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Limit theorems ...
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Limit Theorems for Stochastic Processes

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To our sons
Olivier, Vincent and Andrei

Introduction

The limit theorems in this book belong to the theory of weak convergence of probability measures on metric spaces.

More precisely, our main aim is to give a systematic exposition of the theory of convergence in law for those stochastic processes that are semimartingales.

The choice of the class of semimartingales as our chief object of study has two reasons. One is that this class is broad enough to accomodate most common processes: discrete-time processes, diffusions, many Markov processes, point processes, solutions of stochastic differential equations, ... Our second reason is that we have in our hands a very powerful tool for studying these processes, namely the stochastic calculus. Since the theory of semimartingales, and related topics as random measures, are not usually associated with limit theorems, we decided to write a rather complete account of that theory, which is covered in the first two chapters. In particular, we devote much space to a careful and detailed exposition of the notion of characteristics of a semimartingale, which extends the well-known "Lévy-Khintchine triplet" for processes with independent increments (drift term, variance of the Gaussian part, Lévy measure), and which plays a particularly important rôle in limit theorems.

The meaning of $X^n \xrightarrow{\mathscr{L}} X$ (that is, the sequence (X^n) of processes converges in law to the process X) is not completely straightforward. The first idea would be to use "finite-dimensional convergence", which says that for any choice $t_1, \dots, t_p$ of times, then $(X^n_{t_1}, \dots, X^n_{t_p})$ goes in law to $(X_{t_1}, \dots, X_{t_p})$. This is usually unsatisfactory because it does not ensure convergence in law of such simple functionals as $\inf(t: X^n_t > a)$ or $\sup_{s \leq 1} X^n_s$, etc... In fact, since the famous paper [199] of Prokhorov, the traditional mode of convergence is weak convergence of the laws of the processes, considered as random elements of some functional space. Because semimartingales are right-continuous and have left-hand limits, here the fundamental functional space will always be the "Skorokhod space" $\mathbb{D}$ introduced by Skorokhod in [223]: this space can be endowed with a complete separable metric topology, and $X^n \xrightarrow{\mathscr{L}} X$ will always mean weak convergence of the laws, relative to that topology.

How does one prove that $X^n \xrightarrow{\mathscr{L}} X$?, and in which terms is it suitable to express the conditions? The method proposed by Prokhorov goes as follows:

$$\begin{array}{ccc} \text{(i)} & & \text{(ii)} \\ \left|\begin{array}{l}\text{Tightness of the}\\ \text{sequence } (X^n)\end{array}\right| & + & \left|\begin{array}{l}\text{Convergence of finite-}\\ \text{dimensional distributions}\end{array}\right| \end{array}$$

$$\begin{array}{c} \text{(iii)} \\ + \left|\begin{array}{l}\text{Characterization of } (X) \text{ by}\\ \text{finite-dimensional distributions}\end{array}\right| \Rightarrow X^n \xrightarrow{\mathscr{L}} X \end{array}$$

(as a matter of fact, this is even an equivalence; and of course (iii) is essentially trivial). Sometimes, we will make use of this method. However, it should be emphazised that very often step (ii) is a very difficult (or simply impossible) task to accomplish (with a notable exception concerning the case where the limiting process has independent increments). This fact has led to the development of other strategies; let us mention, for example, the method based upon the "embedding theorem" of Skorokhod, or the "approximation and σ-topological spaces methods" of Borovkov, which allows one to prove weak convergence for large classes of functionals and which are partly based upon (ii). Here we expound the strategy called "martingale method", initiated by Stroock and Varadhan, and which goes as follows:

$$\text{(i)} + \begin{array}{c}\text{(ii}'\text{)}\\ \left|\begin{array}{l}\text{Convergence of triplets}\\ \text{of characteristics}\end{array}\right|\end{array} + \begin{array}{c}\text{(iii}'\text{)}\\ \left|\begin{array}{l}\text{Characterization of } (X) \text{ by the}\\ \text{triplet of characteristics}\end{array}\right|\end{array} \Rightarrow X^n \xrightarrow{\mathscr{L}} X.$$

Here the difficult step is (iii′): we devote a large part of Chapter III to the explicit statement of the problem (called "martingale problem") and to some partial answers.

In both cases, we need step (i): in Chapter VI we develop several tightness criteria especially suited to semimartingales, we also use this opportunity to expose elementary—and less elementary—facts about the Skorokhod topology, in particular for processes indexed by the entire half-line $\mathbb{R}_+$.

