural optimization. Lagrange multiplier equations and the extremum conditions of a penalty function in general, and the interior and exterior forms of that function in particular, are included as alternative bases for derivation of the sensitivity equations. Numerical examples verify the algorithms and illustrate evaluation of the sensitivities of optimal structures to parameters such as load, allowable stress and overall geometrical shape and the influence of individual conflicting objectives (e.g., weight and cost) in multiobjective optimization. An additional application of the sensitivity analysis, to provide a basis for a formal decomposition in hierarchical optimization problems, is also discussed.

Sensitivity Equations

There are at least two ways of deriving equations for the unknown sensitivities. One way is to start from the Lagrange multiplier equations of the constrained minimum; the other way begins with the extremum conditions of a penalty function. In either case the same general functional relationships are recognized. Namely, the objective function:

$$F(x_1, x_2, \ldots x_n, p_1, p_2, \ldots p_k) \tag{1a}$$

and constraints active at the optimum point

$$g_j(x_1, x_2, \ldots x_n, p_1, p_2, \ldots p_k) \tag{1b}$$

and implicitly

$$x_i(p_1, p_2, \ldots p_k) \tag{1c}$$

There is no need to distinguish among the equality and inequality constraints in the original formulation of the optimization problem being analyzed for sensitivity, because all constraints active at the constrained optimum point may be regarded as equality constraints.

Sensitivity Equations Derived from the Lagrange Multiplier Equations

The familiar Lagrange multiplier equations satisfied at a constrained minimum are:

$$\dot{F}^{(i)} + \dot{g}^{(i)}\lambda = 0, \; i = 1 \rightarrow n \tag{2a}$$

$$g_j = 0, \; j = 1 \rightarrow m \tag{2b}$$

The equations may be differentiated with respect to the parameter p_k using the chain-differentiation rule for composite functions along with the functional relationships in Eq. (1). The result of the differentiation is:

$$\dot{F}'^{(i)} + \sum_{q=1}^{n} \ddot{F}^{(i,q)} x'_q$$

$$+ \left(\dot{g}'^{(i)} + \left(\sum_{q=1}^{n} \ddot{g}^{(i,q)} x'_q \right) \right) \lambda$$

$$+ \dot{g}^{(i)} \lambda' = 0, \; i = 1 \to n \tag{3a}$$

$$g'_j + \sum_{i=1}^{n} \dot{g}_j^{(i)} x'_i = 0, \quad j = 1 \to m \tag{3b}$$

Equation (3) can be converted to a uniform matrix notation by collecting terms and using an auxiliary matrix Z:

$$\underset{(n+m)\times(n+m)}{\begin{bmatrix} \underset{n\times n}{[\ddot{F} + Z]} & \underset{n\times m}{[\dot{g}]} \\ \underset{m\times n}{[\dot{g}]^T} & \underset{m\times m}{[0]} \end{bmatrix}} \underset{(n+m)\times 1}{\begin{Bmatrix} \bar{X}' \\ \lambda' \end{Bmatrix}} + \underset{(n+m)\times 1}{\begin{Bmatrix} \underset{n\times 1}{\{\dot{F}'\}} + \underset{n\times m}{[\dot{g}']}\underset{m\times 1}{\{\lambda\}} \\ \underset{m\times 1}{\{g'\}} \end{Bmatrix}} = 0 \tag{4a}$$

where the dimensions of vectors and matrices are inscribed for clarity, and where Z is defined as a square, $n \times n$, matrix whose i,q-element is

$$Z^{(i,q)} = \sum_{j=1}^{m} \frac{\partial^2 g_j}{\partial x_i \, \partial x_q} \lambda_j \tag{4b}$$

Equation (4), whose terms are evaluated at the constrained minimum point, constitute a set of $n + m$ simultaneous linear algebraical equations for unknown derivatives of the optimum solution $\bar{X}'$ (n elements) and λ' (m elements), the latter being auxiliary quantities.

Once the sensitivity derivatives $\bar{x}'_i$ are obtained, the sensitivity derivative of the objective function is determined as total derivative of the composite function F:

$$d\bar{F}/dp = \bar{F}' + \sum_i \dot{\bar{F}}^{(i)} \bar{x}'_i \tag{5}$$

Solution of the Sensitivity Equations

The ways to obtain a solution for $\bar{X}'$ (and λ') from Eq. (4) may be categorized in a number of cases.

<u>Case 1: All Submatrices in Eq. (4) Exist.</u> This basic case corresponds to a constrained minimum defined by a nonlinear objective function and nonlinear constraint functions. The Lagrange multipliers are available as a by-product of the optimization solution. Derivative vectors $\dot{g}_j$ must be linearly independent in order for the matrix of coefficients in Eq. (4) to be nonsingular. A Gram matrix test,[6] in addition to physical insight, may be used to identify redundant constraints to be eliminated. No ill conditioning difficulties were observed when solving Eq. (4) for test problems using a standard Gaussian elimination solution algorithm. A particular case of $m = n$ causes $\bar{X}'$ in Eq. (4) to decouple from λ' (see discussion of Case 4).

<u>Case 2: Multipliers λ_i Not Available From the Optimum Solution.</u> The Lagrange multipliers λ_i must be known in order to construct the matrix of coefficients (they enter matrix Z and the vector of free terms in Eq. (4)). If the optimum being analyzed for sensitivity is obtained by a method that does not yield the λ-values as part of the solution, these values have to be computed as part of the input to Eq. (4) (for an exception see Eq. (8) and the discussion of Case 4). One well-known relationship[7] that may be used for computing the λ values is

$$\lambda = -\left[\dot{g}^T \dot{g}\right]^{-1} \dot{g}^T \dot{F} \tag{6}$$

where the matrix $\dot{g}$ contains the constraints active at the optimum. It is important again that only the linearly independent constraints among the active set be included in Eq. (6) as mentioned in the discussion of Case 1.

<u>Case 3: Linearity of Constraints or Objective Function.</u> The linearity of constraints at the optimum eliminates the matrix Z, while the linearity of the objective function renders $\ddot{F} = 0$. However, if the linearity of the constraints and objective function do not occur simultaneously, the term $\ddot{F} + Z$ does not vanish and the unknowns $\bar{X}'$ and λ' remain coupled in Eq. (4).

This case also includes the constrained minima in which some of the active constraints are side constraints. For the purposes of sensitivity analysis, such constraints should be reformulated as inequality constraints

$$\begin{aligned} g_j &= 1 - x_i/x_{il} \le 0 \quad \text{or} \\ g_j &= x_i/x_{iu} - 1 \le 0 \end{aligned} \tag{7}$$

for lower and upper bounds, respectively. This formulation leads to derivatives with respect to x_i, x_{il} and x_{iu} and, also permits sensitivity analysis with respect to the side constraint parameters such as x_{il} and x_{iu}.

<u>Case 4: Linear Constraints and Linear Objective Function.</u> Under this condition $\ddot{F} = 0$ and $Z = 0$, hence the term $\ddot{F} + Z$ vanishes and the $\bar{X}'$ and λ' vectors decouple in Eq. (4) so that:

$$\underset{m\times n}{[\dot{g}^T]}\underset{n\times 1}{\{\bar{X}'\}} + \underset{m\times 1}{\{g'\}} = \{0\} \tag{8}$$

an equation that does not contain the Lagrange multipliers λ. The solvability of Eq. (8) depends on the dimensions m and n and requires consideration of the following subcases that are likely to occur in practice.

Subcase 1: The $\dot{g}^T$ is a square matrix, $m = n$, and there are no null rows and columns. Consequently, Eq. (8) is determined so that a unique solution can be obtained for $\bar{X}'$. Typically, this subcase occurs when a nonlinear mathematical programing problem is solved by a sequence of steps, each step consisting of finding a constrained minimum of the problem that is locally linearized and subjected to move limits expressed in form of Eq. (7). Such a minimum falls on a full vertex of the linearized feasible domain, hence $m = n$.

Subcase 2: The $\dot{g}^T$ is a rectangular matrix with $m < n$ rendering Eq. (8) underdetermined. Despite this, the sensitivity analysis can still be pursued by seeking m values of $\bar{x}_i'$ such that $d\bar{F}/dp$ is maximized. Mathematically, this calls for augmenting Eq. (8) with an equation for $d\bar{F}/dp$ (Eq. (5)), and solving the resulting linear programing problem that yields m values of $x' \neq 0$ and $(n - m)$ values of $x' = 0$. The corresponding minimization of $d\bar{F}/dp$ can also be carried out, thus estimating the range of values of the objective function sensitivity.

