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JYOICHI KANEKO

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# $q$-SELBERG INTEGRALS AND MACDONALD POLYNOMIALS

By Jyoichi KANEKO

ABSTRACT. – We consider a Jackson integral with special integrand ($q$-Selberg integral) and give an explicit formula of a system of $q$-difference equations satisfied by it. We also define a kind of hypergeometric function having series expansions in terms of Macdonald polynomials and show that this function satisfies a $q$-difference equation formed by summing up equations of the $q$-difference system above after multiplying each by a suitable factor. We can thus conclude the $q$-Selberg integral to be the hypergeometric function in our sense. This implies, in particular, the $q$-integration formula of Macdonald polynomials due to Kadell [Kad2]. These results reproduce our previous ones [Kan2] if we put $q = t^{\alpha}$ and let $t \to 1$.

## 1. Introduction

The purpose of this paper is to give $q$-analogues of our previous results in [Kan2]. Fix $q$ with $0 < q < 1$ and set $(x)_\infty = (x;q)_\infty = \prod_{i=0}^{\infty}(1 - xq^i)$ and $(x)_a = (x;q)_a = (x)_\infty/(xq^a)_\infty$. For $x = (x_1, \ldots, x_m) \in \mathbb{C}^m$ and $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ put

$$\Phi(t) = \prod_{j=1}^{n} t_j^{\alpha+(j-1)(1-2\gamma)} \frac{(qt_j)_\infty}{(q^\beta t_j)_\infty} \prod_{1\le i<j\le n} \frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty} f(x,t) \tag{1.1}$$

$$f(x,t) = \prod_{1\le i\le m, 1\le j\le n} \frac{(x_i t_j)_\infty}{(q^\mu x_i t_j)_\infty} \tag{1.2}$$

$$\Phi_0(t) = \Phi(t) D(t) \tag{1.3}$$

where $D(t) = \prod_{1\le i<j\le n}(t_i - t_j)$.

For $\xi \in (\mathbb{C}^*)^n$ we put $[0, \xi\infty]_q = \{(q^{s_1}\xi_1, \ldots, q^{s_n}\xi_n) \mid (s_1, \ldots, s_n) \in \mathbb{Z}^n\}$. The Jackson integral of a function $f$ on $(\mathbb{C}^*)^n$ over $[0, \xi\infty]_q$ is defined by

$$\int_{[0,\xi\infty]_q} f(t_1, \ldots, t_n)\tilde{\omega} = (1-q)^n \sum_{s_i \in \mathbb{Z}} f(\xi_1 q^{s_1}, \ldots, \xi_n q^{s_n})$$

$$\tilde{\omega} = \frac{d_q t_1}{t_1} \wedge \cdots \wedge \frac{d_q t_n}{t_n}$$

provided it exists. Similarly, the integral over $[0,\xi]=[0,\xi_1]\times\cdots\times[0,\xi_n]$ is defined by

$$\int_{[0,\xi]} f(t_1,\ldots,t_n)\tilde{\omega}=(1-q)^n\sum_{s_i\in\mathbb{Z}_{\geq 0}} f(\xi_1 q^{s_1},\ldots,\xi_n q^{s_n}).$$

We consider the integral

$$(1.4)\qquad {}_qS_{n,m}(\alpha,\beta,\gamma,\mu;x_1,\ldots,x_m;\xi)\ \ ({}_qS_{n,m}(x)\text{ for short})=\int_{[0,\xi\infty]_q}\Phi_0(t)\tilde{\omega}.$$

Write $q^{\mathbb{Z}}=\{q^k;k\in\mathbb{Z}\}$. We assume the following condition which assures that $\Phi(t)$ has no poles on $[0,\xi\infty]_q$.

$$(\mathcal{C}_1)\qquad \begin{cases} q^{\gamma}\xi_j/\xi_i\notin q^{\mathbb{Z}}\text{ for }1\leq i\leq j\leq n\ \text{ or }2\gamma-1\in\mathbb{Z}_{\geq 0};\\ q^{\beta}\xi_j\notin q^{\mathbb{Z}}\text{ for }1\leq j\leq n\ \text{ or }\beta-1\in\mathbb{Z}_{\geq 0};\\ q^{\mu}x_i\xi_j\notin q^{\mathbb{Z}}\text{ for }1\leq i\leq m,1\leq j\leq n.\end{cases}$$

For convergence of the integral we assume also

$$(\mathcal{C}_2)\qquad \begin{cases} \operatorname{Re}\alpha+n-1>4(n-1)\max\{\operatorname{Re}\gamma,0\},\\ \operatorname{Re}\alpha+n-1+\operatorname{Re}\beta-1+m\operatorname{Re}\mu<-2(n-1)|\operatorname{Re}\gamma|.\end{cases}$$

For the proof of convergence under the conditions $(\mathcal{C}_1),(\mathcal{C}_2)$, *see* the Appendix A.

One of our main results (Theorem 4.11) states that if $\mu=1$ or $-\gamma$, then ${}_qS_{n,m}(x)$ has an explicit series expansion in terms of $A$-type Macdonald polynomials [Ma2] (in the case $\mu=-\gamma$, we need to choose $\xi=\xi_F=:(1,q^{\gamma},\ldots,q^{(n-1)\gamma})$). This precisely corresponds to our previous result that $S_{n,m}(x):=\lim_{q=t^{\alpha},t\to 1}{}_qS_{n,m}(x)$ has an explicit series expansion in terms of Jack polynomials [Kan2, Theorem 5, p. 1106] (*see* also [Ko]).

In the case $f(x,t)\equiv 1\,(m=0)$, ${}_qS_{n,0}(\alpha,\beta,\gamma;\xi)$ has been evaluated by K. Aomoto in [Ao2] (*cf.* also [Ao3]):

$$\begin{aligned}(1.5)\qquad &{}_qS_{n,0}(\alpha,\beta,\gamma;\xi)\\ &=q^{\frac{n(n-1)^2}{2}\gamma}\prod_{j=1}^{n}\xi_j^{\alpha-2(j-1)\gamma}\frac{\vartheta(\xi_j q^{\alpha+\beta-(n-1)\gamma})\vartheta(q^{\beta+(j-1)\gamma})\vartheta(q^{j\gamma})}{\vartheta(q^{\alpha+\beta-(n-j)\gamma})\vartheta(\xi_j q^{\beta})\vartheta(q^{\gamma})}\\ &\quad\times\prod_{1\leq i<j\leq n}\frac{\vartheta(\xi_j/\xi_i)}{\vartheta(q^{\gamma}\xi_j/\xi_i)}\\ &\quad\times\prod_{j=1}^{n}\frac{\Gamma_q(\alpha+n-1-(n+j-2)\gamma)\Gamma_q(\beta+(j-1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\alpha+\beta+n-1-(n-j)\gamma)\Gamma_q(\gamma)}.\end{aligned}$$

Here $\vartheta(x)$ denotes the Jacobi elliptic theta function $(x)_\infty(q/x)_\infty(q)_\infty$ and $\Gamma_q(x)$ denotes the $q$-gamma function $(1-q)^{1-x}(q)_\infty/(q^x)_\infty$. We notice that when $n=1$, this integral is nothing but the Ramanujan's ${}_1\psi_1$ sum [As1]. In Appendix B we shall give a self-contained proof of (1.5) based on $q$-difference equation (the $m = 1$ case of Theorem 4.11). If $\xi = \xi_F$, one can simplify the formula (1.5) to get

$$(1.6) \qquad {}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F) = q^{A_n}\prod_{j=1}^{n}\frac{\Gamma_q(\alpha+n-1-(n+j-2)\gamma)\Gamma_q(\beta+(j-1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\alpha+\beta+n-1-(n-j)\gamma)\Gamma_q(\gamma)},$$

where $A_n = \sum_{j=1}^{n}(\alpha-2(j-1)\gamma+n-1)(j-1)\gamma$. If $\gamma$ is equal to a positive integer $k$, (1.6) reduces to the Askey-Habsieger-Kadell's formula [As, H, Kad1] (*see* Proposition 5.2) :

$$\int_{[0,1]^n}\prod_{j=1}^{n}t_j^x\frac{(qt_j)_\infty}{(q^yt_j)_\infty}\prod_{1\le i<j\le n}t_i^{2k}\Big(q^{1-k}\frac{t_j}{t_i}\Big)_{2k}\tilde{\omega} = q^{kx\binom{n}{2}+2k^2\binom{n}{3}}\prod_{i=1}^{n}\frac{\Gamma_q(x+(n-j)k)\Gamma_q(y+(n-j)k)\Gamma_q(1+jk)}{\Gamma_q(x+y+(2n-j-1)k)\Gamma_q(1+k)},$$

where $\alpha$ and $\beta$ are identified with $x+(n-1)(2k-1)$ and $y$ respectively.

In Section 2 we show that, when $\mu = 1$ or $-\gamma$, ${}_qS_{n,m}(x)$ satisfies a system of $q$-difference equations (Theorem 2.3, (2.26)). This system tends to the holonomic system of differential equations in [Kan2, (9), p. 1088] when $q \to 1$. In Section 3 we define a kind of $q$-hypergeometric function ${}_r\Phi_s^{(q,t)}(a_1,\ldots,a_r;b_1,\ldots,b_s;x_1,\ldots,x_m)$ using $A$-type Macdonald polynomials. By setting $q = t^\alpha$ and $t \to 1$, ${}_r\Phi_s^{(q,t)}(x)$ reduces to the hypergeometric function ${}_rF_s^{(\alpha)}(x)$ defined by using Jack polynomials [Kan2, Ko]. We notice that ${}_2F_1^{(\alpha)}$ is a special case of $BC$-type hypergeometric function of Heckman-Opdam [BO]. In Section 4 we prove that ${}_2\Phi_1^{(q,t)}(a,b;c;x)$ satisfies a $q$-difference equation formed by summing up equations of (2.26) multiplied each by a suitable factor. Therefore the uniqueness properties of solutions of this summed-up equation assures us that ${}_qS_{n,m}(x)$ with $\mu = 1$ or $-\gamma$ is nothing but ${}_2\Phi_1^{(q,t)}(a,b;c;x)$ if we adjust $(q,t)$ and $a,b,c$ suitably (Theorem 4.11). As a consequence we obtain an $q$-integration formula of Macdonald polynomials (Theorem 5.1). In the special case that $\xi = \xi_F$ and $\gamma = k$, a positive integer, this was conjectured and proved by Kadell [Kad2] in a different way from ours (though some details of the proof have been omitted in our copy of [Kad2]). This integration formula in turn gives explicit formulae of the values of ${}_2\Phi_1^{(q,t)}(a,b;c;x)$ at special points (Proposition 5.4). In a separate paper [Kan3] we shall show that Theorem 4.11 implies the constant term identities due to Forrester, Zeilberger and Cooper [F, Z, C].

In a recent preprint [BC], Barsky and Carpentier have given a different proof of our previous result [Kan2, Theorem 5] by employing a new method of G. Anderson. It would be interesting to know whether their argument has $q$-analogous counterpart.

Part of the results of this paper were announced in [Kan1]. The author thanks Prof. K. Aomoto for inspiring discussions and the referee for helpful remarks and suggestions.

## 2. $q$-difference system

2.1. Let $T_{q,t_i} = T_i$ denote the $q$-shift operator on the $i$-th coordinate: $T_i\varphi(t) = \varphi(t_1, \dots, qt_i, \dots, t_n)$ and set

$$\frac{\partial \varphi}{\partial_q t_i} = \frac{(T_i - 1)\varphi}{(q-1)t_i}.$$

Note that

$$\frac{\partial(\varphi\psi)}{\partial_q t_i} = \frac{\partial \varphi}{\partial_q t_i}\psi + T_i\varphi \frac{\partial \psi}{\partial_q t_i}$$

which will be of frequent use. Put $b_i(t) = T_i\Phi(t)/\Phi(t)$. In particular

$$b_1(t) = q^{\alpha}\frac{1-q^{\beta}t_1}{1-qt_1}\prod_{j=2}^{n}\frac{t_1 - q^{-\gamma}t_j}{t_1 - q^{\gamma-1}t_j}\prod_{k=1}^{m}\frac{1-q^{\mu}x_k t_1}{1-x_k t_1}.$$

Define the covariant $q$-difference operator $\nabla_i$ by

$$\nabla_i\varphi(t) = \varphi(t) - b_i(t)T_i\varphi(t).$$

Clearly

$$\int_{[0,\xi\infty]_q}\Phi(t)\varphi(t)\tilde{\omega} = \int_{[0,\xi\infty]_q} T_i(\Phi(t)\varphi(t))\tilde{\omega},$$

provided the integral is convergent. Hence we have

$$\int_{[0,\xi\infty]_q}\Phi(t)\nabla_i\varphi(t)\tilde{\omega} = 0.$$

Let $\mathfrak{S}_n$ denote the symmetric group of degree $n$ and for $\sigma \in \mathfrak{S}_n$ define

$$(\sigma\varphi)(t) = \varphi(t_{\sigma(1)}, \dots, t_{\sigma(n)}).$$

Put

$$U_\sigma(t) = \sigma\Phi(t)/\Phi(t).$$

Then we have

$$U_\sigma(t) = \prod_{\substack{1\le i<j\le n \\ \sigma^{-1}(i)>\sigma^{-1}(j)}}\left(\frac{t_j}{t_i}\right)^{2\gamma-1}\frac{\vartheta(q^{\gamma}t_j/t_i)}{\vartheta(q^{1-\gamma}t_j/t_i)}.$$

By using $\vartheta(qx) = -1/x\,\vartheta(x)$ one can easily verify that $T_iU_\sigma(t) = U_\sigma(t)$ for every $i$. We assert that

$$\int_{[0,\xi\infty]_q}\Phi(t)\sigma(\nabla_1\varphi(t))\tilde{\omega} = 0$$

provided the integral is convergent. In fact

$$\begin{aligned} 0 &= \int_{[0,\xi\infty]_q} \sigma(\Phi(t))\sigma(\nabla_1\varphi(t))\tilde{\omega} \\ &= U_\sigma(\xi)\int_{[0,\xi\infty]_q} \Phi(t)\sigma(\nabla_1\varphi(t))\tilde{\omega}. \end{aligned}$$

Hence for the alternation $\mathcal{A}\varphi = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma\cdot(\sigma\varphi)$, we obtain the fundamental

$$\int_{[0,\xi\infty]_q} \Phi(t)\mathcal{A}(\nabla_1\varphi(t))\tilde{\omega} = 0. \tag{2.1}$$

For complex $Q$ we shall denote by ${}_QD_n(t) = {}_QD(t)$ the product $\prod_{1\le i<j\le n}(t_i - Qt_j)$. The following lemma ([Kad1, (4.10), p. 976], *cf.* also [Ma1, chapter 3, (1.3)]) is crucial to our calculations.

LEMMA 2.1. – *Let $M \subset \{1,\ldots,n\}$. Then*

$$\mathcal{A}\Big(\prod_{j\in M} t_j\, {}_QD(t)\Big) = Q^{e(M)}\frac{(Q;Q)_{|M|}\,(Q;Q)_{n-|M|}}{(1-Q)^n}e_{|M|}(t)D(t) \tag{2.2}$$

*where $e(M) = |\{(i,j)|1\le i<j\le n, i\notin M, j\in M\}|$ and $e_r(t)$ denotes the elementary symmetric function of degree $r$.*

This lemma implies

$$\sum_{M\in\{1,\ldots,n\},|M|=r} Q^{e(M)} = \frac{(Q;Q)_n}{(Q;Q)_r(Q;Q)_{n-r}} \tag{2.3}$$

$$\sum_{M\in\{2,\ldots,n\},|M|=r} Q^{e(M)} = Q^r\frac{(Q;Q)_{n-1}}{(Q;Q)_r(Q;Q)_{n-r-1}}. \tag{2.4}$$

LEMMA 2.2.

$$\begin{aligned} &\mathcal{A}\big({}_QD(t)\prod_{k=2}^n(1-xt_k)\big) \\ &\quad = D(t)\left\{\frac{(Q;Q)_{n-1}}{(1-Q)^{n-1}}(-x)\frac{d(\prod_{k=1}^n(1-xt_k))}{d_Qx} + \frac{(Q;Q)_n}{(1-Q)^n}\prod_{k=1}^n(1-xt_k)\right\}. \end{aligned} \tag{2.5}$$

$$\mathcal{A}\Big(\prod_{k=2}^{n}(t_1 - Q^{-1}t_k)\prod_{2\le h<k\le n}(t_h - Qt_k)\prod_{k=2}^{n}(1 - xt_k)\Big) \tag{2.6}$$
$$= Q^{-(n-1)}\frac{(Q;Q)_{n-1}}{(1-Q)^n}D(t)\Big\{\prod_{k=1}^{n}(1 - xt_k) - \prod_{k=1}^{n}(Q - xt_k)\Big\}.$$

*Proof.* – From (2.2) and (2.4) we have

$$\mathcal{A}\big({}_QD(t)\sum_{M\in\{2,\ldots,n\},|M|=r}\prod_{j\in M}t_j\big) = Q^r(1 - Q^{n-r})\frac{(Q;Q)_{n-1}}{(1-Q)^n}e_r(t)D(t),$$

from which follows (2.5). For the proof of (2.6), observe that

$$\text{LHS of } (2.6) = Q^{-(n-1)}\mathcal{A}\big({}_QD(t)\prod_{k=1}^{n-1}(1 - xt_k)\big).$$

Then (2.6) follows at once since (2.2) and (2.3) imply

$$\mathcal{A}\big({}_QD(t)\sum_{M\in\{1,\ldots,n-1\},|M|=r}\prod_{j\in M}t_j\big) = (1 - Q^{n-r})\frac{(Q;Q)_{n-1}}{(1-Q)^n}e_r(t)D(t).$$

Put

$$A_i(x_1,\ldots,x_m;t) = \prod_{j=1,j\ne i}^{m}\frac{tx_i - x_j}{x_i - x_j}.$$

Expansion in partial fractions gives

$$\prod_{j=1}^{m}\frac{x_j - tz}{x_j - z} = (1-t)\sum_{j=1}^{m}\frac{x_jA_j(x;t)}{x_j - z} + t^m. \tag{2.7}$$

Replacing $z$ by $1/z$ and $m$ by $m-1$, we have also

$$\prod_{j=1,j\ne i}^{m}\frac{1 - x_jz/t}{1 - x_jz} = (t-1)t^{1-m}\sum_{j=1,j\ne i}^{m}\frac{A_j(x;t)(x_j - x_i)}{(1 - x_jz)(tx_j - x_i)} + t^{1-m}. \tag{2.8}$$

Specializing $z$ suitably in (2.7) and (2.8), we obtain

$$\sum_{i=1}^{m}A_i(x;t) = \frac{1-t^m}{1-t} \tag{2.9}$$

$$\sum_{j=1,j\neq i}^{m} A_j(x;t)\frac{x_j-x_i}{tx_j-x_i} = \frac{1-t^{m-1}}{1-t} \tag{2.10}$$

$$\sum_{j=1,j\neq i}^{m} A_j(x;t)\frac{x_j}{tx_j-x_i} = \frac{A_i(x;t)-t^{m-1}}{1-t} \tag{2.11}$$

$$\sum_{j=1,j\neq i}^{m} A_j(x;t)\frac{x_i}{tx_j-x_i} = \frac{A_i(x;t)-1}{1-t}. \tag{2.12}$$

Multiplying both sides of (2.8) by $(1-x_iz)^{-1}$ and using

$$\frac{x_j-x_i}{(1-x_iz)(1-x_jz)} = \frac{x_j}{1-x_jz} - \frac{x_i}{1-x_iz}$$

and (2.12), we get

$$\begin{aligned}\frac{1}{1-x_iz}\prod_{j=1,j\neq i}^{m}\frac{1-x_jz/t}{1-x_jz} &= (t-1)t^{1-m} \\ &\times \sum_{j=1,j\neq i}^{m}\frac{A_j(x;t)x_j}{(1-x_jz)(tx_j-x_i)} + t^{1-m}\frac{A_i(x;t)}{1-x_iz}.\end{aligned} \tag{2.13}$$

In what follows $Q$ stands for $q^\gamma$.