The limit theorems themselves are presented in Chapters VII, VIII and IX (the reader can consult [166] for slightly different aspects of the same theory). Conditions insuring convergence always have a similar form, for simple situations (as convergence of processes with independent increments) as well as for more complicated ones (convergence of semimartingales to a semimartingale). Roughly speaking, they say that the triplets of characteristics of X^n converge to the triplet of characteristics of X. As a matter of fact, these conditions are more extensions of two sets of results that are apparently very far apart: those concerning convergence of rowwise independent triangular arrays, as in the book [65] of Gnedenko and Kolmogorov; and those concerning convergence of Markov processes (and especially of diffusion processes, in terms of their coefficients), as in the book [233] of Stroock and Varadhan.

Beside limit theorems, the reader will find apparently disconnected results, which concern absolute continuity for a pair of measures given on a filtered space, and contiguity of sequences of such pairs. In fact, one of our motivations for including such material is that we wanted to give some statistically-oriented applications of our limit theorems (a second motivation is that we indeed find this subject interesting on its own): that is done in Chapter X, where we study convergence of likelihood ratio processes (in particular asymptotic normality) and the so-called "statistical invariance principle" which gives limit theorems under contiguous alternatives.

In order to prepare for these results, we need a rather deep study of contiguity: this is done in Chapter V, in which Hellinger integrals and what we call Hellinger processes are widely used. Hellinger processes are introduced in Chapter IV, which also contains necessary and sufficient conditions for absolute continuity and singularity in terms of the behaviour of those Hellinger processes. Finally, let us mention that some material about convergence in variation is also included in Chapter V.

Within each chapter, the numbering is as follows: 3.4 means statement number 4 in Section 3. When referring to a statement in a previous chapter, say Chapter II, we write II.3.4.

In addition to the usual indexes (Index of Symbols; Index of Terminology), the reader will find in the Index of Topics a reference to all the places in this book where we write about a specific subject: for example, a reader interested only in point processes should consult the Index of Topics first. Finally, all the conditions on the triplets of characteristics which appear in our limit theorems are listed in the Index of Conditions for Limit Theorems.

Parts of this work were performed while one or other author was enjoying the hospitality of the Steklov Mathematical Institute or the Université Pierre et Marie Curie, Paris VI. We are grateful for having had these opportunities.

Paris and Moscow,
June 1987

Jean Jacod
Albert N. Shiryaev

Basic Notation

$\mathbb{R} = (-\infty, +\infty)$ = the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $\bar{\mathbb{R}} = [-\infty, +\infty]$
$\bar{\mathbb{R}}_+ = [0, \infty]$
$\mathbb{Q}$ = the set of rational numbers, $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$
$\mathbb{N} = \{0, 1, 2, \dots\}$ = the set of integers, $\mathbb{N}^* = \{1, 2, 3, \dots\}$
$\mathbb{C}$ = the set of complex numbers
$\mathbb{R}^d$ = the Euclidian d-dimensional space
$|x|$ = the Euclidian norm of $x \in \mathbb{R}^d$, or the modulus of $x \in \mathbb{C}$
$x \cdot y$ = the scalar product of $x \in \mathbb{R}^d$ with $y \in \mathbb{R}^d$
$a \vee b = \sup(a, b)$, $a \wedge b = \inf(a, b)$
$x^+ = x \vee 0$, $x^- = (-x) \vee 0$ for $x \in \mathbb{R}$
1_A = the indicator function of the set A
A^c = the complement of the set A
ε_a = the Dirac measure sitting at point a
a. s. = almost surely
$\lim_{s \uparrow t} = \lim_{s \to t, s \leq t}$, $\lim_{s \uparrow\uparrow t} = \lim_{s \to t, s < t}$
$\lim_{s \downarrow t} = \lim_{s \to t, s \geq t}$, $\lim_{s \downarrow\downarrow t} = \lim_{s \to t, s > t}$
$\otimes$ = tensor product (of spaces, of σ-fields)
$[x]$ = the integer part of $x \in \mathbb{R}_+$
$\mathrm{Re}(x)$, $\mathrm{Im}(x)$ = real and imaginary parts of $x \in \mathbb{C}$
$\ll$ absolute continuity between measures
$\sim$ equivalence between measures
$\perp$ singularity between measures
$\{\cdots\}$ denotes a set

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