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A POLLING MODEL WITH RETRIAL CUSTOMERS

Christos Langaris
University of Ioannina

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Abstract A polling model with n stations and switchover times is considered. The customers are of n different types, arrive to the system according to the Poisson distribution in batches of random size, and if they find the server unavailable, they start to make retrials until succeed to find a position for service. Each batch may contain customers of different types while the numbers of customers belonging to each type in a batch are distributed according to a multivariate general distribution. The server, upon polling a station, stays there for an exponential period of time and if a customer asks for service before this time expires, the customer is served and a new "stay period" begins. Finally the service times and the switchover times are both arbitrarily distributed with different distributions for the different stations.

For such a model we obtain formulas for the expected number of retrial customers in each station in a steady state. Our results can be easily adapted to hold for zero switchover times and also in the case of the ordinary exhaustive service polling model with (without) switchover times and correlated batch arrivals. In all cases mentioned above (retrial model, exhaustive model, switchover times, zero switchover times) to find the expected queue lengths we need finally to solve a set of only n linear equations ($O(n^3)$ arithmetic operations to compute the coefficients).

Tables of numerical values are finally obtained and used to observe the system performance when we vary the values of the parameters.

1. Introduction

A polling model is a system of n queueing stations accessed by a single server in a prescribed order. This kind of systems has been proved useful particularly to model maintenance processes, multiprocessor computers, communication networks and manufacturing systems. There are many varietes of polling models depending on the service disciplines (exhaustive, gated, limited etc.), the existence or not of switchover times between stations, the capacity of the buffers, the order in which the server polls the stations etc. For a complete survey on the earliest works in polling systems see Takagi [26]. More recently we have to mention the works of Resing [22], Eisenberg [8], Srinivasan et al [24] and Altman & Yechiali [2].

As far as we know, in all studies of polling models appeared till now in the literature the customers are assumed to form, upon arrival, a queue in each station and to wait there until the server selects them for service. Thus the customer in such a model does not have the chance- when he finds, upon arrival, the server busy in one of the stations, or performing a switchover time - to leave the system and to retry individually for service later.

Queueing systems with retrial customers have received considerable attention recently and are widely used in computer and communication networks and in telephone switching systems. They are characterized by the fact that an arriving customer, who finds the server busy, leaves the system and repeats his demand after a random amount of time. A complete description of situations where such queues arise, and extensive reviews of the earliest work on the subject may be found in Yang & Templeton [27] and in Falin [9]. We have also to mention here the works of Kulkarni & Choi [17], Falin & Fricker [10], Grishechkin [14], Falin, Artalejo & Martin [11], Langaris & Moutzoukis [18], Moutzoukis & Langaris [21].

Both kind of models - polling models and systems with retrial customers - have been used separately to model complex situations particularly in communication networks. Thus in the field of Local Area Networks (LANs) one can find a number of models handled as polling systems with queued up customers (Bux [6], Ferguson & Aminetzah [12], Fournier & Rosberg [13], Borst [4]), while in the same area, various models with specific protocols have been discribed and analysed as retrial systems with a single station (Kulkarni [16], Choi, Shin & Ahn [7]).

In the work here, we have tried to combine these two characteristics, and to study a polling model with retrial customers. Thus in the model considered, there are n types of customers (one for each station) arriving in batches of random size (a batch may contain customers of all types) and making retrials in each station until they find the server available. There is only one server who polls the stations in a cyclic order and stays in each one of them for a random amount of time awaiting for customers seeking service. There is finally a switchover time when the server passes from a station to the next. For this model, we will describe a method to obtain the mean number of retrial customers in each station (and the mean waiting time, through Little's formula, consequently).

An interesting feature of the approach used is that we can obtain immediately the corresponding quantities (queue length, waiting time) in the case of zero switchover times, simply by replacing in the obtained final formulas, the mean and the second moment of the switchover time by zero. Moreover by sending the mean retrial time and in the sequel the mean "stay period" to zero we arrive at the corresponding formulae of the ordinary exhaustive service polling model with or without switchover times and correlated batch arrivals.

The mean queue lengths (waiting times), in all models described above are found, by solving finally a set of only n linear equations, while the number of arithmetic operations required to derive the coefficients of these equations is $O(n^3)$ or less. Note that, for the exhaustive (and gated) polling model with switchover times and single independent arrivals, Sarkar & Zangwill [23] derived (using the concept of "system time") the expected waiting times, by solving a set of n linear equations (for the variances of the cycle times) too. The mean waiting time in the corresponding model with correlated batch arrivals has been obtained in Levy & Sidi [19] as a solution of a set of n^3 linear equations while Boxma [5] derived for the same model a pseudoconservation law for the mean waiting times.

The paper is organised as follows. After the full description of the model in Section 2, a system of equations satisfied by the steady state probabilities are obtained in Section 3. In Section 4 these equations are used to derive expressions for the mean number of retrial customers in each station. The case of the exhaustive service polling model with correlated batch arrivals is investigated in Section 5. Finally in Section 6 numerical results are obtained for the retrial model and used to observe the system performance under changes in the values of the parameters.

2. The Model

Consider a system consisting of n infinite capacity queueing stations S_i $i = 1,2,...,n$ arranged in a cyclic order. There is only one server who visits the stations in a prescribed cyclic order $S_1, S_2, ..., S_n, S_1, S_2, ...$

Customers arrive into the system according to the Poisson distribution with parameter λ in batches of random size. Each batch may contain customers of different types P_i $i = 1,2,...,n$ and a P_i customer asks always for service at the S_i station. If we denote by X_i $i = 1,2,...,n$ the number of P_i customers in an arbitrary batch, then we define

$$g(\mathbf{x}) = \Pr(X_1 = x_1, X_2 = x_2, ..., X_n = x_n), \qquad g(\mathbf{0}) = 0$$

$$G(\mathbf{z}) = \sum_{\mathbf{x} \geq \mathbf{0}} g(\mathbf{x})\mathbf{z}^{\mathbf{x}}, \qquad g_j = \frac{\partial G(\mathbf{z})}{\partial z_j}\Big|_{\mathbf{z}=\mathbf{1}}, \qquad g_{ij} = \frac{\partial^2 G(\mathbf{z})}{\partial z_j \partial z_i}\Big|_{\mathbf{z}=\mathbf{1}}, \qquad g_i^{(2)} = \frac{\partial^2 G(\mathbf{z})}{\partial z_i^2}\Big|_{\mathbf{z}=\mathbf{1}},$$

where in general $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{z}^{\mathbf{x}} = z_1^{x_1} z_2^{x_2} ... z_n^{x_n}$.

If an arriving batch of customers finds the server in the S_i station and idle then one of the P_i customers of the batch (if any) commences service immediately and the remaining P_j customers $j = 1, 2, ..., n$ join the retrial group of the S_j station respectively and seek for service individually after an exponentially distributed (parameter μ_j for the S_j station) amount of time.

If an arriving batch finds the server either busy in one of the stations or performing a switchover time then all customers of the batch join the corresponding retrial groups.

The server upon polling a station stays there for an exponential amount of time (parameter a_j for the S_j station). If a customer arrives (either from outside or from the retrial group) before this time expires then the customer is served and afterwords a new exponential "stay period" begins. When for the first time the "stay period" ends before an arrival occurs, the server switches to the next station. A switch from one station to another always requires a switchover time.

The service time of a customer in the S_i station is distributed according to an arbitrary distribution with distribution function (D.F) $U_i(x)$, probability density function (p.d.f.) $u_i(x)$ with finite mean $\bar{u}_i$ and second moment $\bar{u}_i^{(2)}$ while the server switchover time between the $(i-1)^{th}$ and i^{th} station is assumed to be also arbitrarily distributed with D.F. $V_i(x)$, p.d.f. $v_i(x)$ and finite mean $\bar{v}_i$ and second moment $\bar{v}_i^{(2)}$ for all $i = 1, 2, ..., n$. All the processes defined above are assumed to be independent to each other.

3. System State Analysis

We will start our analysis by considering firstly a more general model with arbitrarily distributed "stay periods" instead of the exponential ones. Thus we assume that the "stay period" of the server in station S_i $i = 1, 2, ..., n$ follows a general distribution with p.d.f. $b_i(t)$, D.F. $B_i(t)$ and finite mean $\bar{b}_i$.

Let now $L_i(t)$ be the number of customers in the retrial group of station S_i (the one in service-if any- is not included) at time t, and

$$\mathbf{L}(t) = (L_1(t), L_2(t), ..., L_n(t)).$$

Define

$$\xi(t) = \begin{cases} l_i & \text{if the server is working in } S_i \text{ at } t \\ c_i & \text{if the server is staying idle in } S_i \text{ at } t \\ v_i & \text{if the server is switching from } S_{i-1} \text{ to } S_i \text{ at } t, \end{cases}$$

and let

$$\begin{aligned} p_i(\mathbf{k}, x, t)dx &= \Pr(\xi(t) = l_i,\ \mathbf{L}(t) = \mathbf{k},\ x < \overline{U}_i(t) \le x + dx) \\ q_i(\mathbf{k}, x, t)dx &= \Pr(\xi(t) = c_i,\ \mathbf{L}(t) = \mathbf{k},\ x < \overline{B}_i(t) \le x + dx) \qquad (3.1) \\ d_i(\mathbf{k}, x, t)dx &= \Pr(\xi(t) = v_i,\ \mathbf{L}(t) = \mathbf{k},\ x < \overline{V}_i(t) \le x + dx), \end{aligned}$$

where $\overline{U}_i(t), \overline{B}_i(t), \overline{V}_i(t)$ are at t, the elapsed service time, the elapsed "stay period" and the elapsed switchover time respectively. It is easy to see that for $x > 0$ and for all $i = 1, 2, ..., n$

$$p_i(\mathbf{k}, x + dx, t + dx) = (1 - \lambda dx - \hat{u}_i(x)dx)p_i(\mathbf{k}, x, t) + \lambda dx \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{k}} g(\mathbf{k} - \mathbf{m})p_i(\mathbf{m}, x, t)$$

$$\begin{aligned} q_i(\mathbf{k}, x + dx, t + dx) &= (1 - \lambda dx - \hat{b}_i(x)dx - k_i\mu_i dx)q_i(\mathbf{k}, x, t) \qquad (3.2) \\ &+ \lambda dx \sum_{\mathbf{m}_i^*=\mathbf{0}}^{\mathbf{k}_i^*} g(\mathbf{m}_i^*)q_i(\mathbf{k} - \mathbf{m}_i^*, x, t) \end{aligned}$$

$$d_i(\mathbf{k}, x+dx, t+dx) = (1-\lambda dx - \hat{v}_i(x)dx)d_i(\mathbf{k},x,t) + \lambda dx \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{k}} g(\mathbf{k}-\mathbf{m})d_i(\mathbf{m},x,t),$$

where we use $\mathbf{m}_i^* = (m_1, ..., m_{i-1}, 0, m_{i+1}, ..., m_n)$ and for any p.d.f. $f(t)$ with D.F. $F(t)$ we denote by $\hat{f}(t)$ the age-specific failure rate i.e. $\hat{f}(t) = \frac{f(t)}{1-F(t)}$.