This subcase may arise in a nonlinear problem whose constrained minimum occurs not at a full vertex of the feasible domain (where it would be $m = n$), but is defined, at least in part, by tangency of the curved constaint boundary hypersurfaces to a constant objective function hypersurface (e.g., three-bar truss discussed later in this report). In many cases, the hypersurface curvatures represented by the second derivatives with respect to the design variables may not be available among the results of the optimization procedure, and the cost of computing them may be prohibitve. Therefore, they may have to be omitted (set to zero) in the input to the optimum sensitivity analysis. Omitting the unavailable second derivatives in effect replaces the hypersurfaces of the objective function and constraints with hyperplanes intersecting at the constrained minimum point but not forming a full vertex there. Consequently, that point coordinate definition in the design space is lost and Eq. (8) becomes undetermined.

Sensitivity Equations Derived from Extremum Conditions of a Penalty Function

In a broad class of methods known as Sequential Unconstrained Minimization Techniques (SUMT), the objective function is augmented by a penalty term containing the constraints and an approximation to the constrained minimum is determined asymptotically by generating a series of unconstrained minima of a penalty function.[2,7] Sensitivity equations analogous to Eq. (4) may be obtained for a SUMT-determined constrained minimum by differentiating the extremum conditions of a penalty function with respect to a parameter.

There are many formulations of the penalty functions currently in use, for example; interior, exterior[7] and a quadratic extended[9] penalty function formulations. Therefore, a penalty function in its most general form is expressed as

$$\phi = F + rP \tag{9}$$

The penalty term P is a function of the constraints:

$$P(g_j),\; j = 1 \rightarrow m \tag{10}$$

The extremum conditions for ϕ are:

$$\dot{\phi}^{(i)} = \dot{F}^{(i)} + r \sum_{j=1}^{m} \frac{\partial P}{\partial g_j} \dot{g}_j^{(i)} = 0,\; i = 1 \rightarrow n \tag{11}$$

Differentiation of Eq. (11) with respect to parameter p_k yields

$$\sum_{q=1}^{n} \left[\left(\ddot{F}^{(i,q)} + r \sum_{j=1}^{m}\left(\frac{\partial^2 P}{\partial^2 g_j} \dot{g}_j^{(i)}\dot{g}_j^{(q)} + \frac{\partial P}{\partial g_j} \ddot{g}_j^{(i,q)}\right)\right)\bar{x}_q'\right] + \dot{F}'^{(i)} + r \sum_{j=1}^{m}\left(\frac{\partial^2 P}{\partial g_j^2} g_j'\dot{g}_j^{(i)} + \frac{\partial P}{\partial g_j} \dot{g}_j'^{(i)}\right) = 0,\; i = 1 \rightarrow n \tag{12}$$

Equation (12) represents a set of n simultaneous linear equations for n unknown values $\bar{x}_q'$.

For a specific case of an interior penalty function in a frequently used form

$$P = -\sum_{j=1}^{m} 1/g_j \tag{13}$$

Eq. (11) becomes

$$\dot{\phi}^{(i)} = \dot{F}^{(i)} + r\left(\sum_{j=1}^{m} g_j^{-2}\dot{g}_j^{(i)}\right) = 0 \tag{14}$$

and Eq. (12) yields

$$\underset{n\times n}{[M]}\underset{n\times 1}{\{\bar{X}'\}} + \underset{n\times 1}{\{R\}} = 0 \tag{15}$$

where:

$$M_{iq} = \ddot{F}^{(i,q)} - 2r \sum_{j=1}^{m} g_j^{-3} \dot{g}_j^{(i)} \dot{g}_j^{(q)} + r \sum_{j=1}^{m} g_j^{-2} \ddot{g}_j^{(i,q)}$$

$$R_i = \dot{F}'^{(i)} - 2r \sum_{j=1}^{m} g_j^{-3} g_j' \dot{g}_j^{(i)} + r \sum_{j=1}^{m} g_j^{-2} \dot{g}_j'^{(i)}$$

At this point one may be tempted to take advantage of the fact that $g_j^{-2} << g_j^{-3}$ for active constraints, which are very small by definition, and to neglect terms with g_j^{-2} (second derivatives would become eliminated from the calculations) in Eq. (15)-a simplification that has been introduced into optimization practice by Ref. 9. However, such a simplification cannot be made in this case because it renders Eq. (15) singular.

Turning now to an exterior penalty function whose penalty term is:

$$P = \sum_{j=1}^{m} (\langle g_j \rangle)^2; \quad \langle g_j \rangle = \begin{cases} g_j, & \text{if } g_j > 0 \\ 0., & \text{if } g_j \leq 0 \end{cases} \tag{16}$$

one obtains Eq. (12) in the form

$$\underset{n\times n}{[M]} \underset{n\times 1}{\{\bar{X}'\}} + \underset{n\times 1}{\{R\}} = 0 \tag{17}$$

where

$$M_{iq} = \ddot{F}^{(i,q)} + 2r \sum_{j=1}^{m} \left(\dot{g}_j^{(i)} \dot{g}_j^{(q)} + g_j \ddot{g}_j^{(i,q)} \right)$$

$$R_i = \dot{F}'^{(i)} + 2r \sum_{j=1}^{m} \left(g_j' \dot{g}_j^{(i)} + g_j \dot{g}_j'^{(i)} \right)$$

and all constraint functions are in the $\langle g_j \rangle$ form defined by Eq. (16).

Other forms of penalty functions can be treated in the same manner (e.g., Ref. 4).

Numerical values of the objective and constraint functions, their derivatives, and, in a general case, the value of r are necessary for generating sensitivity equations such as Eq. (15). The value of r may not be available for the particular case of a constrained minimum being analyzed for sensitivity. However, it can be computed for the given set of constrained minimum point coordinates from Eq. (11) if that point's location is slightly off the constraint boundary in the feasible region (a typical outcome of a SUMT using an interior penalty function). If the location is slightly off into the unfeasible region (a typical result of the use of an exterior penalty function) the corresponding extremum condition equations for an exterior penalty function can be used to calculate r. Any equation arbitrarily chosen among n equations involved may be used to calculate the single unknown r, but the accuracy of this calculation will obviously benefit from choosing an r that minimizes the sum of the squares of residuals of Eq. (11), thus solving these equations in a least squares sense. The algebraic structure of the sensitivity equations for the penalty function makes their solution independent of the constant r in cases for which $\ddot{F}^{(i,q)} \equiv 0$ and $\dot{F}'^{(i)} \equiv 0$. These cases include, for example, sensitivity of minimum structural mass with respect to parameters other than the specific mass and overall geometrical dimensions.

In both Eqs. (4) and (15), only the right-hand side vector contains the mixed derivatives $(\dot{\cdot}')$ that depend on the choice of parameter p with respect to which the sensivity analysis is to be carried out. This property obviously reduces the amount of numerical labor required in problems with many parameters.

Cross-Applications of Eqs. (4) and (12)

Although the origin of Eq. (4) seems to suggest that it is appropriate for a sensitivity analysis of constrained minima obtained by one of the direct minimization methods (i.e., usable-feasible directions technique) and, by the same token, Eq. (12) seems to apply naturally in sensitivity analysis of constrained minima found by an SUMT approach, these equations can be cross-applied. In such cross-applications, Eq. (6) can be used to determine the λ-values and Eq. (11) to compute the r-value, so that one can apply Eq. (4) (and the corresponding solution subcase equations) to analyze the sensitivity of the SUMT-obtained constrained minima, and, conversely, Eq. (12) can be used when a constrained minimum is determined by a direct search algorithm.

Potential Applications

The primary application that inspired the method discussed herein is the enhancement of the results of optimization with sensitivity data, thus increasing, at a relatively small additional cost, the information about the object being optimized, including a possibility of extrapolation with respect to incremented parameters. However, the method appears capable of additional applications which are described in this section.

Extrapolation

The sensitivity derivatives provide an estimate of the change in the objective function and design variables corresponding to a small change of a parameter. Using the first two terms of the Taylor series expanded about the optimum design point x_0:

$$F_e = \bar{F}_0 + (d\bar{F}/dp)\,\Delta p \tag{18a}$$

$$x_{ei} = \bar{x}_{oi} + \bar{x}_i' \Delta p; \qquad i = 1 \rightarrow n \tag{18b}$$

Alternatively, for F computed as $\bar{F}_o = f(\bar{x}_{oi})$ at optimum:

$$F_e = f(x_{ei}) \tag{18c}$$

which suggests recomputing F using the x_{ei} estimates obtained from Eq. (18b).

