Applications of Mathematics

Lubomír Kubáček

Nonlinear error propagation law

Applications of Mathematics, Vol. 41 (1996), No. 5, 329–345

Persistent URL: http://dml.cz/dmlcz/134330

NONLINEAR ERROR PROPAGATION LAW

Lubomír Kubáček, Olomouc

(Received September 6, 1995)

Summary. The error propagation law is investigated in the case of a nonlinear function of measured data with non-negligible uncertainty.

Keywords: error propagation law, variance, bias

AMS classification: 62F10

1. Introduction

One of the frequently occuring problems in experimental sciences is the following. A value $f(\mu_1,\ldots,\mu_n)$ of a function $f(.)\colon \mathbb{R}^n$ (n-dimensional Euclidean space) $\to \mathbb{R}^1$ must be determined. However, the values $\mu_1,\ldots,\mu_n$ are unknown and only their estimates are at our disposal. What is the bias and the variance of the random variable $f(\hat{\mu})$, where $\hat{\mu} = (\hat{\mu}_1,\ldots,\hat{\mu}_n)'$ ($'$ denotes the transposition) is an estimator of $\mu = (\mu_1,\ldots,\mu_n)'$?

The exact solution is well known when e.g. the joint density function $h(.)\colon \mathbb{R}^n \to \mathbb{R}^1$ of the random vector $\hat{\mu}$ is given and the following procedure is justified (in more detail cf. e.g. [5], p. 51):

Let $\mathbf{f} = (f, f_2,\ldots,f_n)'\colon \mathbb{R}^n \to \mathbb{R}^n$, where $f(.)$ is the considered function and $f_2,\ldots,f_n$ are auxiliary functions such that

$$|J(y)| = \left|1/\det[\partial\mathbf{f}(x)/\partial x']\big|_{y=\mathbf{f}(x)}\right|, \quad y \in \mathbb{R}^n$$

Supported by grant No. 2/1226/95 of the Grant Agency for Science of the Slovak republic and by internal grant No. 311 03 001 of Palacký University, Olomouc

is nonzero and continuous. Then the density function $h_1(.)$ of the random variable $f(\hat{\mu})$ is

$$h_1(y_1) = \int_{\mathbb{R}^{n-1}} h[x_1(y), \dots, x_n(y)] |J(y)| \, \mathrm{d}y_2 \dots \mathrm{d}y_n, \\ y = (y_1, \dots, y_n)' \in \mathbb{R}^n$$

and the bias b and the variance V are

$$b = \int_{\mathbb{R}^1} y_1 h_1(y_1) \, \mathrm{d}y_1 - f(\mu_1, \dots, \mu_n),$$
$$V = \int_{\mathbb{R}^1} \left[y_1 - \int_{\mathbb{R}^1} y_1 h_1(y_1) \, \mathrm{d}y_1 \right]^2 h_1(y_1) \, \mathrm{d}y_1.$$

If the structure of the function $f(.)$ is not simple and/or the number n is great, then the outlined procedure cannot be used in practice. The procedure is complicated even in the case of the known jacobian $|J(y)|$, since the integration must be performed in numerical way; this is tedious and sometimes not sufficiently reliable.

The best way how to solve the problem seems to be simulation. However, the results should be checked in another way to ensure their numerical reliability.

In many cases, the value V is substituted by

$$\tilde{V} = \partial f(x)/\partial x' \big|_{x=\hat{\mu}} \operatorname{Var}(\hat{\mu}) \partial f(x)/\partial x \big|_{x=\hat{\mu}}$$

and the value b is neglected.

The last relation is called the error propagation law; cf. [1], [2], [3], [7] (18.5–6.(c)).

Results obtained in this way may serve as a check of the simulation. However, in the case of nonlinearity of the function $f(.)$, results obtained by the error propagation law are approximate. It need not be clear whether differences between the result obtained by the simulations and the result obtained from the error propagation law can be explained by the approximate character of the error propagation law or not.

To give some comments on this problem is the aim of the paper. Formulae given in the paper enable us sometimes to solve the problem without simulations. Even if some of them seem to be huge, they enable us to develop an algorithm for numerical evaluation of them or, at least, to use several of the first terms of the series in order to check the reliability of the simulations.

2. Notation and preliminary statements

Let $f(.)\colon \mathbb{R}^n \to \mathbb{R}^1$ be either a function which can be expressed by an infinite Taylor series on some domain $\mathcal{R}$, or a polynomial of an arbitrary (finite) degree.

Let the notation $\Delta^s f(\mu), s = 1, 2, \ldots$ have the following meaning:

$$\Delta^1 f(\mu) = \sum_{i=1}^{n} \left(\partial f(x)/\partial x_i\big|_{x=\mu}\right) \Delta\mu_i,$$

$$\Delta^2 f(\mu) = \sum_{i=1}^{n} \left(\partial^2 f(x)/\partial x_i^2\big|_{x=\mu}\right) (\Delta\mu_i)^2 + \sum_{i\neq j} \left(\partial^2 f(x)/\partial x_i \partial x_j\big|_{x=\mu}\right) \Delta\mu_i \Delta\mu_j,$$

$$\Delta^3 f(\mu) = \frac{3!}{3!} \sum_{i=1}^{n} \left(\partial^3 f(x)/\partial x_i^3\big|_{x=\mu}\right) (\Delta\mu_i)^3 + \frac{3!}{2!1!} \sum_{i\neq j} \left(\partial^3 f(x)/\partial x_i^2 \partial x_j\big|_{x=\mu}\right)$$
$$\times (\Delta\mu_i)^2 \Delta\mu_j + \frac{3!}{1!1!1!} \sum_{i\neq j, i\neq k, j\neq k} \left(\partial^3 f(x)/\partial x_i \partial x_j \partial x_k\big|_{x=\mu}\right) \Delta\mu_i \Delta\mu_j \Delta\mu_k$$
$$\ldots \text{etc.}$$

Let $\otimes$ denote the Kronecker multiplication [6] while $a^{j\otimes}$ means $a \otimes a \otimes \ldots \otimes a$ (j-times).