For $x = 0$ finally

$$(3.3)\quad p_i(\mathbf{k},0,t) = \lambda \sum_{\mathbf{m}=\mathbf{0}}^{\mathbf{k}} g(\mathbf{k}-\mathbf{m}+\mathbf{1}_i)\int_0^\infty q_i(\mathbf{m},x,t)dx + (k_i+1)\mu_i \int_0^\infty q_i(\mathbf{k}+\mathbf{1}_i,x,t)dx$$

$$(3.4)\qquad q_i(\mathbf{k},0,t) = \int_0^\infty p_i(\mathbf{k},x,t)\hat{u}_i(x)dx + \int_0^\infty d_i(\mathbf{k},x,t)\hat{v}_i(x)dx$$

$$(3.5)\qquad d_i(\mathbf{k},0,t) = \int_0^\infty q_{i-1}(\mathbf{k},x,t)\hat{b}_{i-1}(x)dx,$$

where we denote by $\mathbf{1}_i$ the n dimensional null row vector with a unit in the i^{th} position and of course if $i = 1$ then $i-1 = n$.

Remark: If we put $v_i(x) = \hat{v}_i(x) \equiv \delta_{x0}$, where δ_{x0} is Kronecker's delta, in (3.4) and replace the second integral of the right hand side (which becomes now $d_i(\mathbf{k},0,t)$) with the integral in (3.5), then it is easy to see that the first two relations in (3.2), relation (3.3) and the new relation (3.4) describe in fact the model with zero switchover times. Note that in this case the probability $d_i(\mathbf{k},0,t)$ in (3.5) corresponds to the point in time at which the server polls station i.

Assuming now that a steady state exists and defining

$$P_i(\mathbf{z},x) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \lim_{t\to\infty} p_i(\mathbf{k},x,t)\mathbf{z}^{\mathbf{k}}$$
$$Q_i(\mathbf{z},x) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \lim_{t\to\infty} q_i(\mathbf{k},x,t)\mathbf{z}^{\mathbf{k}}$$
$$D_i(\mathbf{z},x) = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} \lim_{t\to\infty} d_i(\mathbf{k},x,t)\mathbf{z}^{\mathbf{k}},$$

we obtain from (3.2)

$$(3.6)\qquad \frac{\partial P_i(\mathbf{z},x)}{\partial x} + [\lambda(1-G(\mathbf{z})) + \hat{u}_i(x)]P_i(\mathbf{z},x) = 0,$$

$$(3.7)\qquad \frac{\partial Q_i(\mathbf{z},x)}{\partial x} + \mu_i z_i \frac{\partial Q_i(\mathbf{z},x)}{\partial z_i} + [\lambda(1-G(\mathbf{z}_i^*)) + \hat{b}_i(x)]Q_i(\mathbf{z},x) = 0,$$

$$(3.8)\qquad \frac{\partial D_i(\mathbf{z},x)}{\partial x} + [\lambda(1-G(\mathbf{z})) + \hat{v}_i(x)]D_i(\mathbf{z},x) = 0.$$

Differential equations (3.6) and (3.8) can be solved immediately and give

$$(3.9)\qquad P_i(\mathbf{z},x) = P_i(\mathbf{z},0)\exp\{-(\lambda-\lambda G(\mathbf{z}))x - \int_0^x \hat{u}_i(y)dy\}, \qquad i = 1,2,...,n$$

$$(3.10)\qquad D_i(\mathbf{z},x) = D_i(\mathbf{z},0)\exp\{-(\lambda-\lambda G(\mathbf{z}))x - \int_0^x \hat{v}_i(y)dy\}, \qquad i = 1,2,...,n,$$

while to handle equation (3.7) we have to consider (Gross & Harris [15],p.115) the system

$$\frac{dz_i}{\mu_i z_i} = \frac{dx}{1} = -\frac{dQ_i}{[\lambda(1-G(\mathbf{z}_i^*)) + \hat{b}_i(x)]Q_i}. \tag{3.11}$$

By solving this system in the usual way, we arrive at

$$z_i = Ue^{\mu_i x}, \qquad Q_i(\mathbf{z},x) = V\exp\{-(\lambda - \lambda G(\mathbf{z}_i^*))x - \int_0^x \hat{b}_i(y)dy\},$$

where U and V are constants. The general solution of (3.7) is now of the form $V = F(U)$ i.e.

$$Q_i(\mathbf{z},x) = F(z_i e^{-\mu_i x})\exp\{-(\lambda - \lambda G(\mathbf{z}_i^*))x - \int_0^x \hat{b}_i(y)dy\}$$

and so by putting $x = 0$ in the above equation we obtain the unknown function F as $F(w) = Q_i((z_1,...,z_{i-1},w,z_{i+1},...,z_n),0)$. Thus finally

$$Q_i(\mathbf{z},x) = Q_i((z_1,...,z_{i-1},z_i e^{-\mu_i x},z_{i+1},...,z_n),0)\exp\{-(\lambda - \lambda G(\mathbf{z}_i^*))x - \int_0^x \hat{b}_i(y)dy\}. \tag{3.12}$$

To find now the unknown quantities in (3.9), (3.10) and (3.12) we will use the boundary conditions (3.3)-(3.5). Let us define

$$\overline{Q}_i(\mathbf{z}) = \int_0^\infty Q_i(\mathbf{z},x)dx\ . \tag{3.13}$$

Then from (3.3) by forming the generating functions we obtain after manipulations

$$\mu_i \frac{\partial \overline{Q}_i(\mathbf{z})}{\partial z_i} + \lambda \frac{G(\mathbf{z}) - G(\mathbf{z}_i^*)}{z_i}\overline{Q}_i(\mathbf{z}) = P_i(\mathbf{z},0), \qquad i = 1,2,...,n \tag{3.14}$$

while from (3.4), using (3.9) and (3.10)

$$P_i(\mathbf{z},0)u_i^*(\lambda - \lambda G(\mathbf{z})) + D_i(\mathbf{z},0)v_i^*(\lambda - \lambda G(\mathbf{z})) = Q_i(\mathbf{z},0), \tag{3.15}$$

with $u_i^*(\cdot)$, $v_i^*(\cdot)$ the Laplace transforms (L.T) of $u_i(\cdot)$, $v_i(\cdot)$ respectively. Finally from (3.5)

$$D_i(\mathbf{z},0) = \int_0^\infty Q_{i-1}(\mathbf{z},x)\hat{b}_{i-1}(x)dx. \tag{3.16}$$

Let us consider now the generating function $\overline{Q}_i(\mathbf{z})$. By substituting $Q_i(\mathbf{z},x)$ from (3.12) to (3.13) and evaluate the integral we arrive at

$$\overline{Q}_i(\mathbf{z}) = \sum_{\mathbf{k}=0}^{\infty} \frac{1 - b_i^*(\lambda - \lambda G(\mathbf{z}_i^*) + k_i\mu_i)}{\lambda - \lambda G(\mathbf{z}_i^*) + k_i\mu_i} q_i(\mathbf{k},0)\mathbf{z}^{\mathbf{k}}\ , \tag{3.17}$$

with $b_i^*(\cdot)$ the L.T. of $b_i(\cdot)$. In a similar manner (using (3.12) again)

$$\int_0^\infty Q_i(\mathbf{z},x)\hat{b}_i(x)dx = \sum_{\mathbf{k}=\mathbf{0}}^{\infty} b_i^*(\lambda - \lambda G(\mathbf{z}_i^*) + k_i\mu_i)q_i(\mathbf{k},0)\mathbf{z}^{\mathbf{k}}\ . \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\mu_i z_i \frac{\partial \overline{Q}_i(\mathbf{z})}{\partial z_i} + \lambda(1 - G(\mathbf{z}_i^*))\overline{Q}_i(\mathbf{z}) = Q_i(\mathbf{z},0) - \int_0^\infty Q_i(\mathbf{z},x)\hat{b}_i(x)dx. \tag{3.19}$$

Substituting finally $Q_i(\mathbf{z},0)$ and $P_i(\mathbf{z},0)$ from (3.19) and (3.14) to (3.15) and using (3.16), we arrive at

$$\begin{aligned}(3.20)\quad &\mu_i(z_i-u_i^*(\lambda-\lambda G(\mathbf{z})))\tfrac{\partial\overline{Q}_i(\mathbf{z})}{\partial z_i}+[\lambda-\lambda G(\mathbf{z}_i^*)-\lambda\tfrac{G(\mathbf{z})-G(\mathbf{z}_i^*)}{z_i}u_i^*(\lambda-\lambda G(\mathbf{z}))]\overline{Q}_i(\mathbf{z})\\ &=D_i(\mathbf{z},0)v_i^*(\lambda-\lambda G(\mathbf{z}))-D_{i+1}(\mathbf{z},0).\end{aligned}$$