2.2. CASE OF $\mu = 1$. – In this case we see

$$f(x,t) = \prod_{1\le j\le m, 1\le k\le n}(1-x_jt_k).$$

Put

$$\varphi_i(x,t) = \frac{1-t_1}{1-x_it_1}\prod_{j=1,j\neq i}^{m}\frac{1-q^{-1}x_jt_1}{1-x_jt_1}\,{}_QD(t),$$

so that

$$b_1(t)T_1\varphi_i(x,t) = q^{\alpha+n-1}\frac{1-q^\beta t_1}{1-x_it_1}\prod_{k=2}^{n}(t_1-Q^{-1}t_k)\prod_{2\le h<k\le n}(t_h-Qt_k).$$

We want to calculate $\mathcal{A}(\varphi_i)$ and $\mathcal{A}(b_1T_1\varphi_i)$. Since

$$1-t_1 = \frac{1-x_it_1}{x_i} + 1 - \frac{1}{x_i}$$

we see

$$\mathcal{A}(\varphi_i) = \frac{1}{x_i}\mathcal{A}\Big(\prod_{j=1,j\neq i}^{m} \frac{1-q^{-1}x_jt_1}{1-x_jt_1}{}_QD(t)\Big) + \Big(1-\frac{1}{x_i}\Big)\mathcal{A}\Big(\frac{1}{1-x_it_1}\prod_{j=1,j\neq i}^{m} \frac{1-q^{-1}x_jt_1}{1-x_jt_1}{}_Q\mathcal{D}(t)\Big). \tag{2.14}$$

Substituting $t = q$ and $z = t_1$ in (2.8) and (2.13) gives

$$\prod_{j=1,j\neq i}^{m} \frac{1-q^{-1}x_jt_1}{1-x_jt_1} = (q-1)q^{1-m}\sum_{j=1,j\neq i}^{m} \frac{A_j(x;q)(x_j-x_i)}{(1-x_jt_1)(qx_j-x_i)} + q^{1-m} \tag{2.15}$$

$$\frac{1}{1-x_it_1}\prod_{j=1,j\neq i}^{m} \frac{1-q^{-1}x_jt_1}{1-x_jt_1} = (q-1)q^{1-m}\sum_{j=1,j\neq i}^{m} \frac{A_j(x;q)x_j}{(1-x_jt_1)(qx_j-x_i)} + q^{1-m}\frac{A_i(x;q)}{1-x_it_1}. \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14) and applying (2.5), (2.10) and (2.11), we have

$$\begin{aligned}\frac{f(x,t)}{D(t)}\mathcal{A}(\varphi_i) = {}& \frac{(Q;Q)_n}{(1-Q)^n}f(x,t)\\ &+ q^{1-m}(1-q)\frac{(Q;Q)_{n-1}}{(1-Q)^{n-1}}\sum_{j=1,j\neq i}^{m}\frac{x_j(x_j-1)}{qx_j-x_i}A_j(x;q)\frac{\partial f(x,t)}{\partial_Q x_j}\\ &+ q^{1-m}\frac{(Q;Q)_{n-1}}{(1-Q)^{n-1}}(1-x_i)A_i(x;q)\frac{\partial f(x,t)}{\partial_Q x_i}.\end{aligned}$$

Hence

$$\begin{aligned} T_{Q,x_i}&\Big(\frac{f(x,t)}{D(t)}\mathcal{A}(\varphi_i)\Big)\\ &= \frac{(Q;Q)_{n-1}}{(1-Q)^{n-1}}\Big\{\frac{(1-Q^n)}{1-Q}\Big(f(x,t)+(Q-1)x_i\frac{\partial f}{\partial_Q x_i}\Big)\\ &\quad + q^{1-m}(1-q)\sum_{j=1,j\neq i}^{m}\frac{x_j(x_j-1)}{qx_j-Qx_i}T_{Q,x_i}(A_j(x;q))\\ &\quad \Big(\frac{\partial f}{\partial_Q x_j}+(Q-1)x_i\frac{\partial^2 f}{\partial_Q x_i\partial_Q x_j}\Big)\\ &\quad + q^{1-m}(1-Qx_i)T_{Q,x_i}(A_i(x;q))\Big(\frac{\partial f}{\partial_Q x_i}+(Q-1)x_i\frac{\partial^2 f}{\partial_Q x_i^2}\Big)\Big\}.\end{aligned} \tag{2.17}$$

Next we calculate $\mathcal{A}(b_1T_1\varphi_i)$. Since

$$1-q^\beta t_1=q^\beta\frac{1-x_it_1}{x_i}+1-\frac{q^\beta}{x_i},$$

we see

$$\begin{aligned}\mathcal{A}(b_1T_1\varphi_i)&=q^{\alpha+\beta+n-1}\frac{1}{x_i}\mathcal{A}\Big(\prod_{k=2}^{n}(t_1-Q^{-1}t_k)\prod_{2\le h<k\le n}(t_h-t_k)\Big)\\&\quad+q^{\alpha+n-1}\Big(1-\frac{q^\beta}{x_i}\Big)\mathcal{A}\Big((1-x_it_1)^{-1}\prod_{k=2}^{n}(t_1-Q^{-1}t_k)\prod_{2\le h<k\le n}(t_h-t_k)\Big).\end{aligned}$$

Applying (2.6), we have

$$\begin{aligned}\frac{f(x,t)}{D(t)}\mathcal{A}(b_1T_1\varphi_i)&=q^{\alpha+\beta+n-1}Q^{-(n-1)}\frac{(Q;Q)_n}{(1-Q)^n}\frac{f(x,t)}{x_i}\\&\quad+q^{\alpha+n-1}Q^{-(n-1)}\frac{(Q;Q)_{n-1}}{(1-Q)^{n-1}}\\&\quad\times\Big(1-\frac{q^\beta}{x_i}\Big)\prod_{\substack{1\le j\le m,j\ne i\\1\le k\le n}}(1-x_jt_k)\Big\{\prod_{k=1}^{n}(1-x_it_k)-\prod_{k=1}^{n}(Q-x_it_k)\Big\}.\end{aligned}$$

Hence

$$\begin{aligned}(2.18)\ &T_{Q,x_i}\Big(\frac{f(x,t)}{D(t)}\mathcal{A}(b_1T_1\varphi_i)\Big)\\&=q^{\alpha+\beta+n-1}Q^{-n}\frac{(Q;Q)_n}{(1-Q)^n}\frac{1}{x_i}\Big(f(x,t)+(Q-1)x_i\frac{\partial f}{\partial_Q x_i}\Big)\\&\quad+q^{\alpha+n-1}Q^{-(n-1)}\frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big(1-\frac{q^\beta}{Qx_i}\Big)\\&\Big((Q-1)x_i\frac{\partial f}{\partial_Q x_i}+(1-Q^n)f(x,t)\Big)\\&=q^{\alpha+n-1}Q^{-(n-1)}\frac{(Q;Q)_{n-1}}{(1-Q)^{n-1}}\Big\{\frac{1-Q^n}{1-Q}f(x,t)+(q^\beta Q^{n-1}-x_i)\frac{\partial f}{\partial_Q x_i}\Big\}.\end{aligned}$$

It is clear from (2.1) that

$$\int_{[0,\xi\infty]_q}T_{Q,x_i}(\Phi(t)\mathcal{A}(\nabla_1\varphi(t)))\tilde{\omega}=0. \tag{2.19}$$

Substituting (2.17) and (2.18) into (2.19), we arrive at ($S = {}_qS_{n,m}(x)$)

$$
\begin{aligned}
(2.20)\ 0 = {} & q^{-(\alpha+n-1)}Q^{-1}x_i(1-Qx_i)T_{Q,x_i}(A_i(x;q))\frac{\partial^2 S}{\partial_Q x_i^2} \\
& + q^{-(\alpha+n-1)}(1-q)\sum_{j=1,j\neq i}^{m}\frac{Q^{-1}x_ix_j(1-x_j)}{Qx_i-qx_j}T_{Q,x_i}(A_j(x;q))\frac{\partial^2 S}{\partial_Q x_i\partial_Q x_j} \\
& + \Big\{\frac{q^{m-1}-c}{1-Q}+\frac{1}{1-Q}\big((1-a)(1-b)q^{m-1} \\
& - (q^{m-1}-abQ)\big)q^{-\beta}Qx_i\Big\}q^{\beta}Q^{-1}\frac{\partial S}{\partial_Q x_i} \\
& + \frac{1-q}{1-Q}\Big\{\frac{1-T_{Q,x_i}(A_i(x;q))}{1-q}(c-abQq^{-\beta}Qx_i)q^{\beta}Q^{-1}\frac{\partial S}{\partial_Q x_i} \\
& - \sum_{j=1,j\neq i}^{m}\frac{q^{-(\alpha+n-1)}Q^{-1}x_j(1-x_j)}{Qx_i-qx_j}T_{Q,x_i}(A_j(x;q))\frac{\partial S}{\partial_Q x_j}\Big\} \\
& - \frac{(1-a)(1-b)q^{m-1}}{(1-Q)^2}S
\end{aligned}
$$

where

$$a = Q^{-n}, b = q^{-(\alpha+n-1)}Q^{n-1}, c = q^{-(\alpha+\beta+n-1)}.$$

Now we change the variables:

$$x_i = q^{\beta}Q^{-1}y_i, \qquad i = 1,\ldots,m.$$

Note that

$$T_Q, x_i = T_Q, y_i, \qquad \frac{\partial}{\partial_Q x_i} = q^{-\beta}Q\frac{\partial}{\partial_Q y_i},$$

and

$$A_i(x;q) = A_i(y;q).$$

(2.20) is transformed into

$$
\begin{aligned}
(2.21)\ & y_i(c-abQy_i)T_{Q,y_i}(A_i(y;q))\frac{\partial^2 S}{\partial_Q y_i^2} + (1-q) \\
& \times\sum_{j=1,j\neq i}^{m}\frac{y_iy_j(c-aby_j)}{Qy_i-qy_j}T_{Q,y_i}(A_j(y;q))\frac{\partial^2 S}{\partial_Q y_i\partial_Q y_j} \\
& + \Big\{\frac{q^{m-1}-c}{1-Q}+\frac{1}{1-Q}\big((1-a)(1-b)q^{m-1}-(q^{m-1}-abQ)\big)y_i\Big\}\frac{\partial S}{\partial_Q y_i}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1-q}{1-Q}\Big\{\frac{1-T_{Q,y_i}(A_i(y;q))}{1-q}(c-abQy_i)\frac{\partial S}{\partial_Q y_i} \\
&-\sum_{j=1,j\neq i}^{m}\frac{y_j(c-aby_j)}{Qy_i-qy_j}T_{Q,y_i}(A_j(y;q))\frac{\partial S}{\partial_Q y_j}\Big\} \\
&-\frac{(1-a)(1-b)q^{m-1}}{(1-Q)^2}S=0.
\end{aligned}
$$

2.3. Case of $\mu=-\gamma$. – In this case we have

$$f(x,t)=\prod_{1\le j\le m,1\le k\le n}\frac{(x_jt_k)_\infty}{(Q^{-1}x_jt_k)_\infty}.$$

Put

$$\varphi_i(x,t)=\frac{1-t_1}{1-(qQ)^{-1}x_it_1}\,{}_QD(t),$$

so that

$$b_1(t)T_1\varphi_i(x,t)=q^{\alpha+n-1}\frac{1-q^{\beta}t_1}{1-Q^{-1}x_it_1}\prod_{k=2}^{n}\frac{t_1-q^{-\gamma}t_k}{t_1-q^{\gamma}t_k}\prod_{j=1}^{m}\frac{1-Q^{-1}x_jt_1}{1-x_jt_1}\,{}_QD(t).$$

Then one can proceed in a similar way as in the case of $\mu=1$. We have (we omit the details of calculation)

$$\frac{f(x,t)}{D(t)}\mathcal{A}(\varphi_i)=\frac{(Q;Q)_n}{(1-Q)^n}\frac{qQ}{x_i}f(x,t)+\frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big(1-\frac{qQ}{x_i}\Big)\big(T_{q^{-1},x_i}f(x,t)-Q^nf(x,t)\big),$$

so that

(2.22)
$$T_{q,x_i}\Big(\frac{f(x,t)}{D(t)}\mathcal{A}(\varphi_i)\Big)=\frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big\{(1-Q^n)f(x,t)+(q-1)Q(1-Q^{n-1}x_i)\frac{\partial f}{\partial_q x_i}\Big\}.$$

We have also

$$
\begin{aligned}
&\frac{f(x,t)}{D(t)}\mathcal{A}(b_1T_1\varphi_i) \\
&\quad=q^{\alpha+n-1}Q^{2-m-n}\Big\{Q^{m-1}\frac{(Q;Q)_n}{(1-Q)^n}f(x,t) \\
&\quad-(1-q)Q^n\frac{(Q;Q)_{n-1}}{(1-Q)^n}(q^{\beta}-x_i)A_i(x;Q)\frac{\partial f}{\partial_q x_i} \\
&\quad-(1-q)Q^n\frac{(Q;Q)_{n-1}}{(1-Q)^{n-1}}\sum_{j=1,j\neq i}^{m}\frac{x_j(q^{\beta}-x_j)}{x_i-Qx_j}A_j(x;Q)\frac{\partial f}{\partial_q x_j}\Big\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2.23)\quad & T_{q,x_i}\Big(\frac{f(x,t)}{D(t)}\mathcal{A}(b_1T_1\varphi_i)\Big)\\
&= q^{\alpha+n-1}Q^{2-m-n}\frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big\{(1-Q^n)Q^{m-1}\Big(f(x,t)+(q-1)x_i\frac{\partial f}{\partial_q x_i}\Big)\\
&- q(1-q)Q^n(q^{\beta-1}-x_i)T_{q,x_i}(A_i(x;Q))\Big(\frac{\partial f}{\partial_q x_i}+(q-1)x_i\frac{\partial^2 f}{\partial_q x_i^2}\Big)\\
&- (1-q)Q^n(1-Q)\sum_{j=1,j\neq i}^{m}\frac{x_j(q^\beta-x_j)}{qx_i-Qx_j}T_{q,x_i}(A_j(x;Q))\\
&\times\Big(\frac{\partial f}{\partial_q x_j}+(q-1)x_i\frac{\partial^2 f}{\partial_q x_i\partial_q x_j}\Big)\Big\}.
\end{aligned}
$$

Substituting (2.22) and (2.23) into (2.19) gives the equation corresponding to (2.20):

$$
\begin{aligned}
(2.24)\quad 0 &= q^{\alpha+n}Qx_i(q^{\beta-1}-x_i)T_{q,x_i}(A_i(x;Q))\frac{\partial^2 S}{\partial_q x_i^2}\\
&+ q^{\alpha+n-1}(1-Q)\sum_{j=1,j\neq i}^{m}\frac{Qx_ix_j(q^\beta-x_j)}{qx_i-Qx_j}T_{q,x_i}(A_j(x;Q))\frac{\partial^2 S}{\partial_q x_i\partial_q x_j}\\
&+ \Big\{\frac{Q^m}{1-q}+\frac{Q^{m-1}}{1-q}(-Q^n-q^{\alpha+n-1}Q^{-(n-1)}+q^{\alpha+n-1}Q)x_i\Big\}\frac{\partial S}{\partial_q x_i}\\
&- \frac{q^{\alpha+n}Q}{1-q}(q^{\beta-1}-x_i)T_{q,x_i}(A_i(x;Q))\frac{\partial S}{\partial_q x_i}\\
&- \frac{1-Q}{1-q}q^{\alpha+n-1}Q\sum_{j=1,j\neq i}^{m}\frac{x_j(q^\beta-x_j)}{qx_i-Qx_j}T_{q,x_i}(A_j(x;Q))\frac{\partial S}{\partial_q x_j}\\
&- \frac{(1-Q^n)(1-q^{\alpha+n-1}Q^{-(n-1)})Q^{m-1}}{(1-q)^2}S.
\end{aligned}
$$

Hence changing variables as

$$x_i = Qy_i,\quad i=1,\dots,m$$

yields

$$
\begin{aligned}
(2.25)\quad & y_i(c-abqy_i))T_{q,y_i}(A_i(y;Q))\frac{\partial^2 S}{\partial_q y_i^2}\\
&+ (1-Q)\sum_{j=1,j\neq i}^{m}\frac{y_iy_j(c-aby_j)}{qy_i-Qy_j}T_{q,y_i}(A_j(y;Q))\frac{\partial^2 S}{\partial_q y_i\partial_q y_j}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{\frac{Q^{m-1}-c}{1-q}+\frac{1}{1-q}\big((1-a)(1-b)Q^{m-1}-(Q^{m-1}-abq)\big)y_i\Big\}\frac{\partial S}{\partial_q y_i}\\
&+\frac{1-Q}{1-q}\Big\{\frac{1-T_{q,y_i}(A_i(y;Q))}{1-Q}(c-abqy_i)\frac{\partial S}{\partial_q y_i}\\
&\quad-\sum_{j=1,j\neq i}^{m}\frac{y_j(c-aby_j)}{qy_i-Qy_j}T_{q,y_i}(A_j(y;Q))\frac{\partial S}{\partial_q y_j}\Big\}\\
&-\frac{(1-a)(1-b)Q^{m-1}}{(1-q)^2}S=0
\end{aligned}
$$

where

$$
a=Q^n,\qquad b=q^{\alpha+n-1}Q^{-(n-1)},\qquad c=q^{\alpha+\beta+n-1}.
$$

We have thus proved

THEOREM 2.3. – *Assume $\mu=1$ or $-\gamma$. Then ${}_qS_{n,m}(\alpha,\beta,\gamma,\mu;x;\xi)$ satisfies the following system of q-difference equations ($T_i=T_{q,x_i}$).*

$$
\begin{aligned}
(2.26)\quad &x_i(c-abqx_i)T_i(A_i(x;t))\frac{\partial^2 S}{\partial_q x_i^2}\\
&+(1-q)\sum_{j=1,j\neq i}^{m}\frac{x_ix_j(c-abx_j)}{qx_i-tx_j}T_i(A_j(x;t))\frac{\partial^2 S}{\partial_q x_i\partial_q x_j}\\
&+\Big\{\frac{t^{m-1}-c}{1-q}+\frac{1}{1-q}\big((1-a)(1-b)t^{m-1}-(t^{m-1}-abq)\big)x_i\Big\}\frac{\partial S}{\partial_q x_i}\\
&+\frac{1-t}{1-q}\Big\{\frac{1-T_i(A_i(x;t))}{1-t}(c-abqx_i)\frac{\partial S}{\partial_q x_i}\\
&\quad-\sum_{j=1,j\neq i}^{m}\frac{x_j(c-abx_j)}{qx_i-tx_j}T_i(A_j(x;t))\frac{\partial S}{\partial_q x_j}\Big\}\\
&-\frac{(1-a)(1-b)t^{m-1}}{(1-q)^2}S=0,\quad i=1,\dots,m,
\end{aligned}
$$

*where if $\mu=1$, then put $(q,t)=(Q,q), a=Q^{-n}, b=q^{-(\alpha+n-1)}Q^{n-1}, c=q^{-(\alpha+\beta+n-1)}$ and change $x_i$ with $q^{-\beta}Qx_i, 1\le i\le m$. If $\mu=-\gamma$, then put $(q,t)=(q,Q)$, $a=Q^n, b=q^{\alpha+n-1}Q^{-(n-1)}, c=q^{\alpha+\beta+n-1}$ and change $x_i$ with $Q^{-1}x_i, 1\le i\le m$.*

*Remark.* – In the theorem above the change of variables means that we change only the variables of the $q$-difference equations. We *do not* change the variables of the unknown function $S$. The same remark applies also to the following Theorem 2.4 and 2.5.

2.4. VARIANTS. – One can calculate a system of $q$-difference equations satisfied by the integral ${}_qS_{n,m}(\alpha,\beta,\gamma;\mu;x_1^{-1},\dots,x_m^{-1};\xi)$ provided $\mu=1$ or $-\gamma$ in the same way as in the case of Theorem 2.3. If $\mu=1$, then put

$$\varphi_i(x,t)=\frac{1-t_1}{1-t_1/x_i}{}_QD(t).$$

We have

$$\frac{f(x,t)}{D(t)}\mathcal{A}(\varphi_i)=\frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big\{(1-Q^n)x_if(x,t)+(1-x_i)\big(T_{Q^{-1},x_i}f(x,t)-Q^nf(x,t)\big)\Big\}$$

and

$$\begin{aligned}&\frac{f(x,t)}{D(t)}\mathcal{A}(b_1T_1\varphi_i)\\&\quad=q^{\alpha+n-1}Q^{1-n}\frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big\{(1-Q^n)f(x,t)+(1-Q)Q^nx_i(1-q^\beta x_i)A_i(x;q)\frac{\partial f}{\partial_Qx_i}\\&\quad+(1-q)(1-Q)Q^n\sum_{j=1,j\neq i}^m\frac{x_ix_j(1-q^\beta x_j)}{x_i-qx_j}A_j(x;q)\frac{\partial f}{\partial_Qx_j}\Big\}.\end{aligned}$$

Substituting these into (2.19) gives

$$\begin{aligned}(2.27)\quad 0&=q^{\alpha+n-1}Q^2x_i^2(1-q^\beta Qx_i)T_{Q,x_i}(A_i(x;q))\frac{\partial^2S}{\partial_Qx_i^2}\\&+q^{\alpha+n-1}(1-q)Q^2\sum_{j=1,j\neq i}^m\frac{x_i^2x_j(1-q^\beta x_j)}{Qx_i-qx_j}T_{Q,x_i}(A_j(x;q))\frac{\partial^2S}{\partial_Qx_i\partial_Qx_j}\\&+\frac{x_i}{1-Q}\Big\{Q(Q^{n-1}-x_i)+q^{\alpha+n-1}Q^{-(n-1)}(1-Q^n)\Big\}\frac{\partial S}{\partial_Qx_i}\\&-\frac{q^{\alpha+n-1}Q^2}{1-Q}x_i(1-q^\beta Qx_i)T_{Q,x_i}(A_i(x;q))\frac{\partial S}{\partial_Qx_i}\\&-\frac{(1-q)q^{\alpha+n-1}Q^2}{1-Q}\sum_{j=1,j\neq i}^m\frac{x_ix_j(1-q^\beta x_j)}{Qx_i-qx_j}T_{Q,x_i}(A_j(x;q))\frac{\partial S}{\partial_Qx_j}\\&+\frac{(1-Q^n)(1-q^{\alpha+n-1}Q^{-(n-1)})}{(1-Q)^2}S.\end{aligned}$$

If $\mu=-\gamma$, then put

$$\varphi_i(x,t)=\frac{1-t_1}{1-(qQ)^{-1}t_1/x_i}\prod_{j=1,j\neq i}^m\frac{1-q^{-1}t_1/x_j}{1-(qQ)^{-1}t_1/x_j}{}_QD(t).$$

We have

$$\frac{f(x,t)}{D(t)}\mathcal{A}(\varphi_i) = \frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big\{(1-Q^n)f(x,t) - (1-q)x_i(1-qQx_i)A_i(x;Q)\frac{\partial f}{\partial_q x_i} - (1-q)(1-Q)\sum_{j=1,j\neq i}^{m}\frac{x_ix_j(1-qQx_j)}{x_i-Qx_j}A_j(x;Q)\frac{\partial f}{\partial_q x_j}\Big\}$$

and

$$\frac{f(x,t)}{D(t)}\mathcal{A}(b_1\mathcal{T}_1\varphi_i) = q^{\alpha+n-1}Q^{-(n-1)}\frac{(Q;Q)_{n-1}}{(1-Q)^n}\Big\{(1-Q^n)q^\beta x_i f(x,t) + (1-q^\beta x_i)\frac{f(x,t)}{\prod_{k=1}^n(1-t_k/x_i)}\left(\prod_{k=1}^n(1-t_k/x_i) - \prod_{k=1}^n(Q-t_k/x_i)\right)\Big\}.$$

Substituting these into (2.19) yields

$$\begin{aligned}0 &= qx_i^2(1-q^2Qx_i)T_{q,x_i}(A_i(x;Q))\frac{\partial^2 S}{\partial_q x_i^2}\\ &+(1-Q)\sum_{j=1,j\neq i}^{m}\frac{qx_i^2x_j(1-qQx_j)}{qx_i-Qx_j}T_{q,x_i}(A_j(x;Q))\\ &\times\frac{\partial^2 S}{\partial_q x_i\partial_q x_j} + \frac{x_i}{1-q}\Big\{q^{\alpha+n-1}Q^{-(n-1)}(1-q^{\beta+1}Q^nx_i)\\ &-\big(1-Q^n+(1-q^2Qx_i)q\big)\Big\}\frac{\partial S}{\partial_q x_i}\\ &+\frac{qx_i(1-q^2Qx_i)}{1-q}\big(1-T_{q,x_i}(A_i(x;Q))\big)\frac{\partial S}{\partial_q x_i}\\ &-\frac{1-Q}{1-q}\sum_{j=1,j\neq i}^{m}\frac{qx_ix_j(1-qQx_j)}{qx_i-Qx_j}T_{q,x_i}(A_j(x;Q))\frac{\partial S}{\partial_q x_j}\\ &+\frac{(1-Q^n)(1-q^{\alpha+n-1}Q^{-(n-1)})}{(1-q)^2}S.\end{aligned}$$

From (2.27) and (2.28) we can conclude

THEOREM 2.4. – *Assume $\mu = 1$ or $-\gamma$. Then ${}_qS_{n,m}(\alpha,\beta,\gamma,\mu;x_1^{-1},\dots,x_m^{-1};\xi)$ satisfies the following system of q-difference equations.*

$$x_i(c-abqx_i)T_i(A_i(x;t))\frac{\partial^2 S}{\partial_q x_i^2} + (1-q)\sum_{j=1,j\neq i}^{m}\frac{x_ix_j(c-abx_j)}{qx_i-tx_j}T_i(A_j(x;t))\frac{\partial^2 S}{\partial_q x_i\partial_q x_j}$$

$$+\Big\{\frac{at^{m-1}-c/q+ac/q-c}{1-q}+\frac{1}{1-q}(abq-a^2t^{m-1})x_i\Big\}\frac{\partial S}{\partial_q x_i}$$
$$+\frac{1-t}{1-q}\Big\{\frac{1-T_i(A_i(x;t))}{1-t}(c-abqx_i)\frac{\partial S}{\partial_q x_i}-\sum_{j=1,j\neq i}^{m}\frac{x_j(c-abx_j)}{qx_i-tx_j}T_i(A_j(x;t))\frac{\partial S}{\partial_q x_j}\Big\}$$
$$-\frac{(1-a)(at^{m-1}-c/q)}{(1-q)^2}\frac{1}{x_i}S=0,\qquad i=1,\ldots,m,$$

*where if $\mu=1$, then put $(q,t)=(Q,q)$, $a=Q^{-n}, b=q^{-(\alpha+n-1)}Q^{n-1}, c=q^{-(\alpha+\beta+n-1)}$ and change $x_i$ with $q^{-\beta}Qx_i, 1\le i\le m$. If $\mu=-\gamma$, then put $(q,t)=(q,Q)$, $a=Q^n, b=q^{\alpha+n-1}Q^{-(n-1)}, c=q^{\alpha+\beta+n-1}$ and change $x_i$ with $Q^{-1}x_i, 1\le i\le m$.*

From this theorem one can derive the following theorem by straightforward calculation.