Let $(\partial/\partial x)^{s\otimes} f(\mu), s = 1, 2, \ldots$ have the following meaning:

$$(\partial/\partial x)^{1\otimes} f(\mu) = \partial f(x)/\partial x\big|_{x=\mu} = g,$$

$$(\partial/\partial x)^{2\otimes} f(\mu) = \left[(\partial^2 f(x)/\partial x_1 \partial x', \partial^2 f(x)/\partial x_2 \partial x', \ldots, \partial^2 f(x)/\partial x_n \partial x')\big|_{x=\mu}\right]'$$
$$= \operatorname{vec}(H), \ldots, \text{etc.}$$

The symbol g denotes $\partial f(x)/\partial x\big|_{x=\mu}$ and H means $\frac{\partial^2 f(x)}{\partial x \partial x'}\big|_{x=\mu}$.

Lemma 2.1. *Let $\mu, \mu + e \in \mathcal{R}$. Then*

$$f(\mu + e) = f(\mu) + \sum_{j=1}^{\infty} \frac{1}{j!} \left[\left(\frac{\partial}{\partial x'}\right)^{j\otimes} f(\mu)\right] e^{j\otimes}.$$

Proof. With respect to the assumption we can write

$$f(\mu) = f(\mu) + \sum_{j=1}^{\infty} \frac{1}{j!} \Delta^j f(\mu).$$

Let $a, b \in \mathbb{R}^n$. Since $(a'b)^j = (a')^{j\otimes} b^{j\otimes}$, we have

$$\left[\frac{\partial}{\partial x_1} e_1 + \ldots + \frac{\partial}{\partial x_n} e_n\right]^j f(x)\big|_{x=\mu}$$
$$= \Delta^j f(\mu) = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)^{j\otimes} f(x)\big|_{x=\mu} e^{j\otimes}.$$

□

The symbol $\mathrm{Tr}(A)$ means the trace of the matrix A; in the following the relationship $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\,\mathrm{Tr}(B)$ will be utilized.

Corollary 2.2. *The expressions $\Delta^s f(\mu)$, $s = 1, 2, 3, 4$, can be written also in the form*

$$\Delta^1 f(\mu) = \frac{\partial f(x)}{\partial x'}\Big|_{x=\mu} e$$
$$\Delta^2 f(\mu) = \mathrm{Tr}\left[\partial^2 f(x)/\partial x \partial x'\big|_{x=\mu} ee'\right]$$
$$\Delta^3 f(\mu) = \mathrm{Tr}\left(\left[\frac{\partial}{\partial x'} \otimes \left(\frac{\partial}{\partial x}\frac{\partial}{\partial x'}\right)\right] f(x)\big|_{x=\mu} \{e \otimes [ee']\}\right)$$
$$\Delta^4 f(\mu) = \mathrm{Tr}\left(\left[\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x'}\right) \otimes \left(\frac{\partial}{\partial x}\frac{\partial}{\partial x'}\right)\right] f(x)\big|_{x=\mu} \{[ee' \otimes ee']\}\right).$$

Let $s \in \mathbb{R}^n$, $s's = 1$,

$$\varphi_s(t) = f(\mu) + \sum_{j=1}^{\infty} \frac{1}{j!}\left[\left(\frac{\partial}{\partial x'}\right)^{j\otimes} f(\mu)\right] s^{j\otimes} t^j, t \in \mathbb{R}^1, \tag{2.1}$$

and let $\varrho(s)$ be the radius of convergence of the function $\varphi(.)$.

Example 2.3. (i) Let $f(x) = \sqrt{x}$, $x \in [0, \infty)$ and $\mu \in [0, \infty)$ be fixed. The domain $\mathcal{R}$ for the function

$$\varphi(e) = f(\mu + e) = \sqrt{\mu + e} = \sqrt{\mu} \sum_{j=0}^{\infty} \binom{-1/2}{j} \frac{e^j}{\mu^j}$$

is $\mathcal{R} = (-\mu, +\mu)$.

(ii) Let $f_1(x_1, x_2) = \frac{x_1}{x_2}$, $x_1 \in \mathbb{R}^1$, $x_2 \in \mathbb{R}^1 - \{0\}$ and $\mu_1 \in \mathbb{R}^1$, $\mu_2 \in \mathbb{R}^2 - \{0\}$ be fixed. The domain $\mathcal{R}$ for the series

$$\varphi(e_1, e_2) = f(\mu_1 + e_1, \mu_2 + e_2) = \frac{\mu_1}{\mu_2}\left(1 + \frac{e_1}{\mu_1}\right)\left(1 - \frac{e_2}{\mu_2} + \frac{e_2^2}{\mu_2^2} - \ldots\right)$$

is $\mathcal{R} = (-\infty, +\infty) \times (-\mu_2, \mu_2)$ for $\mu_2 > 0$ and $(-\infty, +\infty) \times (\mu_2, -\mu_2)$ for $\mu_2 < 0$. An exception occurs for $s = (\cos\alpha, \sin\alpha)'$, where $\tan\alpha = \mu_2/\mu_1$; here $\varrho(s) = (-\infty, +\infty)$.

If e_0 is a point where the series

$$\sum_{j=1}^{\infty} \frac{1}{j!} \left[\left(\frac{\partial}{\partial x'} \right)^{j\otimes} f(\mu) \right] e_0^{j\otimes}$$

converges, then the series

$$\sum_{j=1}^{\infty} \frac{1}{j!} \left[\left(\frac{\partial}{\partial x'} \right)^{j\otimes} f(\mu) \right] e^{j\otimes}$$

converges uniformly on the domain

$$\{u\colon u \in \mathbb{R}^n, |u_i| < |e_{0,i}|, i = 1, \ldots, n\}.$$

(in detail cf. [4], Chpt. 12).

Assumption: Let either (i) or (ii) be satisfied:

(i) A function $f(.)\colon \mathbb{R}^n \to \mathbb{R}^1$ can be expressed by an infinite Taylor series on a domain

$$\mathcal{R} = \{u\colon u \in \mathbb{R}^n, \|u\| < \varrho(u/\|u\|)\}$$

and S_j is the support of a probability measure, given by a distribution function $F_j(.)\colon \mathbb{R}^1 \to \mathbb{R}^1$ of the jth component of the random vector $\varepsilon = \hat{\mu} - \mu$. Let $S_j \subset [a_j, b_j]$, where $-\infty < a_j < b_j < \infty$, $j = 1, \ldots n$ and $S = X_{j=1}^{n} S_j \subset \mathcal{R}$.

(ii) A function $f(.)\colon \mathbb{R}^n \to \mathbb{R}^1$ is a polynomial of an arbitrary (finite) degree and all statistical moments of the random vector $\hat{\mu}$ exist (in this case S can be even $\mathbb{R}^n$).