Optimization with Multiple Objectives

Sensitivity analysis may be used to evaluate the influence of each of the multiple objectives on a constrained minimum. In a multiple objective optimization one may use a composite objective function in the form of a weighted sum of the single objectives:

$$F = a_1F_1 + a_2F_2 + \dots a_iF_i + \dots a_hF_h \tag{19}$$

where the weighting factors a_i represent the relative importance of the individual objectives F_i.

Since each weighting factor a_i is a problem parameter, the sensivities of the optimum with respect to the a_i factors can be obtained to determine the influence these factors have on the optimum and the trends associated with changes to the a_i values. It is conceivable that in well behaved problems it may be possible to use the derivatives with respect to a_i for extrapolation of $\bar{F}$ and $\bar{x}_i$ values to estimate their magnitudes for $a_i = 1$, and $a_j = 0$, for $j \neq i$. This, in effect, would provide estimates of single objective optimizations for each of the "h" objectives F_i, at a computational cost not much larger than the cost of a single optimization.

Activation or Deactivation of a Constraint

The λ' sensitivity derivatives generated in the course of the sensitivity analysis may be used to evaluate the increment in parameter p that will render an active constraint inactive. Considering the active constraint g_j, and the corresponding λ_j' value, a simple extrapolation of λ_j to zero:

$$\lambda_{ej} = \lambda_{oj} + \lambda_j' \Delta p = 0 \tag{20a}$$

yields an estimate

$$\Delta p = - \lambda_{oj}/\lambda_j' \tag{20b}$$

for the increment Δp of parameter p sufficient to remove the constraint g_j from the active constraint set. Analogously, one may estimate an increment Δp needed for an inactive constraint, whose value g_j is less than $-\varepsilon$ at the constrained optimum, to become active:

$$(g_j)_{new} = -\varepsilon = (g_j)_{old} + (g_j' + \sum_{i=1}^{n} \dot{g}_j^{(i)} \bar{x}_i') \Delta p \tag{21}$$

where the $\bar{x}_i'$ values are available from the sensitivity analysis and the values of g_j' and $\dot{g}_j^{(i)}$ would have to be computed additionally for the previously inactive constraint g_j.

Decomposition of a Large Optimization Problem into Several Subproblems

Another application is a decomposition of large multivariable optimization problems into a number of smaller subproblems. Suppose that the vector (X) of n design variables in an optimization problem is partitioned into X_A and X_B parts of n_A and n_B respective lengths, and that an optimization is performed with n_A variables, while holding elements of (X_B) fixed as parameters of the problem. Applying the optimum sensitivity analysis, the partial derivatives of the objective function and elements of (X_A) with respect to elements of (X_B) can be obtained. These derivatives, used in a first order Taylor series, permit expression of the objective function and each element of (X_A) as approximate linear functions of (X_B) thus eliminating (X_A) from the problem in the vicinity of the design point where the derivatives were evaluated. This capability to eliminate a group of variables may be used as means for formal decomposition of a large optimization problem. To be amenable for such decomposition, the problem should have a hierarchical structure of variables. This means that for each subsystem at level "i + 1" a vector (X_A) can be identified as containing the subsystem "local" variables and (X_B) can be recognized as being composed of the variables that a subsystem of a higher level i imposes on the subsystem at the lower level "i + 1". Carried out systematically to the system level, that is to the highest level of i = 1, this decomposition would eliminate all subsystem local variables replacing them with their linear approximate relationships to the variables of the highest level.

At the present stage of development, applications beyond the straightforward sensitivity evaluation and extrapolation (see the numerical examples) require further numerical experimentation to assess their usefulness.

Numerical Examples

This section contains numerical examples which demonstrate the sensitivity analysis in the context of structural optimization, verify correctness of the analysis, and provide a measure of usefulness of sensitivity derivatives in estimating the effect of problem parameters on the optimal objective function and design variables. The examples include a tubular column and a three-bar-truss for which closed form solutions are obtained, a ten-bar truss that requires use of a finite element analysis, and a thin-walled beam characterized by strongly nonlinear constraints for local buckling.

All the numerical examples are defined by Fig. 1 and detailed data* provided in the Appendix.

Verification of the Sensitivity Equations

A tubular thin-walled column and a three-bar truss,[10] shown in Fig. 1a and 1b, offer a possibility to verify the sensitivity analysis in a closed form. Each of these cases poses a two-design variable problem for which the closed form expressions for the objective function and constraints are given in the Appendix.

Tubular Column. Closed form expressions (Eq. (A2)) define the constrained minimum formed by vertices of constraints 1 and 3 described in the Appendix. Differentiation of these expressions with respect to parameter P yields a set of sensitivity derivatives. Another set of the sensitivity derivatives is obtained from Eq. (4), whose terms are derived analytically from the objective and constraint functions given in the Appendix. By the way of verification of Eq. (4), the two sets of the sensitivity derivatives are shown to be in an exact agreement in Table 1A.

Three-Bar Truss. For the three-bar truss, the constrained minimum formulas (Eq. (A5)) describe the minimum at the point where the objective function contour (straight line) is tangent to constraint 11 (Eq. (A4)) as shown in Ref. 10, Figs. 1-7. The sensitivity derivatives displayed in Table 1B (Key 1) are calculated from Eq. (4) whose terms are derived analytically from Eqs. (A3) and (A4). For verification, these derivatives agree exactly with the results of direct differentiation (Key 2). Derivatives with respect to P were derived from Eq. (A5). Shown also in Table 1B is the good agreement between results obtained from Eq. (4) (Key 1) and Eq. (15) (Key 3).

The upper and lower estimates of $d\bar{F}/dp$ (Key 4 and 5), obtained from the linearized Case 4/Subcase 2 (see Eq. (8)) are both very close to the exact value despite that the corresponding design variable derivatives are not. Moreover, the interval between these estimates is very narrow and it does contain the exact value of $d\bar{F}/dp$. However, it was not attempted within this study to prove that this is a general property of the solution of Case 4 rather than a fortuitous result.

Results for a truss configuration with $\alpha = 60°$ are given in Table 1C. Since, when closed form solutions are sought for derivatives with respect to α, expressions corresponding to Eq. (A5) are algebraically complex, their differentiation (Key 6) is additionally verified by a finite difference approximation.

The linearity of the truss constraints and objective function with respect to load P and ratio of the load to the allowable stress (P/σ_a) provides an additional verification of the corresponding sensitivity derivatives. Because of that linearity, the relative changes of cross-sectional areas are equal to the relative change of the load and could have been predicted by simple insight. The same is not true, however, for the tubular column whose nonlinear relationship between the design variables and the load are introduced by the buckling constraints, and for the parameter α in the truss where the nonlinearity is of a geometrical origin. It is in such nonlinear cases that the sensitivity analysis method discussed herein finds its usefulness.

Extrapolation of the Optimal Solution

Examples of extrapolation defined by Eqs. (18a), (18b), and (18c) are given for the three-bar truss, a ten-bar truss and a thin-walled beam.

A Three-Bar Truss Example. An optimum solution was considered as a function of the angle α (Fig. 1b). Results collected in Table 2 for two truss configurations, with $\alpha = 45°$ and $60°$, show:

1. In rows 1 and 6: the optimal solution obtained by a usable-feasible directions method coupled with a finite element program. The use of numerical methods for optimization and analysis accounts for the small differences between these results and the exact solution given in Table 1B and in Ref. 10, for $\alpha = 45°$.

2. In rows 2 to 5 and 7: extrapolated values are compared with the results of reoptimizations from the initial point ($A_1 = A_2 = 1$) and from the optimum point (row 1). Extrapolation is carried out using the sensitivity derivatives from Table 1, and formulas (18a) and (18b). In one case, in row 7, a result from Eq. (18c) is also given.

The extrapolation results are very good for small increments of 5 percent; they gradually deteriorate for larger increments, significantly so for the smaller of the two design variables (A_2). However, they remain very good for the larger variable (A_1) and excellent for the objective function (mass) throughout the comparison range, even for 45°/60° change of α in both directions. One of the results, given in row 7, shows the extrapolation of the objective function by Eq. (18c) as being significantly better than the one obtained from Eq. (18a).

The number of optimization iterations in reoptimization from the optimum solution point varied from a little less than, to about one half the number of iterations needed for reoptimization from the initial point ($A_1 = A_2 = 1$). This underscores the reduction of the computing cost realized by substituting the sensitivity analysis for reoptimization with incremented parameters.