THEOREM 2.5. – *Assume $\mu=1$, or $-\gamma$. Then $(x_1\cdots x_m)^{\mu n}{}_qS_{n,m}(\alpha,\beta,\gamma,\mu;x_1^{-1},\ldots,x_m^{-1};\xi)$ satisfies the system* (2.26) *in which if $\mu=1$, then put $(q,t)=(Q,q)$, $a=Q^{-n}$, $b=q^{-(\alpha+n-1)}Q^{n-1}, c=q^{-(\alpha+\beta+n-1)}$ and change $x_i$ with $q^{-\beta}Qx_i, 1\le i\le m$. If $\mu=-\gamma$, then put $(q,t)=(q,Q)$, $a=Q^n, b=q^{\alpha+n-1}Q^{-(n-1)}, c=q^{\alpha+\beta+n-1}$ and change $x_i$ with $Q^{-1}x_i, 1\le i\le m$.*

## 3. $q$-Hypergeometric functions

3.1. MACDONALD POLYNOMIALS. – We first recall the definition of Macdonald polynomials [Ma2]. Let $\lambda=(\lambda_1,\lambda_2,\ldots)$ be a *partition* and $\lambda'=(\lambda_1',\lambda_2',\ldots)$ the conjugate partition. The number $\lambda_1'$ of parts of $\lambda$ is denoted by $\ell(\lambda)$, called the length of $\lambda$. If $\lambda$ has $m_1$ parts equal to 1, $m_2$ parts equal to 2, and so on, we write $\lambda=(1^{m_1}2^{m_2}\cdots)$ and denote $\prod_{r\ge1}(r^{m_r}m_r!)=z_\lambda$. We write $|\lambda|=\sum\lambda_i$. If $\mu$ is another partition, then write $\mu\le\lambda$ when $|\lambda|=|\mu|$ and $\mu_1+\cdots+\mu_i\le\lambda_1+\cdots+\lambda_i$ for all $i$. Given a partition $\lambda=(\lambda_1,\ldots,\lambda_m)$ of length $\le m$, the monomial symmetric polynomial $m_\lambda(x_1,\ldots,x_m)$ is defined by

$$m_\lambda(x_1,\ldots,x_m)=\sum x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_m^{\lambda_m},$$

where the sum is over all distinct monomials obtainable from $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_m^{\lambda_m}$ by permutations of the $x$'s. In particular when $\lambda=(r)$ we have the $r$-th power sum:

$$m_{(r)}=p_r(x_1,\ldots,x_m)=\sum_{i=1}^{m}x_i^r.$$

For each partition $\lambda$, we set $p_\lambda=p_{\lambda_1}p_{\lambda_2}\cdots$.

Let $q,t$ be independent indeterminates and $\mathbb{Q}(q,t)$ be the field of rational functions in $q$ and $t$. We have the fundamental ([Ma2, (2.8)]).

THEOREM 3.1. – *For each partition $\lambda$ of length $\le m$ there exists a unique symmetric polynomial $P_\lambda(x_1,\ldots,x_m;q,t)$ with coefficients in $\mathbb{Q}(q,t)$ satisfying*

$$P_\lambda=\sum_{\mu\le\lambda}u_{\lambda\mu}m_\mu \tag{3.1}$$

*where* $u_{\lambda\mu} \in \mathbb{Q}(q,t)$ *and* $u_{\lambda\lambda} = 1$; *and*

$$D_1^{(q,t)} P_\lambda = e_\lambda P_\lambda \tag{3.2}$$

*where* $D_1^{(q,t)}$ *and* $e_\lambda(q,t)$ *are defined by*

$$D_1^{(q,t)} = \sum_{i=1}^{m} A_i(x;t)\, x_i \frac{\partial}{\partial_q x_i}, \qquad e_\lambda(q,t) = \sum_{i=1}^{m} \frac{1-q^{\lambda_i}}{1-q} t^{m-i}. \tag{3.3}$$

We understand that $P_\lambda(x_1,\dots,x_m) = 0$ if $\ell(\lambda) \geq m+1$. One can readily verify that $P_\lambda(x_1,\dots,x_r,0,\dots,0) = P_\lambda(x_1,\dots,x_r)$.

Denote the ring of symmetric polynomials in $x_1,\dots,x_m$ over the field $F = \mathbb{Q}(q,t)$ by $\Lambda_{m,F}$. Define a scalar product $\langle\ ,\ \rangle$ on $\Lambda_{m,F}$ by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda(q,t)$$

where

$$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

Note that $D_1^{(q,t)}$ has the following properties ([Ma2, (2.7.1)-(2.7.3)]):

$$D_1^{(q,t)} m_\lambda = \sum_{\mu \leq \lambda} c_{\lambda\mu} m_\mu$$

for each partition $\lambda$ of length $\leq m$;

$$\langle D_1^{(q,t)} f, g \rangle = \langle g, D_1^{(q,t)} g \rangle$$

for all $f, g \in \Lambda_{m,F}$;

$$\lambda \neq \mu \Rightarrow c_{\lambda\lambda} \neq c_{\mu\mu}.$$

From these properties one can deduce that the condition (3.2) in the Theorem 3.1 can be replaced by

$$\langle P_\lambda, P_\mu \rangle = 0 \qquad if \quad \lambda \neq \mu.$$

We shall need a kind of specialization formula of $P_\lambda$. Let $u$ be a new indeterminate and define a homomorphism

$$\epsilon_{u,t} : \Lambda_{m,F} \to F[u]$$

by

$$\epsilon_{u,t}(p_r) = \frac{1-u^r}{1-t^r} \tag{3.4}$$

for each $r \geq 1$. Then we see

$$\epsilon_{t^m,t}(f) = f(1, t, \ldots, t^{m-1}).$$

Consider the *diagram* of $\lambda$ in which the rows and columns are arranged as in a matrix, with the *i*th row consisting of $\lambda_i$ boxes. For each square $s = (i, j)$ in the diagram of $\lambda$, let

$$\begin{aligned} a(s) &= \lambda_i - j, & a'(s) &= j - 1, \\ l(s) &= \lambda'_j - i, & l'(s) &= i - 1, \end{aligned}$$

and put

$$h_\lambda(q,t) = \prod_{s\in\lambda}(1 - q^{a(s)}t^{l(s)+1}), \qquad h'_\lambda(q,t) = \prod_{s\in\lambda}(1 - q^{a(s)+1}t^{l(s)})$$

and

$$b_\lambda = b_\lambda(q,t) = \frac{h_\lambda(q,t)}{h'_\lambda(q,t)}.$$

One has different expressions of $h_\lambda$ and $h'_\lambda$.

PROPOSITION 3.2. – *Let $\lambda$ be a partition of length $\leq n$. Then*

$$h'_\lambda(q,t) = (q)_\infty^n \left(\prod_{i=1}^n (q^{\lambda_i+1}t^{n-i})_\infty\right)^{-1} \prod_{1\leq i<j\leq n} \frac{(q^{\lambda_i-\lambda_j+1}t^{j-i})_\infty}{(q^{\lambda_i-\lambda_j+1}t^{j-i-1})_\infty}. \tag{3.5}$$

$$h_\lambda(q,t) = (t)_\infty^n \left(\prod_{i=1}^n (q^{\lambda_i}t^{n-i+1})_\infty\right)^{-1} \prod_{1\leq i<j\leq n} \frac{(q^{\lambda_i-\lambda_j}t^{j-i+1})_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i})_\infty}. \tag{3.6}$$

*Proof.* – We prove (3.5). One can prove (3.6) in the same way. Put

$$C = \{i \,|\, \lambda_{i+1} < \lambda_i\},$$

so that

$$\prod_{s\in\lambda}(1 - q^{a(s)+1}t^{l(s)}) = \prod_{i\in C}\prod_{r=0}^{i-1}\prod_{j=\lambda_{i+1}+1}^{\lambda_i}(1 - q^{\lambda_{i-r}-j+1}t^r).$$

Observe that

$$\prod_{1\le i<j\le n}(q^{\lambda_i-\lambda_j+1}t^{j-i})_\infty=\prod_{r=1}^{n-1}\prod_{i=r+1}^{n}(q^{\lambda_{i-r}-\lambda_i+1}t^r)_\infty$$
$$=(q)_\infty^{-n}\prod_{r=0}^{n-1}\prod_{i=r+1}^{n}(q^{\lambda_{i-r}-\lambda_i+1}t^r)_\infty,$$
$$\prod_{1\le i<j\le n}(q^{\lambda_i-\lambda_j+1}t^{j-i-1})_\infty=\prod_{r=0}^{n-2}\prod_{i=r+1}^{n-1}(q^{\lambda_{i-r}-\lambda_{i+1}+1}t^r)_\infty$$
$$=\left(\prod_{r=0}^{n-1}(q^{\lambda_{n-r}+1}t^r)_\infty\right)^{-1}\prod_{r=0}^{n-1}\prod_{i=r+1}^{n}(q^{\lambda_{i-r}-\lambda_{i+1}+1}t^r)_\infty.$$

Hence we get

$$\text{RHS of (3.5)}=\prod_{r=0}^{n-1}\prod_{i=r+1}^{n}\frac{(q^{\lambda_{i-r}-\lambda_i+1}t^r)_\infty}{(q^{\lambda_{i-r}-\lambda_{i+1}+1}t^r)_\infty}$$
$$=\prod_{i\in C}\prod_{r=0}^{i-1}\frac{(q^{\lambda_{i-r}-\lambda_i+1}t^r)_\infty}{(q^{\lambda_{i-r}-\lambda_{i+1}+1}t^r)_\infty}$$
$$=\prod_{i\in C}\prod_{r=0}^{i-1}\prod_{j=\lambda_{i+1}+1}^{\lambda_i}(1-q^{\lambda_{i-r}-j+1}t^r),$$

as desired.

We define the *generalized factorial* $(a)_\lambda^{(q,t)}$ by

$$(a)_\lambda^{(q,t)}=\prod_{s\in\lambda}(t^{l'(s)}-q^{a'(s)}a).$$

The following explicit formula ([Ma2, (5.3)]) is essential:

THEOREM 3.3. – *We have*

$$\epsilon_{u,t}(P_\lambda(q,t))=\frac{(u)_\lambda^{(q,t)}}{h_\lambda(q,t)}.$$

*Remark.* – This formula is equivalent to the $q$-binomial theorem for the $q$-hypergeometric functions defined in the next subsection (*see* Theorem 3.5). We shall treat this problem in a separate paper.

We shall also need ([Ma2, (3.11)]):

$$\sum_\lambda b_\lambda(q,t)P_\lambda(x;q,t)P_\lambda(y;q,t)=\prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty},\tag{3.7}$$

or its dual [Ma2, (3.12)]

$$\sum_{\lambda} P_\lambda(x;q,t)P_{\lambda'}(y;t,q) = \prod_{i,j}(1+x_iy_j). \tag{3.8}$$

This is a consequence of [Ma2, Sect. 5, p. 159]:

$$\langle P_\lambda, P_\lambda\rangle = b_\lambda(q,t)^{-1}. \tag{3.9}$$

3.2. $q$-HYPERGEOMETRIC FUNCTION. – For a partition $\lambda$, denote $b(\lambda) = \sum (i-1)\lambda_i = \sum \lambda_i'(\lambda_i'-1)/2$.

DEFINITION 3.4. – Let $a_1,\ldots,a_r$ and $b_1,\ldots,b_s$ be complex numbers such that $(b_j)_\lambda^{(q,t)} \neq 0, 1 \leq j \leq m$ for any partition of length $\leq m$. The $q$-hypergeometric function ${}_r\Phi_s^{(q,t)}(a_1,\ldots,a_r;b_1,\ldots,b_s;x_1,\ldots,x_m)$ is defined by

$$ {}_r\Phi_s^{(q,t)}(a_1,\ldots,a_r;b_1,\ldots,b_s;x) = \sum_{\lambda} \frac{\prod_{i=1}^r (a_i)_\lambda^{(q,t)}\left\{(-1)^{|\lambda|}q^{b(\lambda')}\right\}^{s+1-r}}{\prod_{i=1}^s (b_j)_\lambda^{(q,t)} h'_\lambda(q,t)} P_\lambda(x;q,t). \tag{3.10}$$

As a consequence of the factor $\left\{(-1)^{|\lambda|}q^{b(\lambda')}\right\}^{s+1-r}$, it follows that

$$\begin{aligned} &\lim_{a\to\infty} {}_{r+1}\Phi_s^{(q,t)}(a_1,\ldots,a_r,a;b_1,\ldots,b_s;x_1/a,\ldots,x_m/a) \\ &\quad = {}_r\Phi_s^{(q,t)}(a_1,\ldots,a_r;b_1,\ldots,b_s;x). \end{aligned} \tag{3.11}$$

When $m=1$, ${}_r\Phi_s^{(q,t)}(x)$ reduces to the ordinary $q$-hypergeometric function ${}_r\phi_s(x)$ (*cf.* [An, GR]), being independent of $t$.

THEOREM 3.5. – *We have*

$$ {}_1\Phi_0^{(q,t)}(a;-;x_1,\ldots,x_m) = \prod_{i=1}^m \frac{(ax_i;q)_\infty}{(x_i;q)_\infty}. \tag{3.12}$$

*Proof.* – Note first that [Ma2, (2.6)]:

$$\prod_{i,j}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \sum_\lambda z_\lambda(q,t)^{-1}p_\lambda(x)p_\lambda(y). \tag{3.13}$$

In particular we have

$$\prod_{i=1}^m \frac{(ax_i;q)_\infty}{(x_i;q)_\infty} = \sum_\lambda z_\lambda(q,a)^{-1}p_\lambda(x).$$

It follows from (3.7) and (3.13) that

$$\sum_{\lambda} b_\lambda(q,t) P_\lambda(x;q,t) P_\lambda(y;q,t) = \sum_{\lambda} z_\lambda(q,t)^{-1} p_\lambda(x) p_\lambda(y) \tag{3.14}$$

so

$$\sum_{|\lambda| \le l} b_\lambda(q,t) P_\lambda(x;q,t) P_\lambda(y;q,t) = \sum_{|\lambda| \le l} z_\lambda(q,t)^{-1} p_\lambda(x) p_\lambda(y) \tag{3.15}$$

for arbitrary $l$. In view of Theorem 3.3 and

$$\epsilon_{a,t}(p_\lambda(y)) = \prod_{i=1}^{\ell(\lambda)} \frac{1-a^{\lambda_i}}{1-t^{\lambda_i}},$$

applying $\epsilon_{a,t}$ to both sides of (3.15) considered as polynomials in $y$ yields

$$\sum_{|\lambda| \le l} \frac{(a)_\lambda^{(q,t)}}{h'_\lambda(q,t)} P_\lambda(x) = \sum_{|\lambda| \le l} z_\lambda(q,a)^{-1} p_\lambda(x). \tag{3.16}$$

Since $l$ is arbitrary, this clearly gives (3.12).

COROLLARY 3.6

$${}_0\Phi_0^{(q,t)}(x_1,\dots,x_m) = \prod_{i=1}^{m} (x_i;q)_\infty. \tag{3.17}$$

*Proof.* – This follows at once from (3.11) because

$$\lim_{a\to\infty} \prod_{i=1}^{m} \frac{(x_i;q)_\infty}{(x_i/a;q)_\infty} = \prod_{i=1}^{m} (x_i;q)_\infty.$$

Next we consider the convergence of the series. We assume $0 < t < 1$.

LEMMA 3.7. – *Let* $\|x\| = \max\{|x_1|,\dots,|x_m|\}$. *There exists a positive constant* $C$ *depending only on* $q, t$ *and* $m$ *such that*

$$|P_\lambda(x;;q,t)| \le C\left(h_\lambda^{-1} h'_\lambda\right)^{1/2} (m\,\|x\|)^{|\lambda|}.$$

*Proof.* – Put $|\lambda| = d$ and write

$$P_\lambda = \sum_{|\mu|=d} a_\mu p_\mu$$

so that

$$\langle P_\lambda, P_\lambda \rangle = \sum_{|\mu|=d} a_\mu^2 z_\mu(q,t).$$

By Cauchy's inequality we have

$$|P_\lambda|^2 \leq \Big( \sum_{|\mu|=d} a_\mu^2 z_\mu(q,t) \Big) \Big( \sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q,t)} \Big).$$

It follows from (3.9) that

$$\sum_{|\mu|=d} a_\mu^2 z_\mu(q,t) = h_\lambda^{-1} h'_\lambda.$$

Put

$$C_1 = \max_{\ell(\lambda) \leq m} \frac{1-t^{\lambda_i}}{1-q^{\lambda_i}}$$

so that

$$\sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q,t)} \leq C_1 \sum_{|\mu|=d} z_\mu^{-1} p_\mu^2(|x_1|, \ldots, |x_m|).$$

Since $\sum_{|\mu|=d} z_\mu^{-1} p_\mu = \sum_{|\mu|=d} m_\mu$ ([Ma2, p. 17]), we obtain

$$\sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q,t)} \leq C_1 m^d \binom{m+d-1}{d} ||x||^{2d}.$$

Note that

$$\binom{m+d-1}{d} = \frac{(d+1)(d+2)\cdots(d+m-1)}{(m-1)!} \leq d^{m-1} \frac{m!}{(m-1)!}.$$

Put $C_2 = \max_d m^{-d} d^{m-1}$. This gives

$$\sum_{|\mu|=d} \frac{|p_\mu|^2}{z_\mu(q,t)} \leq C_1 C_2 m^{2d+1} ||x||^{2d}.$$

Hence by setting $C = \big(m C_1 C_2\big)^{1/2}$, we arrive at the desired inequality.

THEOREM 3.8. – *We have*

(1) *If* $r \leq s$, *then the series* (3.10) *converges absolutely for all* $x \in \mathbb{C}^m$.

(2) *If* $r = s+1$, *then the series* (3.10) *converges absolutely for* $||x|| < m^{-1}$.