As a consequence of Assumption the random variable $f(\hat{\mu})$ possesses all statistical moments and the series

$$\sum_{j=1}^{\infty} \frac{1}{j!} \left[\left(\frac{\partial}{\partial x'} \right)^{j\otimes} f(\mu) \right] e^{j\otimes}$$

converges uniformly on S, converges absolutely for any $e \in S$, can be integrated and differentiated (with respect to e) term by term, put to the second power and the resulting series converges on S as well. This follows from the consideration in [4], Chpt. 12.

In the following the symbol E denotes the mean value under the probability measure considered. Let (in more detail cf. [9])

$$E(\varepsilon) = 0,\ E(\varepsilon\varepsilon') = \Sigma,\ E[\varepsilon \otimes (\varepsilon\varepsilon')] = \varphi,\ E[(\varepsilon\varepsilon') \otimes (\varepsilon\varepsilon')] = \psi$$

and

$$E(\varepsilon^{j\otimes}) = \delta_j, \quad j = 1, \ldots.$$

The notation $\operatorname{vec}(A_{m,n})$ means the (mn)-dimensional column vector given by the columns of the $m \times n$ matrix A ordered one under another.

Under the given notation we have

$$\delta_2 = \operatorname{vec}(\Sigma), \quad \delta_3 = \operatorname{vec}(\varphi) = \operatorname{vec}(\varphi'), \quad \delta_4 = \operatorname{vec}(\psi).$$

If $\varepsilon \sim N_n(0, \Sigma)$, i.e., the random vector ε is normally distributed with the mean value $E(\varepsilon) = 0$ and with the covariance matrix $E(\varepsilon\varepsilon') = \Sigma$, then $\{\Sigma\}_{i,j} = \sigma_{i,j}$, $i, j = 1, \dots, n$. Then

$$\delta_2 = \operatorname{vec}(\Sigma), \ \delta_3 = 0, \ \{\delta_4\}_{i,j,k,l} = \sigma_{i,j}\sigma_{k,l} + \sigma_{i,k}\sigma_{j,l} + \sigma_{i,l}\sigma_{j,k}.$$

Here $\{\delta_4\}_{i,j,k,l} = E(\varepsilon_i\varepsilon_j\{\varepsilon\varepsilon\}_{k,l})$, $i, j, k, l = 1, \dots, n$ (in more detail cf. [8], p. 75).

3. Determination of the bias and the variance of $f(\hat{\mu})$

Taking into account Lemma 2.1 and Assumption we can write

$$f(\hat{\mu}) = f(\mu + \varepsilon) = \sum_{j=0}^{\infty} a_j' \varepsilon^{j\otimes}, \tag{3.1}$$

where $a_j' = \frac{1}{j!}\left(\frac{\partial}{\partial x'}\right)^{j\otimes} f(x)\big|_{x=\mu}$, $a_0 = f(\mu)$, $\varepsilon^{0\otimes} = 1$ (if (ii) from Assumption holds, then obviously $a_j' = 0$ for j greater than an integer N).

Lemma 3.1. *Let ε be the random vector considered and let $C_{i,j} = \operatorname{cov}(\varepsilon^{i\otimes}, \varepsilon^{j\otimes})$. Then $\operatorname{vec}(C_{i,j}) = \delta_{i+j} - \delta_i \otimes \delta_j$.*

Proof is obvious. □

Lemma 3.2. *Let A and B be any $r \times (ni)$ and $s \times (nj)$, respectively, matrices. Then*

$$\operatorname{vec}[\operatorname{cov}(A\varepsilon^{i\otimes}, B\varepsilon^{j\otimes})] = (B \otimes A)(\delta_{i+j} - \delta_i \otimes \delta_j).$$

Proof. Since $\operatorname{vec}(UVX) = (X' \otimes U)\operatorname{vec}(V)$ for any matrices U, V, X of proper dimensions, we can write

$$\begin{aligned}\operatorname{vec}[\operatorname{cov}(A\varepsilon^{i\otimes}, B\varepsilon^{j\otimes})] &= \operatorname{vec}[A\operatorname{cov}(\varepsilon^{i\otimes}, \varepsilon^{j\otimes})B'] \\ &= (B \otimes A)(\delta_{i+j} - \delta_i \otimes \delta_j)\end{aligned}$$

in view of Lemma 3.1. □

Theorem 3.3. *Under the given Assumption*
(i)

$$b = \sum_{j=2}^{\infty} a_j' \delta_j,$$

(ii)

$$V = \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} (a_r' \otimes a_j')(\delta_{j+r} - \delta_j \otimes \delta_r).$$

Proof. (i) By virtue of (3.1) we can write

$$b = E[f(\hat{\mu})] - f(\mu) = a_0' + a_1'\delta_1 + \sum_{j=2}^{\infty} a_j'\delta_j - f(\mu).$$

Since $a_0' = f(\mu)$ and $\delta_1 = 0$, (i) is proved.
(ii)

$$\begin{aligned} \operatorname{Var}[f(\hat{\mu})] &= \operatorname{cov}\left(\sum_{j=1}^{\infty} a_j' \varepsilon^{j\otimes}, \sum_{k=1}^{\infty} a_k' \varepsilon^{k\otimes} \right) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (a_k' \otimes a_j')(\delta_{j+k} - \delta_j \otimes \delta_k). \end{aligned}$$

Here Lemma 3.2 and (3.1) are taken into account. □

Remark 3.4. The terms $a_2'\delta_2$, $a_3'\delta_3$ and $a_4'\delta_4$ can be rewritten as

$$\begin{aligned} a_2'\delta_2 &= \frac{1}{2} \operatorname{Tr}(H\Sigma) \\ a_3'\delta_3 &= \frac{1}{6} \operatorname{Tr}\left(\left\{ \left[\frac{\partial}{\partial x'} \otimes \left(\frac{\partial}{\partial x} \frac{\partial}{\partial x'} \right) f(x) \right]_{x=\mu} \right\} \varphi \right), \\ a_4'\delta_4 &= \frac{1}{24} \operatorname{Tr}\left(\left\{ \left[\left(\frac{\partial}{\partial x} \frac{\partial}{\partial x'} \right) \otimes \left(\frac{\partial}{\partial x} \frac{\partial}{\partial x'} \right) f(x) \right]_{x=\mu} \right\} \psi \right). \end{aligned}$$