A Ten-Bar Truss Example. The truss shown in Fig. 1c often used in the optimization literature (e.g., Ref. 11) as one of the standard reference examples for algorithm testing, is chosen to illustrate a case too large to be handled by a closed form solution. Consequently, a stiffness-based, finite element method, augmented with an analytical technique (e.g., Ref. 7, 11, or 12) to generate first and second derivatives of displacements and

*The three-bar and ten-bar truss are intended to be examined in context of the standard reference examples that were published using the U. S. Customary Units. Therefore, they are presented in the same units for consistency, while the SI units are used in the other examples.

stresses with respect to the design variables, was used as the analysis program. This program was coupled with a general purpose optimization program[13] to obtain an optimum solution. The optimum sensitivity algorithm based on Eq. (4) was implemented as a general purpose program and executed as a postprocessor to the optimization procedure.

The objective function was the material weight, and the ten cross-sectional areas (numbered in Fig. 1c) were design variables. Constraints were imposed as minimum limits on the variable values, and as an allowable stress (see Appendix).

Optimum solution and sensitivity derivatives with respect to the truss depth H are collected in Table 3. The nonzero derivatives are negative, because the increase of H decreases the forces in the horizontal and diagonal rods. Considering variations of H, Table 4 presents comparisons of reoptimization results for increments of H versus extrapolation results obtained by Eqs. (18a) and (18b) using sensitivity derivatives from Table 3. The comparison shows that the relative errors of extrapolation do not exceed 2.5 percent for the design variables and 1.3 percent for the objective function for 20 percent increment of H. The comparison is presented graphically in Fig. 2 for the objective function and one typical variable (A_1). The graph shows that the extrapolation practically coincides with the reoptimization for up to 10 percent change of H. For a 20 percent change of H, extrapolation overestimates the decrements of weight and typical variable by about 18 percent and 12 percent, respectively.

For this structure, it would be difficult to predict by physical insight alone how the objective function would change with the increase of H, because the decreases of cross-sections are counteracted by the increases of lengths in the diagonal and vertical rods.

<u>A Thin-Walled Box Beam Example</u>. The beam shown in Fig. 1d provides an example with a high degree of nonlinearity because of the constraints which include local buckling and the equality constraints imposed on the cross-sectional area and moment of inertia. In this problem the values of variables B, H, T_1, T_2, and T_3 are sought that make A and I equal to prescribed values A_p and I_p while minimizing the objective function $F = \Omega$, where

$$\Omega = \sum_j (\langle g_j \rangle)^2 \tag{22a}$$

$$\langle g_j \rangle = \begin{cases} g_j, \text{ if } g_j > 0 \\ 0.00, \text{ if } g_j \leq 0 \end{cases} \tag{22b}$$

and the constraints g_j are the stress constraints (see Appendix, Eqs. (A17) and (A18)). The quantity Ω provides a single measure of unsatisfaction of the constraints, and because of the power factor in Eq. (22a), it is continuous up to its first derivatives with respect to the design variables, if the constraints are continuous. This continuity is required for the optimizer (Ref. 13) used in the study. Thus, the optimization problem is

$$\begin{aligned} &\min_{X_i} F, \text{ subject to} \\ &A = A_p,\ I = I_p,\ F = \Omega \ \text{ defined by} \\ &\quad \text{Eq. (22), and} \\ &g_j \leq 0 \end{aligned} \tag{23}$$

where inequality constraints g_j are the minimum gage and other geometrical constraints (see Appendix, Eqs. (A11), (A12) and (A13)). This somewhat unusual formulation of a structural optimization problem has a meaning of proportioning the detailed dimensions of a cross-section to achieve a least violation of the constraints, while conforming to the prescribed cross-section stiffness properties in tension (A_p) and bending (I_p), and is motivated by its use in a multilevel decomposition described in the section on "Potential Results." In that decomposition, the A_p and I_p values are to be the system level variables imposed on a beam treated as a subsystem governed by local variables B, H, T_1, T_2, T_3.

As explained in the Appendix, the variables B and T_2 are eliminated from the problem by solving the equality constraint equations $A = A_p$ and $I = I_p$, so that only the variables H, T_1 and T_3 are left as free variables. Similarly, equilibrium equations are used to express the right-end forces in terms of the left-end, statically independent forces. The stress and geometry constraints are expressed as functions of the end forces and design variables as shown in the Appendix. Because of the algebraic complexities of these functions, all the derivative terms in Eq. (4) are computed by finite differences.

The optimum solution and the sensitivity derivatives of the independent local variables and the objective function with respect to the bending moment N_3 and the moment of inertia I_p are displayed in Table 5. The reoptimization reference results for variations of N_3 and I_p and the corresponding extrapolation results are collected in Table 6. The table shows the extrapolation estimates to be quite good throughout the parameter variation range with respect to absolute values of the objective function and design variables as well as their increments. A typical sample of the extrapolation and reoptimization results for the box beam is plotted in Fig. 3 against the variations of the parameter I_p. The graph shows the extrapolation's capability to account for practically the entire change of the objective function F and the design variable H (the beam depth) corresponding to a parameter increment of up to 20 percent. The objective function was extrapolated using Eqs. (18a) and (18c) (F^* and F^{**}, respectively in Table 6) with the latter giving much better estimates (this trend was already observed in Table 2, row 7).

The relatively good accuracy of the extrapolation results for the beam problem suggests its potential usefulness in the previously described concept of a multilevel decomposition.

Optimization with Conflicting Objectives

The three-bar truss example is used to test the capability of the sensitivity analysis to determine the influences of several, possibly conflicting, objectives. The three-bar truss (Fig. 1b) is used again as an example. The truss is modified so that the rods corresponding to cross-section A_1 are assumed to be made of steel $\sigma_a = 36667.$ lb/in^2, $\rho = 0.282$ lb/in^3, and unit volume cost $c = 5.$ \$/lb; the center rod (A_2) is assumed to be made of titanium of $\sigma_a = 40000.$, $\rho = 0.160$ and $c = 11.$ Minimization of the mass (F_1) and cost (F_2) objectives is considered, with the realization that the two objectives will conflict because titanium is lighter but also more costly than steel.

First, an optimization is carried out with a composite objective function $F = a_1 F_1 + a_2 \beta F_2$ from initial point $A_1 = A_2 = 1.$, and the weighting factors $a_1 = a_2 = 1.$ that reflect an approximately even "importance" subjectively assigned to the two objectives (the constant β equalizes numerical magnitudes of the two terms in F at the initial point). The results of the optimization are given in Table 7, row 1. The reader may find it interesting to compare these results with the optimum solution point given for a single material truss in Table 1B. Next, sensitivity derivatives with respect to weighting ("importance") factors are obtained (Table 7, rows 2 and 3), treating the factors as parameters of the problem, to determine the trends that would be followed should the relative importance of the two objectives change. These trends are extrapolated to the extremes of mass-only and cost-only in rows 4 and 5, and are verified by the results of full optimizations for each objective separately, starting from the initial point. The objective function extrapolation results shown are obtained by Eq. (18c) which, in this case, provided again a better accuracy than Eq. (18a). Comparison of rows 4 and 5 with row 1 shows that:

(1) Single objective optimization yields an objective value reduced relative to the value of the same objective obtained from multiobjective optimization (an expected result). In this case the reductions are small, but significant, 2.52 percent for the mass and 1.95 percent for the cost.

(2) The extrapolation underpredicts the mass reduction by about 9 percent and overpredicts the cost reduction by about 36 percent.

The verification shows the previously observed tendency for good prediction of the larger of the design variables and even better prediction for the objective function. Remarkably, the disappearance of the expensive titanium rod in the cost-only optimization is predicted very well by the extrapolation. It appears that the approach has the potential of providing useful estimates of the results of many single objective optimizations by extrapolating from the results obtained by executing only one optimization with a composite objective function.

Sensitivity Analysis Accuracy as a Function of the Optimum Solution Degree of Convergence

The relative error of extrapolation obviously depends on the accuracy of the sensitivity derivatives which, in turn, are influenced by the degree of convergence of the optimum solution. Although, theoretically, solution convergence is rigorously prescribed by the Kuhn-Tucker conditions, in practice the optimization procedures usually terminate by less rigorous, "practical," criteria. For example, the optimizer[13] used in this study may stop when all the constraints are satisfied and when the last n iterations (n was assumed 5) produced relative differences between two consecutive objective functions smaller than a tolerance C. To be able to use the optimum sensitivity analysis with confidence, one needs to know how strongly the sensitivity derivatives depend on the convergence controls represented by, for example, the tolerance C.