(3) *If* $r > s+1$, *then the series* (3.10) *does not converge absolutely for* $x \neq (0, \ldots, 0)$ *unless it terminates.*

*Proof.* – We compare the series (3.10) with the series

$$
{}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;z)=\sum_{d=0}^{\infty}\frac{\prod_{k=1}^{r}(a_k;q)_d\Big\{(-1)^d q^{d(d-1)/2}\Big\}^{s+1-r}}{\prod_{l=1}^{s}(b_l;q)_d}\frac{z^d}{(q;q)_d}
$$

which is known to have radius of convergence $\rho=\infty$ if $r\le s$, $\rho=1$ if $r=s+1$, $\rho=0$ if $r>s+1$ unless it terminates [GR, p. 5]. Put

$$
a_{ki}=a_k t^{-(i-1)},\quad b_{li}=b_l t^{-(i-1)}.
$$

Note first that

$$
(h_\lambda h'_\lambda)^{-1}\left(\prod_{i=1}^{m}(q;q)_{\lambda_i}\right)^2\le(1-t)^{-m}
$$

and

$$
t^{(r-s)b(\lambda)}\le R^{|\lambda|}
$$

where we put $R=1$ if $r\ge s$, and $R=t^{(r-s)m}$ if $r<s$. Then by Lemma 3.7 we have

$$
\begin{aligned}
&\sum_\lambda\left|\frac{\prod_{i=1}^{r}(a_i)_\lambda^{(q,t)}\Big\{(-1)^{|\lambda|}q^{b(\lambda')}\Big\}^{s+1-r}}{\prod_{i=1}^{s}(b_j)_\lambda^{(q,t)}h'_\lambda(q,t)}P_\lambda\right|\\
&\quad\le(1-t)^{-m/2}\sum_\lambda q^{(s+1-r)b(\lambda')}t^{(r-s)b(\lambda)}\left|\frac{\prod_{k=1}^{r}\prod_{i=1}^{m}(a_{ki};q)_{\lambda_i}}{\prod_{l=1}^{s}\prod_{i=1}^{m}(b_{li};q)_{\lambda_i}}\left(\frac{h_\lambda}{h'_\lambda}\right)^{1/2}\frac{P_\lambda}{\prod_{i=1}^{m}(q;q)_{\lambda_i}}\right|\\
&\quad\le(1-t)^{-m/2}C\sum_\lambda t^{(r-s)b(\lambda)}\left|\frac{\prod_{k=1}^{r}\prod_{i=1}^{m}(a_{ki};q)_{\lambda_i}q^{(s+1-r)b(\lambda')}}{\prod_{l=1}^{s}\prod_{i=1}^{m}(b_{li};q)_{\lambda_i}}\frac{(m||x||)^{|\lambda|}}{\prod_{i=1}^{m}(q;q)_{\lambda_i}}\right|\\
&\quad\le(1-t)^{-m/2}C\prod_{i=1}^{m}\sum_{\lambda_i=0}^{\infty}\left|\frac{\prod_{k=1}^{r}(a_{ki};q)_{\lambda_i}q^{(s+1-r)\lambda_i(\lambda_i-1)/2}}{\prod_{l=1}^{s}(b_{li};q)_{\lambda_i}}\frac{(Rm||x||)^{\lambda_i}}{\prod_{i=1}^{m}(q;q)_{\lambda_i}}\right|.
\end{aligned}
$$

This completes the proof of the assertions (1) and (2).

For the proof of (3), suppose $x_i\ne0, 1\le i\le a$, and $x_i=0$ otherwise. Note that

$$
{h'_\lambda}^{-1}\prod_{i=1}^{m}(q;q)_{\lambda_i}\ge\big((1-q)(1-qt^m)^{-1}\big)^{|\lambda|}.
$$

Put $\theta = (1-q)(1-qt^m)^{-1}$. Picking up the terms with $\lambda = (d, d, \ldots, d) = (d^a)$ (so that $P_\lambda = x_1 \cdots x_a$ by (3.1)), we obtain

$$\begin{aligned}
&\sum_\lambda \left| \frac{\prod_{i=1}^r (a_i)_\lambda^{(q,t)} \left\{(-1)^{|\lambda|} q^{b(\lambda')}\right\}^{s+1-r}}{\prod_{i=1}^s (b_j)_\lambda^{(q,t)} h'_\lambda(q,t)} P_\lambda \right| \\
&\geq \sum_\lambda t^{(r-s)b(\lambda)} \left| \frac{\prod_{k=1}^r \prod_{i=1}^m (a_{ki};q)_{\lambda_i} q^{(s+1-r)b(\lambda')}}{\prod_{l=1}^s \prod_{i=1}^m (b_{li};q)_{\lambda_i}} \frac{\theta^{|\lambda|} P_\lambda}{\prod_{i=1}^m (q;q)_{\lambda_i}} \right| \\
&\geq \sum_{d=0}^\infty \left| \frac{\prod_{k=1}^r \prod_{i=1}^a (a_{ki};q)_d \, q^{a(s+1-r)d(d-1)/2}}{\prod_{l=1}^s \prod_{i=1}^a (b_{li};q)_d} \frac{|t^{(r-s)a(a-1)/2} \theta^a x_1 \cdots x_a|^d}{((q;q)_d)^a} \right|.
\end{aligned}$$

This last series is easily shown to be divergent, thereby completing the proof of (3).

## 4. $q$-Difference system of $q$-hypergeometric function

4.1. SUMMED-UP EQUATION. – As in the case of $q = 1$ [Kan2], we shall consider the $q$-difference equation formed by summing the $q$-difference equations, multiplied by $A_i(x;t)$ each, in the system (2.26). First we introduce auxiliary operators:

$$D_2^{(q,t)} = \frac{1-q}{1-t} \sum_{1 \leq i < j \leq m} A_{ij}(x;t) x_i x_j \frac{\partial^2}{\partial_q x_i \partial_q x_j} - \frac{1}{1-t} \sum_{1 \leq i \neq j \leq m} A_{ij}(x;t) x_i \frac{\partial}{\partial_q x_i}$$

where

$$A_{ij}(x;t) = t \prod_{k=1, k \neq i,j}^m \frac{(tx_i - x_k)(tx_j - x_k)}{(x_i - x_k)(x_j - x_k)},$$

and

$$\varepsilon = \varepsilon_m = \sum_{i=1}^m A_i(x;t) \frac{\partial}{\partial_q x_i}.$$

It is known that [Ma3]

$$D_2^{(q,t)} P_\lambda = f_\lambda(q,t) P_\lambda \tag{4.1}$$

where

$$\begin{aligned}
f_\lambda &= \frac{1}{(1-q)(1-t)} \sum_{1 \leq i < j \leq m} t^{2m-i-j} (q^{\lambda_i + \lambda_j} - 1) \\
&= \sum_{1 \leq i < j \leq m} \left\{ \frac{(1 - t^{m-i} q^{\lambda_i})(1 - t^{m-j} q^{\lambda_j})}{(1-q)(1-t)} - \frac{(1-t^{m-i})(1-t^{m-j})}{(1-q)(1-t)} \right\} + \frac{1-m}{1-t} e_\lambda.
\end{aligned}$$

Denote by $\Lambda_m^r$ the vector space of symmetric homogeneous polynomials of degree $r$ in $x_1, \dots, x_m$ with coefficients in $\mathbb{Q}(q,t)$.

LEMMA 4.1. – *$\varepsilon$ defines a linear mapping from $\Lambda_m^r$ to $\Lambda_m^{r-1}$ for every $r$.*

*Proof.* – Since the $p_\lambda(x_1, \dots, x_m)'s$ with $|\lambda| = r$ and $\ell(\lambda) \le m$ form a basis of $\Lambda_m^r$, it suffices to show $\varepsilon p_\lambda \in \Lambda_m^{r-1}$. But this easily boils down to prove the case of $\lambda = (r)$. We see

$$\varepsilon p_r = \frac{1-q^r}{1-q} \sum_{i=1}^m A_i x_i^{r-1} = \frac{1-q^r}{1-q^{r-1}} D_1 \, p_{r-1}.$$

Hence the lemma follows immediately from (3.2) and the fact that the $P_\lambda(x_1, \dots, x_m)'s$ with $|\lambda| = r-1$ and $\ell(\lambda) \le m$ form a basis of $\Lambda_m^{r-1}$.

Let us denote by $L_m = L_m^{(q,t)}$ the $q$-difference operator formed by summing the $q$-difference operators, multiplied by $A_i = A_i(x;t)$ each, in the system (2.26):

$$\begin{aligned}
(4.2)\quad L_m = &\sum_{i=1}^m x_i(c - abqx_i) A_i T_i(A_i) \frac{\partial^2}{\partial_q x_i^2} + (1-t) \\
&\times \sum_{1 \le i \ne j \le m} \frac{x_i x_j (c - abx_j)}{qx_i - tx_j} A_i T_i(A_j) \frac{\partial^2}{\partial_q x_i \partial_q x_j} \\
&+ \sum_{i=1}^m \Big\{ \frac{t^{m-1} - c}{1-q} + \frac{1}{1-q} \big( (1-a)(1-b)t^{m-1} - (t^{m-1} - abq) \big) x_i \Big\} A_i \frac{\partial}{\partial_q x_i} \\
&+ \frac{1-t}{1-q} \Bigg\{ \sum_{i=1}^m \frac{1 - T_i(A_i)}{1-t} (c - abqx_i) A_i \frac{\partial}{\partial_q x_i} \\
&- \sum_{1 \le i \ne j \le m} \frac{x_j (c - abx_j)}{qx_i - tx_j} A_i T_i(A_j) \frac{\partial}{\partial_q x_j} \Bigg\} \\
&- \frac{(1-a)(1-b)t^{m-1}}{(1-q)^2} \frac{1-t^m}{1-t}.
\end{aligned}$$

Now we can state the following crucial lemma.

LEMMA 4.2. – *We have*

$$\begin{aligned}
(4.3)\ L_m = &\frac{c}{1-q}(D_1 \varepsilon - \varepsilon D_1) - ab\left( D_1^2 - \frac{1-t^2}{t(1-q)} D_2 \right) + \frac{t^{m-1}}{1-q} \varepsilon \\
&+ \frac{1}{1-q} \left\{ \frac{2ab(1-t^m)}{1-t} - (a+b)t^{m-1} \right\} D_1 - \frac{(1-a)(1-b)t^{m-1}}{(1-q)^2} \frac{1-t^m}{1-t}.
\end{aligned}$$

*Proof.* – We show that

$$(4.4)\quad \frac{1}{1-q}(D_1\varepsilon - \varepsilon D_1)$$
$$= \sum_{i=1}^{m} x_i A_i T_i(A_i)\frac{\partial^2}{\partial_q x_i^2} + (1-t)\sum_{1\le i<j\le m} x_i x_j \left\{\frac{A_iT_i(A_j)}{qx_i - tx_j} + \frac{A_jT_j(A_i)}{qx_j - tx_i}\right\}\frac{\partial^2}{\partial_q x_i \partial_q x_j}$$
$$- \frac{1}{1-q}\sum_{i=1}^{m} A_iT_i(A_i)\frac{\partial}{\partial_q x_i} - \frac{1-t}{1-q}\sum_{1\le i\ne j\le m}\frac{x_j}{qx_i - tx_j}A_iT_i(A_j)\frac{\partial}{\partial_q x_j}$$

$$(4.5)\quad D_1^2 - \frac{1-t^2}{t(1-q)}D_2$$
$$= \sum_{i=1}^{m} q x_i^2 A_i T_i(A_i)\frac{\partial^2}{\partial_q x_i^2}$$
$$+ (1-t)\sum_{1\le i<j\le m} x_i x_j\left\{\frac{x_jA_iT_i(A_j)}{qx_i - tx_j} + \frac{x_iA_jT_j(A_i)}{qx_j - tx_i}\right\}\frac{\partial^2}{\partial_q x_i\partial_q x_j}$$
$$+ \frac{1}{1-q}\sum_{i=1}^{m}\Bigg\{-qx_iA_iT_i(A_i) - (1-t)\sum_{1\le j\le m, j\ne i}\frac{x_i^2}{qx_j - tx_i}A_jT_j(A_i)$$
$$+ \frac{2-(t^{m-1}+t^m)}{1-t}x_iA_i\Bigg\}\frac{\partial}{\partial_q x_i}.$$

One can check (4.3) without difficulty assuming (4.4) and (4.5). It is clear that the coefficient of $\partial^2/\partial_q x_i^2$ in $D_1\varepsilon - \varepsilon D_1$ is $(1-q)x_iA_iT_i(A_i)$. For the coefficient of $\partial^2/\partial_q x_i \partial_q x_j$, it suffices to observe

$$\begin{aligned}
&\text{the coefficient of } \partial^2/\partial_q x_i\partial_q x_j \text{ in } D_1\varepsilon - \varepsilon D_1\\
&= x_iA_iT_i(A_j) + x_jA_jT_j(A_i) - x_jA_iT_i(A_j) - x_iA_jT_j(A_i)\\
&= (x_i - x_j)(A_iT_i(A_j) - A_jT_j(A_i))\\
&= (x_i - x_j)\frac{A_{ij}}{t}\left\{\frac{(tx_i - x_j)(tx_j - qx_i)}{(x_i - x_j)(x_j - qx_i)} - \frac{(tx_j - x_i)(tx_i - qx_j)}{(x_j - x_i)(x_i - qx_j)}\right\}\\
&= \frac{A_{ij}}{t}\frac{(1-q)(1-t)(q-t)x_ix_j(x_i + x_j)}{(qx_i - x_j)(qx_j - x_i)}.
\end{aligned}$$

and

$$\begin{aligned}
&\frac{A_iT_i(A_j)}{qx_i - tx_j} + \frac{A_jT_j(A_i)}{qx_j - tx_i}\\
&\quad = \frac{A_{ij}}{t}\left\{\frac{tx_i - x_j}{(x_i - x_j)(qx_i - x_j)} + \frac{tx_j - x_i}{(x_j - x_i)(qx_j - x_i)}\right\}\\
&\quad = \frac{A_{ij}}{t}\frac{(q-t)(x_i + x_j)}{(qx_i - x_j)(qx_j - x_i)}.
\end{aligned}$$

We have also

$$\begin{aligned}
&\text{the coefficient of } \partial/\partial_q x_i \text{ in } D_1\varepsilon - \varepsilon D_1 \\
&= \sum_{j=1}^{m} A_j x_j \frac{\partial A_i}{\partial_q x_j} - \sum_{j=1}^{m} A_j \frac{\partial (A_i x_i)}{\partial_q x_j} \\
&= -A_i T_i(A_i) - \sum_{1\le j\le m, j\ne i} (x_i - x_j) A_j \frac{\partial A_i}{\partial_q x_j}.
\end{aligned}$$

Since

$$\begin{aligned}
(x_i - x_j)\frac{\partial A_i}{\partial_q x_j} &= \prod_{k=1,k\ne i,j} \frac{tx_i - x_k}{x_i - x_k}\Big(\frac{tx_i - qx_j}{x_i - qx_j} - \frac{tx_i - x_j}{x_i - x_j}\Big)\frac{x_i - x_j}{(q-1)x_j} \\
&= \prod_{k=1,k\ne i,j} \frac{tx_i - x_k}{x_i - x_k}\frac{(1-t)x_i}{qx_j - x_i} \\
&= \frac{(1-t)x_i}{qx_j - tx_i} T_j(A_i),
\end{aligned}$$

the proof of (4.4) is complete.

The coefficient of $\partial^2/\partial_q x_i^2$ in $D_1^2 - \frac{1-t^2}{t(1-q)} D_2$ is clearly as in (4.5). For the coefficient of $\partial^2/\partial_q x_i \partial_q x_j$, note that

$$\begin{aligned}
&x_i x_j (A_i T_i(A_j) + A_j T_j(A_i)) \\
&= \frac{A_{ij}}{t}\frac{x_i x_j}{x_i - x_j}\frac{(qx_i - tx_j)(tx_i - x_j)(qx_j - x_i) - (qx_j - tx_i)(tx_j - x_i)(qx_i - x_j)}{(qx_i - x_j)(qx_j - x_i)}
\end{aligned}$$

and

$$\begin{aligned}
&(qx_i - tx_j)(tx_i - x_j)(qx_j - x_i) - (qx_j - tx_i)(tx_j - x_i)(qx_i - x_j) \\
&= (qx_i - x_j + (1-t)x_j)(tx_i - x_j)(qx_j - x_i) - (qx_j - x_i + (1-t)x_i)(tx_j - x_i)(qx_i - x_j) \\
&= (1+t)(x_i - x_j)(qx_i - x_j)(qx_j - x_i) \\
&\quad + (1-t)\{x_j(qx_j - x_i)(tx_i - x_j) - x_i(qx_i - x_j)(tx_j - x_i)\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\text{the coefficient of } \partial^2/\partial_q x_i \partial_q x_j \text{ in } D_1^2 - \frac{1-t^2}{t(1-q)} D_2 \\
&= (1-t)\frac{A_{ij}}{t}\frac{x_i x_j\{x_j(qx_j - x_i)(tx_i - x_j) - x_i(qx_i - x_j)(tx_j - x_i)\}}{(x_i - x_j)(qx_i - x_j)(qx_j - x_i)} \\
&= (1-t)x_i x_j\Big\{\frac{x_j A_i T_i(A_j)}{qx_i - tx_j} + \frac{x_i A_j T_j(A_i)}{qx_j - tx_i}\Big\}.
\end{aligned}$$

Since

the coefficient of $\partial/\partial_q x_i$ in $D_1^2 - \dfrac{1-t^2}{t(1-q)}D_2$

$$= \frac{1}{1-q}\Big\{\frac{1-t^m}{1-t}x_iA_i - qx_iA_iT_i(A_i) - \sum_{1\le j\le m, j\ne i} x_iA_jT_j(A_i) + \frac{1+t}{t}\sum_{1\le j\le m, j\ne i} x_iA_{ij}\Big\},$$

we conclude the proof of (4.5) from

$$\begin{aligned}
&\sum_{1\le j\le m, j\ne i}\{A_jT_j(A_i) - \frac{1+t}{t}A_{ij} - (1-t)\frac{x_i}{qx_j - tx_i}A_jT_j(A_i)\}\\
&= \sum_{1\le j\le m, j\ne i} A_j \prod_{k=1,k\ne i,j}^{m}\frac{tx_i - x_k}{x_i - x_k}\Big\{\frac{tx_i - qx_j}{x_i - qx_j} - (1+t)\frac{x_j - x_i}{tx_j - x_i} - (1-t)\frac{x_i}{qx_j - x_i}\Big\}\\
&= \sum_{1\le j\le m, j\ne i} A_j \prod_{k=1,k\ne i,j}^{m}\frac{tx_i - x_k}{x_i - x_k}\,\frac{tx_i - x_j}{tx_j - x_i}\\
&= -A_i\sum_{1\le j\le m, j\ne i} A_j\frac{x_j - x_i}{tx_j - x_i}\\
&= -\frac{1-t^{m-1}}{1-t}A_i
\end{aligned}$$

where the last equality follows from (2.10).

### 4.2. Generalized binomial coefficients.

DEFINITION 4.3. – For any partitions $\lambda$ and $\mu$ of length $\le m$, the generalized binomial coefficient $\binom{\lambda}{\mu}_m$ is defined by

$$\varepsilon\Big(\frac{t^{b(\lambda)}}{\epsilon_{t^m,t}(P_\lambda)}P_\lambda(x_1,\dots,x_m)\Big) = \sum_\mu \binom{\lambda}{\mu}_m \frac{t^{b(\mu)}}{\epsilon_{t^m,t}(P_\mu)}P_\mu(x_1,\dots,x_m).$$

*Remark.* – If we put $q = t^\alpha$ and let $t \to 1$, then it is readily seen that $\binom{\lambda}{\mu}_m$ reduces to the generalized binomial coefficient defined by using Jack polynomials [Kan2, p. 1096]. In this case the following theorem has been announced by [L] and proved by [Kan2].

Denote $\mu \subset \lambda$ if $\mu_i \le \lambda_i$ for every $i$.

THEOREM 4.4. – (1) $\binom{\lambda}{\mu}_m \ne 0$ *if and only if* $\mu \subset \lambda$ *and* $|\mu| = |\lambda| - 1$.

(2) *the* $\binom{\lambda}{\mu}_m$*'s are independent of the dimension* $m$ *in the sense that* $\binom{\lambda}{\mu}_r = \binom{\lambda}{\mu}_s$ *provided* $r, s \le \ell(\lambda)$.

We leave the proof to Section 6. We write $\binom{\lambda}{\mu}$ dropping the subscript $m$. For each partition $\lambda$, we put $\lambda_{(i)} = (\lambda_1,\dots,\lambda_{i-1},\lambda_i+1,\lambda_{i+1},\dots)$ and $\lambda^{(i)} = (\lambda_1,\dots,\lambda_{i-1},\lambda_i-1,\lambda_{i+1},\dots)$ and call them admissible if the parts are in nonincreasing order. We shall write

$\lambda_{(i,j)} = (\lambda_{(i)})_{(j)}, \lambda^{(i,j)} = (\lambda^{(i)})^{(j)}$. By Theorem 4.4 (1), $\binom{\lambda}{\mu} = 0$ unless $\lambda = \mu_{(i)}$ (or $\mu = \lambda^{(i)}$) for some $i$. Hence, in view of $b(\lambda) - b(\lambda^{(i)}) = i - 1$, we have

$$\varepsilon P_\lambda(x_1, \ldots, x_m) = c_\lambda h'_\lambda \sum_{i=1}^{m} \binom{\lambda}{\lambda^{(i)}} \frac{t^{1-i}}{c_{\lambda^{(i)}} h'_{\lambda^{(i)}}} P_{\lambda^{(i)}}(x_1, \ldots, x_m) \tag{4.6}$$

where we have put

$$c_\lambda = c_\lambda(q, t, m) = \frac{\epsilon_{t^m,t}(P_\lambda)}{h'_\lambda} = \frac{(t^m)_\lambda^{(q,t)}}{h_\lambda} h'_\lambda.$$

The summation in (4.6) is over all $i$ such that $\lambda^{(i)}$ is admissible. This convention will be used in all future summations involving $\lambda_{(i)}$ or $\lambda^{(i)}$.

Proposition 4.5. – *The formal series*

$$S(x_1, \ldots, x_m) = \sum_\lambda \gamma_\lambda \frac{P_\lambda(x_1, \ldots, x_m)}{h'_\gamma}$$

*satisfies the summed-up equation $L_m(S) = 0$ if and only if the coefficients $\gamma_\lambda$ satisfy the following recurrence relations*

$$\begin{aligned} &\frac{1}{(1-q)c_\lambda} \sum_{i=1}^{m} c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} t^{m+1-2i}(t^{i-1} - q^{\lambda_i} c)\gamma_{\lambda_{(i)}} \\ &\quad + \left\{ -ab\left(e_\lambda^2 - \frac{1-t^2}{t(1-q)} f_\lambda\right) + \left(\frac{2ab(1-t^m)}{(1-q)(1-t)} - \frac{(a+b)t^{m-1}}{1-q}\right) e_\lambda \right. \\ &\quad \left. - \frac{(1-a)(1-b)t^{m-1}(1-t^m)}{(1-q)^2(1-t)} \right\} \gamma_\lambda = 0. \end{aligned} \tag{4.7}$$

*Proof.* – One can easily verify that, using (3.2), (4.1) and (4.6), the left-hand side of (4.7) is the coefficient of $P_\lambda / h'_\lambda$ in $L_m(S)$.