A Modified Model: Let us consider now a different "staying" policy in which the server is not allowed to repeat the "stay period" each time a customer completes service but now the **total time** he is allowed to spend in an idle mode, from the instant he polls the station until the instant he leaves it, is an arbitrarily distributed r.v. with p.d.f $b_i(t)$ and D.F. $B_i(t)$ for the i^{th} station respectively. In this case, let us replace the first and second of (3.1) by

$$p_i'(\mathbf{k},y,x,t)dxdy=\Pr[\xi(t)=l_i,\ \mathbf{L}(t)=\mathbf{k},\ y<\overline{B}_i'(t)\le y+dy,\ x<\overline{U}_i(t)\le x+dx],$$

$$q_i'(\mathbf{k},y,x,t)dxdy=\Pr[\xi(t)=c_i,\ \mathbf{L}(t)=\mathbf{k},\ y<\overline{B}_i'(t)\le y+dy,\ x<\overline{B}_i''(t)\le x+dx]$$

where now $\overline{B}_i'(t)$ is the total time the server already spent in an idle mode before commencing his more recent service and $\overline{B}_i''(t)$ is the elapsed "stay period" counting from the last epoch the server becomes idle. If now we repeat the previous analysis then, for $x>0$, we obtain

$$\begin{aligned}&P_i'(\mathbf{z},y,x)=P_i'(\mathbf{z},y,0)\exp\{-(\lambda-\lambda G(\mathbf{z}))x-\int_0^x\hat{u}_i(w)dw\}\\ (3.21)\quad &D_i'(\mathbf{z},x)=D_i'(\mathbf{z},0)\exp\{-(\lambda-\lambda G(\mathbf{z}))x-\int_0^x\hat{v}_i(y)dy\}\\ &Q_i'(\mathbf{z},y,x)=Q_i'((z_1,...,z_{i-1},z_ie^{-\mu_ix},z_{i+1},...,z_n),y,0)\exp\{-(\lambda-\lambda G(\mathbf{z}_i^*))x-\int_0^x\hat{b}_i(y+u)du\},\end{aligned}$$

while from the boundary conditions

$$\begin{aligned}&\mu_i\frac{\partial\bar{Q}_i'(\mathbf{z})}{\partial z_i}+\lambda\frac{G(\mathbf{z})-G(\mathbf{z}_i^*)}{z_i}\bar{Q}_i'(\mathbf{z})=P_i'(\mathbf{z},0)\\ &Q_i'(\mathbf{z},y,0)=P_i'(\mathbf{z},y,0)u_i^*(\lambda-\lambda G(\mathbf{z})),\qquad y>0\\ (3.22)\quad &Q_i'(\mathbf{z},0,0)=D_i'(\mathbf{z},0)v_i^*(\lambda-\lambda G(\mathbf{z}))\\ &D_i'(\mathbf{z},0)=\int_0^\infty\int_0^\infty Q_{i-1}'(\mathbf{z},y,x)\hat{b}_{i-1}(x+y)dxdy.\end{aligned}$$

with $\bar{Q}_i'(\mathbf{z})\equiv\int_0^\infty\int_0^\infty Q_i'(\mathbf{z},y,x)dxdy$, $P_i'(\mathbf{z},0)\equiv\int_0^\infty P_i'(\mathbf{z},y,0)dy$. If now we write for $\bar{Q}_i'(\mathbf{z})$ and $\int_0^\infty\int_0^\infty Q_i'(\mathbf{z},y,x)\hat{b}_i(x+y)dxdy$ expressions similar to (3.17) and (3.18) we arrive again, after manipulations, at the basic relation (3.20) (with $\bar{Q}_i'(\mathbf{z})$, $D_\cdot'(\mathbf{z},0)$ instead of $\bar{Q}_i(\mathbf{z})$, $D_\cdot(\mathbf{z},0)$ respectively).

Note here that from the second and third of (3.22)

$$Q_i'(\mathbf{z},0)\equiv\int_0^\infty Q_i'(\mathbf{z},y,0)dy=D_i'(\mathbf{z},0)v_i^*(\lambda-\lambda G(\mathbf{z}))+P_i'(\mathbf{z},0)u_i^*(\lambda-\lambda G(\mathbf{z}))$$

which is (3.15). Comparing (3.21), (3.22) with the corresponding relations in the original model, one realizes that the differences between the two models lie in fact in the formulae giving $Q_i'(\mathbf{z},y,x)$, $D_i'(\mathbf{z},0)$, where the term $\hat{b}_\cdot(y+\cdot)$ does not allow a simple integration with respect to y. It is now clear that if we assume exponential "stay periods" then this term becomes a constant and, as it is expected, the two models become completely equivalent.

Exponential stay periods: Let us assume now that $b_i(t) = a_i e^{-a_i t}$ i.e. we assume exponential "stay periods". In this case from (3.9) and (3.10) we obtain

$$\overline{P}_i(\mathbf{z}) \equiv \int_0^\infty P_i(\mathbf{z},x)dx = P_i(\mathbf{z},0)\frac{1-u_i^*(\lambda-\lambda G(\mathbf{z}))}{\lambda-\lambda G(\mathbf{z})} \tag{3.23}$$

$$\overline{D}_i(\mathbf{z}) \equiv \int_0^\infty D_i(\mathbf{z},x)dx = D_i(\mathbf{z},0)\frac{1-v_i^*(\lambda-\lambda G(\mathbf{z}))}{\lambda-\lambda G(\mathbf{z})}, \tag{3.24}$$

while from the boundary conditions (3.14), (3.16) we get

$$\mu_i\frac{\partial\overline{Q}_i(\mathbf{z})}{\partial z_i} + \lambda\frac{G(\mathbf{z})-G(\mathbf{z}_i^*)}{z_i}\overline{Q}_i(\mathbf{z}) = P_i(\mathbf{z},0), \qquad i=1,2,...,n \tag{3.25}$$

$$D_i(\mathbf{z},0) = a_{i-1}\overline{Q}_{i-1}(\mathbf{z}). \tag{3.26}$$

Substituting finally from (3.26) to (3.20) we arrive, for all $i=1,2,...,n$, at

$$\begin{aligned}\mu_i(z_i - u_i^*(\lambda-\lambda G(\mathbf{z})))\tfrac{\partial\overline{Q}_i(\mathbf{z})}{\partial z_i} + [\lambda-\lambda G(\mathbf{z}_i^*) - \lambda\tfrac{G(\mathbf{z})-G(\mathbf{z}_i^*)}{z_i}u_i^*(\lambda-\lambda G(\mathbf{z})) + a_i]\overline{Q}_i(\mathbf{z})\\ = a_{i-1}v_i^*(\lambda-\lambda G(\mathbf{z}))\overline{Q}_{i-1}(\mathbf{z}).\end{aligned} \tag{3.27}$$

Note that, as one can see from (3.21), (3.22), relations (3.23)-(3.27) are also satisfied by $\bar{P}_i'(\mathbf{z})$, $\bar{Q}_i'(\mathbf{z})$, $\bar{D}_i'(\mathbf{z})$, where $\bar{P}_i'(\mathbf{z}) \equiv \int_0^\infty\int_0^\infty P_i'(\mathbf{z},y,x)dydx$.

Note also that by putting $v_i^*(\cdot) = 1$ in (3.23)-(3.27) we obtain immediately the corresponding formulae for the model with zero switchover times.

4. Mean Queue Lengths in steady state

We will try here to derive expressions for the expected number of retrial customers in each station for the case of exponential "stay periods".

Before starting our analysis we will obtain a necessary condition for the stability (boundedness in probability of the total amount of work in the system at any time t) of our model. Let us define

$$\rho_i = \lambda g_i \bar{u}_i, \qquad i=1,2,...,n\ , \qquad \rho = \sum_{i=1}^n \rho_i,$$

then

Theorem 4.1 A necessary condition for stability is

$$\rho < 1.$$

Proof: The proof follows the steps of that in Theorem 3.1 in Altman & Levy [3]. Suppose that our system is stable and $\rho \geq 1$. Consider, as an alternative system (System 0), the ordinary exhaustive service polling model without switchover periods. System 0 is a system with greedy (the server never idles at a nonempty queue) and exhaustive service policy and one can easily understand (see Liu & Nain [20]) that at every moment t the amount of unfinished work at this system is less than or equal to that of our original retrial model. Moreover it is known (see for example Altman et al [1]) that a necessary (and sufficient) condition for the stability of System 0 is $\rho < 1$. Thus, for $\rho \geq 1$, the amount of work in System 0 converges in distribution to infinity and this is true consequently for our original retrial model too. It contradicts of course to the hypothesis that the retrial model is stable, and this proves the theorem.

From here on and all through the following sections we will assume $\rho< 1$. We continue the analysis proving the following lemma.

Lemma 4.1 The generating functions $\bar{P}_i(\mathbf{z}), \bar{Q}_i(\mathbf{z}), \bar{D}_i(\mathbf{z})$ at the point $\mathbf{z} = \mathbf{1}$ are given by

$$\bar{P}_i(\mathbf{1}) = \rho_i \tag{4.1}$$

$$\bar{Q}_i(\mathbf{1}) = \frac{1-\rho}{a_i \sum_{m=1}^{n}(\bar{v}_m + \frac{1}{a_m})} \tag{4.2}$$

$$\bar{D}_i(\mathbf{1}) = \frac{\bar{v}_i(1-\rho)}{\sum_{m=1}^{n}(\bar{v}_m + \frac{1}{a_m})}. \tag{4.3}$$

Proof: We will prove the lemma under more general assumptions on the system characteristics using arguments similar to that used in section 5 of Altman et al [1].

Let us consider the more general model with arbitrary stationary: total stay periods, switchover times, interarrival times with batch arrival rate λ (g_i is the mean number of P_i customers per arrival) and service times with mean $\bar{u}_i$ for the i^{th} queue. Suppose also that the workload is stationary and that the model admits a steady state.

Let finally $... < T_{-1} < T_0 \leq 0 < T_1 < T_2 < ...$ be the time epoches the server arrives at queue 1, $N(t)$ the point process that counts the number of these arrivals until time t, P^0 the Palm probability related to N, E^0 the expectation with respect to (w.r.t.) P^0, $\nu > 0$ the intensity of $N(t)$ and $C_n = T_{n+1} - T_n$ the n^{th} cycle time. We assume that the cycle times are stationary and $EC_0 < \infty$.

The slope of the workload $W_k(t)$ in queue k at time t is equal to -1 if the server is working in queue k at t and equal to 0 in all other cases. Thus using Miyazawa's rate conservation principle and observing that the mean magnitude of the jump of $W_k(t)$ is $g_k\bar{u}_k$, we realize that

$$\bar{P}_i(\mathbf{1}) = P(\xi(0) = l_i) = -EW_i^+(0) = \lambda g_i \bar{u}_i,$$

where $\xi(t)$ has been defined in page 3 and $W_i^+(\cdot)$ means the right derivativeof $W_i(\cdot)$. Thus (4.1) holds.

Define now the sequences $\{\sigma_m^i\}, \{\gamma_m^i\}, \{\delta_m^i\}$ $m = 0, 1, 2, ...$ where σ_m^i, γ_m^i is the time spent by the server at the m^{th} visit to queue i, in a busy mode and in an idle mode respectively and δ_m^i is the switchover time between $(i-1)^{th}$ and i^{th} queue (at the m^{th} visit to queue i). Following the steps of Proposition 5.2 in Altman et al [1] we arrive easily at

$$\bar{P}_i(\mathbf{1}) = P(\xi(0) = l_i) = \nu E^0 \sigma_0^i, \qquad \bar{Q}_i(\mathbf{1}) = P(\xi(0) = c_i) = \nu E^0 \gamma_0^i$$

$$\bar{D}_i(\mathbf{1}) = P(\xi(0) = v_i) = \nu E^0 \delta_0^i \tag{4.4}$$

But the cycle time C_0 can be written as $C_0 = \sum_{i=1}^{n}(\sigma_0^i + \gamma_0^i + \delta_0^i)$ and as $\nu E^0 C_0 = 1$ we obtain

$$\nu = \frac{1-\rho}{\sum_{i=1}^{n}(E^0\gamma_0^i + E^0\delta_0^i)}. \tag{4.5}$$

Using (4.4) and (4.5) we arrive, in the case of our original retrial model, at (4.2), (4.3) and the lemma has been proved.