In the case of normality ((ii) from Assumption must be satisfied) $a_3'\delta_3 = 0$ and

$$a_4'\delta_4 = \frac{1}{24} \sum_{i,j,k,l} \left(\frac{\partial^4 f(x)}{\partial x_i \partial x_j \partial x_k \partial x_l} \Big|_{x=\mu} \right) (\sigma_{i,j}\sigma_{k,l} + \sigma_{i,k}\sigma_{j,l} + \sigma_{i,l}\sigma_{j,k}).$$

Corollary 3.5. *Let $f(.)$ be a polynomial of the fourth order and $\hat{\mu} \sim N_n(\mu, \Sigma)$. Then the exact formula for b is*

$$b = \frac{1}{2} \operatorname{Tr}(H\Sigma) + \frac{1}{24} \sum_{i,j,k,l} \left(\frac{\partial^4 f(x)}{\partial x_i \partial x_j \partial x_k \partial x_l} \Big|_{x=\mu} \right) (\sigma_{i,j}\sigma_{k,l} + \sigma_{i,k}\sigma_{j,l} + \sigma_{i,l}\sigma_{j,k}).$$

Corollary 3.6. *If the function $f(.)$ is a polynomial of the second order, then the exact formula for V is*

(i) *in the general case*

$$V = g'\Sigma g + \left[\left(\frac{\partial}{\partial x'} \otimes \frac{\partial}{\partial x'}\right) f(x)\right]\Big|_{x=\mu} \varphi \frac{\partial f(x)}{\partial x}\Big|_{x=\mu}$$

$$\frac{1}{4}\left[\left(\frac{\partial}{\partial x'} \otimes \frac{\partial}{\partial x'}\right) f(x)\right]\Big|_{x=\mu} \{\psi - \mathrm{vec}(\Sigma)[\mathrm{vec}(\Sigma)]'\} \left[\left(\frac{\partial}{\partial x'} \otimes \frac{\partial}{\partial x'}\right) f(x)\right]\Big|_{x=\mu};$$

(ii) *in the case of normality*

$$V = g'\Sigma g + \frac{1}{4}\sum_{i,j,k,l} H_{i,j}H_{k,l}(\sigma_{i,k}\sigma_{j,l} + \sigma_{i,l}\sigma_{j,k}).$$

4. Determination of the bias and the variance of $\tilde{V}$

Let $\xi = \frac{\partial f(x)}{\partial x}\big|_{x=\hat{\mu}}$ and $\eta = \xi - E(\xi)$. Then $\tilde{V} = \xi'\Sigma\xi$ is an estimator of $\mathrm{Var}[f(\hat{\mu})]$. The bias $b(\tilde{V})$ of the estimator $\xi'\Sigma\xi$ is

$$b(\tilde{V}) = E(\xi'\Sigma\xi) - \mathrm{Var}[f(\hat{\mu})].$$

Let

$$A_j' = \frac{1}{j!}\left(\left[\left(\frac{\partial}{\partial x'}\right)^{j\otimes} \frac{\partial f(x)}{\partial x}\right]\Big|_{x=\mu}\right), \quad j = 0, 1, 2, \ldots$$

Obviously

$$A_0' = \frac{\partial f(x)}{\partial x}\Big|_{x=\mu} = g, A_1' = H, \xi = \sum_{j=0}^{\infty} A_j'\varepsilon^{j\otimes} \quad \text{and} \quad \eta = \sum_{j=1}^{\infty} A_j'(\varepsilon^{j\otimes} - \delta_j).$$

Lemma 4.1. $E(\xi'\Sigma\xi) = E(\xi')\Sigma E(\xi) + \mathrm{Tr}[\Sigma \,\mathrm{Var}(\xi)]$.

Proof is obvious. □

Lemma 4.2.
(i) $E(\xi) = \sum_{j=0}^{\infty} A_j'\delta_j$,
(ii) $\mathrm{vec}[\mathrm{Var}(\xi)] = \sum_{r=1}^{\infty}\sum_{j=1}^{\infty}(A_r' \otimes A_j')(\delta_{j+r} - \delta_j \otimes \delta_r)$.

Proof is obvious. □

Theorem 4.3. *The bias* $b(\tilde{V}) = E(\xi'\Sigma\xi) - \mathrm{Var}[f(\hat{\mu})]$ *is*

$$b(\tilde{V}) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \delta_j' A_j \Sigma A_k' \delta_k$$

$$+\sum_{r=1}^{\infty}\sum_{j=1}^{\infty}[\delta_2'(A_r' \otimes A_j') - a_r' \otimes a_j'](\delta_{j+r} - \delta_j \otimes \delta_r).$$

Proof. It follows from Lemma 4.1, Lemma 4.2, the relationship

$$\mathrm{Tr}[\Sigma\,\mathrm{Var}(\xi)] = [\mathrm{vec}(\Sigma)]'\,\mathrm{vec}[\mathrm{Var}(\xi)] = \delta_2' \sum_{r=1}^{\infty}\sum_{j=1}^{\infty}(A_r' \otimes A_j')(\delta_{j+r} - \delta_j \otimes \delta_r)$$

and Theorem 3.3 (ii). □

Corollary 4.4. *If the function* $f(.)$ *is a polynomial of the second order, then the exact formula for the bias* $b(\tilde{V})$ *is*

(i) *in the general case*

$$b(\tilde{V}) = \mathrm{Tr}(H\Sigma H\Sigma) - \frac{1}{2}\{g' \otimes [\mathrm{vec}(H)]'\}\,\delta_3$$

$$-\frac{1}{2}\{[\mathrm{vec}(H)]' \otimes g'\}\,\delta_3 - \frac{1}{4}\{[\mathrm{vec}(H)]' \otimes [\mathrm{vec}(H)]'\}\,(\delta_4 - \delta_2 \otimes \delta_2),$$

(ii) *in the case of normality*

$$b(\tilde{V}) = \mathrm{Tr}(H\Sigma H\Sigma) - \frac{1}{4}\sum_{i,j,k,l} H_{i,j}H_{k,l}(\sigma_{i,k}\sigma_{j,l} + \sigma_{i,l}\sigma_{j,k}).$$

Lemma 4.5.

$$\mathrm{Var}(\xi'\Sigma\xi) = [\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'E(\xi^{4\otimes}) - \left\{[\mathrm{vec}(\Sigma)]'E(\xi^{2\otimes})\right\}^2.$$