Note that for $r \le m$ we have

$$S(x_1, \ldots, x_r) := S(x_1, \ldots, x_r, 0, \ldots, 0) = \sum_\lambda \gamma_\lambda \frac{P_\lambda(x_1, \ldots, x_m)}{h'_\gamma}. \tag{4.8}$$

The recurrence relation (4.7) implies the following uniqueness property.

Corollary 4.6. – *Assume that $(c)_\lambda^{(q,t)} \neq 0$ for any partition $\lambda$ of length $\le m$. If the formal series* (4.8) *satisfies $L_r(S(x_1, \ldots, x_r)) = 0$ for every $r \le m$, and $S(0, \ldots, 0) = 0$, then $S(x_1, \ldots, x_m) \equiv 0$.*

*Proof.* – Note that the coefficient of $\gamma_{(i)}$ of (4.7) is not zero because of the assumption $(c)_\lambda^{(q,t)} \neq 0$. We prove $S(x_1, \ldots, x_r) = 0$ by induction on $r$, the case $r = 1$ being immediate from (4.7). Clearly it suffices to show $S(x_1, \ldots, x_m) = 0$ assuming $S(x_1, \ldots, x_r) = 0$ for $r \leq m - 1$, *i.e.* $\gamma_\lambda = 0$ if $\ell(\lambda) \leq m - 1$. For $\gamma_\kappa$ with $\kappa_m = 1$, put $\lambda = \kappa^{(m)}$ or $\lambda_{(m)} = \kappa$. Substituting this $\lambda$ into (4.7) immediately shows that $\gamma_{\lambda_{(m)}}$ is a linear combination of $\gamma_\lambda$ and $\gamma_{\lambda_{(i)}}$, $i < m$. Thus we find $\gamma_{\lambda_{(m)}} = 0$. The general case follows by induction on $\kappa_m$.
Let us denote by $(2.26)_m$ the system (2.26) to express its dimensional dependence.

LEMMA 4.7. – *If $S(x_1, \ldots, x_m)$ is a solution of* $(2.26)_m$, *then* $S(x_1, \ldots, x_r)$, $1 \leq r \leq m$, *is a solution of* $(2.26)_r$.

*Proof.* – Clearly it suffices to prove the case of $r = m - 1$. Substitute $S(x_1, \ldots, x_m)$ into $(2.26)_m$ with $i \neq m$ and put $x_m = 0$. Then one finds that $S(x_1, \ldots, x_{m-1})$ satisfies that the system $(2.26)_{m-1}$ multiplied by $t$.
This lemma implies that if $S(x_1, \ldots, x_m)$ is a solution of $(2.26)_m$, then $L_r(S(x_1, \ldots, x_r)) = 0$ for $r \leq m$. Hence by Corollary 4.6 we obtain

LEMMA 4.8. – *Assume that $(c)_\lambda^{(q,t)} \neq 0$ for any partition $\lambda$ of length $\leq m$. If $S(x_1, \ldots, x_m)$ is a solution of $(2.26)_m$ and $S(0, \ldots, 0) = 0$, then $S(x_1, \ldots, x_m) \equiv 0$.*

We next provide some formulas for the $\binom{\lambda}{\mu}$'s.

LEMMA 4.9. – *We have*

$$\sum_{i=1}^{m} c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} = \frac{1 - t^m}{(1-q)(1-t)} c_\lambda \tag{4.9}$$

$$\sum_{i=1}^{m} q^{\lambda_i} t^{m-i} c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} = \left\{ \frac{1 - t^m}{(1-q)(1-t)} - e_\lambda \right\} t^{m-1} c_\lambda \tag{4.10}$$

$$\begin{aligned}\sum_{i=1}^{m} \left(q_i^\lambda t^{m-i}\right)^2 c_{\lambda_{(i)}} \binom{\lambda_{(i)}}{\lambda} = &\left\{ \frac{t^{2m-2}(1 - t^m)}{(1-q)(1-t)} - \frac{2t^{m-1}(1-t^m)}{1 - te_\lambda} \right. \\ &\left. + (1-q)t^{m-1}{e_\lambda}^2 + (t-1)(t^{m-1} + t^{m-2}) f_\lambda \right\} c_\lambda.\end{aligned} \tag{4.11}$$

*Proof.* – We first show

$$\sum_{i=1}^{m} A_i x_i = t^{m-1} m_1 \tag{4.12}$$

$$\sum_{i=1}^{m} A_i x_i^2 = t^{m-1} m_{(2)} + t^{m-2}(t-1) m_{(1,1)} \tag{4.13}$$

$$\sum_{1\le i\ne j\le m} A_{ij}(x;t)x_i = \frac{t^{m-1}(1-t^{m-1})}{1-t}m_1 \tag{4.14}$$

$$\sum_{1\le i<j\le m} A_{ij}(x;t)x_ix_j = t^{2m-3}m_{(1,1)} \tag{4.15}$$

where $m_1 = m_{(1)} = x_1 + \cdots + x_m$. Replacing $z$ with $1/z$ in (2.7) yields

$$\prod_{i=1}^{m} \frac{t-zx_i}{1-zx_i} = (t-1)\sum_{i=1}^{m} \frac{zx_iA_i(x;t)}{1-zx_i} + t^m.$$

Differentiating both sides once (resp. twice) with respect to $z$ and then setting $z=0$ gives (4.12) (resp. (4.13)). By (2.10) and (2.7) with $z = tx_i$ we have

$$\begin{aligned}
&\sum_{j=1,j\ne i}^{m} \prod_{k=1,k\ne i,j}^{m} \frac{tx_j-x_k}{x_j-x_k}\frac{x_i-x_j}{tx_i-x_j} = \sum_{j=1,j\ne i}^{m} \prod_{k=1,k\ne i,j}^{m} \frac{tx_j-x_k}{x_j-x_k}\left\{\frac{1}{t}+\frac{t-1}{t}\frac{x_j}{x_j-tx_i}\right\}\\
&= \frac{1}{t}\frac{1-t^{m-1}}{1-t} + \frac{t-1}{t}\frac{1}{1-t}\left\{-t^{m-1} + \prod_{k=1,k\ne i}^{m}\frac{x_k-t^2x_i}{x_k-tx_i}\right\}\\
&= \frac{1}{t}\left\{\frac{1-t^m}{1-t} - \prod_{k=1,k\ne i}^{m}\frac{x_k-t^2x_i}{x_k-tx_i}\right\}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\sum_{1\le i\ne j\le m} A_{ij}(x;t)x_i\\
&= t\sum_{i=1}^{m} A_ix_i\left\{\sum_{j=1,j\ne i}^{m}\prod_{k=1,k\ne i,j}^{m}\frac{tx_j-x_k}{x_j-x_k}\frac{x_i-x_j}{tx_i-x_j}\right\}\\
&= \frac{1-t^m}{1-t}\sum_{i=1}^{m}A_ix_i - \sum_{i=1}^{m}A_i(x;t^2)x_i\\
&= \frac{1-t^m}{1-t}t^{m-1}m_1 - t^{2m-2}m_1\\
&= \frac{t^{m-1}(1-t^{m-1})}{1-t}m_1.
\end{aligned}$$

The proof of (4.15) is similar to that of (4.14) and we omit it.

Setting $a=0$ in (3.12) yields

$$\sum_{\lambda} \frac{t^{b(\lambda)}}{h'_\lambda}P_\lambda(x_1,\ldots,x_m) = \prod_{i=1}^{m}\frac{1}{(x_i;q)_\infty}. \tag{4.16}$$

It follows from (2.9) that

$$\varepsilon\Big(\prod_{i=1}^{m}\frac{1}{(x_i;q)_\infty}\Big)=\frac{1}{1-q}\Big(\sum_{i=1}^{m}A_i\Big)\prod_{i=1}^{m}\frac{1}{(x_i;q)_\infty}$$
$$=\frac{1-t^m}{(1-q)(1-t)}\prod_{i=1}^{m}\frac{1}{(x_i;q)_\infty}.$$

Hence applying $\varepsilon$ to both sides of (4.16), we have

$$\frac{1-t^m}{(1-q)(1-t)}\sum_{\lambda}\frac{t^{b(\lambda)}}{h'_\lambda}P_\lambda(x_1,\dots,x_m)=\sum_{\lambda}c_\lambda\Big(\sum_{i=1}^{m}\binom{\lambda}{\lambda^{(i)}}\frac{t^{b(\lambda^{(i)})}}{c_{\lambda^{(i)}}h'_{\lambda^{(i)}}}P_{\lambda^{(i)}}\Big).$$

Equating coefficients of $P_\lambda$ in both sides yields (4.9).

For the proof of (4.10), note that by (4.12) and (2.9) we see

$$D_1\Big(\frac{1}{\prod_{i=1}^{m}(x_i;q)_\infty}\Big)=\frac{1}{1-q}\Big(\Big(\sum_{i=1}^{m}A_ix_i\Big)\prod_{i=1}^{m}\frac{1}{(x_i;q)_\infty}\Big)$$
$$=\frac{t^{m-1}}{1-q}\frac{m_1}{\prod_{i=1}^{m}(x_i;q)_\infty}$$

and

$$\varepsilon\Big(\frac{m_1}{\prod_{i=1}^{m}(x_i;q)_\infty}\Big)=\Big\{\sum_{i=1}^{m}A_i+\frac{1}{1-q}\sum_{i=1}^{m}T_i(m_1)A_i\Big\}\frac{1}{\prod_{i=1}^{m}(x_i;q)_\infty}$$
$$=\Big\{\frac{1-t^m}{1-t}+\Big(\frac{1-t^m}{(1-q)(1-t)}-t^{m-1}\Big)m_1\Big\}\frac{1}{\prod_{i=1}^{m}(x_i;q)_\infty},$$

so that we obtain

$$\varepsilon D_1\Big(\frac{1}{\prod_{i=1}^{m}(x_i;q)_\infty}\Big)=\frac{t^{m-1}}{1-q}\Big\{\frac{1-t^m}{1-t}+\Big(\frac{1-t^m}{(1-q)(1-t)}-t^{m-1}\Big)m_1\Big\}\prod_{i=1}^{m}\frac{1}{(x_i;q)_\infty}.$$

Hence applying $\varepsilon D_1$ to both sides of (4.16) gives

$$\sum_{\lambda}\frac{t^{b(\lambda)}e_\lambda}{h'_\lambda}\varepsilon P_\lambda(x_1,\dots,x_m)=\frac{t^{m-1}(1-t^m)}{(1-q)(1-t)}\sum_{\lambda}\frac{t^{b(\lambda)}}{h'_\lambda}P_\lambda(x_1,\dots,x_m)$$
$$+\Big(\frac{1-t^m}{(1-q)(1-t)}-t^{m-1}\Big)\sum_{\lambda}\frac{t^{b(\lambda)}e_\lambda}{h'_\lambda}P_\lambda(x_1,\dots,x_m).$$

Substituting (4.6) in the left-hand side and equating coefficients of $P_\lambda$ of both sides gives (4.10) (use $e_{\lambda_{(i)}}=e_\lambda+t^{m-i}q^{\lambda_i}$ and then (4.9)).

By virtue of (4.14) and (4.15) we can derive

$$D_2\Big(\prod_{i=1}^m \frac{1}{(x_i;q)_\infty}\Big) = \frac{t^{m-1}}{(1-q)(1-t)}\Big\{t^{m-2}m_{(1,1)} - \frac{1-t^{m-1}}{1-t}m_1\Big\}\prod_{i=1}^m \frac{1}{(x_i;q)_\infty}.$$

Also by (4.12) and (4.13) we have

$$\varepsilon\Big(\frac{m_{(1,1)}}{\prod_{i=1}^m (x_i;q)_\infty}\Big) = \Big\{\frac{1-t^{m-1}}{1-t}m_1 + \Big(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1}+t^{m-2})\Big)m_{(1,1)}\Big\} \times \prod_{i=1}^m \frac{1}{(x_i;q)_\infty}.$$

Hence applying $\varepsilon D_2$ to both sides of (4.16), we get

$$\begin{aligned}
&\sum_\lambda \frac{t^{b(\lambda)}f_\lambda}{h'_\lambda}\varepsilon P_\lambda(x) \\
&= \frac{t^{m-1}}{(1-q)(1-t)}\Big\{t^{m-2}\varepsilon\Big(\frac{m_{(1,1)}}{\prod_{i=1}^m (x_i;q)_\infty}\Big) - \frac{1-t^{m-1}}{1-t}\varepsilon\Big(\frac{m_1}{\prod_{i=1}^m (x_i;q)_\infty}\Big)\Big\} \\
&= \frac{t^{m-1}}{(1-q)(1-t)}\Big\{t^{m-2}\Big(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1}+t^{m-2})\Big)\frac{m_{(1,1)}}{\prod_{i=1}^m (x_i;q)_\infty} \\
&\quad + \frac{1-t^{m-1}}{1-t}\Big(t^{m-2}+t^{m-1} - \frac{1-t^m}{(1-q)(1-t)}\Big)\frac{m_1}{\prod_{i=1}^m (x_i;q)_\infty} \\
&\quad - \frac{(1-t^{m-1})(1-t^m)}{(1-t)^2}\frac{1}{\prod_{i=1}^m (x_i;q)_\infty}\Big\} \\
&= \Big(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1}+t^{m-2})\Big)\sum_\lambda \frac{t^{b(\lambda)}f_\lambda}{h'_\lambda}P_\lambda(x) \\
&\quad - \frac{t^{m-1}(1-t^{m-1})(1-t^m)}{(1-q)(1-t)^3}\sum_\lambda \frac{t^{b(\lambda)}}{h'_\lambda}P_\lambda(x).
\end{aligned}$$

Substituting (4.6) into the left-hand side and equating coefficients of $P_\lambda$ of both sides yields

$$\sum_\lambda f_{\lambda_{(i)}}c_{\lambda_{(i)}}\binom{\lambda_{(i)}}{\lambda} = \Big\{\Big(\frac{1-t^m}{(1-q)(1-t)} - (t^{m-1}+t^{m-2})\Big)f_\lambda - \frac{t^{m-1}(1-t^{m-1})(1-t^m)}{(1-q)(1-t)^3}\Big\}c_\lambda.$$

Since

$$f_{\lambda_{(i)}} = f_\lambda + \Big\{(1-q)e_\lambda - \frac{1-t^m}{1-t}\Big\}\frac{t^{m-i}q^{\lambda_i}}{1-t} + \frac{(t^{m-i}q^{\lambda_i})^2}{1-t},$$

one can simplify the left-hand side by using (4.9) and (4.10) and this completes the proof of (4.11).

4.3. $q$-DIFFERENCE SYSTEM. – We now state one of the main results of this paper.

THEOREM 4.10. – *The hypergeometric function* ${}_2\Phi_1^{(q,t)}(a,b;c;x_1,\ldots,x_m)$ *is the unique solution of the summed-up equation* $L_m(S)=0$ *subject to the following condition:*

(a) $S(x)$ *is a symmetric function in* $x_1,\ldots,x_m$.

(b) $S(x)$ *is analytic at the origin with* $S(0)=1$.

(c) $S(x_1,\ldots,x_r,0,\ldots,0)$ *is a solution of* $L_r(S)=0$ *for every* $r\le m$.

*Proof.* – The uniqueness is immediate from Corollary 4.6. Put $\gamma_\lambda = (a)_\lambda^{(q,t)}(b)_\lambda^{(q,t)}/(c)_\lambda^{(q,t)}$. Then we see

$$\gamma_{\lambda_{(i)}} = \gamma_\lambda(t^{i-1}-q^{\lambda_i}a)(t^{i-1}-q^{\lambda_i}b)(t^{i-1}-q^{\lambda_i}c)^{-1}.$$

By virtue of (4.7), the proof that $L_m({}_2\Phi_1^{(q,t)})=0$ (and also (c)) boils down to show

$$\frac{1}{(1-q)c_\lambda}\sum_{i=1}^{m} c_{\lambda_{(i)}}\binom{\lambda_{(i)}}{\lambda}t^{m+1-2i}(t^{i-1}-q^{\lambda_i}a)(t^{i-1}-q^{\lambda_i}a) - ab\Big(e_\lambda^2-\frac{1-t^2}{t(1-q)}f_\lambda\Big) + \Big(\frac{2ab(1-t^m)}{(1-q)(1-t)}-\frac{(a+b)t^{m-1}}{1-q}\Big)e_\lambda - \frac{(1-a)(1-b)t^{m-1}(1-t^m)}{(1-q)^2(1-t)} = 0.$$

But this is an immediate consequence of Lemma 4.9.

Next we compare the hypergeometric series ${}_2\Phi_1^{(q,t)}(a,b;c;x_1,\ldots,x_m)$ with the $q$-Selberg integral ${}_qS_{n,m}(\alpha,\beta,\gamma,\mu;x;\xi)$ with $\mu=1$ or $-\gamma$. If $\mu=1$, then ${}_qS_{n,m}(x)$, being a polynomial, is analytic at the origin. But if $\mu=-\gamma$, then in general ${}_qS_{n,m}(x)$ has poles at $x_i=\xi_j^{-1}q^{\gamma+k}, k\in\mathbb{Z}$, so that the origin is an essential singularity. In this case we choose $\xi=\xi_F=(1,Q,\ldots,Q^{n-1})$. Then the integral ${}_qS_{n,m}(x;\xi_F)$ over $[0,\xi_F]_q$ is analytic at the origin because the integral is done only over the set $<\xi_F>$ consisting of the points such that $t_1=q^{\nu_1}, t_2/t_1=q^{\gamma+\nu_2},\ldots,t_n/t_{n-1}=q^{\gamma+\nu_n}$ for each $\nu_j\in\mathbb{Z}_{\ge 0}$ (this is the so called "$\alpha$-stable cycle" in [Ao1]). Combining Theorem 2.3, Lemma 4.8 and Theorem 4.10, we now obtain

THEOREM 4.11. – *We have* ($q^\beta Q^{-1}x=(q^\beta Q^{-1}x_1,\ldots,q^\beta Q^{-1}x_m)$ *etc.*)

(4.17)
$$ {}_qS_{n,m}(\alpha,\beta,\gamma,1;q^\beta Q^{-1}x;\xi) = C\cdot {}_2\Phi_1^{(Q,q)}(Q^{-n},q^{-(\alpha+n-1)}Q^{n-1};q^{-(\alpha+\beta+n-1)};x)$$

(4.18)
$$ {}_qS_{n,m}(\alpha,\beta,\gamma,-\gamma;Qx;\xi_F) = C_F\cdot {}_2\Phi_1^{(q,Q)}(Q^n,q^{\alpha+n-1}Q^{-(n-1)};q^{\alpha+\beta+n-1};x)$$

*where* $C={}_qS_{n,0}(\alpha,\beta,\gamma;\xi), C_F={}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F)$.

The condition that ${}_2\Phi_1^{(q,t)}(a,b;c;x_1,\ldots,x_m)$ satisfies the system (2.26) is equivalent to an infinite system of *polynomial equations* in $q,t,a,b,c$. The formula (4.18) implies that these equations hold when $t=q^\gamma, a=q^{n\gamma}, b=q^{\alpha+n-1-(n-1)\gamma}, c=q^{\alpha+\beta+n-1}$. Since $n,\alpha,\beta,\gamma$ are arbitrary, these equations hold for any $q,t,a,b,c$. Thus we arrive at

THEOREM 4.12. – *The hypergeometric series* ${}_2\Phi_1^{(q,t)}(a,b;c;x_1,\ldots,x_m)$ *is the unique solution of the system* (2.26) *subject to the following conditions:*

(a) $S(x)$ *is a symmetric function in* $x_1,\ldots,x_m$.

(b) $S(x)$ *is analytic at the origin with* $S(0)=1$.

## 5. Consequences

5.1. INTEGRATION FORMULA OF MACDONALD POLYNOMIALS.

Put

$$ {}_qD(\alpha,\beta,\gamma;t)=\prod_{j=1}^{n} t_j^{\alpha+(j-1)(1-2\gamma)}\frac{(qt_j)_\infty}{(q^\beta t_j)_\infty}\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty}D(t). $$

Theorem 4.11 implies the following integration formula.

THEOREM 5.1.

$$ \begin{aligned} &\int_{[0,\xi\infty]_q} P_\lambda(t;q,Q)\,{}_qD(\alpha,\beta,\gamma;t)\tilde{\omega}\\ &\quad={}_qS_{n,0}(\alpha,\beta,\gamma;\xi)\frac{(Q^n)_\lambda^{(q,Q)}(q^{\alpha+n-1}Q^{-(n-1)})_\lambda^{(q,Q)}}{h_\lambda(q,Q)(q^{\alpha+\beta+n-1})_\lambda^{(q,Q)}}\\ &\quad={}_qS_{n,0}(\alpha,\beta,\gamma;\xi)P_\lambda(1,Q,\ldots,Q^{n-1};q,Q)\frac{(q^{\alpha+n-1}Q^{-(n-1)})_\lambda^{(q,Q)}}{(q^{\alpha+\beta+n-1})_\lambda^{(q,Q)}}. \end{aligned} $$

*Proof.* – Replacing $x$ by $-q^\beta Q^{-1}x$, $\lambda$ by $\lambda'$ and setting $y=t=(t_1,\ldots,t_n)$, $(q,t)=(Q,q)$ in (3.8), we have

$$ \prod_{1\le i\le m,1\le j\le n}(1-q^\beta Q^{-1}x_it_j)=\sum_\lambda(-q^\beta Q^{-1})^{|\lambda'|}P_{\lambda'}(x;Q,q)P_\lambda(t;q,Q). $$

Substituting this into (4.17) yields

$$ \begin{aligned} \text{(5.1)}\qquad &\sum_\lambda(-q^\beta Q^{-1})^{|\lambda'|}P_{\lambda'}(x;Q,q)\int_{[0,\xi\infty]_q}P_\lambda(t;q,Q)\,{}_qD(\alpha,\beta,\gamma;t)\tilde{\omega}\\ &\quad=C\cdot{}_2\Phi_1^{(Q,q)}(Q^{-n},q^{-(\alpha+n-1)}Q^{n-1};q^{-(\alpha+\beta+n-1)};x)\\ &\quad=C\sum_{\lambda'}\frac{(Q^{-n})_{\lambda'}^{(Q,q)}(q^{-(\alpha+n-1)}Q^{n-1})_{\lambda'}^{(Q,q)}}{(q^{-(\alpha+\beta+n-1)})_{\lambda'}^{(Q,q)}h'_{\lambda'}(Q,q)}P_{\lambda'}(x;Q,q). \end{aligned} $$

Note that in general we have

$$ (a)_\lambda^{(q,t)}=(-a)^{|\lambda|}(a^{-1})_{\lambda'}^{(t,q)},\qquad h'_\lambda(q,t)=h_{\lambda'}(t,q). $$

Hence equating the coefficients of $P_{\lambda'}(x;Q,q)$ in (5.1) immediately gives the desired first equality. The second equality is a direct consequence of Theorem 3.3.