Define now, for all $i = 1, 2, ..., n$

$$L_j^{P_i} = E(L_j \; ; \; \xi = l_i), \qquad L_j^{Q_i} = E(L_j \; ; \; \xi = c_i), \qquad L_j^{D_i} = E(L_j \; ; \; \xi = v_i),$$

where L_j represents the number of retrial customers in station j (in a steady state) and $\xi = \lim_{t\to\infty} \xi(t)$ is defined in page 3. Let also $R_{ij} = \lambda g_j(1 + a_i\bar{v}_i + \lambda\bar{u}_i\pi_i)$ with $\pi_i = (1 - G(\mathbf{1}_i^*))$. We shall prove the following

Theorem 4.2 The mean numbers of retrial customers $L_j^{Q_i}, L_j^{D_i}$, $i,j=1,2,...,n$ are given by

$$(4.6)\qquad L_i^{Q_i}=\frac{\lambda}{\mu_i}[g_i-\frac{(1-\rho)\pi_i}{a_i\sum_{m=1}^n(\bar{v}_m+\frac{1}{a_m})}],\qquad i=1,2,...,n$$

$$(4.7)\quad L_j^{Q_i}=\frac{a_j}{a_i}L_j^{Q_j}+\frac{1}{a_i}\sum_{m=j+1}^{i}(\lambda g_j\mu_m\bar{u}_m L_m^{Q_m}+\frac{a_1R_{mj}}{a_m}\bar{Q}_1(\mathbf{1})),\qquad i,j=1,2,...,n,\ i\neq j$$

$$(4.8)\qquad L_j^{D_i}=\bar{v}_i a_{i-1}L_j^{Q_{i-1}}+\frac{\lambda g_j\bar{v}_i^{(2)}}{2}a_1\bar{Q}_1(\mathbf{1}),\qquad i,j=1,2,...,n$$

where $\bar{Q}_1(\mathbf{1})$ is known from (4.2).

Proof: By putting $\mathbf{z}=\mathbf{1}$ in (3.27) we obtain $a_i\bar{Q}_i(\mathbf{1})=a_{i-1}\bar{Q}_{i-1}(\mathbf{1})$ or

$$(4.9)\qquad \bar{Q}_i(\mathbf{1})=\frac{a_1}{a_i}\bar{Q}_1(\mathbf{1}),\qquad i=1,2,...,n$$

while from (3.23), it is easy to see that

$$(4.10)\qquad \bar{P}_i(\mathbf{1})=\bar{u}_iP_i(\mathbf{1},0),\qquad i=1,2,...,n.$$

Then from (3.25) using (4.10) we obtain

$$(4.11)\qquad \bar{P}_i(\mathbf{1})=\bar{u}_i\mu_iL_i^{Q_i}+\lambda\bar{u}_i\pi_i\bar{Q}_i(\mathbf{1}).$$

Using finally (4.1),(4.2) in (4.11) we arrive at (4.6) easily.

Differentiating now relation (3.27) w.r.t. z_j $(j\neq i)$ at the point $\mathbf{z}=\mathbf{1}$ we obtain

$$(4.12)\qquad -\lambda\mu_ig_j\bar{u}_iL_i^{Q_i}+a_iL_j^{Q_i}-a_{i-1}L_j^{Q_{i-1}}=\lambda g_j[1+a_i\bar{v}_i+\lambda\bar{u}_i\pi_i]\bar{Q}_i(\mathbf{1})$$

$$i,j=1,2,...,n,\qquad i\neq j.$$

and so by putting $j-k$ instead of i in (4.12) we arrive at

$$(4.13)\qquad L_j^{Q_{j-k}}=\frac{a_{j-k-1}}{a_{j-k}}L_j^{Q_{j-k-1}}+\frac{1}{a_{j-k}}(\lambda g_j\mu_{j-k}\bar{u}_{j-k}L_{j-k}^{Q_{j-k}}+\frac{a_1R_{j-kj}}{a_{j-k}}\bar{Q}_1(\mathbf{1})).$$

Putting $k=-1$ in the above relation we obtain (4.7) for $i=j+1$. Putting $k=-2$, and using (4.7) with $i=j+1$, we obtain (4.7) for $i=j+2$. It is clear now that we can obtain, in the same way, relation (4.7) for all $i,j=1,2,...,n$, $i\neq j$.

To derive finally relation (4.8) we will use (3.26) and (3.24). Thus from (3.26)

$$\frac{\partial D_i(\mathbf{z},0)}{\partial z_j}\Big|_{\mathbf{z}=\mathbf{1}}=a_{i-1}\frac{\partial\bar{Q}_{i-1}(\mathbf{z})}{\partial z_j}\Big|_{\mathbf{z}=\mathbf{1}}=a_{i-1}L_j^{Q_{i-1}},$$

and it is easy to see that

$$\frac{\partial v_i^*(\lambda-\lambda G(\mathbf{z}))}{\partial z_j}\Big|_{\mathbf{z}=\mathbf{1}}=\lambda g_j\bar{v}_i,\qquad \frac{\partial^2 v_i^*(\lambda-\lambda G(\mathbf{z}))}{\partial z_j^2}\Big|_{\mathbf{z}=\mathbf{1}}=\lambda^2g_j^2\bar{v}_i^{(2)}+\lambda\bar{v}_ig_j^{(2)}.$$

By taking now derivatives in (3.24) we obtain (4.8) and the theorem has been proved.

To finish our analysis we have to evaluate $L_j^{P_i} = \frac{\partial \bar{P}_i(\mathbf{z})}{\partial z_j} |_{\mathbf{z}=\mathbf{1}}$. From (3.25),

$$\frac{\partial P_i(\mathbf{z},0)}{\partial z_j} |_{\mathbf{z}=\mathbf{1}} = \mu_i \frac{\partial^2 \bar{Q}_i(\mathbf{z})}{\partial z_i \partial z_j} |_{\mathbf{z}=\mathbf{1}} + \lambda w_{ij} \bar{Q}_i(\mathbf{1}) + \lambda \pi_i \frac{\partial \bar{Q}_i(\mathbf{z})}{\partial z_j} |_{\mathbf{z}=\mathbf{1}} , \tag{4.14}$$

where

$$w_{ij} = \begin{cases} g_i - \pi_i & j = i \\ g_j - \frac{\partial G(\mathbf{z}_i^*)}{\partial z_j} |_{\mathbf{z}=\mathbf{1}} & j \neq i. \end{cases} \tag{4.15}$$

By taking now derivatives in (3.23) w.r.t. z_j at $\mathbf{z} = \mathbf{1}$ and using (4.10) and (4.1) we finally arrive at

$$L_j^{P_i} = \mu_i \bar{u}_i L_{ij}^{Q_i} + \lambda \bar{u}_i \pi_i L_j^{Q_i} + \lambda \bar{u}_i w_{ij} \bar{Q}_i(\mathbf{1}) + \frac{\lambda g_j \bar{u}_i^{(2)} \rho_i}{2\bar{u}_i}, \tag{4.16}$$

where $L_{ij}^{Q_i} = \frac{\partial^2 \bar{Q}_i(\mathbf{z})}{\partial z_i \partial z_j} |_{\mathbf{z}=\mathbf{1}}$. Thus to evaluate $L_j^{P_i}$ we need the n^2 quantities $L_{ij}^{Q_i}$ for all $i, j = 1, 2, ..., n$. We will try to find them in the sequel.

Differentiating relation (3.27) twice w.r.t. z_i at $\mathbf{z} = \mathbf{1}$ we arrive at

$$[2\mu_i(1-\rho_i) + a_i] L_{ii}^{Q_i} - a_{i-1} L_{ii}^{Q_{i-1}} = H_1(i), \qquad i = 1, 2, ..., n, \tag{4.17}$$

with

$$\begin{aligned} H_1(i) = {} & [2\lambda w_{ii} + 2\lambda^2 \pi_i g_i \bar{u}_i + \mu_i(\lambda^2 g_i^2 \bar{u}_i^{(2)} + \lambda \bar{u}_i g_i^{(2)})] L_i^{Q_i} + 2a_{i-1} \lambda g_i \bar{v}_i L_i^{Q_{i-1}} + [\lambda(g_i^{(2)} \\ & - 2w_{ii}) + 2\lambda w_{ii} \rho_i + \lambda \pi_i (\lambda^2 g_i^2 \bar{u}_i^{(2)} + \lambda \bar{u}_i g_i^{(2)}) + a_i(\lambda^2 g_i^2 \bar{v}_i^{(2)} + \lambda \bar{v}_i g_i^{(2)})] \bar{Q}_i(\mathbf{1}) \end{aligned} \tag{4.18}$$

while by differentiating (3.27) twice w.r.t. z_j $(j \neq i)$ at $\mathbf{z} = \mathbf{1}$ we obtain

$$-2\mu_i \lambda g_j \bar{u}_i L_{ij}^{Q_i} + a_i L_{jj}^{Q_i} - a_{i-1} L_{jj}^{Q_{i-1}} = H_2(i,j), \qquad \begin{matrix} i, j = 1, 2, ..., n \\ i \neq j \end{matrix} , \tag{4.19}$$

with

$$\begin{aligned} H_2(i,j) = {} & \mu_i(\lambda^2 g_j^2 \bar{u}_i^{(2)} + \lambda \bar{u}_i g_j^{(2)}) L_i^{Q_i} + 2\lambda g_j (1 + \lambda \pi_i \bar{u}_i) L_j^{Q_i} + 2\lambda a_{i-1} g_j \bar{v}_i L_j^{Q_{i-1}} \\ & + \lambda [g_j^{(2)}(1 + \lambda \pi_i \bar{u}_i + a_i \bar{v}_i) + 2\lambda w_{ij} g_j \bar{u}_i + \lambda g_j^2 (\lambda \pi_i \bar{u}_i^{(2)} + a_i \bar{v}_i^{(2)})] \bar{Q}_i(\mathbf{1}). \end{aligned} \tag{4.20}$$