Results that shed some light on that dependence are shown in Tables 8, 9, 10 for the three-bar and ten-bar trusses and for the beam. The results show consistently the influence of C to be rather weak, especially for the objective function derivatives. The relatively large error of the derivative of T_1 ($\partial T_1/\partial N_3$) in Table 10 appears to have little significance because this particular derivative value is practically a numerical zero as shown in Table 5.

Since the tolerance C influences rather strongly the number of iterations (indicated in the tables), it appears that one may save some computational cost and obtain sufficiently accurate sensitivity information by not pursuing the optimization procedure too far beyond the aforementioned threshold on its convergence path.

Summary of the Numerical Examples

Results of the examples verify the numerical solutions of the sensitivity Eqs. (4) and (15) by analytical solutions for the simple cases of a tubular column and a three-bar truss. The three-bar truss results suggest also that in some cases a linearized, inexpensive analysis may provide very good estimates for the objective function sensitivity derivative $d\bar{F}/dp$. Additional verification of the sensitivity equations is seen in the agreement of the extrapolation results and the reoptimization results for small variations of the parameters in the ten-bar truss and the thin-walled beam.

For all cases tested, the extrapolation based on the sensitivity derivatives shows the ability to predict the reoptimization results with a relative error of the order of a few percent for a parameter increment range order of 20 percent, and the relative error appears to be a rather weak function of the degree of convergence of the optimum solution within practical convergence limits. For smaller increments, below 10 percent, the relative error of extrapolation is small enough for the accuracy needed in most engineering calculations. This is apparently due to the fact (apparent in Figs. 2 and 3) that although the relative variations of the objective and constraint functions may be of the same order as the relative increments of the parameters that cause them, they are, at least for the cases tested, nearly linear functions of these parameters in the vicinity of the optimum despite the nonlinearity of the optimization problems themselves.

The extrapolation predictions are generally better for numerically larger, more significant variables, and better for the objective function than for the variables. In most, but not all, cases tested, the extrapolation via the function recomputation using the extrapolated variables (Eq. (18c)) yielded an accuracy better than extrapolation via the objective function derivatives (Eq. (18a)).

Particularly interesting are the extrapolation results for the shape parameters for three- and ten-bar trusses (Tables 2 and 4) since they suggest a potential application for decoupling the overall shape variables from the cross-section dimension variables in structural optimization.

Still another potential use of the sensitivity analysis is in extrapolation of a single optimization with a composite objective function to the extremes corresponding to single objective optimizations, as illustrated by the three-bar truss case with the conflicting objectives of cost and weight.

In the foregoing examples, the extrapolation accuracy benefitted from the lack of slightly satisfied constraints with their boundaries near the optimum solution. If such constraints existed, their boundaries could have been penetrated in the process of extrapolation, thus introducing a discontinuity associated with new constraints being brought into the active constraint set. It is obviously important to be alert for such discontinuities, which can be detected by Eqs. (20a) (20b), and (21), when extrapolating from an optimum solution point.

Concluding Remarks

Methods for determining the sensitivity derivatives of a constrained minimum solution to the problem parameters are discussed and the governing equations are derived. The equations directly yield the derivatives of the optimum design variables and of the objective function with respect to the parameters that are constants of the problem. For example, derivatives of optimal cross-sectional dimensions and structural mass of a structure can be obtained with respect to allowable stress, load, overall structural dimensions, etc. The derivatives or, in other words, the sensitivity data are obtained by solving a set of linear algebraical equations, thus eliminating the need to repeat the optimization for incremented values of the parameters. While the primary application of the sensitivity analysis is to determine trends at the solution point, it has potential uses in optimization with multiple objectives, activation or deactivation of constraints, and a formal decomposition of large optimization problems.

The paper refers to structural optimization for verification of the sensitivity solutions and for examples of some of the many potential applications. The examples for a tubular column and a truss demonstrate determination of sensitivity with respect to load and shape parameters, and include extrapolation of the optimum solution for incremented values of the parameters. Results are also presented for a ten-bar truss analyzed by a finite-element method. Included as a test structure is a thin-walled beam that introduces a high degree of nonlinearity through inclusion of local buckling constraints. The results show that a practically significant extrapolation accuracy may be obtained for a reasonably broad range of parameter changes. The results also show that accuracy does not depend strongly on the degree of convergence of the optimum solution from which the sensitivity derivatives are obtained as long as that optimum is within practical convergence bounds.

An example of optimization of a two-material truss with conflicting objectives of mass and cost points to the usefulness of the sensitivity analysis to predict trends and to extrapolate with respect to the individual objectives from the basis of a single optimization with a composite objective function.

Finally, while the examples used in the paper are structural optimization problems, the equations for the sensitivity derivatives are entirely general since they are derived from the basic Lagrange multiplier equations or, alternatively, from extremum conditions for a penalty function. Consequently, the optimum sensitivity analysis applies to optimization problems in any discipline, and also to interdisciplinary systems.

Appendix

Details of the Numerical Examples

This Appendix contains the detailed information and numerical data needed for definition of each numerical example.

Tubular Column

Numerical data for the column shown in Fig. 1a are as follows: $L = 5\text{m}$, $P = 500$ kN, material is steel, $E = 20.6\ \text{MN/cm}^2$, $c = 0.2$, $\sigma_a = 39.2\ \text{kN/cm}^2$.

Design variables are t and R. The objective function is the material volume $F \cong 2\pi RtL$. Constraints are (1) stress $g_1 = \sigma/\sigma_a - 1 \leq 0$; (2) column buckling $g_2 = P/P_{cr} - 1 \leq 0$; and (3) cylinder wall buckling $g_3 = \sigma/\sigma_{cr} - 1 \leq 0$ where

$$\sigma = P/A,\ P_{cr} = \pi^2 EI/4L^2,\ \ \sigma_{cr} = cEt/R \qquad \text{(A1)}$$

$$A \cong 2\pi Rt;\ I \cong \pi R^3 t$$

From the constraint expressions in conjunction with Eq. (A1), one obtains formulas for the vertex of constraints 1 and 3:

$$t = (P/2\pi cE)^{1/2},\ \ R = (E^{1/2}/\sigma_a)(c/2\pi)^{1/2}\,P^{1/2} \qquad \text{(A2)}$$

Three-Bar Truss

Numerical data for the truss shown in Fig 1b are as follows: $L = 10$ in., $L' = \sqrt{2}\ L$ regardless of the value of angle α, $P = 20$ kip, material is an Al-alloy, $\sigma_a = 20$ ksi, and $\rho = 0.1$ lb/in.^{3}.

Design variables are A_1 and A_2, with $A_3 = A_1$. The objective function is material weight

$$F = \rho L(2\sqrt{2}A_1 + A_2) \tag{A3}$$

and constraints are on stress in rods 1, 2, and 3 for two loading cases, P_1 and P_2, with $P_1 = P_2 = P$ and orientation of the forces P_1 and P_2 not affected by changes of the angle α.

The constraint functions are

$$g_{u,v} = |\sigma_{u,v}|/\sigma_a - 1 \le 0 \tag{A4}$$

where

$$\sigma_{11} = P_y \sin \alpha/L'k + P_x/2A_1 \cos \alpha$$

$$\sigma_{21} = P_y/Lk,\ \sigma_{31} = P_y \sin \alpha/L'k - P_x/2A_1 \cos \alpha$$

$$k = A_1\ 2\sin^2\alpha/L' + A_2/L$$

$$P_x = -(\sqrt{2}/2)P,\ P_y = (\sqrt{2}/2)P$$

Subscripts u and v define rod number and load case number, respectively. Due to the structure symmetry, only the constraints for stresses shown above need to be included. For the optimum located[10] at a tangent point of a contour of the objective function and the function $g_{11} = 0$, one can use Eqs. (A3) and (A4) to derive closed form expressions for optimum A_1 and A_2. For the given data and the particular case of $\alpha = 45°$, these expressions are

$$A_1 = (P/\sigma_a)\,\frac{3 + \sqrt{3}}{6},\quad A_2 = (P/\sigma_a)\,\frac{1}{\sqrt{6}} \tag{A5}$$

Ten-Bar Truss

For the truss shown in Fig. 1c, the dimensions are $L = 360$ in., with $H = L$ initially, one loading case is assumed with $P = 10^5$ lbf, and the material is an Al-alloy of $E = 10^7$ psi, $\sigma_a = 25$ ksi, and $\rho = 0.1$ lbf/in.^{3}. Design variables are ten rod cross-sections; the minimum limit on A is $A_{min} = 0.1$ in.^{2}; and constraints are imposed on stress in the j-th rod: $g_j = |\sigma_j|/\sigma_a - 1 \le 0$. The objective function is the structural weight.