We next show Theorem 5.1 implies the integration formula of Kadell [Kad2]. Assume $\gamma = k$, a positive integer. Put $x = \alpha + (n-1)(1-2k), y = \beta$.

PROPOSITION 5.2. – *Assume* $\mathrm{Re}(x) > 0$, $y \neq 0, -1, -2, \ldots$. *We have*

$$\begin{aligned}
(5.2)\qquad & \int_{[0,1]^n} P_\lambda(t;q,q^k) \prod_{j=1}^n t_j^x \frac{(qt_j)_\infty}{(q^y t_j)_\infty} \prod_{1\le i<j\le n} t_i^{2k}(q^{1-k}t_j/t_i)_{2k}\,\tilde{\omega} \\
&= q^{kx\binom{n}{2}+2k^2\binom{n}{3}} \frac{(q^{nk})_\lambda^{(q,q^k)}}{h_\lambda(q,q^k)} \\
&\quad \times \prod_{i=1}^n \frac{\Gamma_q(ik+1)\Gamma_q(x+(n-i)k+\lambda_i)\Gamma_q(y+(n-i)k)}{\Gamma_q(k+1)\Gamma_q(x+y+(2n-i-1)k+\lambda_i)} \\
&= q^{kx\binom{n}{2}+2k^2\binom{n}{3}} P_\lambda(1,q^k,\ldots,q^{(n-1)k};q,q^k) \\
&\quad \cdot \prod_{i=1}^n \frac{\Gamma_q(ik+1)\Gamma_q(x+(n-i)k+\lambda_i)\Gamma_q(y+(n-i)k)}{\Gamma_q(k+1)\Gamma_q(x+y+(2n-i-1)k+\lambda_i)}.
\end{aligned}$$

*Proof.* – Observe that $\prod_{1\le i<j\le n} t_i^{2k-1}(q^{1-k}t_j/t_i)_{2k-1}$ is antisymmetric. Using Lemma 2.1, (4.17) and Theorem 5.1, we have

LHS of (5.2)

$$\begin{aligned}
&= \int_{<\xi_F>} P_\lambda(t;q,q^k) \prod_{j=1}^n t_j^x \frac{(qt_j)_\infty}{(q^y t_j)_\infty} \mathcal{A}\Big(\prod_{1\le i<j\le n}(t_i - q^k t_j)\Big) t_i^{2k-1}(q^{1-k}t_j/t_i)_{2k-1}\tilde{\omega} \\
&= \frac{(q^k;q^k)_n}{(1-q^k)^n} \int_{<\xi_F>} P_\lambda(t;q,q^k)_q D(x+(n-1)(2k-1),y,k;t)\tilde{\omega} \\
&= \frac{(q^k;q^k)_n}{(1-q^k)^n} q^{A_n} \frac{(q^{nk})_\lambda}{h_\lambda(q,q^k)} \frac{\prod_{(i,j)\in\lambda}(1-q^{x+(n-i)k+j-1})}{\prod_{(i,j)\in\lambda}(1-q^{x+y+(2n-i-1)k+j-1})} \\
&\quad \cdot \prod_{i=1}^n \frac{\Gamma_q(ik)\Gamma_q(x+(n-i)k)\Gamma_q(y+(n-i)k)}{\Gamma_q(k)\Gamma_q(x+y+(2n-i-1)k)}.
\end{aligned}$$

Hence (5.2) is immediate from the formulas

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$$

and

$$A_n = \sum_{j=1}^n (\alpha - 2(j-1)k + n - 1)(j-1)k = kx\binom{n}{2} + 2k^2\binom{n}{3}.$$

Kadell [Kad2] has given a different proof of Proposition 5.2 in a slightly different expression:

$$\text{LHS of (5.2)} = q^{B_n}\frac{(q^k;q^k)_n}{(1-q^k)^n}\prod_{1\le i<j\le n}\frac{(\lambda_i-\lambda_j+(j-i)k))_k}{(1-q)^k}$$
$$\times\prod_{i=1}^{n}\frac{\Gamma_q(x+(n-i)k+\lambda_i)\Gamma_q(y+(n-i)k)}{\Gamma_q(x+y+(2n-i-1)k+\lambda_i)},$$

where $B_n = k\sum_{i=1}^n (i-1)\lambda_i + kx\binom{n}{2} + 2k^2\binom{n}{3}$. This is checked by utilizing (3.6):

$$h_\lambda(q,q^k) = (q^k)_\infty^n\prod_{i=1}^n (q^{\lambda_i+(n-i+1)k})_\infty^{-1}\prod_{1\le i<j\le n}(q^{\lambda_i-\lambda_j+(j-i)k})_k^{-1}$$

and observing

$$(q^{nk})_\lambda^{(q,q^k)}(q^{\lambda_i+(n-i+1)k})_\infty = q^{k\sum_{i=1}^n (i-1)\lambda_i}(q^k)_\infty^n.$$

5.2. Integral representation of ${}_r\Phi_s^{(q,t)}(a,b;c;x)$.

Let $a_1,\dots,a_r$ and $b_1,\dots,b_s$ be such that $(b_j)_\lambda^{(q,t)}\ne 0$ for any $j$ and any partition $\lambda$ of length $\le m$. Assume $m\le n$, and put

$$(5.3)\qquad {}_r\Phi_s^{(q,t)}(a_1,\dots,a_r;b_1,\dots,b_s;x_1,\dots,x_m;y_1,\dots,y_n)$$
$$=\sum_\lambda \frac{\prod_{i=1}^r (a_i)_\lambda^{(q,t)}\left\{(-1)^{|\lambda|}q^{b(\lambda')}\right\}^{s+1-r}}{\prod_{i=1}^s (b_j)_\lambda^{(q,t)} h'_\lambda(q,t)P_\lambda(1,t,\dots,t^{n-1})}P_\lambda(x;q,t)P_\lambda(y;q,t).$$

This series converges in a neighborhood of origin only if $r\le s+1$ and its proof is similar to that of Theorem 3.8.

Proposition 5.3. – *Let $a_{r+1}=q^\varepsilon$ and $b_{s+1}=q^\eta$ and put $\alpha=\varepsilon+(n-1)(\gamma-1),\beta=\eta-\varepsilon-(n-1)\gamma$. We have*

$$(5.4)\qquad {}_{r+1}\Phi_{s+1}^{(q,t)}(a_1,\dots,a_{r+1};b_1,\dots,b_{s+1};x_1,\dots,x_m)$$
$$= {}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F)^{-1}$$
$$\times\int_{[0,\xi_F\infty]_q}{}_r\Phi_s^{(q,t)}(a_1,\dots,a_r;b_1,\dots,b_s;x_1,\dots,x_m;y_1,\dots,y_n)$$
$$\times\, {}_qD(\alpha,\beta,\gamma;y)\tilde{\omega}$$

*provided the right-hand side is convergent.*

*Proof.* – This is an immediate consequence of Theorem 5.1 because

$$\frac{(q^{\alpha+n-1}Q^{-(n-1)})_\lambda^{(q,Q)}}{(q^{\alpha+\beta+n-1})_\lambda^{(q,Q)}} = \frac{(q^\varepsilon)_\lambda^{(q,Q)}}{(q^\eta)_\lambda^{(q,Q)}}.$$

PROPOSITION 5.4. – *Let $a = q^\delta, b = q^\varepsilon, c = q^\eta$ and $\alpha = \varepsilon + (n-1)(\gamma-1)$, $\beta = \eta - \varepsilon - (n-1)\gamma$. We have*

(5.5)
$$_2\Phi_1^{(q,Q)}(q^{-N}, b; c; q, qQ, \ldots, qQ^{n-1}) = q^{Nn(\alpha+(n-1)(1-\gamma))} \prod_{i=1}^{n} \frac{(q^{\beta+(i-1)\gamma})_N}{(q^{\alpha+\beta+n-1-(i-1)\gamma})_N},$$

*where $\delta = -N, N \in \mathbb{Z}_{\geq 0}$ and*

$$\begin{aligned}(5.6)\qquad & {}_2\Phi_1^{(q,Q)}(a, b; c; c/ab, cQ/ab, \ldots, cQ^{n-1}/ab)\\ &= \prod_{i=1}^{n} \frac{\Gamma_q(\beta - \delta + (i-1)\gamma)\Gamma_q(\alpha + \beta + n - 1 - (i-1)\gamma)}{\Gamma_q(\beta + (i-1)\gamma)\Gamma_q(\alpha + \beta - \delta + n - 1 - (i-1)\gamma)}\end{aligned}$$

*provided the left-hand side is convergent.*

*Proof.* – By Theorem 3.5 we have

$$(5.7)\qquad {}_1\Phi_0^{(q,Q)}(q^{-N}; q, qQ, \ldots, qQ^{n-1}; y_1 \ldots, y_n) = \prod_{j=1}^{n} (q^{-N+1}y_i)_N,$$

$$(5.8)\qquad {}_1\Phi_0^{(q,Q)}(a; c/ab, cQ/ab, \ldots, cQ^{n-1}/ab; y_1 \ldots, y_n) = \prod_{j=1}^{n} \frac{(q^\beta y_i)_\infty}{(q^{\beta-\gamma}y_i)_\infty}.$$

Substituting (5.7) (resp. (5.8)) in the right-hand side of (5.4) with $m = n, r = 1, s = 0$ and $a_1 = q^{-N}$ (resp. $a_1 = a$) gives

$$\begin{aligned}&{}_2\Phi_1^{(q,Q)}(q^{-N}, b; c; q, qQ, \ldots, qQ^{n-1}) = q^{Nn(\alpha+(n-1)(1-\gamma))} \frac{{}_qS_{n.0}(\alpha, \beta + N, \gamma; q^{-N}\xi_F)}{{}_qS_{n.0}(\alpha, \beta, \gamma; \xi_F)}\\ &= q^{Nn(\alpha+(n-1)(1-\gamma))} \frac{{}_qS_{n.0}(\alpha, \beta + N, \gamma; \xi_F)}{{}_qS_{n.0}(\alpha, \beta, \gamma; \xi_F)},\\ &{}_2\Phi_1^{(q,Q)}(a, b; c; c/ab, cQ/ab, \ldots, cQ^{n-1}/ab) = \frac{{}_qS_{n.0}(\alpha, \beta - \delta, \gamma; \xi_F)}{{}_qS_{n.0}(\alpha, \beta, \gamma; \xi_F)}.\end{aligned}$$

Hence the proof follows from the explicit formula (4.19).

## 6. Proof of Theorem 4.4

6.1. SKEW MACDONALD POLYNOMIALS. – For any partition $\lambda, \mu, \nu$ define rational functions $f^\lambda_{\mu\nu}(q,t)$ by

$$f^\lambda_{\mu\nu} = f^\lambda_{\mu\nu}(q,t) = \frac{< P_\lambda, P_\mu P_\nu >}{< P_\lambda, P_\lambda >}.$$

Equivalently,

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu}(q,t) P_\lambda. \tag{6.1}$$

Clearly $f^\lambda_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$. Moreover it holds that $f^\lambda_{\mu\nu} = 0$ unless $\lambda \supset \mu$ and $\lambda \supset \nu$ [Ma2, (4.2)].

If $\lambda, \mu$ are partitions, define *skew Macdonald polynomials* $P_{\lambda/\mu}$ by

$$P_{\lambda/\mu} = b_\lambda^{-1} b_\mu \sum_\nu b_\nu f^\lambda_{\mu\nu}(q,t) P_\nu. \tag{6.2}$$

Hence $P_{\lambda/\mu} = 0$ unless $\lambda \supset \mu$. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be two sequences of independent indeterminates. Then we have [Ma2, (4.5)]:

$$P_\lambda(x,y) = \sum_\mu P_{\lambda/\mu}(x) P_\mu(y). \tag{6.3}$$

Put

$$\widetilde{f^\lambda_{\mu\nu}} = \frac{b_\mu b_\nu}{b_\lambda} f^\lambda_{\mu\nu}.$$

Setting $x = x_m, y = (x_1, \ldots, x_{m-1})$ in (6.3), we get

$$P_\lambda(x_1, \ldots, x_m) = \sum_\mu \Big( \sum_r \widetilde{f^\lambda_{\mu(r)}} \Big) x_m^r P_\mu(x_1, \ldots, x_{m-1}). \tag{6.4}$$

If $\lambda \supset \mu$ then the skew shape $\lambda/\mu$ (regarded as a difference $\lambda - \mu$ of diagrams) is called a *horizontal $r$-strip* (resp. *vertical $r$-strip*) if $|\lambda/\mu| = r$ and no two distinct squares of $\lambda/\mu$ lie in the same column (resp. row). Then $f^\lambda_{\mu(r)} \neq 0$ (resp. $f^\lambda_{\mu(1^r)} \neq 0$) if and only if $\lambda \supset \mu$ and $\lambda/\mu$ is a horizontal (resp. vertical) $r$-strip [Ma2, (4.8)]. Moreover they can be explicitly evaluated as follows. For each square $s$ and each partition $\lambda$, define

$$b_\lambda(s) = b_\lambda(s; q, t) = \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}} \tag{6.5}$$

if $s \in \lambda$, and $b_\lambda(s) = 1$ if $s \notin \lambda$. If $S$ is any set of squares (contained in the diagram of $\lambda$ or not), put

$$b_\lambda(S) = \prod_{s \in S} b_\lambda(s). \tag{6.6}$$

Now let $\lambda, \mu$ be partitions such that $\lambda \supset \mu$ and $\lambda/\mu$ is a horizontal $r$-strip. Let $C_{\lambda/\mu}$ (resp. $R_{\lambda/\mu}$) denote the union of the columns (resp. rows) that contain squares of $\lambda/\mu$. Then [Ma2, (5.12)]:

$$f^\lambda_{\mu(r)} = b_{(r)}^{-1} b_\lambda(C_{\lambda/\mu})/b_\mu(C_{\lambda/\mu}). \tag{6.7}$$

Observe that

$$\widetilde{f^\lambda_{\mu(1)}} = b_{(1)} \frac{b_\mu(R_{\lambda/\mu})}{b_\lambda(R_{\lambda/\mu})} = \frac{1-t}{1-q} \frac{b_\mu(R_{\lambda/\mu})}{b_\lambda(R_{\lambda/\mu})}. \tag{6.8}$$

If $\lambda, \mu$ be partitions such that $\lambda \supset \mu$ and $\lambda/\mu$ is a vertical $r$-strip, then applying the duality theorem [Ma2, (3.5)] (*cf.* [Ma3, Chap.6, (7.9)]) to (6.7), we obtain

$$f^\lambda_{\mu(1^r)} = b_\lambda(\bar{R}_{\lambda/\mu})/b_\mu(\bar{R}_{\lambda/\mu}), \tag{6.9}$$

where $\bar{R}_{\lambda/\mu}$ denotes the union of rows that do not contain squares of $\lambda/\mu$.

6.2. Lemmas. – Put

$$\widetilde{\binom{\lambda}{\mu}}_m = \prod_{s \in \lambda} (1 - q^{a'(s)} t^{m-l'(s)}) \Big( \prod_{s \in \mu} (1 - q^{a'(s)} t^{m-l'(s)}) \Big)^{-1} h_\mu h_\lambda^{-1} \binom{\lambda}{\mu}_m, \tag{6.10}$$

so that from Definition 4.3 we have

$$\varepsilon P_\lambda(x_1, \ldots, x_m) = \sum_{|\mu|=|\lambda|-1} \widetilde{\binom{\lambda}{\mu}}_m P_\mu(x_1, \ldots, x_m). \tag{6.11}$$

Lemma 6.1. – *Let $\lambda$ and $\mu$ be partitions of $\ell(\lambda) \le m$ and $\ell(\mu) \le m-1$. We have*

$$\widetilde{\binom{\lambda}{\mu}}_m = t \widetilde{\binom{\lambda}{\mu}}_{m-1} + \widetilde{f^\lambda_{\mu(1)}}, \ if \ \ell(\lambda) \le m-1,$$

$$\widetilde{\binom{\lambda}{\mu}}_m = \widetilde{f^\lambda_{\mu(1)}}, \ if \ \ell(\lambda) = m.$$

*Proof.* – Setting $x_m = 0$ in (6.11) yields

$$t\varepsilon P_\lambda(x_1,\ldots,x_{m-1}) + \frac{\partial P_\lambda(x_1,\ldots,x_m)}{\partial_q x_m}\bigg|_{x_m=0} = \sum_\mu \widetilde{\binom{\lambda}{\mu}}_m P_\mu(x_1,\ldots,x_{m-1}),$$

in which we see by (6.4) that

$$\frac{\partial P_\lambda(x_1,\ldots,x_m)}{\partial_q x_m}\bigg|_{x_m=0} = \sum_{|\mu|=|\lambda|-1} \widetilde{f^{\lambda}_{\mu(1)}} P_\mu(x_1,\ldots,x_{m-1}).$$

Hence equating the coefficients of $P_\mu(x_1,\ldots,x_{m-1})$ in both sides gives the desired formulas.

For each partition $\lambda$, define $\lambda_* = (\lambda_1-1,\lambda_2-1,\ldots,\lambda_{\ell(\lambda)}-1)$. One can readily derive from Theorem 3.1 that if $\ell(\lambda) = m$, then

$$P_\lambda(x_1,\ldots,x_m) = \Big(\prod_{i=1}^m x_i\Big) P_{\lambda_*}(x_1,\ldots,x_m). \tag{6.12}$$

LEMMA 6.2. – *For partitions $\lambda$ and $\mu$ of length $= m$, we have*

$$\widetilde{\binom{\lambda}{\mu}}_m = q\widetilde{\binom{\lambda_*}{\mu_*}}_m + \widetilde{f^{\mu}_{\lambda_*(1^{m-1})}}.$$

*Proof.* – It follows from (6.12) that

$$\varepsilon P_\lambda(x_1,\ldots,x_m) = \varepsilon\Big(\prod_{i=1}^m x_i\Big) P_{\lambda_*}(x_1,\ldots,x_m) + q\Big(\prod_{i=1}^m x_i\Big)\varepsilon P_{\lambda_*}(x_1,\ldots,x_m). \tag{6.13}$$

We assert in general that

$$\varepsilon e_r(x_1,\ldots,x_m) = \frac{1-t^{m-r+1}}{1-t} e_{r-1}(x_1,\ldots,x_m). \tag{6.14}$$

In fact

$$\begin{aligned}
\sum_{i=1}^m A_i \frac{\partial}{\partial_q x_i} e_r(x_1,\ldots,x_m) &= \sum_{i=1}^m A_i e_{r-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m)\\
&= \sum_{i=1}^m A_i\big(e_{r-1}(x_1,\ldots,x_m) - x_i e_{r-2}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_m)\big)\\
&= \frac{1-t^m}{1-t} e_{r-1}(x_1,\ldots,x_m) - D_1 e_{r-1}(x_1,\ldots,x_m)\\
&= \frac{1-t^{m-r+1}}{1-t} e_{r-1}(x_1,\ldots,x_m).
\end{aligned}$$

Substituting (6.14) with $r = m$ into (6.13) and equating the coefficients of $P_\mu(x_1, \ldots, x_m)$ of both sides gives the desired formula.

LEMMA 6.3. – *We have*

$$\begin{aligned}(6.15)\qquad &\sum_{\lambda,\mu} f^\lambda_{\nu(1^r)} \widetilde{\binom{\lambda}{\mu}}_m P_\mu(x_1, \ldots, x_m)\\ &\quad= \Big(\frac{1-t^m}{1-t} + qe_\nu\Big) e_{r-1} P_\nu(x_1, \ldots, x_m)\\ &\quad\quad - D_1\Big(e_{r-1}P_\nu(x_1, \ldots, x_m)\Big) + e_r \varepsilon P_\nu(x_1, \ldots, x_m).\end{aligned}$$

*Proof.* – The left-hand side is nothing but $\varepsilon\big(e_r P_\nu(x_1, \ldots, x_m)\big)$. On the other hand using (6.14) and that

$$\begin{aligned}T_{q,x_i} e_r(x_1, \ldots, x_m) &= e_r(x_1, \ldots, x_m) + (q-1)x_i e_{r-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)\\ &= e_r(x_1, \ldots, x_m) + qx_i e_{r-1}(x_1, \ldots, x_m) - x_i T_{q,x_i} e_{r-1}(x_1, \ldots, x_m),\end{aligned}$$

we have

$$\begin{aligned}&\varepsilon\big(e_r P_\nu(x_1, \ldots, x_m)\big)\\ &\quad= (\varepsilon e_r) P_\nu(x_1, \ldots, x_m) + \sum_{i=1}^m A_i T_{q,x_i}(e_r) \frac{\partial}{\partial_q x_i} P_\nu(x_1, \ldots, x_m)\\ &\quad= \frac{1-t^{m-r+1}}{1-t} e_{r-1} P_\nu(x_1, \ldots, x_m) + e_r \varepsilon P_\nu(x_1, \ldots, x_m) + q e_{r-1} D_1 P_\nu(x_1, \ldots, x_m)\\ &\quad\quad - \sum_{i=1}^m A_i x_i T_{q,x_i}(e_{r-1}) \frac{\partial}{\partial_q x_i} P_\nu(x_1, \ldots, x_m).\end{aligned}$$

Hence the formula (6.15) is immediate from

$$\begin{aligned}D_1\big(e_{r-1}P_\nu(x_1, \ldots, x_m)\big) &= \frac{t^{m-r+1} - t^m}{1-t} e_{r-1} P_\nu(x_1, \ldots, x_m)\\ &\quad + \sum_{i=1}^m A_i x_i T_{q,x_i}(e_{r-1}) \frac{\partial}{\partial_q x_i} P_\nu(x_1, \ldots, x_m).\end{aligned}$$

LEMMA 6.4.

$$\begin{aligned}(6.17)\qquad &\sum_{\lambda,\mu} f^\lambda_{\nu(2)} \widetilde{\binom{\lambda}{\mu}}_m P_\mu(x_1, \ldots, x_m)\\ &\quad= \frac{1}{1-qt}\big\{(1+q)(1-t^m) - (1-q^2)e_\nu\big\} e_1 P_\nu(x_1, \ldots, x_m)\\ &\quad\quad + \frac{t(1-q^2)}{1-qt} D_1\big(e_1 P_\nu(x_1, \ldots, x_m)\big) + P_{(2)} \varepsilon P_\nu(x_1, \ldots, x_m).\end{aligned}$$

*Proof.* – One can readily deduce from Theorem 3.1 that

$$P_{(2)} = m_{(2)} + \frac{(1+q)(1-t)}{1-qt} m_{(1,1)},$$

so

$$\frac{\partial P_{(2)}}{\partial_q x_i} = \frac{t(1-q^2)}{1-qt} x_i + \frac{(1+q)(1-t)}{1-qt} e_1, \tag{6.18}$$

from which we get

$$\varepsilon P_{(2)} = \frac{(1+q)(1-qt^m)}{1-qt} e_1. \tag{6.19}$$

We have

$$\varepsilon(P_{(2)}P_\nu) = \varepsilon(P_{(2)})P_\nu + P_{(2)}\varepsilon(P_\nu) + (q-1)\sum_{i=1}^m A_i x_i \frac{\partial P_{(2)}}{\partial_q x_i} \frac{\partial P_\nu}{\partial_q x_i} \tag{6.20}$$

and by (6.16) with $r = 2$ that

$$(q-1)\sum_{i=1}^m A_i x_i^2 \frac{\partial P_\nu}{\partial_q x_i} = D_1(e_1 P_\nu) - (t^{m-1} + e_\nu) e_1 P_\nu. \tag{6.21}$$

Substituting (6.18) and (6.19) into (6.20) and then applying (6.21) yields (6.17).