If now we put $j - k$ instead of i in (4.19) we obtain

$$a_{j-k} L_{jj}^{Q_{j-k}} = 2\lambda \mu_{j-k} g_j \bar{u}_{j-k} L_{j-kj}^{Q_{j-k}} + a_{j-k-1} L_{jj}^{j-k-1} + H_2(j-k, j), \tag{4.21}$$

$$k = 1, 2, ..., j-1, j-n, j-n+1, ..., -1.$$

Summing the above equations for all k and adding the result in (4.17) we arrive, for all $i = 1, 2, ..., n$, at

$$2\mu_i L_{ii}^{Q_i} - 2\lambda g_i \sum_{m=1}^{n} \mu_m \bar{u}_m L_{mi}^{Q_m} = H_1(i) + \sum_{\substack{m=1 \\ m \neq i}}^{n} H_2(m, i), \tag{4.22}$$

which are the first n equations in the n^2 unknowns $L_{ij}^{Q_i}$. To derive the remaining $(n-1)n$ equations we will use again relation (3.27). Differentiating twice (3.27) w.r.t. z_i and z_j $(i \neq j)$ we get

$$[\mu_i(1-\rho_i) + a_i] L_{ij}^{Q_i} - \mu_i \lambda g_j \bar{u}_i L_{ii}^{Q_i} - a_{i-1} L_{ij}^{Q_{i-1}} = H_3(i,j), \qquad \begin{matrix} i, j = 1, 2, ..., n \\ i \neq j \end{matrix} \tag{4.23}$$

with

$$(4.24)\quad \begin{aligned} H_3(i,j) = \; & [\mu_i(\lambda^2 g_i g_j \bar{u}_i^{(2)} + \lambda \bar{u}_i g_{ij}) + \lambda g_j(1+\lambda\pi_i\bar{u}_i)]L_i^{Q_i} + \lambda(w_{ii}+\pi_i\rho_i)L_j^{Q_i} \\ & + a_{i-1}\lambda\bar{v}_i(g_i L_j^{Q_{i-1}} + g_j L_i^{Q_{i-1}}) + \lambda[g_{ij}(1+\lambda\pi_i\bar{u}_i + a_i\bar{v}_i) - w_{ij}(1-\rho_i) \\ & + \lambda g_i g_j(\lambda\pi_i\bar{u}_i^{(2)} + a_i\bar{v}_i^{(2)}) + w_{ii}\lambda g_j\bar{u}_i]\bar{Q}_i(\mathbf{1}), \end{aligned}$$

while differentiating (3.27) twice w.r.t. z_j and z_k ($j \neq k \neq i$) we obtain

$$(4.25)\quad -\mu_i\lambda g_j\bar{u}_i L_{ik}^{Q_i} - \mu_i\lambda g_k\bar{u}_i L_{ij}^{Q_i} + a_i L_{jk}^{Q_i} - a_{i-1}L_{jk}^{Q_{i-1}} = H_4(i,j,k), \quad \begin{matrix} i,j,k=1,\dots,n \\ i\neq j\neq k \end{matrix}.$$

with

$$(4.26)\quad \begin{aligned} H_4(i,j,k) = \; & \lambda\mu_i(\lambda g_k g_j\bar{u}_i^{(2)} + \bar{u}_i g_{jk})L_i^{Q_i} + \lambda(1+\lambda\pi_i\bar{u}_i)(g_j L_k^{Q_i} + g_k L_j^{Q_i}) \\ & + \lambda a_{i-1}\bar{v}_i(g_j L_k^{Q_{i-1}} + g_k L_j^{Q_{i-1}}) + \lambda[g_{jk}(1+\lambda\pi_i\bar{u}_i + a_i\bar{v}_i) \\ & + \lambda\bar{u}_i(w_{ij}g_k + w_{ik}g_j) + \lambda g_k g_j(\lambda\pi_i\bar{u}_i^{(2)} + a_i\bar{v}_i^{(2)})]\bar{Q}_i(\mathbf{1}). \end{aligned}$$

From (4.25), by putting $k-r$ instead of i, we obtain

$$a_{k-r}L_{kj}^{Q_{k-r}} = \lambda\mu_{k-r}\bar{u}_{k-r}(g_j L_{k-rk}^{Q_{k-r}} + g_k L_{k-rj}^{Q_{k-r}}) + a_{k-r-1}L_{kj}^{Q_{k-r-1}} + H_4(k-r,j,k),$$
$$r = 1,2,\dots,k-j-1.$$

Summing the above relations for all r and adding the result in (4.23) we arrive at

$$(4.27)\quad [\mu_i(1-\rho_i)+a_i]L_{ij}^{Q_i} - \mu_i\lambda g_j\bar{u}_i L_{ii}^{Q_i} - a_j L_{ji}^{Q_j} - \lambda\sum_{m=j+1}^{i-1}\mu_m\bar{u}_m(g_i L_{mj}^{Q_m} + g_j L_{mi}^{Q_m})$$
$$= H_3(i,j) + \sum_{m=j+1}^{i-1} H_4(m,j,i), \qquad \begin{matrix} i,j=1,2,\dots,n \\ i\neq j \end{matrix}.$$

with $\sum_{m=i}^{i-1} A_m \equiv 0$ for any quantity A_m. Equations (4.22) and (4.27) constitute a system of n^2 equations from which the n^2 unknowns $L_{ij}^{Q_i}$ $i,j=1,2,\dots,n$ can be found.

From here on we will try to make the computations, required to derive L_j^P, simpler, by reducing the number of equations in (4.22) and (4.27). Let us define

$$\hat{N}_i = \sum_{m=1}^{n}\mu_m\bar{u}_m L_{mi}^{Q_m}, \qquad A_{ij} = \mu_j(a_i + \mu_i(1-\rho_i)) + a_j\mu_i$$

$$(4.28)\quad K_{ij} = H_3(i,j) + H_3(j,i) + \sum_{m=j+1}^{i-1} H_4(m,j,i) + \sum_{m=i+1}^{j-1} H_4(m,i,j)$$

$$F_{ij} = \frac{\lambda g_j\bar{u}_i}{2}\,(H_1(i) + \sum_{\substack{m=1\\ m\neq i}}^{n} H_2(m,i)) + H_3(i,j) + \sum_{m=j+1}^{i-1} H_4(m,j,i)$$

$$M_{ij} = \mu_i\bar{u}_i(\mu_j F_{ij} + a_j K_{ij}), \qquad Y_{ij} = \mu_j\bar{u}_j[(\mu_i(1-\rho_i)+a_i)K_{ij} - \mu_i F_{ij}],$$

and consider the two sets of indices $\mathcal{A}_n = \{(k,i) : k = 1,2,\dots,in(\frac{n-1}{2}),\ i = 1,2,\dots,n\}$, $\mathcal{B}_n = \{(k,i) : k = \frac{n}{2},\ i = 1,2,\dots,\frac{n}{2},\ n \text{ even}\}$ where $in(m)$ means the integer part of m. Consider also for all $(k,i) \in \mathcal{A}_n \cup \mathcal{B}_n$ the recurrent relations

$$h_{i-r}^{(i,i-k)} = \frac{1}{A_{ii-k}}[c_{ii-k}^r + \mu_i\mu_{i-k}(\lambda\bar{u}_i g_{i-k}\sum_{m=i-k+1}^{i-r} h_{i-r}^{(m,i)} + \rho_i\sum_{m=i-r}^{i-1} h_{i-r}^{(m,i-k)})]$$
$$h_{i-r}^{(i-k,i)} = \frac{1}{A_{ii-k}}[d_{ii-k}^r - \mu_i\mu_{i-k}(\rho_{i-k}\sum_{m=i-k+1}^{i-r} h_{i-r}^{(m,i)} + \lambda\bar{u}_{i-k}g_i\sum_{m=i-r}^{i-1} h_{i-r}^{(m,i-k)})]$$

(4.29) $\qquad r = 0,1,2,...,k$

$$h_0^{(i,i-k)} = \frac{1}{A_{ii-k}}[M_{ii-k} + \lambda\mu_{i-k}\mu_i\bar{u}_i \sum_{m=i-k+1}^{i-1} (g_{i-k}h_0^{(m,i)} + g_i h_0^{(m,i-k)})]$$
$$h_0^{(i-k,i)} = \frac{1}{A_{ii-k}}[Y_{ii-k} - \lambda\mu_i\mu_{i-k}\bar{u}_{i-k} \sum_{m=i-k+1}^{i-1} (g_{i-k}h_0^{(m,i)} + g_i h_0^{(m,i-k)})],$$

where $h_i^{(i,i)} \equiv 0$, $\sum_{m=i}^{i-1} A_m \equiv 0$ for all i and for any quantity A_m, and

$$c_{ij}^r = \begin{cases} \lambda\mu_i\bar{u}_i g_j(a_j+\mu_j\rho_i) & r=0 \\ \mu_i\rho_i a_j & r=k \\ 0 & otherwise \end{cases}, \quad d_{ij}^r = \begin{cases} \mu_j\rho_j(\mu_i+a_i-2\rho_i\mu_i) & r=0 \\ \lambda g_i\mu_j\bar{u}_j(\mu_i(1-\rho_i)+a_i) & r=k \\ 0 & otherwise. \end{cases}$$

Note here that the indices are moving in cyclic order, i.e. if for example $n = 10$, $i = 1$, $r = 3$, $k = 5$ then $\mu_{i-r} \equiv \mu_{-2} \equiv \mu_{n-2} \equiv \mu_8$, $\mu_{i-k} \equiv \mu_6$, $\bar{u}_{i-k} \equiv \bar{u}_6$, $h_{i-r}^{(i,i-k)} \equiv h_8^{(1,6)}$, $\sum_{m=i-k+1}^{i-r} h_{i-r}^{(m,i)} \equiv \sum_{m=-3}^{-2} h_8^{(m,1)} \equiv h_8^{(7,1)} + h_8^{(8,1)}$ and so on.

Remark: For $k = 1$, one can easily see, that all sums in (4.29) are empty and so the quantities $h^{(\cdot,\cdot)}$ are, for $k = 1$ and all i, r, completely known. Using these quantities in (4.29) again, we get immediately the $h^{(\cdot,\cdot)}$'s for $k = 2$ and all i, r (each sum has now at most one, completely known, term). Continuing in the same way we obtain recursively the quantities $h^{(\cdot,\cdot)}$ for all k, i, r. Let us denote now the sums as

$$\hat{e}_1^{(k)}(i,r) \equiv \sum_{m=i-k+1}^{i-r} h_{i-r}^{(m,i)}, \qquad \hat{e}_2^{(k)}(i,r) \equiv \sum_{m=i-r}^{i-1} h_{i-r}^{(m,i-k)}. \tag{4.30}$$

Then the sums, appeared in (4.30), have together no more than k ($\leq in(\frac{n}{2})$) terms and, it is easy to see that, for all i, r, k

$$\hat{e}_1^{(k)}(i,r) = \hat{e}_1^{(k-1)}(i,r) + h_{i-r}^{[i-(k-1),\ i]}, \qquad \hat{e}_1^{(k)}(i,k) = 0$$

$$\hat{e}_2^{(k)}(i,r) = \hat{e}_2^{(k-1)}(i-1,r-1) + h_{(i-1)-(r-1)}^{[(i-1),\ (i-1)-(k-1)]}, \qquad \hat{e}_2^{(k)}(i,0) = 0.$$

Thus when we pass from $k-1$ to k in (4.29) we need in fact to perform only one addition (of two completely known, from the case $k-1$, terms) to construct the new sum $\hat{e}^{(k)}(i,r)$. Consequently, to calculate each $h^{(\cdot,\cdot)}$ in (4.29) we need to perform at most seven multiplication/divisions and four addition/subtractions and so the number of arithmetic operations required to obtain all $h^{(\cdot,\cdot)}$ is $O(n^3)$ or less. Similar observations hold for $h_0^{(\cdot,\cdot)}$ where now the number of arithmetic operations is $O(n^2)$ or less.