Box Beam

Detailed data for the box beam shown in Fig. 1d are as follows: $E = 7.38$ MN/cm^2, $\sigma_a = 13.79$ kN/cm^2, $\tau_a = 8.62$ kN/cm^2, $\nu = 0.3$, $N_1 = 44.48$ kN, $N_2 = 2.22$ kN, $N_3 = 451.9$ kN cm, $A_p = 9.68$ cm^2, $I_p = 83.25$ cm^4, $L = 50.80$ cm, $\varepsilon = 0.5$ and the side constraints are in cm: $2.54 \le H \le 25.4$, $2.54 \le B \le 25.4$, $0.127 \le T_1 \le 2.54$, $0.127 \le T_2 \le 2.54$, $0.127 \le T_3 \le 2.54$.

<u>Elimination of the Dependent Variable</u>. Based on the beam cross-sectional dimensions shown in Fig. 1d, the area A and moment of inertia I of the cross-section about the y-axis are:

$$A = 2HT_2 + (T_1 + T_3)(B - 2T_2) \tag{A6}$$

$$y_G = \frac{BT_1(H - \frac{T_1}{2}) + BT_3\frac{T_3}{2} + 2T_2(H - T_1 - T_3)(T_3 + \frac{1}{2}(H - T_1 - T_3))}{B(T_1 + T_3) + 2(H - T_1 - T_3)T_2} \tag{A7}$$

$$I = B\,\frac{T_1^{\,3}}{12} + BT_1(H - \frac{T_1}{2} - y_G)^2$$

$$+ B\,\frac{T_3^{\,3}}{12} + BT_3(T_3/2 - y_G)^2$$

$$+ 2\left\{T_2\,\frac{[(H - T_1 - T_3)]^3}{12}\right.$$

$$\left. + T_2(H - T_1 - T_3)(T_3 + \tfrac{1}{2}(H - T_1 - T_3) - y_G)^2\right\} \tag{A8}$$

Choosing arbitrarily the variables B and T_2 as dependent variables, one obtains from Eq. (A6), (A7) and (A8) a quadratic equation for B and T_2 in terms of A, I, H, T_1, T_3 for which one solution is

$$B = (-\alpha_1 - (\alpha_1^{\,2} - 4\alpha_2\alpha_0)^{1/2})/2\alpha_2 \tag{A9}$$

$$T_2 = (1/(2(H - T_1 - T_3)))(A - B(T_1 + T_3)) \tag{A10}$$

where

$$\alpha_0 = \frac{12I}{[H - (T_1 + T_3)]^2} - A$$

$$\alpha_1 = \frac{-2H^2(T_1 + T_3) - 2H(T_1 + T_3)^2 + 12HT_1T_3}{[H - (T_1 + T_3)]^2}$$

$$\alpha_2 = \frac{3H^2(T_1 - T_3)^2}{A[H - (T_1 + T_3)]^2}$$

The other possible solution (having a positive sign of the square root term in Eq. (A9) leads to another local optimum design point. It was found experimentally that in this case the Eq. (A9) solution provides a better design (lower objective function value) than the other solution. Therefore, only the Eq. (A9) solution was used.

Constraint Definitions. Two groups of constraints are defined: one for geometry and one for strength. The geometry constraints are as follows:

1. Upper and lower limits

$$g = 1 - x_i/x_{iL} \le 0 \quad \text{and} \quad g = x_i/x_{iu} - 1 \le 0 \tag{A11}$$

respectively, where x_i's correspond to dimensions B, H, T_1, T_2, T_3.

2. Limits assuring that the box cross-section remains hollow

$$g = 1 - \frac{H - T_1 - T_3}{3(T_{1L} + T_{3L})} \le 0 \quad \text{and}$$

$$g = 1 - \frac{B - 2T_2}{3(2T_{2L})} \le 0 \tag{A12}$$

where T_{1L}, T_{2L}, T_{3L} are thickness minimum gages, and the coefficient 3 is a judicious factor.

3. A real number solution requirement for Eq. (A9):

$$g = -((\alpha_1^2 - 4\alpha_0\alpha_2) - \varepsilon) \le 0 \tag{A13}$$

where ε is a small positive constant.

The strength constraints are stress and local buckling constraints evaluated by engineering beam bending theory formulas and approximate closed form solutions for local buckling of a thin-walled beam. These constraints are computed at nine points strategically distributed over both end cross-section contours as shown in Fig. 1d. The constraint formulas are shown below. Normal stresses due to bending and axial force are at points 1, 2, 4, 5:

at 1

$$\sigma_{b1} = \frac{+N_3(H - y_G)}{I} - \frac{N_1}{A}$$

at 2

$$\sigma_{b2} = \frac{-N_3}{I} y_G - \frac{N_1}{A} \tag{A14}$$

at 4

$$\sigma_{b4} = \frac{-(N_2L - N_3)(H - y_G)}{I} - \frac{N_1}{A}$$

at 5

$$\sigma_{b5} = \frac{+(N_2L - N_3)y_G}{I} - \frac{N_1}{A}$$

Shearing stress at point 3 is

$$\tau = N_2Q/(2IT_2) \tag{A15}$$

where $Q = T_2(H - y_G - T_1)^2 + BT_1(H - T_1/2 - y_G)$. Each of the above stress values is constrained to an allowable value by

$$g = \frac{\text{stress}}{\text{allowable stress}} - 1 \le 0 \tag{A16}$$

Critical local buckling stresses are (Ref. 14):

$$\sigma_{cr} = \frac{\pi^2 k_c E}{12(1 - \nu^2)} (T_1/B)^2 \quad \text{at points 1 and 4,}$$

$$\sigma_{cr} = \frac{\pi^2 k_c E}{12(1 - \nu^2)} (T_3/B)^2 \quad \text{at points 2 and 5 with } k_c = 5.5 \text{ (Ref. 15).}$$

These critical stresses are combined with the stresses from Eq. (A14) into stability constraints

$$g = (\text{compressive stress})/(\text{critical stress}) - 1 \le 0 \tag{A17}$$

The stability constraint for the vertical web under the combined action of normal and shearing stresses is computed at points 6, 7, 8, and 9 according to an interaction formula:

$$g = \left(\frac{|\sigma_{bi}|}{\sigma_{cr_1}}\right)^{1.75} + \frac{-\sigma_2}{\sigma_{cr_2}} + \left(\frac{\tau}{\sigma_{cr_3}}\right)^2 - 1 \leq 0$$

$i = 6,7,8,9,$ (A18)

where

$$\sigma_{cr_j} = \frac{\pi^2 E}{12(1 - \nu^2)} \; k_{cj}(T_2/B)^2, \qquad j = 1,2,3,$$

with $k_{c1} = 33$, $k_{c2} = 4$ and $k_{c3} = 7.5$ (Ref. 15, C5.15, C5.2, and C5.11, respectively); σ_{bi} is normal stress due to bending moment only given by the first term in each of the equations (A14), σ_2 is normal stress due to axial force, $\sigma_2 = -N_1/A$, and shearing stress τ is defined by Eq. (A15).

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Table 1 Optima and sensitivity derivatives for tubular column and three-bar truss (Figs. 1a and 1b)

KEY: 1. from Eq. (4); 2. from direct differentiation; 3. from Eq. (15); 4 and 5. case 4, subcase 2, maximization and minimization of $d\bar{F}/dp$, respectively; 6. central finite difference for $\Delta\alpha$ = 0.01°, in addition to analytical differentiation.