6.3. Now we turn to the proof of Theorem 4.4. We shall prove a stronger assertion: For any partitions $\lambda$ and $\mu$ of length $\leq m$, we have

$$\binom{\lambda}{\mu}_m = t^{b(\lambda)-b(\mu)} \frac{h'_\lambda}{h'_\mu} f^\lambda_{\mu(1)}, \tag{6.22}$$

which is, in view of (6.8) and (6.10), equivalent to

$$\widetilde{\binom{\lambda}{\mu}}_m = \begin{cases} \frac{1-q^{\lambda_i-1}t^{m-i+1}}{1-q} \frac{b_\mu(R_{\lambda/\mu})}{b_\lambda(R_{\lambda/\mu})}, & \text{if } \mu = \lambda^{(i)}, \\ 0, & \text{if } \mu \not\subset \lambda. \end{cases} \tag{6.23}$$

Here note that $R_{\lambda/\mu}$ is the $i$-th row of $\lambda$. We shall denote the $i$-row (resp. $j$-th column) of $\lambda$ by $R_{\lambda,i}$ (resp. $C_{\lambda,j}$) and write $b_\lambda(R_i)$ (resp. $b_\lambda(C_i)$) for $b_\lambda(R_{\lambda,i})$ (resp. $b_\lambda(C_{\lambda,i})$). Note that $b_\lambda(R_{\kappa,i}) = b_\lambda(R_i)$ provided $\kappa \supset \lambda$. (6.23) is rewritten as

$$\widetilde{\binom{\lambda}{\mu}}_m \begin{cases} \frac{1-q^{\lambda_i-1}t^{m-i+1}}{1-q} \frac{b_\mu(R_i)}{b_\lambda(R_i)}, & \text{if } \mu = \lambda^{(i)}, \\ 0, & \text{if } \mu \not\subset \lambda. \end{cases} \tag{6.24}$$

We prove (6.24) by induction on the dimension $m$. The case $m = 1$ is easy to check: Put $\lambda = (r), \mu = (r-1)$. Then clearly $\widetilde{\binom{\lambda}{\mu}}_m = (1-q^r)/(1-q)$ holds. On the other hand we see

$$b_\lambda(R_1) = \frac{(t;q)_r}{(q;q)_r}, b_\mu(R_1) = \frac{(t;q)_{r-1}}{(q;q)_{r-1}},$$

and therefore (6.24) follows at once. We assume that (6.24) holds in the dimensions $\leq m-1$. This implies that (6.24) holds in the case $\ell(\lambda) \leq m-1$. In fact by Lemma 6.1 we see that $\widetilde{\binom{\lambda}{\mu}}_m = 0$ unless $\mu \subset \lambda$. Moreover if $\mu = \lambda^{(i)}$ and $\ell(\lambda) \leq m-1$, then by means of (6.8) we have

$$\begin{aligned}\widetilde{\binom{\lambda}{\mu}}_m &= t\frac{1-q^{\lambda_i-1}t^{m-i}}{1-q}\frac{b_\mu(R_i)}{b_\lambda(R_i)} + \frac{1-t}{1-q}\frac{b_\mu(R_i)}{b_\lambda(R_i)}\\ &= \frac{1-q^{\lambda_i-1}t^{m-i+1}}{1-q}\frac{b_\mu(R_i)}{b_\lambda(R_i)}.\end{aligned}$$

If $\ell(\lambda) = m$, then $\mu = \lambda^{(m)}$ and $\lambda_m = 1$. Hence (6.24) is immediate from Lemma 6.1 also in this case.

Next suppose $\ell(\mu) = m$ and $\ell(\lambda) = m$ and that (6.24) holds for $\lambda_*$ and $\mu_*$. Then Lemma 6.2 implies that $\widetilde{\binom{\lambda}{\mu}}_m = 0$ unless $\mu \subset \lambda$. If $\mu = \lambda^{(i)}$, then by (6.9) we have

$$\widetilde{\binom{\lambda}{\mu}}_m = q\frac{1-q^{\lambda_i-2}t^{m-i+1}}{1-q}\frac{b_{\mu_*}(R_i)}{b_{\lambda_*}(R_i)} + \frac{b_\mu(\bar{R}_{\mu/\lambda_*})}{b_{\lambda_*}(\bar{R}_{\mu/\lambda_*})}. \tag{6.25}$$

Observe that

$$\begin{aligned}b_{\mu_*}(R_i) &= b_\mu(R_i)b_\mu((i,1))^{-1} = \frac{1-q^{\lambda_i-1}t^{m-i}}{1-q^{\lambda_i-2}t^{m-i+1}}b_\mu(R_i)\\ b_{\lambda_*}(R_i) &= b_\lambda(R_i)b_\lambda((i,1))^{-1} = \frac{1-q^{\lambda_i}t^{m-i}}{1-q^{\lambda_i-1}t^{m-i+1}}b_\lambda(R_i)\end{aligned}$$

and

$$b_\mu(\bar{R}_{\mu/\lambda_*}) = b_\mu(R_i),\ b_{\lambda_*}(\bar{R}_{\mu/\lambda_*}) = b_{\lambda_*}(R_i).$$

Substituting these into (6.25) gives (6.24). Iterating this argument, we see that the case $\ell(\lambda) = \ell(\mu) = m$ reduces to the case $\ell(\mu) \leq m-1$ (which we have just proved) or the case $\ell(\mu) = m$ and $\ell(\lambda) \leq m-1$.

It now remains to show that $\widetilde{\binom{\lambda}{\mu}}_m = 0$ when $\ell(\lambda) \leq m-1$ and $\ell(\mu) = m$. We prove this by induction on $\ell(\lambda)$ and for fixed $\ell(\lambda)$ on $\lambda_{\ell(\lambda)}$, the case $\lambda = (1)$ being obvious.

Note first that $\widetilde{\binom{\lambda}{\mu}}_m = 0$ for $\lambda$ of $\ell(\lambda) \le p-1, p \le m-1$ implies $\widetilde{\binom{\lambda}{\mu}}_m = 0$ for $\lambda$ of $\ell(\lambda) \le p$ and $\lambda_p = 1$. This follows from (6.15) by setting $r = 1, \nu = \lambda^{(p)}$ (so that $\ell(\nu) = p-1$) and equating the coefficients of $P_\mu$ of both sides. So we have reduced the proof to showing $\widetilde{\binom{\lambda_{(p)}}{\mu}}_m = 0$ for any partition $\lambda$ of $|\lambda| = |\mu|$ and $\ell(\lambda) = p \le m-1$ provided that $\widetilde{\binom{\kappa}{\mu}}_m = 0$ when $\ell(\kappa) \le p-1$ or $\ell(\kappa) = p$ and $\kappa_p \le \lambda_p$.

We divide the proof into several parts treating different cases. First we assume $p \le m-2$ and derive necessary equalities from Lemma 6.3 and Lemma 6.4. We have

$$\widetilde{\binom{\lambda^{(p)}_{(p+1,i)}}{\mu}}_m = 0 \tag{6.26}$$

provided $i \le p-1$. This follows immediately from (6.15) if we set $r = 1, \nu = \lambda^{(p)}_{(i)}$ and compare the coefficients of $P_\mu$ using induction hypothesis. Setting $r = 1$ (resp. $r = 2$) and $\nu = \lambda$ or $\lambda^{(p)}_{(p+1)}$ (resp. $\nu = \lambda^{(p)}$) in (6.15) and comparing coefficients of $P_\mu$ using induction hypothesis and (6.26), we find that

$$f^{\lambda_{(p)}}_{\lambda(1)} \widetilde{\binom{\lambda_{(p)}}{\mu}}_m + f^{\lambda_{(p+1)}}_{\lambda(1)} \widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m = \sum_i f^{\mu}_{\lambda^{(i)}(1)} \widetilde{\binom{\lambda}{\lambda^{(i)}}}_m \tag{6.27}$$

$$\begin{aligned} &f^{\lambda_{(p+1)}}_{\lambda^{(p)}_{(p+1)}(1)} \widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m + f^{\lambda^{(p)}_{(p+1,p+2)}}_{\lambda^{(p)}_{(p+1)}(1)} \widetilde{\binom{\lambda^{(p)}_{(p+1,p+2)}}{\mu}}_m \\ &\quad + f^{\lambda^{(p)}_{(p+1,p+1)}}_{\lambda^{(p)}_{(p+1)}(1)} \widetilde{\binom{\lambda^{(p)}_{(p+1,p+1)}}{\mu}}_m = \sum_i f^{\mu}_{\lambda^{(p,i)}_{(p+1)}(1)} \widetilde{\binom{\lambda^{(p)}_{(p+1)}}{\lambda^{(p,i)(p+1)}}}_m \end{aligned} \tag{6.28}$$

$$\begin{aligned} &f^{\lambda_{(p+1)}}_{\lambda^{(p)}(1^2)} \widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m + f^{\lambda^{(p)}_{(p+1,p+2)}}_{\lambda^{(p)}(1^2)} \widetilde{\binom{\lambda^{(p)}_{(p+1,p+2)}}{\mu}}_m \\ &= \left(\frac{1-t^m}{1-t} + qe_{\lambda^{(p)}} - e_\mu\right) f^{\mu}_{\lambda^{(p)}(1)} + \sum_i f^{\mu}_{\lambda^{(p,i)}(1^2)} \widetilde{\binom{\lambda^{(p)}}{\lambda^{(p,i)}}}_m . \end{aligned} \tag{6.29}$$

Similarly, setting $\nu = \lambda^{(p)}$ in (6.17) gives

$$\begin{aligned} &f^{\lambda_{(p)}}_{\lambda^{(p)}(2)} \widetilde{\binom{\lambda_{(p)}}{\mu}}_m + f^{\lambda_{(p+1)}}_{\lambda^{(p)}(2)} \widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m + f^{\lambda^{(p)}_{(p+1,p+1)}}_{\lambda^{(p)}(2)} \widetilde{\binom{\lambda^{(p)}_{(p+1,p+1)}}{\mu}}_m \\ &= \frac{1}{1-qt}\{(1-q^2)(te_\mu - e_{\lambda^{(p)}} + (1+q)(1-t^m)\} f^{\mu}_{\lambda^{(p)}(1)} \\ &\quad + \sum_i f^{\mu}_{\lambda^{(p,i)}(2)} \widetilde{\binom{\lambda^{(p)}}{\lambda^{(p,i)}}}_m . \end{aligned} \tag{6.30}$$

Here note that, because of induction hypothesis, the generalized binomial coefficients appearing in the right-hand sides of (6.27)–(6.30) are given by (6.24). So we regard these equalities as equations of unknowns $\widetilde{\binom{\lambda_{(p)}}{\mu}}_m$, $\widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m$, $\widetilde{\binom{\lambda^{(p)}_{(p+1,\,p+1)}}{\mu}}_m$, and $\widetilde{\binom{\lambda^{(p)}_{(p+1,\,p+2)}}{\mu}}_m$.

*Remark*. – If $\lambda_p = 1$ (resp. $\lambda_p = 2$), we understand $\widetilde{\binom{\lambda^{(p)}_{(p+1,\,p+1)}}{\mu}}_m$, and $\widetilde{\binom{\lambda^{(p)}_{(p+1,\,p+2)}}{\mu}}_m$ (resp. $\widetilde{\binom{\lambda^{(p)}_{(p+1,\,p+1)}}{\mu}}_m$ ) to be zero and the following argument should be modified accordingly. We leave this task to the reader and assume henceforth that $\lambda_p \geq 3$.

*Case* 1. – $p \leq m-2$ and $\mu \not\subset \lambda^{(p)}_{(p+1,p+2)}$. Observe that the right-hand sides of (6.27)-(6.30) are all vanishing. Hence, for the proof of $\widetilde{\binom{\lambda_{(p)}}{\mu}}_m = 0$, it suffices to show that the determinant of coefficient matrix of equations is not identically zero. For this purpose, set $t = 1$, then we have in general

$$b_\lambda(s) = \frac{1-q^{a(s)}}{1-q^{a(s)+1}}.$$

Hence by (6.8) and (6.9) we find that the coefficients appearing in the equations (6.27)-(6.29) are all equal to one. Also by (6.7) we have

$$b_{(1)}^{-1} f^{\lambda_{(p)}}_{\lambda^{(p)}(2)}\big|_{t=1} = b_{(1)}^{-1} f^{\lambda^{(p)}_{(p+1,p+1)}}_{\lambda^{(p)}(2)}\big|_{t=1} = 1$$

and

$$\begin{aligned}
b_{(1)}^{-1} f^{\lambda_{(p+1)}}_{\lambda^{(p)}(2)})\big|_{t=1} &= (b_{(1)} b_{(2)})^{-1} \frac{b_{\lambda_{(p+1)}}(C_{\lambda_{(p+1)},1} \cup C_{\lambda^{(p)},\lambda_p})}{b_{\lambda^{(p)}}(C_{\lambda_{(p+1)},1} \cup C_{\lambda^{(p)},\lambda_p})}\bigg|_{t=1} \\
&= (b_{(1)} b_{(2)}^{-1})\big|_{t=1} \frac{1-q^{\lambda_p - 1}}{1-q^{\lambda_p}} \frac{1-q^{\lambda_p-1}}{1-q^{\lambda_p-2}} \\
&= \big((1-q)(1-q^{\lambda_p})(1-q^{\lambda_p-2})\big)^{-1}(1-q^2)(1-q^{\lambda_p-1})^2,
\end{aligned}$$

which we denote by $C(q)$. Therefore the determinant of the coefficient matrix (we multiply the equation (6.30) by $b_{(1)}^{-1}$) at $t = 1$ is

$$\begin{vmatrix}
f^{\lambda_{(p)}}_{\lambda(1)} & f^{\lambda_{(p+1)}}_{\lambda(1)} & 0 & 0 \\
0 & f^{\lambda_{(p+1)}}_{\lambda^{(p)}_{(p+1)}(1)} & f^{\lambda^{(p)}_{(p+1,p+1)}}_{\lambda^{(p)}_{(p+1)}(1)} & f^{\lambda^{(p)}_{(p+1,p+2)}}_{\lambda^{(p)}_{(p+1)}(1)} \\
0 & f^{\lambda_{(p+1)}}_{\lambda^{(p)}(1^2)} & 0 & f^{\lambda^{(p)}_{(p+1,p+2)}}_{\lambda^{(p)}(1^2)} \\
b_{(1)}^{-1} f^{\lambda_{(p)}}_{\lambda^{(p)}(2)} & b_{(1)}^{-1} f^{\lambda_{(p+1)}}_{\lambda^{(p)}(2)} & b_{(1)}^{-1} f^{\lambda^{(p)}_{(p+1,p+1)}}_{\lambda^{(p)}(2)} & 0
\end{vmatrix}_{t=1}$$

$$= \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & C(q) & 1 & 0 \end{vmatrix} = C(q) - 1 \not\equiv 0,$$

as desired.

*Case* 2. – $p = m-2$ and $\mu \subset \lambda^{(p)}_{(p+1,p+2)}$. It necessarily follows that $\mu = \lambda^{(p,r)}_{(p+1,p+2)}$ for some $r \leq p$, and therefore the right-hand side of (6.27) is clearly zero. We see also

$$\text{RHS of } (6.29) = f^{\lambda^{(p,r)}_{(p+1,p+2)}}_{\lambda^{(p,r)}(1^2)} \widetilde{\begin{pmatrix} \lambda^{(p)} \\ \lambda^{(p,r)} \end{pmatrix}}.$$

Hence it suffices to show that

$$f^{\lambda^{(p)}_{(p+1,p+2)}}_{\lambda^{(p)}(1^2)} \widetilde{\begin{pmatrix} \lambda^{(p)}_{(p+1,p+2)} \\ \lambda^{(p,r)}_{(p+1,p+2)} \end{pmatrix}}_m = f^{\lambda^{(p,r)}_{(p+1,p+2)}}_{\lambda^{(p,r)}(1^2)} \widetilde{\begin{pmatrix} \lambda^{(p)} \\ \lambda^{(p,r)} \end{pmatrix}}_m. \tag{6.31}$$

By (6.24) we have

$$\begin{aligned} &\widetilde{\begin{pmatrix} \lambda^{(p)} \\ \lambda^{(p,r)} \end{pmatrix}}_m^{-1} \widetilde{\begin{pmatrix} \lambda^{(p)}_{(p+1,p+2)} \\ \lambda^{(p,r)}_{(p+1,p+2)} \end{pmatrix}}_m \\ &= b_{\lambda^{(p,r)}}(R_r)^{-1} b_{\lambda^{(p,r)}_{(p+1,p+2)}}(R_r) b_{\lambda^{(p)}_{(p+1,p+2)}}(R_r)^{-1} b_{\lambda^{(p)}}(R_r) \\ &= b_{\lambda^{(p,r)}}((r,1))^{-1} b_{\lambda^{(p,r)}_{(p+1,p+2)}}((r,1)) b_{\lambda^{(p)}_{(p+1,p+2)}}((r,1))^{-1} b_{\lambda^{(p)}}((r,1)). \end{aligned}$$

On the other hand by (6.9) we see also

$$\begin{aligned} f^{\lambda^{(p)}_{(p+1,p+2)}}_{\lambda^{(p)}(1^2)} &= b_{\lambda^{(p)}}(C_{\lambda^{(p)},1})^{-1} b_{\lambda^{(p)}_{(p+1,p+2)}}(C_{\lambda^{(p)},1}). \\ f^{\lambda^{(p,r)}_{(p+1,p+2)}}_{\lambda^{(p,r)}(1^2)} &= b_{\lambda^{(p,r)}}(C_{\lambda^{(p,r)},1})^{-1} b_{\lambda^{(p,r)}_{(p+1,p+2)}}(C_{\lambda^{(p,r)},1}), \end{aligned}$$

so that

$$\begin{aligned} &\left(f^{\lambda^{(p)}_{(p+1,p+2)}}_{\lambda^{(p)}(1^2)}\right)^{-1} f^{\lambda^{(p,r)}_{(p+1,p+2)}}_{\lambda^{(p,r)}(1^2)} \\ &= b_{\lambda^{(p)}}((r,1)) b_{\lambda^{(p,r)}}((r,1))^{-1} b_{\lambda^{(p,r)}_{(p+1,p+2)}}((r,1)) b_{\lambda^{(p)}_{(p+1,p+2)}}((r,1))^{-1}. \end{aligned}$$

This completes the proof of (6.31).

*Case* 3. – $p = m-1$ and $\mu \not\subset \lambda_{(p+1)}$. In this case by (6.15) with $r=1$ (resp. $r=2$) and $\nu = \lambda$ (resp. $\nu = \lambda^{(p)}$) and induction hypothesis we have for some $s < p$

$$f^{\lambda_{(p)}}_{\lambda(1)} \widetilde{\begin{pmatrix} \lambda_{(p)} \\ \mu \end{pmatrix}}_m + f^{\lambda_{(p+1)}}_{\lambda(1)} \widetilde{\begin{pmatrix} \lambda_{(p+1)} \\ \mu \end{pmatrix}}_m = 0 \tag{6.32}$$

$$(6.33)\qquad f^{\lambda_{(p+1)}}_{\lambda^{(p)}(1^2)}\widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m + f^{\lambda^{(p)}_{(p+1,s)}}_{\lambda^{(p)}(1^2)}\widetilde{\binom{\lambda^{(p)}_{(p+1,s)}}{\mu}}_m = \sum_i f^{\mu}_{\lambda^{(p,i)}(1^2)}\widetilde{\binom{\lambda^{(p)}}{\lambda^{(p,i)}}}_m .$$

Observe that, as $\mu \not\subset \lambda_{(p+1)}$ is assumed, $f^{\mu}_{\lambda^{(p,i)}(1^2)}$ does not vanish only if $\mu$ is of the form

$$(6.34)\qquad \mu = \lambda^{(p,r)}_{(p+1,s)},\ r < p,\ r \neq s,$$

and $i = r$. If not, then one can readily conclude from Lemma 6.2 and induction hypothesis that $\widetilde{\binom{\lambda^{(p)}_{(p+1,s)}}{\mu}}_m = 0$ . Hence $\widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m = 0$ follows from (6.33), so that we obtain $\widetilde{\binom{\lambda_{(p)}}{\mu}}_m = 0$ from (6.32).