Define finally $L_j^{\mathbf{P}} \equiv \sum_{i=1}^{n} L_j^{P_i}$ and

$$e_i = \begin{cases} in(\frac{n}{2}) & i = 1,2,...,in(\frac{n}{2}) \\ in(\frac{n-1}{2}) & i = in(\frac{n}{2})+1,...,n \end{cases}, \qquad \bar{e}_i = \begin{cases} in(\frac{n-1}{2}) & i = 1,2,...,in(\frac{n}{2}) \\ in(\frac{n}{2}) & i = in(\frac{n}{2})+1,...,n, \end{cases}$$

then

Theorem 4.3 For our retrial polling model, $L_j^{\mathbf{P}}$ is given for all $j = 1,2,...,n$ by

$$L_j^{\mathbf{P}} = \hat{N}_j + \sum_{i=1}^{n}(\lambda\bar{u}_i\pi_i L_j^{Q_i} + \lambda\bar{u}_i w_{ij}\bar{Q}_i(\mathbf{1}) + \frac{\lambda g_j \bar{u}_i^{(2)}\rho_i}{2\bar{u}_i}) \tag{4.31}$$

where the n quantities $\hat{N}_j \quad j=1,2,...,n$ can be found as the solution of the system of linear equations

$$(4.32) \quad [(1-\rho_i)-\sum_{\substack{m=1\\m\neq i}}^{n} h_i^{(m,i)}]\hat{N}_i - \sum_{r=1}^{e_i}\hat{N}_{i-r}\sum_{k=r}^{e_i} h_{i-r}^{(i-k,i)} - \sum_{r=1}^{\bar{e}_i}\hat{N}_{i+r}\sum_{k=r}^{\bar{e}_i} h_{i+r}^{(i+k,i)} = \hat{C}_i \,,$$

with $i=1,2,...,n$ and

$$\hat{C}_i = \frac{\bar{u}_i}{2}(H_1(i)+\sum_{\substack{m=1\\m\neq i}}^{n} H_2(m,i)) + \sum_{k=1}^{e_i} h_0^{(i-k,i)} + \sum_{k=1}^{\bar{e}_i} h_0^{(i+k,i)}.$$

Proof: Relation (4.31) can be obtained from (4.16) easily. Also from (4.22)

$$(4.33) \qquad L_{ii}^{Q_i} = \frac{\lambda g_i}{\mu_i}\hat{N}_i + \frac{1}{2\mu_i}(H_1(i)+\sum_{\substack{m=1\\m\neq i}}^{n} H_2(m,i)), \qquad i=1,2,...,n$$

and substituting in (4.27) we arrive at

$$(4.34) \quad [\mu_i(1-\rho_i)+a_i]L_{ij}^{Q_i} - a_j L_{ji}^{Q_j} = \lambda g_j \rho_i \hat{N}_i + \lambda \sum_{m=j+1}^{i-1} \mu_m \bar{u}_m (g_i L_{mj}^{Q_m} + g_j L_{mi}^{Q_m}) + F_{ij},$$

$$i,j = 1,2,...,n, \qquad i\neq j,$$

while from (4.27) by interchanging i and j and adding the obtained equation to (4.27) we get

$$(4.35) \qquad \mu_i L_{ij}^{Q_i} + \mu_j L_{ji}^{Q_j} - \lambda g_i \hat{N}_j - \lambda g_j \hat{N}_i = K_{ij}, \qquad \begin{matrix} i=1,2,...,n-1 \\ j=i+1,i+2,...,n. \end{matrix}$$

Equations (4.34) and (4.35) can be written (with $j=i-1$) in a matrix form as

$$\begin{pmatrix} \mu_i(1-\rho_i)+a_i & -a_{i-1} \\ \mu_i & \mu_{i-1} \end{pmatrix} \begin{pmatrix} L_{ii-1}^{Q_i} \\ L_{i-1i}^{Q_{i-1}} \end{pmatrix} = \begin{pmatrix} \lambda g_{i-1}\rho_i \hat{N}_i + F_{ii-1} \\ \lambda g_i \hat{N}_{i-1} + \lambda g_{i-1}\hat{N}_i + K_{ii-1} \end{pmatrix}$$

and so

$$\begin{aligned} \mu_i \bar{u}_i L_{ii-1}^{Q_i} &= h_i^{(i,i-1)}\hat{N}_i + h_{i-1}^{(i,i-1)}\hat{N}_{i-1} + h_0^{(i,i-1)} \\ \mu_{i-1}\bar{u}_{i-1} L_{i-1i}^{Q_{i-1}} &= h_i^{(i-1,i)}\hat{N}_i + h_{i-1}^{(i-1,i)}\hat{N}_{i-1} + h_0^{(i-1,i)} \end{aligned} \qquad i=1,2,...,n$$

or, by putting $i+1$ instead of i in the first

$$\begin{aligned} \mu_{i+1}\bar{u}_{i+1} L_{i+1i}^{Q_{i+1}} &= h_{i+1}^{(i+1,i)}\hat{N}_{i+1} + h_i^{(i+1,i)}\hat{N}_i + h_0^{(i+1,i)} \\ \mu_{i-1}\bar{u}_{i-1} L_{i-1i}^{Q_{i-1}} &= h_i^{(i-1,i)}\hat{N}_i + h_{i-1}^{(i-1,i)}\hat{N}_{i-1} + h_0^{(i-1,i)} \end{aligned} \qquad i=1,2,...,n,$$

where the quantities $h^{(\cdot,\cdot)}$ are given by (4.29). Now it is clear that we can repeat the procedure using again (4.34) and (4.35) with $j=i-2,i-3,...$ to obtain for all $i=1,2,...,n$,

$$\begin{aligned} \mu_{i+1}\bar{u}_{i+1} L_{i+ki}^{Q_{i+k}} &= h_{i+k}^{(i+k,i)}\hat{N}_{i+k} + h_{i+k-1}^{(i+k,i)}\hat{N}_{i+k-1} + ... + h_i^{(i+k,i)}\hat{N}_i + h_0^{(i+k,i)} & k=1,2,...,\bar{e}_i \\ \mu_{i-1}\bar{u}_{i-1} L_{i-ki}^{Q_{i-k}} &= h_i^{(i-k,i)}\hat{N}_i + h_{i-1}^{(i-k,i)}\hat{N}_{i-1} + ... + h_{i-k}^{(i-k,i)}\hat{N}_{i-k} + h_0^{(i-k,i)} & k=1,2,...,e_i. \end{aligned}$$

Using finally the above relations and the definition of $\hat{N}_i$ (the first of (4.28)) we arrive at the system of linear equations (4.32) and the theorem has been proved.

5. The case of exhaustive service

Let us suppose now that $\mu_i \equiv \mu$, $a_i \equiv a$ for all $i = 1, 2, ..., n$. If we assume that $\mu \to \infty$ then the retrial characteristics of our model are swept out and the model becomes an exhaustive service polling system with "stay periods". If we assume in the sequel that $a \to \infty$ then we get the ordinary polling model with exhaustive service, switchover times and correlated batch arrivals. We will try in the present section to obtain the mean queue lengths $\tilde{L}_j^{Q_i}$, $\tilde{L}_j^{D_i}$, $\tilde{L}_j^{\mathbf{P}}$, of this polling model with exhaustive service, as limits of the corresponding quantities $L_j^{Q_i}$, $L_j^{D_i}$, $L_j^{\mathbf{P}}$ of the retrial system, when $\mu \to \infty$, $a \to \infty$.

From (4.6), (4.7) it is easy to see that

$$(5.1)\quad \begin{aligned} &\lim_{\mu\to\infty} L_i^{Q_i} = 0, \qquad \lim_{\mu\to\infty} \mu L_i^{Q_i} = \lambda g_i - \frac{\lambda(1-\rho)\pi_i}{n+a\sum_{m=1}^{n}\bar{v}_m} \\ &\lim_{\mu\to\infty} L_j^{Q_i} = \frac{\lambda g_j}{a}\sum_{m=j+1}^{i}[\rho_m + (1+a\bar{v}_m)\frac{1-\rho}{n+a\sum_{m=1}^{n}\bar{v}_m}] \\ &\beta_{ij} \equiv \lim_{a\to\infty} a(\lim_{\mu\to\infty} L_j^{Q_i}) = \begin{cases} \lambda g_j \sum_{m=j+1}^{i}[\rho_m + \bar{v}_m \frac{1-\rho}{\sum_{m=1}^{n}\bar{v}_m}] & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \end{aligned}$$

Thus, as it is expected, for all i, j

$$\tilde{L}_j^{Q_i} \equiv \lim_{a\to\infty}\lim_{\mu\to\infty} L_j^{Q_i} = 0,$$

while using (5.1) in (4.8)

$$(5.2)\quad \tilde{L}_j^{D_i} \equiv \lim_{a\to\infty}\lim_{\mu\to\infty} L_j^{D_i} = \frac{\lambda g_j \bar{v}_i^{(2)}}{2}\frac{1-\rho}{\sum_{m=1}^{n}\bar{v}_m} + \bar{v}_i\beta_{i-1j}.$$

If now we use (5.1) in (4.18), (4.20), (4.24) and (4.26), we obtain the quantities $\tilde{H}_{\cdot}(\cdot) \equiv \lim_{a\to\infty}\lim_{\mu\to\infty} H_{\cdot}(\cdot)$ as

$$(5.3)\quad \begin{aligned} \tilde{H}_4(i,j,k) = {} & \lambda^2 g_i(\lambda g_j g_k \bar{u}_i^{(2)} + \bar{u}_i g_{jk}) + \lambda\bar{v}_i(g_j\beta_{i-1k} + g_k\beta_{i-1j}) \\ & + (\lambda^2 g_j g_k \bar{v}_i^{(2)} + \lambda\bar{v}_i g_{jk})\frac{1-\rho}{\sum_{m=1}^{n}\bar{v}_m} \end{aligned}$$

$$\tilde{H}_1(i) = \tilde{H}_4(i,i,i), \qquad \tilde{H}_2(i,j) = \tilde{H}_4(i,j,j), \qquad \tilde{H}_3(i,j) = \tilde{H}_4(i,j,i),$$

and from (4.28) the corresponding quantities, in the case of the exhaustive service model, are

$$(5.4)\quad \begin{aligned} \tilde{K}_{ij} &= \sum_{m=j+1}^{i}\tilde{H}_4(m,j,i) + \sum_{m=i+1}^{j}\tilde{H}_4(m,i,j) \\ \tilde{F}_{ij} &= \frac{\lambda g_j \bar{u}_i}{2}\sum_{m=1}^{n}\tilde{H}_2(m,i) + \sum_{m=j+1}^{i}\tilde{H}_4(m,j,i). \end{aligned}$$