Variable	Optimal value	Key	Sensitivity derivatives		
A. Tubular column					
			$\partial/\partial P$		
		1	1.390		
t, cm	0.139			$\times 10^{-7}$ cm/N	
		2	1.390		
		1	1.459		
R, cm	14.59			$\times 10^{-5}$ cm/N	
		2	1.459		
		1	2.028		
F, cm^3	6371.18			$\times 10^{-3}$ cm^3/N	
		2	2.028		
B. Three-bar truss, α = 45°					
			$\partial/\partial P$		$\partial/\partial\alpha$
		1	3.940		-0.88345
		2	3.940	$\times 10^{-5}$	-0.8849
A_1, in^2	0.7887	3	3.940		
		4	0.0	in^2/lb	in^2/rad
		5	4.662		
		1	2.049		0.97571
		2	2.049	$\times 10^{-5}$	0.9793
A_2, in^2	0.4093	3	2.051		
		4	13.231	in^2/lb	in^2/rad
		5	0.00		
		1	13.190		-1.523
		2	13.190	$\times 10^{-5}$	-1.524
F, lb	2.64	3	13.190		
		4	13.231	in^2/lb	lb/rad
		5	13.186		

Table 1 Concluded

Variable	Optimal value	Key	Sensitivity derivatives	
C. Three-bar truss, $\alpha = 60°$				
			$\partial/\partial\alpha$	
A_1, in^2	0.6023	1	-0.52044	in^2/rad
		6	-0.52044	
A_2, in^2	0.5536	1	0.04373	in^2/rad
		6	0.04373	
F, lb	2.257	1	-1.4283	lb/rad
		6	-1.4283	

Table 2 Comparison of extrapolation and optimization for three-bar truss

	$\Delta\alpha$ %	Weight lb	A_1 in^2	A_2 in^2
Configuration with $\alpha = 45°$				
#1	0	2.629	0.7911	0.3919
2	-5	2.689* 2.690** 2.690***	0.8258 0.8198 0.8256	0.3536 0.3712 0.3547
3	-10	2.749 2.755 2.754	0.8605 0.8701 0.8619	0.3152 0.2934 0.3166
4	-20	2.869 2.899 2.899	0.9299 0.9514 0.9574	0.2386 0.2082 0.1912
#5	+33.3	2.231 2.257 2.257	0.5598 0.6023 0.6050	0.6473 0.5536 0.5460
Configuration with $\alpha = 60°$				
#6	0	2.257	0.6023	0.5536
#7	-25	2.631 2.629 2.631†	0.7385 0.7911 --	0.5422 0.3919 --

*first line: extrapolation; **second line: reoptimization from initial point; ***third line: reoptimization from optimum point

#)NOTE: 1. Rows 5 and 6 correspond to the same configuration with $\alpha = 60°$.
2. Rows 1 and 7 correspond to the same configuration with $\alpha = 45°$.
3. Optimum values in row 1 are from numerical optimization hence they differ slightly from those in Table 1B.

†Obtained from Eq. (18c)

Table 3 Optimum solution and its sensitivity derivatives with respect to parameter H for ten-bar truss (Fig. 1c)

Rod no.	Cross-section area		∂(cross-section area)/∂H	
1	7.9606	(in^2)	$-0.20672061\ 10^{-1}$	(in^2/in)
2	0.1	"	0.0	"
3	8.0676	"	$-0.20095106\ 10^{-1}$	"
4	3.9521	'	$-0.9871375\ 10^{-2}$	"
5	0.1	"	0.0	"
6	0.1	"	0.0	"
7	5.7505	"	$-0.71620975\ 10^{-2}$	"
8	5.5863	"	$-0.75472761\ 10^{-2}$	"
9	5.5699	"	$-0.69817317\ 10^{-2}$	"
10	0.1	"	0.0	"
Weight	1595.92 lb		∂(weight)/∂H = -1.7047630 lb/in	

Table 4 Comparison of extrapolation and reoptimization for variations of parameter H in ten-bar truss

	Increments of H, %			
	0	5	10	20
Rod no.	Cross-section areas, in^2			
1	7.961	7.589* 7.584**	7.216 7.224	6.472 6.635
2	0.1	0.1 0.1	0.1 0.1	0.1 0.1
3	8.068	7.706 7.700	7.344 7.350	6.621 6.736
4	3.952	3.774 3.756	3.597 3.589	3.241 3.280
5	0.1	0.1 0.1	0.1 0.1	0.1 0.1
6	0.1	0.1 0.1	0.1 0.1	0.1 0.1
7	5.751	5.622 5.626	5.493 5.497	5.235 5.289
8	5.586	5.450 5.437	5.315 5.338	5.043 5.144
9	5.570	5.444 5.439	5.319 5.324	5.067 5.121
10	0.1	0.1 0.1	0.1 0.1	0.1 0.1
Weight, lb	1595.92	1565.23 1563.33	1534.55 1535.50	1473.18 1491.94

*first line: extrapolation using Eqs. (18a) and (18b); **second line: reoptimization

Table 5 Optimum solution and derivatives with respect to parameters N_3 (end bending moment) and I_p (moment of inertia corresponding to N_3) for box beam (Fig. 1d)

Variable (Fig. 1d)	Optimal value, cm	Derivative $\partial/\partial N_3$, cm/Ncm	Derivative $\partial/\partial I_p$, cm/cm^4
H	6.807	$-1.0396 \cdot 10^{-6}$	$4.469 \cdot 10^{-2}$
T_1	0.3246	$1.3756 \cdot 10^{-8}$	$2.460 \cdot 10^{-4}$
T_3	0.5415	$-6.0940 \cdot 10^{-7}$	$2.638 \cdot 10^{-3}$
B*	9.425	--	--
T_2*	0.1275	--	--
		1/Ncm	1/cm^4
Objective F	0.2735	$4.6030 \cdot 10^{-6}$	$-1.1848 \cdot 10^{-2}$

*Dependent variables

Table 6 Comparison of extrapolation and reoptimization for variations of parameters N_3 and I_p in box beam

		Perturbed designs					
		Increment of N_3, %			Increment of I_p, %		
Variable	Baseline optimal design	5	10	20	5	10	20
H, cm	-- 6.807	6.784* 6.783**	6.760 6.770	6.713 6.715	6.993 6.987	7.179 7.170	7.551 7.534
T_1, cm	-- 0.3246	0.3249 0.3246	0.3254 0.3254	0.3259 0.3279	0.3256 0.3249	0.3266 0.3264	0.3287 0.3299
T_3, cm	-- 0.5415	0.5276 0.5268	0.5138 0.5199	0.4864 0.4867	0.5525 0.5494	0.5634 0.5611	0.5832 0.5984
B, cm	-- 9.425	9.579 9.590	9.736 9.663	10.08 10.04	9.240 9.288	9.046 9.096	8.646 8.672
T_2, cm	-- 0.1275	0.1275 0.1275	0.1273 0.1273	0.1262 0.1273	0.1280 0.1273	0.1293 0.1278	0.1339 0.1232
F†	-- 0.2735	0.3775 0.3890	0.4815 0.5284	0.6896 0.8912	0.2242 0.2279	0.1749 0.1906	0.0762 0.1320
F‡	-- --	0.3891 --	0.5279 --	0.8364 --	0.2283 --	0.1922 --	0.1373 --

*first line: extrapolation, by Eq. (18a)†, by Eq. (18c)‡;
**second line: reoptimization

Table 7 Use of sensitivity and extrapolation for three-bar truss optimization for mass and cost

	Objective function*	A_1 in^2	A_2 in^2
	Optimum for $a_1 = a_2 = 1$		
1	$F_1 = 4.150$ $\beta F_2 = 3.6963$**	0.4910	0.1480
	Sensitivity derivatives $\partial/\partial a_1$		
2	4.1422***	-0.035805	0.11371
	Sensitivity derivatives $\partial/\partial a_2$		
3	3.7011***	0.032997	-0.10479
	Extrapolation and reoptimization for mass only $a_1 = 1$, $a_2 = 0$		
4	4.0547† 4.0457‡	0.4577 0.4308	0.2526 0.3812
	Extrapolation and reoptimization for cost only $a_1 = 0$, $a_2 = 1$		
5	3.5980† 3.6240‡	0.5265 0.5455	0.03411 0.00001

*)objective function is $F = a_1F_1 + a_2\beta F_2$, F_1 being the mass and F_2 the cost

**)quantity βF_2, instead of F_2 alone, is referred to consistently in the table

***)total derivative, dF/da_i

†first line: extrapolation using Eq. (18c)

‡second line: reoptimization

Table 8 Influence of degree of convergence of optimum solutions on the sensitivity derivatives for three-bar truss

Tolerance C	0.0001	0.001	0.01	0.1
Number of iterations	18	7	5	4
Optimal solutions				
A_1, in^2	0.7944	0.80187	0.78238	0.79026
A_2, in^2	0.38238	0.36502	0.4211	0.39907
Weight, lb	2.62927	2.63306	2.63401	2.63427
Sensitivity derivatives				
$\partial A_1/\partial\alpha$, in^2/rad	0.8993	0.9141	0.8737	0.8394
	2.9	4.6*		1.8
$\partial A_2/\partial\alpha$, in^2/rad	-1.0243	-1.0628	-0.9503	-0.9945
	7.8	11.8		4.7
Weight, $\partial W/\partial\alpha$, lb/rad	1.5188	1.5227	1.5210	1.5210
	-0.01	0.01		0.0