Proof. By the definition we have

$$\begin{aligned}
\mathrm{Var}(\tilde{V}) &= E(\xi'\Sigma\xi\xi'\Sigma\xi) - [E(\xi'\Sigma\xi)]^2 \\
&= E[\mathrm{Tr}(\Sigma\xi\xi')\,\mathrm{Tr}(\Sigma\xi\xi')] - \{E[\mathrm{Tr}(\Sigma\xi\xi')]\}^2 \\
&= E\left\{[\mathrm{vec}(\Sigma)]'\xi^{2\otimes}(\xi^{2\otimes})'\,\mathrm{vec}(\Sigma)\right\} - \left(\left\{[\mathrm{vec}(\Sigma)]'E(\xi^{2\otimes})\right\}\right)^2 \\
&= [\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'E(\xi^{4\otimes}) - \left\{[\mathrm{vec}(\Sigma)]'E(\xi^{2\otimes})\right\}^2.
\end{aligned}$$

□

Lemma 4.6.
(i) $E(\xi^{2\otimes}) = [E(\xi)]^{2\otimes} + E(\eta^{2\otimes})$,
(ii)

$$\begin{aligned} E(\xi^{4\otimes}) = [E(\xi)]^{4\otimes} + E(\eta^{2\otimes}) \otimes [E(\xi)]^{2\otimes} \\ +E[\eta \otimes E(\xi) \otimes \eta \otimes E(\xi)] + E(\xi) \otimes E(\eta^{2\otimes}) \otimes E(\xi) \\ +E(\eta^{3\otimes}) \otimes E(\xi) + E\left\{\eta \otimes [E(\xi)]^{2\otimes} \otimes \eta\right\} \\ +E[E(\xi) \otimes \eta \otimes E(\xi) \otimes \eta] + E[\eta^{2\otimes} \otimes E(\xi) \otimes \eta] \\ +[E(\xi)]^{2\otimes} \otimes E(\eta^{2\otimes}) + E[\eta \otimes E(\xi) \otimes \eta^{2\otimes}] \\ +E(\xi) \otimes E(\eta^{3\otimes}) + E(\eta^{4\otimes}). \end{aligned}$$

P r o o f is straightforward. □

Theorem 4.7.

$$\begin{aligned} \mathrm{Var}(\xi'\Sigma\xi) = 2\{\mathrm{vec}[\Sigma E(\xi)E(\xi')\Sigma]\}'E(\eta^{2\otimes}) \\ + 2\{[E(\xi')\Sigma] \otimes [E(\xi')\Sigma]\}E(\eta^{2\otimes}) \\ + 2\{[E(\xi')\Sigma] \otimes [\mathrm{vec}(\Sigma)]'\}E(\eta^{3\otimes}) \\ + 2\{[\mathrm{vec}(\Sigma)]' \otimes [E(\xi')\Sigma]\}E(\eta^{3\otimes}) \\ + [\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'\{E(\eta^{4\otimes}) - [E(\eta^{2\otimes})]^{2\otimes}\}. \end{aligned}$$

P r o o f. By virtue of Lemma 4.5 we have

$$\mathrm{Var}(\tilde{V}) = [\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'E(\xi^{4\otimes}) - \{[\mathrm{vec}(\Sigma)]'E(\xi^{2\otimes})\}^2.$$

Due to Lemma 4.6 the expression $[\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'E(\xi^{4\otimes})$ consists of 12 terms; it is necessary to rearrange several of them.

As an example let the expression

$$[\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'E[\eta \otimes E(\xi) \otimes \eta \otimes E(\xi)]$$

be taken into account. Then

$$\begin{aligned} [\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'E[\eta \otimes E(\xi) \otimes \eta \otimes E(\xi)] \\ = E\{[\mathrm{vec}(\Sigma)]'[\eta \otimes E(\xi)][\mathrm{vec}(\Sigma)]'[\eta \otimes E(\xi)]\} = \\ = E\{[\mathrm{vec}(\Sigma)]'[\eta \otimes E(\xi)][\eta' \otimes E(\xi)]\,\mathrm{vec}(\Sigma)\} \\ = [\mathrm{vec}(\Sigma)]'[E(\eta\eta') \otimes E(\xi)E(\xi')]\,\mathrm{vec}(\Sigma) \\ = \sum_i \sum_j E(\eta_i\eta_j)\sigma'_{.,i}E(\xi)\sigma'_{.,j}E(\xi) = \{\mathrm{vec}[\Sigma E(\xi)E(\xi')\Sigma]\}'E(\eta^{2\otimes}). \end{aligned}$$

The other 11 terms can be rearranged in a similar way. Then it is sufficient to take into account the relationships

$$\{[\mathrm{vec}(\Sigma)]'E(\xi^{2\otimes})]^2 = \{[\mathrm{vec}(\Sigma)]'[E(\xi)]^{2\otimes} + [\mathrm{vec}(\Sigma)]'E(\eta^{2\otimes})\}^2 =$$
$$= [\mathrm{vec}(\Sigma)\otimes\mathrm{vec}(\Sigma)]'[E(\xi)]^{4\otimes} + 2[\mathrm{vec}(\Sigma)]'[E(\xi)]^{2\otimes}[\mathrm{vec}(\Sigma)]'E(\eta^{2\otimes})$$
$$+[\mathrm{vec}(\Sigma)\otimes\mathrm{vec}(\Sigma)]'[E(\eta^{2\otimes})]^{2\otimes}$$

(Lemma 4.6.(i)) and

$$\mathrm{Var}(\xi'\Sigma\xi) = E(\xi'\Sigma\xi\xi'\Sigma\xi) - [E(\xi'\Sigma\xi)]^2,$$

where Lemma 4.5 and the rewritten expressions from Lemma 4.6 (ii) are used. □

The expression for $\mathrm{Var}(\tilde{V})$ from Theorem 4.7 cannot be used directly. The vector $\left[\sum_{j=0}^{\infty} A_j'\delta_j\right]^{k\otimes}$ must be used instead of $[E(\xi)]^{k\otimes}$, $k = 1, 2$, and

$$E\left\{\left[\sum_{j=1}^{\infty} A_j'(\varepsilon^{j\otimes} - \delta_j)\right]^{k\otimes}\right\}$$

instead of $E(\eta^{k\otimes})$, $k = 1, 2, 3, 4$.