By defining finally $\tilde{h}_{\cdot}^{(\cdot,\cdot)} \equiv \lim_{a\to\infty}\lim_{\mu\to\infty} h_{\cdot}^{(\cdot,\cdot)}$ and taking limits in (4.29) we obtain, for all $(k,i) \in \mathcal{A}_n \cup \mathcal{B}_n$

$$\begin{aligned}\tilde{h}_{i-r}^{(i,i-k)} &= \tilde{c}_{ii-k}^{r} + \tfrac{1}{1-\rho_i}(\lambda\bar{u}_i g_{i-k}\sum_{m=i-k+1}^{i-r}\tilde{h}_{i-r}^{(m,i)} + \rho_i\sum_{m=i-r}^{i-1}\tilde{h}_{i-r}^{(m,i-k)})\\ \tilde{h}_{i-r}^{(i-k,i)} &= \tilde{d}_{ii-k}^{r} - \tfrac{1}{1-\rho_i}(\rho_{i-k}\sum_{m=i-k+1}^{i-r}\tilde{h}_{i-r}^{(m,i)} + \lambda g_i\bar{u}_{i-k}\sum_{m=i-r}^{i-1}\tilde{h}_{i-r}^{(m,i-k)})\end{aligned}$$

(5.5) $$r = 0,1,2,...,k$$

$$\begin{aligned}\tilde{h}_{0}^{(i,i-k)} &= \tfrac{1}{1-\rho_i}[\bar{u}_i\tilde{F}_{ii-k} + \lambda\bar{u}_i\sum_{m=i-k+1}^{i-1}(g_{i-k}\tilde{h}_0^{(m,i)} + g_i\tilde{h}_0^{(m,i-k)})]\\ \tilde{h}_{0}^{(i-k,i)} &= \bar{u}_{i-k}\tilde{K}_{ii-k} - \tfrac{1}{1-\rho_i}[\bar{u}_{i-k}\tilde{F}_{ii-k} + \lambda\bar{u}_{i-k}\sum_{m=i-k+1}^{i-1}(g_{i-k}\tilde{h}_0^{(m,i)} + g_i\tilde{h}_0^{(m,i-k)})],\end{aligned}$$

where $\tilde{h}_i^{(i,i)} \equiv 0$, $\sum_{m=i}^{i-1} A_m \equiv 0$ for all i and for any quantity A_m, and

$$\tilde{c}_{ij}^{r} = \begin{cases}\frac{\lambda u_i g_j \rho_i}{1-\rho_i} & r=0\\ 0 & otherwise\end{cases}, \quad \tilde{d}_{ij}^{r} = \begin{cases}\frac{\rho_j(1-2\rho_i)}{1-\rho_i} & r=0\\ \lambda g_i\bar{u}_j & r=k\\ 0 & otherwise.\end{cases}$$

Note that the observation concerning the cyclic movement of the indices in (4.29) and the Remark following it hold for (5.5) too. Now we are ready to state the theorem

Theorem 5.1 For the exhaustive service polling model, $\tilde{L}_j^{\mathbf{P}}$ is given, for all $j = 1,2,...,n$, by

(5.6) $$\tilde{L}_j^{\mathbf{P}} = \tilde{N}_j + \lambda g_j\sum_{i=1}^{n}\frac{\bar{u}_i^{(2)}\rho_i}{2\bar{u}_i}$$

where the n quantities $\tilde{N}_j$ $j = 1,2,...,n$ can be found as the solution of the system of linear equations

(5.7) $$[(1-\rho_i) - \sum_{\substack{m=1\\m\neq i}}^{n}\tilde{h}_i^{(m,i)}]\tilde{N}_i - \sum_{r=1}^{e_i}\tilde{N}_{i-r}\sum_{k=r}^{e_i}\tilde{h}_{i-r}^{(i-k,i)} - \sum_{r=1}^{\bar{e}_i}\tilde{N}_{i+r}\sum_{k=r}^{\bar{e}_i}\tilde{h}_{i+r}^{(i+k,i)} = \tilde{C}_i\ ,$$

with $i = 1,2,...,n$ and

$$\tilde{C}_i = \frac{\bar{u}_i}{2}\sum_{m=1}^{n}\tilde{H}_2(m,i) + \sum_{k=1}^{e_i}\tilde{h}_0^{(i-k,i)} + \sum_{k=1}^{\bar{e}_i}\tilde{h}_0^{(i+k,i)}.$$

Relations (5.2) and (5.6) allow us to calculate the mean queue length in each station

$$\tilde{L}'_j = \tilde{L}_j^{\mathbf{P}} + \sum_{i=1}^{n}\tilde{L}_j^{D_i}, \qquad j = 1,2,...,n$$

for the **exhaustive service polling model with switchover times and correlated batch arrivals**.

Note that, for this model with correlated batch arrivals, numerical calculations have shown that our results for the mean waiting time (excluding service)

$$E(\tilde{W}_j) = \frac{\tilde{L}'_j}{\lambda g_j}$$

coinside with the results obtained from Levy & Sidi [19] formulae (n^3 linear equations), while if we assume that $G(\mathbf{z}) = \sum_{i=1}^{n} \lambda_i z_i / \sum_{i=1}^{n} \lambda_i$, then we get results for the model with single independent arrivals (λ_i the arrival rate in i^{th} station), i.e. the model of Ferguson & Aminetzah [12] (n^2 linear equations), Sarkar & Zangwill [23] (n linear equations), etc.

Remark: From the observation made in the end of Section 3 and the analysis in Section 4 it is easy to understand that to obtain the corresponding results for the **retrial polling model with zero switchover times and correlated batch arrivals** we have only to put $\bar{v}_i = \bar{v}_i^{(2)} = 0$, in Lemma 4.1, Theorem 4.2 and in (4.18), (4.20), (4.24), (4.26) of Section 4.

In a similar way by putting $\bar{v}_i = \bar{v}_i^{(2)} = 0$ in Section 4 and assuming in the sequel $\mu \to \infty$, $a \to \infty$ we obtain results for the **exhaustive service polling model with zero switchover times and correlated batch arrivals.** In this case it is easy to see that the mean queue length in station j is given by

$$\hat{L}'_j = \hat{L}_j^{\mathbf{P}} = \hat{N}'_j + \sum_{i=1}^{n} \left(\frac{\lambda \bar{u}_i w_{ij}(1-\rho)}{n} + \frac{\lambda g_j \bar{u}_i^{(2)} \rho_i}{2\bar{u}_i} \right)$$

where the quantities $\hat{N}'_j$ satisfy again system (5.7), $\hat{h}^{(\cdot,\cdot)}$ satisfy (5.5), $\hat{K}_{ij}, \hat{F}_{ij}$ are given by (5.4) and

$$\hat{H}_4(i,j,k) = \lambda^2 g_i (\lambda g_j g_k \bar{u}_i^{(2)} + \bar{u}_i g_{jk}) + (\lambda g_{jk} + \lambda w_{ik} T_{ij} + \lambda w_{ij} T_{ik}) \tfrac{1-\rho}{n}$$

$$\hat{H}_1(i) = \hat{H}_4(i,i,i), \qquad \hat{H}_2(i,j) = \hat{H}_4(i,j,j), \qquad \hat{H}_3(i,j) = \hat{H}_4(i,j,i),$$

with $T_{ij} = \lambda g_j \bar{u}_i - \delta_{ij}$ (δ_{ij} is Kronecker's delta). If finally we assume, in the above model, single independent arrivals i.e. if we put above $w_{ij} = g_{ij} = 0$, $\lambda g_i = \lambda_i$, then numerical calculations have shown that our results for the mean waiting time (excluding service), in this case of zero switchover times, coinside with the results obtained from Takagi[25].

Note that, in all models studied in the present work, to find the mean queue lengths one has to calculate first the quantities $h^{(\cdot,\cdot)}$ ($O(n^3)$ arithmetic operations), and in the sequel to solve a set of n linear equations ($O(n^3)$ operations again). Thus the total complexity to obtain the mean queue lenghts is finally $O(n^3)$.

6. Numerical results

The mean number of retrial customers in the j^{th} station of the original retrial polling model is given by

$$\hat{L}_j = L_j^{\mathbf{P}} + \sum_{i=1}^{n} (L_j^{Q_i} + L_j^{D_i}), \qquad j = 1, 2, ..., n, \tag{6.1}$$

and from (4.6), (4.7), (4.8) and (4.31) it is completely known. To observe the way in which this mean value $\hat{L}_j$ is affected when we vary the values of the parameters, we give here Tables 1 and 2. To construct the tables we assumed that $n = 5$, i.e. that we have a polling model with five stations, and that the service times and the switchover times follow exponential distributions,

$$u_i(x) = \frac{1}{\bar{u}_i} e^{-\frac{1}{\bar{u}_i} x}, \qquad v_i(x) = \frac{1}{\bar{v}_i} e^{-\frac{1}{\bar{v}_i} x}, \qquad i = 1, 2, ..., n.$$

We assume further that, if Y denotes the batch size and X_{mi} $i=1,2,...,5$ the number of type i customers in a batch of size m, then

$$\Pr(Y=m)=\frac{1}{2^m},\qquad \Pr(X_{m1}=k_1,...,X_{m5}=k_5)=\frac{m!}{k_1!...k_5!}p_1^{k_1}...p_5^{k_5},\qquad m=1,2,...$$

with $k_1+k_2+...+k_5=m$ and $p_1+p_2+...+p_5=1$.