*Relative difference = $100\% \dfrac{\text{VALUE} - \text{REFERENCE}}{\text{REFERENCE}}$ with REFERENCE being the corresponding value from the column for C = 0.01

Table 9 Influence of degree of convergence of optimum solutions on the sensitivity derivatives for ten-bar truss

Tolerance C	0.003		0.006	0.01	
Number of iterations	32		26	21	
Optimal solutions					
A_1**, in^2	7.9606		7.9152	8.1914	
A_9, in^2	5.5699		5.5811	5.5847	
Weight, lb	1595.92		1595.72	1608.28	
Sensitivity derivatives					
$\partial A_1/\partial H$, in^2/in	-0.020672	0.60*	-0.020549	-0.029937	45.7
$\partial A_9/\partial H$, in^2/in	-0.00698	4.2	-0.00670	-0.006898	3.0
Weight, $\partial W/\partial H$, lb/in	-2.9273	0.10	-2.9253	-3.61609	23.61

*See footnote in Table 8, column for C = 0.006 is REFERENCE column

**Selected cross-sections only to show largest and smallest effect of C on $\partial/\partial H$

Table 10 Influence of degree of convergence of optimum on the sensitivity derivatives for box beam

Tolerance C	0.001	0.005	0.025
Number of iterations	25	19	13
		Optimal solutions	
H, cm	6.806	6.803	6.843
T_1, cm	0.3246	0.3241	0.3287
T_3, cm	0.5410	0.5377	0.5677
B, cm	9.431	9.468	9.111
T_2, cm	0.1275	0.1278	0.1270
Objective F	0.2734	0.2736	0.2759
		Sensitivity derivatives $\partial/\partial N_3$	
$\partial H/\partial N_3$, cm/Ncm	$-1.037 \cdot 10^{-6}$ 5.2	$-9.853 \cdot 10^{-7}$	$-1.458 \cdot 10^{-6}$ 47.9
$\partial T_1/\partial N_3$, cm/Ncm	$1.379 \cdot 10^{-8}$ -9.0	$1.515 \cdot 10^{-8}$	$-1.761 \cdot 10^{-10}$ -101.0
$\partial T_3/\partial N_3$, cm/Ncm	$-6.077 \cdot 10^{-7}$ 5.3	$-5.769 \cdot 10^{-7}$	$-8.630 \cdot 10^{-7}$ 49.6
$\partial F/\partial N_3$, 1/Ncm	$4.602 \cdot 10^{-6}$ 0.0	$4.604 \cdot 10^{-6}$	$4.603 \cdot 10^{-6}$ 0.0
		Sensitivity derivatives $\partial/\partial I_p$	
$\partial H/\partial I_p$, cm/cm^4	$4.466 \cdot 10^{-2}$ 0.4	$4.450 \cdot 10^{-2}$	$4.619 \cdot 10^{-2}$ 3.8
$\partial T_1/\partial I_p$, cm/cm^4	$2.453 \cdot 10^{-4}$ 2.3	$2.399 \cdot 10^{-4}$	$2.977 \cdot 10^{-4}$ 24.1
$\partial T_3/\partial I_p$, cm/cm^4	$2.632 \cdot 10^{-3}$ 4.2	$2.526 \cdot 10^{-3}$	$3.508 \cdot 10^{-3}$ 38.9
$\partial F/\partial I_p$, 1/cm^4	$-11.85 \cdot 10^{-3}$ 0.0	$-11.85 \cdot 10^{-3}$	$-11.84 \cdot 10^{-3}$ 0.0

*See footnote in Table 8; reference column is column for C = 0.005

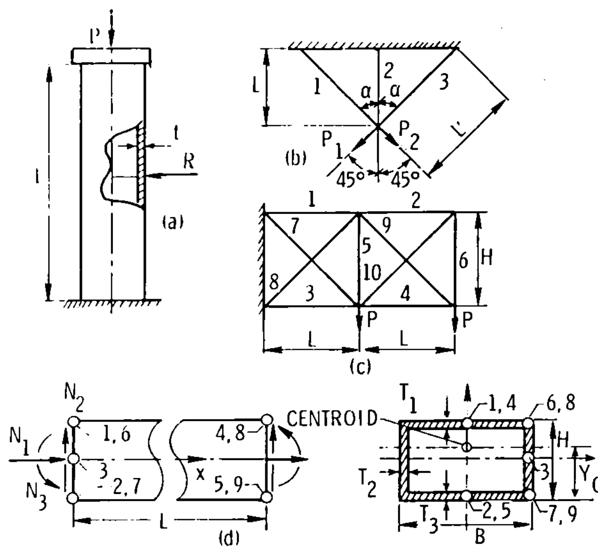

Fig. 1 Numerical examples: (a) tubular column, (b) three-bar truss, (c) ten-bar truss, (d) box beam. Numerical data are given in Appendix.

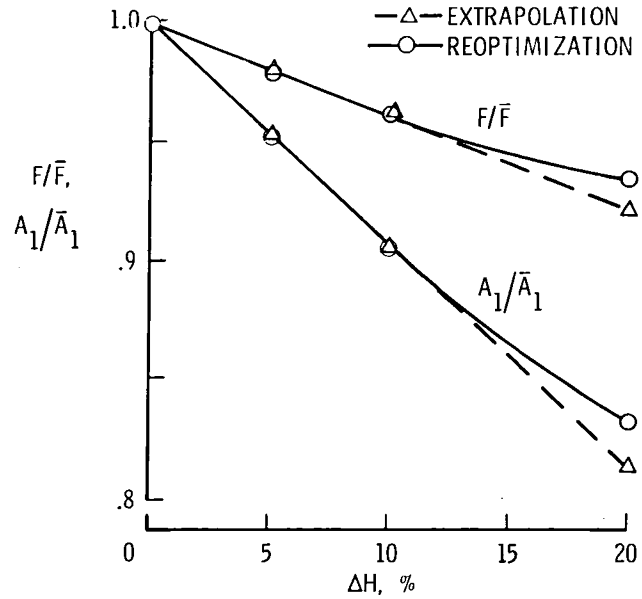

Fig. 2 Ten-bar truss: objective function and one cross-section as functions of parameter H

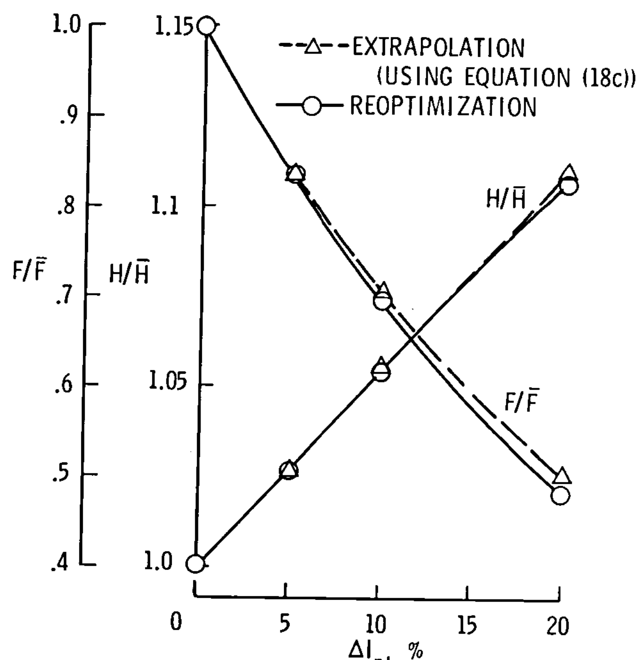

Fig. 3 Box beam: Objective function and width as functions of parameter I_p

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16. Abstract
In the optimum sensitivity problem, one seeks to determine the values of derivatives of the optimal objective function and design variables with respect to those physical quantities which were kept constant as problem parameters during optimization. Examples of these sensitivity derivatives might include derivatives of cross-sectional area and structural mass with respect to allowable stress and derivatives of fuel consumed and wing aspect ratio with respect to aircraft range. Derivation of the sensitivity equations that yield the sensitivity derivatives directly, which avoids the costly and inaccurate "perturb-and-reoptimize" approach, is discussed and solvability of the equations is examined. The equations apply to optimum solutions obtained by direct search methods as well as those generated by procedures of the sequential unconstrained minimization technique (SUMT) class. Applications are discussed for the use of the sensitivity derivatives in extrapolation of the optimal objective function and design variable values for incremented parameters, optimization with multiple objectives, and decomposition of large optimization problems. Several aspects of these applications and verification of the sensitivity equation are presented through numerical examples.

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