Lemma 4.8.

$$E(\eta^{2\otimes}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(A_j'\otimes A_k')(\delta_{j+k} - \delta_j\otimes\delta_k)$$

(cf. Lemma 4.1) and

$$E(\eta^{3\otimes}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(A_j'\otimes A_k'\otimes A_l')\boxed{3},$$

where

$$\boxed{3} = \delta_{j+k+l} - \delta_j\otimes\delta_{k+l} - (I\otimes\delta_k\otimes I)\delta_{j+l} - \delta_{j+k}\otimes\delta_l + 2\delta_j\otimes\delta_k\otimes\delta_l;$$

$$E(\eta^{4\otimes}) - \left[E(\eta^{2\otimes})\right]^2 = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}(A_j'\otimes A_k'\otimes A_l'\otimes A_r'\otimes)\boxed{4},$$

where

$$\boxed{4} = \delta_{j+k+l+r} - \delta_j \otimes \delta_{k+l+r} - (I \otimes \delta_k \otimes I \otimes I)\delta_{j+l+r} + 2\delta_j \otimes \delta_k \otimes \delta_{l+r} -$$
$$-(I \otimes I \otimes \delta_l \otimes I)\delta_{j+k+r} + (\delta_j \otimes I \otimes \delta_l \otimes I)\delta_{k+r} + (I \otimes \delta_k \otimes \delta_l \otimes I)\delta_{j+r} -$$
$$-4\delta_j \otimes \delta_k \otimes \delta_r - \delta_{j+k+l} \otimes \delta_r + \delta_j \otimes \delta_{k+l} \otimes \delta_r + (I \otimes \delta_k \otimes I \otimes \delta_r)\delta_{j+l}$$
$$+2\delta_{j+k} \otimes \delta_l \otimes \delta_r - \delta_{j+k} \otimes \delta_{l+r}.$$

Proof. Since

$$(A_{m,n} \otimes B_{p,r} \otimes C_{s,t})(D_{n,a} \otimes E_{r,b} \otimes F_{t,c}) = (AD) \otimes (BE) \otimes (CF)$$

for any dimensions m, n, p, r, s, t, a, b, c of matrices A, B, C, D, E, F, we can write

$$\varepsilon^{j\otimes} \otimes \delta_k \otimes \varepsilon^{l\otimes} = (I_{jn,jn} \otimes \delta_k \otimes I_{ln,ln})(\varepsilon^{j\otimes} \otimes 1 \otimes \varepsilon^{l\otimes})$$
$$\Rightarrow \quad E(\varepsilon^{j\otimes} \otimes \delta_k \otimes \varepsilon^{l\otimes}) = (I_{jn,jn} \otimes \delta_k \otimes I_{ln,ln})\delta_{j+l}.$$

Now it is sufficient to use the equality

$$\eta^{k\otimes} = \left[\sum_{j=1}^{\infty} A_j'(\varepsilon^{j\otimes} - \delta_j)\right]^{k\otimes}, \quad k = 1,2,3,4$$

and the above given rule for proving the assertions. The procedure is elementary though tedious and therefore it is left to the reader. □

Theorem 4.9. *Let Assumption be satisfied. Then*

$$\mathrm{Var}(\xi'\Sigma\xi) = 2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}(\delta_j' \otimes \delta_k')(A_k \otimes A_j)(\Sigma \otimes \Sigma)(A_l' \otimes A_r')(\delta_{l+r} - \delta_l \otimes \delta_r)$$
$$+2\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}(\delta_j' \otimes \delta_k')(A_j \otimes A_k)(\Sigma \otimes \Sigma)(A_l' \otimes A_r')(\delta_{l+r} - \delta_l \otimes \delta_r)$$
$$+2\left\{\left(\sum_{j=0}^{\infty}\delta_j' A_j \Sigma\right) \otimes [\mathrm{vec}(\Sigma)]'\right\}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}(A_k' \times A_l' \otimes A_r')\boxed{3}$$
$$+2\left\{[\mathrm{vec}(\Sigma)]' \otimes (\sum_{j=0}^{\infty}\delta_j' A_j \Sigma)\right\}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}(A_k' \times A_l' \otimes A_r')\boxed{3}$$
$$+[\mathrm{vec}(\Sigma) \otimes \mathrm{vec}(\Sigma)]'\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}(A_j' \otimes A_k' \otimes A_l' \otimes A_r')\boxed{4}.$$

Proof. The expression can be obtained directly from Theorem 4.7 and Lemma 4.8. □

Corollary 4.10. *Let $f(.)$ be a polynomial of the second order. Then*
(i) *if $\hat{\mu}$ is normally distributed, then the exact formula is*

$$\operatorname{Var}\left(\frac{\partial f(x)}{\partial x'}\Big|_{x=\hat{\mu}}\Sigma\frac{\partial f(x)}{\partial x}\Big|_{x=\hat{\mu}}\right) = 4g'\Sigma H\Sigma H\Sigma g + 2\operatorname{Tr}[(H\Sigma)^4].$$

(ii) *In the general case,*

$$\operatorname{Var}\left(\frac{\partial f(x)}{\partial x'}\Big|_{x=\hat{\mu}}\Sigma\frac{\partial f(x)}{\partial x}\Big|_{x=\hat{\mu}}\right) = 4g'\Sigma H\Sigma H\Sigma g + 4[\operatorname{vec}(\Sigma)]'(H\otimes H)\varphi H\Sigma g$$
$$+[\operatorname{vec}(\Sigma)]'\otimes[\operatorname{vec}(\Sigma)]'H^{4\otimes}(\delta_4-\delta_2\otimes\delta_2).$$

5. Some comments to the determination of the standard deviation

The expressions $\sqrt{\operatorname{Var}[f(\hat{\mu})]}$ and $\sqrt{\xi'\Sigma\xi}$, respectively, are required more frequnetly than $\operatorname{Var}[f(\hat{\mu})]$ and $\xi'\Sigma\xi$. Thus it should be stated something on the statistical behaviour of $\sqrt{\xi'\Sigma\xi}$.

Lemma 5.1. *Let $\varphi_s(t)$, $t\in\mathbb{R}^1$, be the series from (2.1). Let for a given function $\varphi(.)\colon \mathbb{R}^1\to\mathbb{R}^1$ the radius of covergence of its Taylor series*

$$\varphi(z) = h_0 + h_1 z + h_2 z^2 + \dots$$

be $\kappa > 0$. Let $|\varphi_s(0)| < \kappa$. Then
(i) *the series $\varphi[\varphi_s(t)] = \sum_{j=0}^{\infty} h_j[\varphi_s(t)]^j$ converges on the interval $(t_1(s), t_2(s))$, where*

$$t_1(s) = \inf\{t\colon |\varphi_s(t)| < \kappa\},$$
$$t_2(s) = \sup\{t\colon |\varphi_s(t)| < \kappa\}.$$

(ii) *Let*

$$\mathcal{R}_\varphi = \{x\colon x\in\mathbb{R}^n, x = (1-\alpha)t_1(x/\|x\|) + \alpha t_2(x/\|x\|), 0<\alpha<1\}$$

and $S\subset\mathcal{R}_\varphi$.