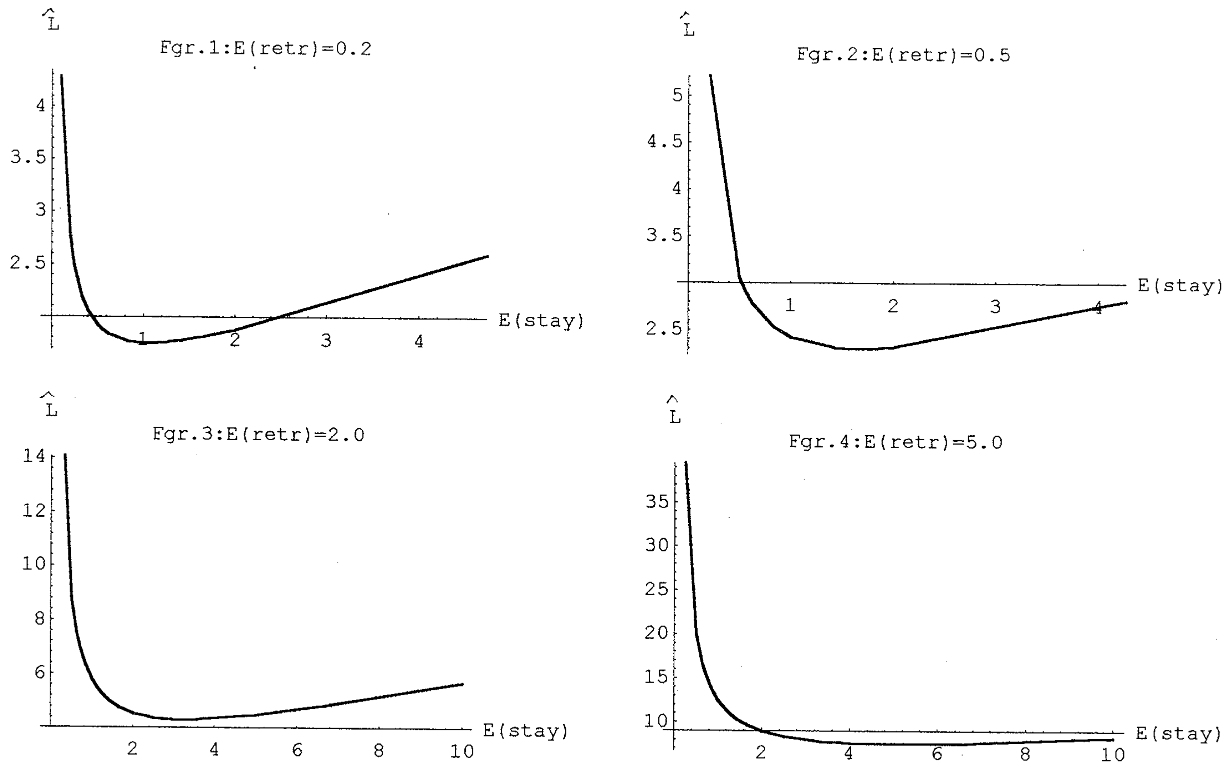
$\hat{\mu}_i \backslash \hat{a}_i$		**0.1**	**0.2**	**0.5**	**1**	**2**	**5**	**20**
0.2	$\lambda=0.1$	1.47	0.95	0.66	0.60	0.64	0.90	2.45
	$\lambda=0.2$	4.29	2.77	1.93	1.75	1.88	2.67	7.21
	$\lambda=0.4$	110.49	71.27	49.64	45.59	49.54	71.07	189.81
0.5	$\lambda=0.1$	3.12	1.81	1.05	0.82	0.78	1.00	2.53
	$\lambda=0.2$	9.11	5.29	3.06	2.42	2.32	2.98	7.44
	$\lambda=0.4$	236.45	137.23	79.60	63.55	61.50	79.43	196.37
1	$\lambda=0.1$	5.87	3.24	1.69	1.20	1.03	1.17	2.65
	$\lambda=0.2$	17.16	9.49	4.95	3.54	3.05	3.48	7.83
	$\lambda=0.4$	446.38	247.16	129.53	93.49	81.43	93.36	207.30
2	$\lambda=0.1$	11.36	6.10	2.97	1.96	1.52	1.51	2.91
	$\lambda=0.2$	33.25	17.88	8.73	5.78	4.53	4.49	8.61
	$\lambda=0.4$	866.25	467.03	229.40	153.35	121.30	121.23	229.17
10	$\lambda=0.1$	55.30	28.99	13.23	8.01	5.47	4.19	4.96
	$\lambda=0.2$	161.94	85.04	38.96	23.71	16.30	12.57	14.85
	$\lambda=0.4$	4225.18	2225.96	1028.33	632.29	440.23	344.17	404.10

Table 1: Values of $\hat{L}_i$ for $\bar{u}_i=1.2$, $\bar{v}_i=2.0$ and $p_i=0.2$ $i=1,2,...,5$.

Table 1 gives values of $\hat{L}_i$ ($\equiv\hat{L}$, $i=1,2,...5$) for the symmetric system (all parameters are statistically identical for all stations). We have used the symmetric system so as to have a clearer picture of the way in which our models affected from changes in the mean stay period $\hat{a}_i=1/a_i$, and in the mean retrial time $\hat{\mu}_i=1/\mu_i$, particularly when the arrival rate λ increases. In this table one can observe the large increase of $\hat{L}$ when we pass from $\lambda=0.1$ to $\lambda=0.2$ and particularly to $\lambda=0.4$, an increase which is more apparent for large values of $\hat{\mu}_i$. Thus, for $\hat{a}_i=0.2$ for example, $\hat{L}$ increases from 0.95 to 71.27 when we pass from $\lambda=0.1$ to $\lambda=0.4$ in the case of $\hat{\mu}_i=0.2$, while the corresponding values for $\hat{\mu}_i=10$ are 28.99 and 2225.96. This means that in such kind of models, we should be careful with the permitted input. Sometimes, even small changes in the arrival rate, could increase dramatically the number of the retrial customers in the system.

An interesting phenomenon here is the behavior of $\hat{L}$ when we increase the mean stay period $\hat{a}_i$. It seems that, at the beginning, when the mean stay period increases it helps the retrial customers to find the server idle and to start their service, which results of course to a smaller $\hat{L}$. This behavior continues until $\hat{a}_i$ becomes equal to a critical value. From this point and after any increase to $\hat{a}_i$ ceases to be usefull for the system, on the contrary it prevent the server to depart and to start serving in the next station and so $\hat{L}$ starts to increase again and becomes large for large values of $\hat{a}_i$. Thus there is an optimal value, $\hat{a}_i^*$ say, of $\hat{a}_i$, a value with which we can achieve the minimal possible mean length $\hat{L}$.

To observe in more details this optimal value of $\hat{a}_i$ and the way in which it depends on the mean retrial time $\hat{\mu}_i$ we present here Figures 1-4. In all figures we have used *Mathematica* to plot $\hat{L}$ against $\hat{a}_i$, using for all i, $\lambda=0.2$, $\bar{u}_i=1.2$, $\bar{v}_i=2.0$ and $p_i=0.2$. In Fgr.1, where $\hat{\mu}_i=0.2$, we can see that the optimal value of $\hat{a}_i$ is $\hat{a}_i^*=1.06$ and $\min\hat{L}=1.75$. In Fgr.2, with $\hat{\mu}_i = 0.5$, $\hat{a}_i^* = 1.66$ and $\min \hat{L} = 2.30$, while in Fgr. 3 where the mean retrial time increases to $\hat{\mu}_i = 2.0$, $\hat{a}_i^* = 3.2$ and $\min \hat{L} = 4.30$. In Fgr.4 finally, $\hat{\mu}_i = 5.0$ and we have $\hat{a}_i^* = 5.05$, $\min \hat{L} = 7.52$. In all figures one observes a fast reduction of $\hat{L}$, when $\hat{a}_i$ increases



Figures 1-4: Plot of $\hat{L}$ for $\lambda = 0.2$, $\bar{u}_i = 1.2$, $\bar{v}_i = 2.0$, $p_i = 0.2$, $i = 1, 2, ..., 5$.

$\bar{u}_i \backslash \bar{v}_i$		**0.5**	**1.5**	**3**	**5**	**10**	**20**
	$\hat{L}_1 =$	0.36	0.73	1.29	2.04	3.92	7.68
	$\hat{L}_2 =$	0.71	1.45	2.57	4.07	7.82	15.31
0.2	$\hat{L}_3 =$	1.00	2.10	3.78	6.02	11.62	22.84
	$\hat{L}_4 =$	3.31	6.93	12.40	19.71	37.99	74.57
	$\hat{L}_5 =$	4.01	8.82	16.10	25.81	50.11	98.73
	$\hat{L}_1 =$	0.63	1.22	2.11	3.31	6.29	12.27
	$\hat{L}_2 =$	1.25	2.42	4.19	6.55	12.47	24.30
0.6	$\hat{L}_3 =$	1.78	3.52	6.15	9.67	18.46	36.05
	$\hat{L}_4 =$	5.70	11.39	19.96	31.39	59.99	117.21
	$\hat{L}_5 =$	7.07	14.57	25.85	40.91	78.57	153.92
	$\hat{L}_1 =$	9.54	17.40	29.20	44.93	84.26	162.92
	$\hat{L}_2 =$	18.69	34.10	57.22	88.05	165.12	319.28
1.2	$\hat{L}_3 =$	27.43	50.10	84.13	129.50	242.93	469.80
	$\hat{L}_4 =$	84.70	155.17	260.91	401.91	754.43	1459.47
	$\hat{L}_5 =$	107.86	198.06	333.41	513.89	965.12	1867.58

Table 2: Values of $\hat{L}_i$ for $\lambda = 0.4$, $\hat{\mu} = (0.2, 0.2, 0.5, 1, 2)$, $\mathbf{\hat{a}} = (0.2, 0.2, 0.5, 0.5, 1)$ and $\mathbf{\hat{p}} = (0.05, 0.1, 0.15, 0.3, 0.4)$.

from small values to this optimal value $\hat{a}_i^*$. From this point and after, $\hat{L}$ starts to increase again, but in a smoother way now. The general observation here is that the first thing that we have to do, if we operate a such kind of model, is to discover numerically this optimal $\hat{a}_i^*$, and to arrange the mean time we will allow the server to stay in each station, accordingly.

Table 2, finally, represents values of $\hat{L}_i$ $i = 1, 2, ..., 5$ for an asymmetric system now, with $\lambda = 0.4$ and $\hat{\mu} = (0.2, 0.2, 0.5, 1, 2)$, $\hat{\mathbf{a}} = (0.2, 0.2, 0.5, 0.5, 1)$, $\hat{\mathbf{p}} = (0.05, 0.1, 0.15, 0.3, 0.4)$, i.e. a system with small values of the parameters for the first two and larger for the remaining three stations.

In this table one can observe the way in which the mean retrial lengths $\hat{L}_i$ are affected from changes in the mean service times $\bar{u}_i$ and in the mean switchover times $\bar{v}_i$ (we have used the same value of $\bar{u}_i$ and of $\bar{v}_i$ for all stations). Thus, for $\bar{v}_i = 0.5$, $\hat{L}_1$ increases from 0.36 to 9.54 when we increase the mean service time from 0.2 to 1.2 while the corresponding values for the last station are $\hat{L}_5 = 4.03$ and $\hat{L}_5 = 107.86$ respectively. Similar observations hold for changes in the mean switchover times.

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Christos LANGARIS
University of Ioannina, Dept of Mathematics
45110, Ioannina, Greece