We now assume (6.34). It clearly suffices to show

$$(6.35)\qquad f^{\lambda^{(p)}_{(p+1,s)}}_{\lambda^{(p)}(1^2)}\widetilde{\binom{\lambda^{(p)}_{(p+1,s)}}{\lambda^{(p,r)}_{(p+1,s)}}}_m = f^{\lambda^{(p,r)}_{(p+1,s)}}_{\lambda^{(p,r)}(1^2)}\widetilde{\binom{\lambda^{(p)}}{\lambda^{(p,r)}}}_m .$$

By (6.24) we have

$$(6.36)\qquad \widetilde{\binom{\lambda^{(p)}}{\lambda^{(p,r)}}}_m^{-1}\widetilde{\binom{\lambda^{(p)}_{(p+1,s)}}{\lambda^{(p,r)}_{(p+1,s)}}}_m = b_{\lambda^{(p,r)}}(R_r)^{-1}b_{\lambda^{(p,r)}_{(p+1,s)}}(R_r)b_{\lambda^{(p)}_{(p+1,s)}}(R_r)^{-1}b_{\lambda^{(p)}}(R_r)$$
$$= b_{\lambda^{(p,r)}}(S)^{-1}b_{\lambda^{(p,r)}_{(p+1,s)}}(S)b_{\lambda^{(p)}_{(p+1,s)}}(S)^{-1}b_{\lambda^{(p)}}(S),$$

where $S = (r,1)\cup(r,\lambda_s+1)$ if $r < s$ and $= (r,1)$ if $r > s$. On the other hand by (6.9) we have also

$$f^{\lambda^{(p)}_{(p+1,s)}}_{\lambda^{(p)}(1^2)} = b_{\lambda^{(p)}}(C_1'\cup C_{\lambda_s+1})^{-1}\, b_{(1)}^{-2}\, b_{\lambda^{(p)}_{(p+1,s)}}(C_1'\cup C_{\lambda_s+1})$$
$$f^{\lambda^{(p,r)}_{(p+1,s)}}_{\lambda^{(p,r)}(1^2)} = b_{\lambda^{(p,r)}}(C_1'\cup C_{\lambda_s+1})^{-1}\, b_{(1)}^{-2}\, b_{\lambda^{(p,r)}_{(p+1,s)}}(C_1'\cup C_{\lambda_s+1}),$$

where $C_1' = C_1 \setminus (s,1)$. Observe that

$$b_{\lambda^{(p,r)}}(C_1'\cup C_{\lambda_s+1})^{-1}b_{\lambda^{(p)}}(C_1'\cup C_{\lambda_s+1}) = b_{\lambda^{(p,r)}}(S)^{-1}b_{\lambda^{(p)}}(S)$$
$$b_{\lambda^{(p)}_{(p+1,s)}}(C_1'\cup C_{\lambda_s+1})^{-1}b_{\lambda^{(p,r)}_{(p+1,s)}}(C_1'\cup C_{\lambda_s+1}) = b_{\lambda^{(p,r)}}(S)^{-1}b_{\lambda^{(p)}}(S),$$

to get

$$\left(f^{\lambda^{(p)}_{(p+1,s)}}_{\lambda^{(p)}(1^2)}\right)^{-1} f^{\lambda^{(p,r)}_{(p+1,s)}}_{\lambda^{(p,r)}(1^2)} = \text{RHS of (6.36)}.$$

This completes the proof of (6.35).

*Case* 4. – $p = m-1$ and $\mu \subset \lambda_{(p+1)}$. So $\mu = \lambda^{(r)}_{(p+1)}$ for some $r < p+1$ and $\mu_m = 1$. One can readily derive from (6.15) with $r = 1$ and $\nu = \lambda$ and induction hypothesis that

$$f^{\lambda_{(p)}}_{\lambda(1)} \widetilde{\binom{\lambda_{(p)}}{\mu}}_m + f^{\lambda_{(p+1)}}_{\lambda(1)} \widetilde{\binom{\lambda_{(p+1)}}{\mu}}_m = f^{\mu}_{\lambda^{(r)}(1)} \widetilde{\binom{\lambda}{\lambda^{(r)}}}_m .$$

So it remains only to show

$$f^{\lambda_{(p+1)}}_{\lambda(1)} \widetilde{\binom{\lambda_{(p+1)}}{\lambda^{(r)}_{(p+1)}}}_m = f^{\lambda^{(r)}_{(p+1)}}_{\lambda^{(r)}(1)} \widetilde{\binom{\lambda}{\lambda^{(r)}}}_m .$$

This is concluded, as in the previous cases, from (6.9) and (6.24): It holds that

$$\begin{aligned}
\widetilde{\binom{\lambda}{\lambda^{(r)}}}_m^{-1} \widetilde{\binom{\lambda_{(p+1)}}{\lambda^{(r)}_{(p+1)}}}_m &= \left(f^{\lambda_{(p+1)}}_{\lambda(1)}\right)^{-1} f^{\lambda^{(r)}_{(p+1)}}_{\lambda^{(r)}(1)} \\
&= b_{\lambda_{(p+1)}}((r,1))^{-1} b_{\lambda}((r,1)) b_{\lambda^{(r)}}((r,1))^{-1} b_{\lambda^{(r)}_{(p+1)}}((r,1)).
\end{aligned}$$

We have completed the proof of Theorem 4.4.

## Appendix A. Convergence of the integral

We show that the integral ${}_qS_{n,m}(\alpha,\beta,\gamma,\mu;x_1,\ldots,x_m;\xi)$ converges under the conditions $(\mathcal{C}_1),(\mathcal{C}_2)$. It is immediate that, if $(aq^s)_\infty/(bq^s)_\infty$ has no pole at any $s \in \mathbb{Z}$, then

$$\left|\frac{(aq^s)_\infty}{(bq^s)_\infty}\right| \le \begin{cases} M_1, & s \ge 0, \\ M_2|a/b|^{-s}, & s < 0, \end{cases}$$

where $M_1 = \max_{s\ge 0}|(aq^s)_\infty/(bq^s)_\infty|$, $M_2 = |(a)_\infty/(b)_\infty|\max_{s\ge 0}|(a^{-1})_s/(b^{-1})_s|$. Using this, for $t_j = \xi_j q^{s_j}$ one has

$$\begin{aligned}
\left|\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty}(1-t_j/t_i)\right| &\le \begin{cases} Cte., & s_j - s_i \ge 0, \\ Cte.|q^{2\gamma}|^{s_j - s_i}, & s_j - s_i < 0, \end{cases} \\
\left|\frac{(qt_j)_\infty}{(q^\beta t_j)_\infty}\right| &\le \begin{cases} Cte., & s_j \ge 0, \\ Cte.|q^{\beta-1}|^{s_j}, & s_j < 0, \end{cases} \\
\left|\prod_{i=1}^m \frac{(x_i t_j)_\infty}{(q^\mu x_i t_j)_\infty}\right| &\le \begin{cases} Cte., & s_j \ge 0, \\ Cte.|q^{m\mu}|^{s_j}, & s_j < 0. \end{cases}
\end{aligned}$$

For $s \in \mathbb{Z}$, we put

$$a_s = \begin{cases} 1, & s \ge 0, \\ q^{\beta - 1 + m\mu}, & s < 0. \end{cases}$$

*Case* Re $\gamma \ge 0$. – So $|q^\gamma| \le 1$ and it follows from the inequality above that

$$\left|\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^\gamma t_j/t_i)_\infty}(1-t_j/t_i)\right| \le Cte.|q^{2\gamma}|^{-|s_j|-|s_i|}.$$

Hence

$$|\Phi_0(\xi_1 q^{s_1},\ldots,\xi_n q^{s_n})| \leq Cte. \prod_{j=1}^{n} |q^{(\alpha+n-1-2(j-1)\gamma)s_j-2(n-1)\gamma|s_j|} a_{s_j}|.$$

The condition $(\mathcal{C}_2)$ in the case Re $\gamma \geq 0$ is equivalent to

$$\sum_{s=0}^{\infty} |q^{(\alpha+n-1-4(n-1)\gamma)s}| + \sum_{s=-1}^{-\infty} |q^{(\alpha+n-1+\beta-1+m\mu+2(n-1)\gamma)s}| < \infty.$$

This clearly implies the convergence of the series

$$\text{(A.1)} \qquad \int_{[0,\xi\infty]_q} \Phi_0(t)\tilde{\omega} = (1-q)^n \sum_{s_i\in\mathbb{Z}} \Phi_0(\xi_1 q^{s_1},\ldots,\xi_n q^{s_n}).$$

*Case* Re $\gamma < 0$. – We see

$$\left|\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^{\gamma}t_j/t_i)_\infty}(1-t_j/t_i)\right| \leq Cte.,$$

so that

$$|\Phi_0(\xi_1 q^{s_1},\ldots,\xi_n q^{s_n})| \leq Cte. \prod_{j=1}^{n} |q^{(\alpha+n-1-2(j-1)\gamma)s_j} a_{s_j}|.$$

The condition $(\mathcal{C}_2)$ in the case Re$\gamma < 0$ is equivalent to

$$\sum_{s=0}^{\infty} |q^{(\alpha+n-1)s}| + \sum_{s=-1}^{-\infty} |q^{(\alpha+n-1+\beta-1+m\mu-2(n-1)\gamma)s}| < \infty.$$

This implies the convergence of the series (A.1).

When $\xi = \xi_F$, as the summation in (A.1) is only over $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n$, one can relax the condition $(\mathcal{C}_2)$ into

$$\mathrm{Re}\,\alpha + n - 1 > \max\{2(n-1)\mathrm{Re}\gamma, 0\}.$$

Finally we note that, as $|P_\lambda(x)| \leq Cte.(|x_1| + \ldots + |x_m|)^{|\lambda|}$, the integral $\int_{[0,\xi\infty]_q} P_\lambda(t;q,Q)_q D(\alpha,\beta,\gamma;t)\tilde{\omega}$ converges provided that

$$\mathrm{Re}\,\alpha + n - 1 > 4(n-1)\max\{\mathrm{Re}\,\gamma, 0\},$$
$$\mathrm{Re}\,\alpha + n - 1 + \mathrm{Re}\,\beta - 1 + m\mathrm{Re}\,\mu + |\lambda| < -2(n-1)|\mathrm{Re}\,\gamma|.$$

## Appendix B. Evaluation of ${}_qS_{n,0}(\alpha,\beta,\gamma;\xi)$

We begin by showing that

$$\text{(B.1)}\qquad {}_qS_{n,0}(\alpha+1,\beta,\gamma;\xi)=q^{n(n-1)\gamma/2}\prod_{j=1}^{n}\frac{1-q^{\alpha+n-1-(n+j-2)\gamma}}{1-q^{\alpha+\beta+n-1-(n-j)\gamma}}\,{}_qS_{n,0}(\alpha,\beta,\gamma;\xi).$$

Indeed this is a consequence of the case $m=1$ of (4.17): Equating the coefficients of $x^n$ of both sides gives

$$(-q^{\beta}Q^{-1})^n\,{}_qS_{n,0}(\alpha+1,\beta,\gamma;\xi)=\frac{(Q^{-n};Q)_n}{(Q;Q)_n}\frac{(q^{-(\alpha+n-1)}Q^{n-1};Q)_n}{(q^{-(\alpha+\beta+n-1)};Q)_n}\,{}_qS_{n,0}(\alpha,\beta,\gamma;\xi),$$

and this leads to (B.1) immediately. We first prove (1.6) by induction on $n$, the case $n=1$ being nothing but the $q$-beta integral formula [As1], [GR, p. 19]. We proceed as in [Kad1]. Set

$$\text{(B.2)}\qquad {}_qpr_n(\alpha,\beta,\gamma)=q^{\frac{n(n-1)}{2}\alpha\gamma}\prod_{j=1}^{n}\frac{\Gamma_q(\alpha+n-1-(n+j-2)\gamma)}{\Gamma_q(\alpha+\beta+n-1-(n-j)\gamma)}$$

$$\text{(B.3)}\qquad {}_qQ_n(\alpha,\beta,\gamma)=\frac{{}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F)}{{}_qpr_n(\alpha,\beta,\gamma)}.$$

By (B.1) and the equation $\Gamma_q(\alpha+1)=(1-q^{\alpha})/(1-q)\,\Gamma_q(\alpha)$, we see

$$\text{(B.4)}\qquad {}_qQ_n(\alpha+1,\beta,\gamma)={}_qQ_n(\alpha,\beta,\gamma).$$

We extend ${}_qQ_n(\alpha,\beta,\gamma)$ to all $\alpha$ by this equation.

We assume that $\gamma$ is real and $\gamma>0$, $\operatorname{Re}\alpha+(n-1)(1-2\gamma)>0$. We show that

$$\text{(B.5)}\qquad {}_qQ_n(\alpha,\beta,\gamma)=q^{C_n}\prod_{j=1}^{n}\frac{\Gamma_q(\beta+(j-1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\gamma)}$$

where $C_n=\sum_{j=1}^{n}(-2(j-1)\gamma+n-1)(j-1)\gamma$. Rewriting ${}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F)$ as iterated integral, we have

$$\begin{aligned}{}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F)=&\int_{[0,q^{(n-1)\gamma}]}t_n^{\alpha+(n-1)(1-2\gamma)}\frac{(qt_n)_\infty}{(q^{\beta}t_n)_\infty}\\&\left[\int_{[0,(1,\dots,q^{(n-2)\gamma})]}\prod_{j=1}^{n-1}t_j^{\alpha+(j-1)(1-2\gamma)}\frac{(qt_j)_\infty}{(q^{\beta}t_j)_\infty}\right.\\&\left.\times\prod_{1\le i<j\le n}\frac{(q^{1-\gamma}t_j/t_i)_\infty}{(q^{\gamma}t_j/t_i)_\infty}D(t)\frac{d_qt_1}{t_1}\wedge\cdots\wedge\frac{d_qt_{n-1}}{t_{n-1}}\right]\frac{d_qt_n}{t_n}.\end{aligned}$$

Set $\alpha_0 = (n-1)(2\gamma-1)$. Observe that

$$\lim_{\alpha\to\alpha_0} \frac{1-q^{\alpha-\alpha_0}}{1-q} \int_{[0,q^{(n-1)\gamma}]} t_n^{\alpha-\alpha_0} \frac{(qt_n)_\infty}{(q^\beta t_n)_\infty} \frac{d_q t_n}{t_n} = 1.$$

Hence we obtain

$$\text{(B.6)} \qquad \begin{aligned} \lim_{\alpha\to\alpha_0} \frac{1-q^{\alpha-\alpha_0}}{1-q} {}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F) &= {}_qS_{n-1,0}(\alpha_0+1,\beta,\gamma;\xi_F) \\ &= q^{A_{n-1}} \prod_{j=1}^{n-1} \frac{\Gamma_q((n-j+1)\gamma)\Gamma_q(\beta+(j-1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\beta+(n+j-1)\gamma)\Gamma_q(\gamma)} \end{aligned}$$

where

$$A_{n-1} = \sum_{j=1}^{n-1} \big(\alpha_0 - 2(j-1)\gamma + n - 1\big)(j-1)\gamma = C_n + \frac{n(n-1)}{2}\alpha_0\gamma.$$

On the other hand by (B.2) and (B.3) we have

$$\text{(B.7)} \qquad \begin{aligned} &\lim_{\alpha\to\alpha_0} \frac{1-q^{\alpha-\alpha_0}}{1-q} {}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F) \\ &= \lim_{\alpha\to\alpha_0} \frac{1-q^{\alpha-\alpha_0}}{1-q} {}_qpr_n(\alpha,\beta,\gamma)\, {}_qQ_n(\alpha,\beta,\gamma) \\ &= q^{\frac{n(n-1)}{2}\alpha_0\gamma} \frac{1}{\Gamma_q(\beta+(n-1)\gamma)} \prod_{j=1}^{n-1} \frac{\Gamma_q((n-j)\gamma)}{\Gamma_q(\beta+(n+j-1)\gamma)} {}_qQ_n(\alpha_0,\beta,\gamma). \end{aligned}$$

Equating (B.6) and (B.7) yields

$$\text{(B.8)} \ \begin{aligned} {}_qQ_n(\alpha_0,\beta,\gamma) &= q^{-\frac{n(n-1)}{2}\alpha_0\gamma + A_{n-1}} \Gamma_q(\beta+(n-1)\gamma) \prod_{j=1}^{n-1} \frac{\Gamma_q(\beta+(n+j-1)\gamma)}{\Gamma_q((n-j)\gamma)} \\ &\quad \times \prod_{j=1}^{n-1} \frac{\Gamma_q((n-j+1)\gamma)\Gamma_q(\beta+(j-1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\beta+(n+j-1)\gamma)\Gamma_q(\gamma)} \\ &= q^{C_n} \prod_{j=1}^{n} \frac{\Gamma_q(\beta+(j-1)\gamma)\Gamma_q(j\gamma)}{\Gamma_q(\gamma)}. \end{aligned}$$

This establishes (B.5) when $\alpha = \alpha_0 + k, k \in \mathbb{Z}$. One can show that ${}_qQ_n(\alpha_0,\beta,\gamma)$ is bounded in the strip $\alpha_0 + 1 \le \operatorname{Re}\alpha \le \alpha_0 + 2$ in the exactly same way as [Kad1]. Hence it is bounded for all $\alpha$ by (B.4), and thus (B.5) follows from Liouville's theorem that a bounded entire function is constant. The restriction that $\gamma$ is real and positive is easily removed by analytic continuation.

Now we turn to the proof of (1.5). Set

$$ {}_qS_{n,0}(\alpha,\beta,\gamma;\xi) = c(\xi)\,{}_qS_{n,0}(\alpha,\beta,\gamma;\xi_F). \tag{B.9}$$

By the definition of Jackson integral, we see for any $j$ that

$$ T_{q,\xi_j}c(\xi) = c(\xi). \tag{B.10}$$

Observe that $c(\xi)\prod_{j=1}^n \xi_j^{2(j-1)\gamma-\alpha}$ is meromorphic on $(\mathbb{C}^*)^n$ with simple poles in $\{\xi \mid \prod_{1\le i<j\le n}\vartheta(q^\gamma\xi_j/\xi_i)\prod_{j=1}^n\vartheta(q^\beta\xi_j) = 0\}$. We assert that $c(\xi)$ is vanishing on $\{\xi \mid \xi_j = \xi_i q^k, k \in \mathbb{Z}\}$. By (B.10), it suffices to show that if $\xi_i = \xi_j$, $i<j$, then ${}_qS_{n,0}(\alpha,\beta,\gamma;\xi) = 0$. Let $\sigma_{ij}$ be the transposition of $i$ and $j$. We have

$$\begin{aligned}
\int_{[0,\xi\infty]_q}\Phi_0(t)\tilde{\omega} &= \int_{[0,\sigma_{ij}(\xi)\infty]_q}\Phi_0(t)\tilde{\omega}\\
&= \int_{[0,\xi\infty]_q}\sigma_{ij}(\Phi_0(t))\tilde{\omega}\\
&= -U_{\sigma_{ij}}(\xi)\int_{[0,\xi\infty]_q}\Phi_0(t)\tilde{\omega}.
\end{aligned}$$

Hence our assertion follows from (use $\vartheta(q/x) = \vartheta(x)$)

$$\begin{aligned}
U_{\sigma_{ij}}(\xi) &= \prod_{\substack{1\le k<l\le n\\ \sigma_{ij}(k)>\sigma_{ij}(l)}}\left(\frac{\xi_l}{\xi_k}\right)^{2\gamma-1}\frac{\vartheta(q^\gamma\xi_l/\xi_k)}{\vartheta(q^{1-\gamma}\xi_l/\xi_k)}\\
&= \frac{\vartheta(q^\gamma)}{\vartheta(q^{1-\gamma})}\prod_{i<k<j}\left(\frac{\xi_j}{\xi_k}\right)^{2\gamma-1}\frac{\vartheta(q^\gamma\xi_j/\xi_k)}{\vartheta(q^{1-\gamma}\xi_j/\xi_k)}\prod_{i<l<j}\left(\frac{\xi_l}{\xi_i}\right)^{2\gamma-1}\frac{\vartheta(q^\gamma\xi_l/\xi_i)}{\vartheta(q^{1-\gamma}\xi_l/\xi_i)}\\
&= 1.
\end{aligned}$$

We are now able to write

$$ c(\xi) = \prod_{j=1}^n \xi_j^{\alpha-2(j-1)\gamma}\frac{1}{\vartheta(q^\beta\xi_j)}\prod_{1\le i<j\le n}\frac{\vartheta(\xi_j/\xi_i)}{\vartheta(q^\gamma\xi_j/\xi_i)}f(\xi)$$

where $f(\xi)$ is holomorphic on $(\mathbb{C}^*)^n$. One can derive from (B.10) that

$$ T_{q,\xi_j}f(\xi) = -\frac{1}{q^{\alpha+\beta-(n-1)\gamma}\xi_j}f(\xi).$$

Therefore we conclude that

$$f(\xi) = Cte. \prod_{j=1}^{n} \vartheta(q^{\alpha+\beta-(n-1)\gamma}\xi_j).$$

Since $c(\xi_F) = 1$, we arrive at

$$c(\xi) = q^{\sum_{j=1}^{n}(2(j-1)\gamma-\alpha)} \prod_{j=1}^{n} \xi_j^{\alpha-2(j-1)\gamma} \frac{\vartheta(\xi_j q^{\alpha+\beta-(n-1)\gamma})\vartheta(q^{\beta+(j-1)\gamma})\vartheta(q^{j\gamma})}{\vartheta(q^{\alpha+\beta-(n-j)\gamma})\vartheta(\xi_j q^{\beta})\vartheta(q^{\gamma})}$$
$$\prod_{1\leq i<j\leq n} \frac{\vartheta(\xi_j/\xi_i)}{\vartheta(q^{\gamma}\xi_j/\xi_i)}.$$

Combining this with (B.9) and (1.6) completes the proof of (1.5).

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Jyoichi KANEKO
Graduate School of Mathematics
Kyushu University
Ropponmatsu, Fukuoka 810
Japon