Then $\varphi[f(x)]$, $x\in S$, can be expressed by a series which converges uniformly on S and absolutely for any $x_0\in S$.

Proof. It follows from the consideration in [4], p. 488. □

Under the given Assumption, since (cf. Theorem 3.3)

$$\begin{aligned}\operatorname{Var}[f(\hat{\mu})] &= \sum_{r=1}^{\infty}\sum_{j=1}^{\infty}(a_r' \otimes a_j')(\delta_{j+r} - \delta_j \otimes \delta_r) \\ &= g'\Sigma g + \sum_{r=2}^{\infty}\sum_{j=2}^{\infty}(a_r' \otimes a_j')(\delta_{j+r} - \delta_j \otimes \delta_r),\end{aligned}$$

we can write

$$\begin{aligned}\sqrt{\operatorname{Var}[f(\hat{\mu})]} &= \sqrt{g'\Sigma g}\left(1 + \frac{1}{g'\Sigma g}\sum_{r=2}^{\infty}\sum_{j=2}^{\infty}(a_r' \otimes a_j')(\delta_{j+r} - \delta_j \otimes \delta_r)\right)^{1/2} \\ &= \sqrt{g'\Sigma g}\sum_{k=0}^{\infty}\binom{1/2}{k}\left[\frac{1}{g'\Sigma g}\sum_{r=2}^{\infty}\sum_{j=2}^{\infty}(a_r' \otimes a_j')(\delta_{j+r} - \delta_j \otimes \delta_r)\right]^k\end{aligned}$$

and this series converges iff

$$\left|\frac{\operatorname{Var}[f(\hat{\mu})] - g'\Sigma g}{g'\Sigma g}\right| < 1$$

(cf. Example 2.3 (i)).

For the first orientation the following formulae can serve. For the sake of simplicity the quadratic function $f(.)$ and the statistical moments up to the fourth order only are considered in the following.

Thus

$$\sigma = \sqrt{V} = \sqrt{g'\Sigma g + \operatorname{Tr}[(g' \otimes H)\varphi] + \frac{1}{4}\{\operatorname{Tr}[(H \otimes H)\psi] - [\operatorname{Tr}(H\Sigma)]^2\}};$$

in the case of normality,

$$\begin{aligned}\sigma = \sqrt{V} &= \sqrt{g'\Sigma g + \frac{1}{2}\operatorname{Tr}(H\Sigma H\Sigma)} \\ &= \sqrt{g'\Sigma g} + \frac{1}{4\sqrt{g'\Sigma g}}\operatorname{Tr}[(H\Sigma)^2] - \frac{1}{32(\sqrt{g'\Sigma g})^3}\{\operatorname{Tr}[(H\Sigma)^2]\}^2 + \dots\end{aligned}$$

Analogously

$$\begin{aligned}\tilde{\sigma} = \sqrt{\tilde{V}} &= \sqrt{\xi'\Sigma\xi} = \sqrt{g'\Sigma g}\sqrt{1+U} \\ &= \sqrt{g'\Sigma g}\left(1 + \frac{U}{2} - \frac{U^2}{8} + \dots\right),\end{aligned} \tag{5.1}$$

where

$$U = \frac{1}{g'\Sigma g}\left[g'\Sigma g \sum_{k=1}^{\infty} A_k' \varepsilon^{k\otimes} + \sum_{j=1}^{\infty} (\varepsilon^{j\otimes})' A_j \Sigma g + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (\varepsilon^{j\otimes})' A_j \Sigma A_k' \varepsilon^{k\otimes}\right].$$

Since $U = (\xi'\Sigma\xi - g'\Sigma g)/g'\Sigma g$, thus $\mathrm{Var}(U) = \mathrm{Var}(\tilde{V})/[(g'\Sigma g)^2]$.

The series (5.1) converges in the case that the support S_1 of the distribution of the random variable U satisfies the condition

$$P\{|U| < 1\} = 1.$$

Remark 5.2. In some situations Assumption is not satisfied; e.g. $\hat{\mu}$ is normally distributed and $\mathcal{R} \subset \mathbb{R}^n$ and $\mathcal{R} \neq \mathbb{R}^n$. In this case the given formulae for the bias and variance are not valid. Nevertheless, they can be of some use in the case $P\{\hat{\mu} \notin \mathcal{R}\} < \varepsilon$, for $\varepsilon > 0$ sufficiently small.

E.g. in the case $f(x) = \frac{1}{x}$, $x \in \mathbb{R}^1 - \{0\}$, let $\hat{\mu} \sim N_1(\mu, \sigma^2)$, where $\mu > 0$ and $\frac{\sigma}{\mu} < 1$. Then the expressions (two first terms from the expressions given in Theorem 3.3 are taken into account only)

$$E\left(\frac{1}{\hat{\mu}}\right) = \frac{1}{\mu} + \frac{\sigma^2}{\mu^3} + \dots, \qquad \mathrm{Var}\left(\frac{1}{\hat{\mu}}\right) = \frac{\sigma^2}{\mu^4} + \frac{2\sigma^4}{\mu^6} + \dots,$$

are in good agreement with results obtained by the simulations (5.000 trials) for any $\frac{\sigma}{\mu} < 0.1$.

Analogously as in the case of $\sqrt{\mathrm{Var}[f(\hat{\mu})]}$, the quadratic function $f(.)$ and the statistical moments up to the fourth order only are considered in the following.

Thus

$$\begin{gathered}
\tilde{V} = g'\Sigma g + 2\varepsilon' H\Sigma g + \varepsilon' H'\Sigma H\varepsilon, \\
\mathrm{Var}(\tilde{V}) = 4g'\Sigma H\Sigma H\Sigma g + 4\,\mathrm{Tr}\left\{[(g'\Sigma H)\otimes(H\Sigma H)]\varphi\right\} \\
+ \mathrm{Tr}\left\{[(H\Sigma H)\otimes(H\Sigma H)]\psi\right\} - \left\{\mathrm{Tr}[(H\Sigma)^2]\right\}^2, \\
\tilde{\sigma} = \sqrt{g'\Sigma g}\left\{1 + \frac{1}{g'\Sigma g} g'\Sigma H\varepsilon + \frac{1}{2g'\Sigma g}\varepsilon' H\Sigma H\varepsilon - \frac{1}{2(g'\Sigma g)^2}\varepsilon' H\Sigma g g'\Sigma H\varepsilon + \right. \\
+ \left(-\frac{1}{2}\right)\frac{1}{(g'\Sigma g)^2}\varepsilon' H\Sigma g\varepsilon' H\Sigma H\varepsilon + \frac{1}{2(g'\Sigma g)^3}\varepsilon' H\Sigma g\varepsilon' H\Sigma g\varepsilon' H\Sigma g g'\Sigma H\varepsilon \\
+ \left(-\frac{1}{8}\right)\frac{1}{(g'\Sigma g)^2}\varepsilon' H\Sigma H\varepsilon\varepsilon' H\Sigma H\varepsilon \\
\left. + \frac{3}{4}\varepsilon' H\Sigma g g'\Sigma H\varepsilon\varepsilon' H\Sigma g g'\Sigma H\varepsilon - \frac{15}{24(g'\Sigma g)^4}\varepsilon' H\Sigma g g'\Sigma H\varepsilon\varepsilon' H\Sigma g g'\Sigma H\varepsilon\right],
\end{gathered}$$

$$E(\tilde{\sigma}) = \sqrt{g'\Sigma g}\left[1 + \frac{1}{2g'\Sigma g}\operatorname{Tr}(H\Sigma H\Sigma) - \frac{1}{2(g'\Sigma g)^2}g'\Sigma H\Sigma H\Sigma g + \dots\right],$$

$$b(\tilde{\sigma}) = E(\tilde{\sigma}) - \sigma = \frac{1}{4\sqrt{g'\Sigma g}}\operatorname{Tr}(\Sigma H\Sigma H) -$$

$$-\frac{1}{2(g'\Sigma g)^{3/2}}g'\Sigma H\Sigma H\Sigma g + \frac{1}{32(g'\Sigma g)^{3/2}}\left\{\operatorname{Tr}\left[(\Sigma H)^2\right]\right\}^2 + \dots,$$

$$\operatorname{Var}(\tilde{\sigma}) = \frac{g'\Sigma H\Sigma H\Sigma g}{g'\Sigma g}$$

$$+\frac{\operatorname{Tr}\left[(H\Sigma)^4\right]}{2g'\Sigma g} - 3\frac{g'\Sigma H\Sigma H\Sigma H\Sigma H\Sigma g}{(g'\Sigma g)^2} + \frac{5}{2}\frac{(g'\Sigma H\Sigma H\Sigma g)^2}{(g'\Sigma g)^3} + \dots$$

Example 5.3. (Assumption is not satisfied, cf. Remark 5.2.) Let $Y \sim N_n(X\beta, \sigma^2 V)$, $\beta \in \mathbb{R}^k$, $r(X_{n,k}) = k < n$, $r(V) = n$. Then

$$\hat{\sigma^2} = Y'[V^{-1} - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}]Y/(n-k) \sim \sigma^2\frac{\chi^2_{n-k}}{n-k}.$$

Thus

$$\sqrt{\hat{\sigma^2}} = \sigma\sqrt{1 + \left(\frac{\chi^2_{n-k}}{n-k} - 1\right)}.$$

Let

$$\eta = \frac{\chi^2_{n-k}}{n-k} - 1.$$

Then $E(\eta) = 0$ and $\operatorname{Var}(\eta) = \frac{2}{n-k}$. If $\sqrt{\frac{2}{n-k}}2.5 = 0.5$, i.e., $n - k = 50$, then with respect to the Tchebysheff inequality $P\{|\eta| < 0.5\} \geqslant 0.84$ (in fact $P\{|\eta| < 0.5\} = P\{25 < \chi^2_{50} < 75\} = 0.99$). With respect to the approximation

$$\tilde{\sigma} = \sigma\sqrt{1+\eta} = \sigma\left(1 + \frac{1}{2}\eta - \frac{1}{8}\eta^2 + \frac{1}{8}\eta^3 - \frac{5}{128}\eta^4 + \dots\right)$$

and the well known relationship

$$E(\eta^k) = \frac{1}{\Gamma(\frac{1}{2})}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{2}{f}\right)^j\Gamma\left(\frac{f}{2}+1\right),$$

$(f = n - k)$, we obtain

$$E(\tilde{\sigma}) = \sigma \times 0.9948$$

(the terms up to the fourth order including). The exact well known formula is

$$E(\tilde{\sigma}) = \sigma\sqrt{\frac{2}{f}}\frac{\Gamma\left(\frac{f+1}{2}\right)}{\Gamma\left(\frac{f}{2}\right)} = \sigma \times 0.9950.$$

The agreement seems to be good for practical purposes.

References

[1] *H. J. Bartsch*: Mathematical Formulae. Praha, SNTL, 1965. (In Czech.)
[2] *F. Čechura*: Mine Surveying, Part I, Adjustment Theory. Matice hornicko-hutnická, 1948. (In Czech.)
[3] *F. Čuřík*: Mathematics (Technical Handbook). Praha, ČMT, 1944. (In Czech.)
[4] *G.M. Fichtengolc*: Course of Differential and Integral Calculus. Fizmatgiz, Moscow, 1959. (In Russian.)
[5] *M. Fisz*: Wahrscheinlichkeitsrechnung und mathematische Statistik. VEB, Deutscher Verlag der Wissenschaften, Berlin, 1962.
[6] *F.B. Gantmacher*: The Theory of Matrices. Vols. I and II. Chelsea, New York, 1959.
[7] *G.A. Korn, T.M. Korn*: Mathematical Handbook for Scientists and Engineers. McGraw-Hill, New York-Toronto-London, 1961.
[8] *A. M. Kshirsagar*: Multivariate Analysis. Marcel Dekker Inc., New York, 1972.
[9] *J.R. Magnus, H. Neudecker*: Matrix Differential Calculus with Applications in Statistics and Econometrics. J. Wiley, Chichester-New York-Brisbane-Toronto-Singapore, 1991.

Author's address: *Lubomír Kubáček*, Department of Mathematical Analysis and Applied Mathematics, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic.