Chaotic dynamics of fractional Vallis system for El-Niño Amey S. Deshpande^{1 2} and Varsha Daftardar-Gejji^{3 4}

Abstract

Vallis proposed a simple model for El-Niño weather phenomenon (referred as Vallis system) by adding an additional parameter p to the Lorenz system. He showed that the chaotic behavior of the Vallis system is related to the El-Niño effect. In the present article we study fractional version of Vallis system in depth. We investigate bifurcations and chaos present in the fractional Vallis system along with the effect of variation of system parameter p. It is observed that the range of values of parameter p for which the Vallis system is chaotic, reduces with the reduction of the fractional order.

Further we analyze the incommensurate fractional Vallis system and find the critical value below which the system loses chaos. We also synchronize Vallis system with Bhalekar-Gejji system.

1 Introduction

El-Niño is a weather phenomenon associated with the collection of band of warm ocean water in the east-central equatorial Pacific ocean including the Pacific coast of South America. This phenomenon is a part of El-Niño Southern Oscillation (ENSO) climatic system which is known to have a great impact on the global climate [1]. Various mathematical models exist which describe El-Niño phenomenon [2]. These models range from simplistic to highly complex. One of the simple models is proposed by Vallis in 1986 [3] which consists of set of three autonomous non-linear differential equations (hereafter referred as Vallis system). Vallis system is a modification of Lorenz system [4] with addition of a system parameter p. Vallis proved the existence of chaos in Vallis system and also showed that El-Niño phenomenon is related to the chaotic behavior of the Vallis system [5].

Over the years, integer order Vallis system has been analyzed in various articles [6, 7, 8]. The chaos and periodicity of Vallis model has been investigated in depth for the integer order model over B - c parameter space [9].

Fractional order Vallis system is not explored much. Only a couple of papers have appeared recently. Alkahtani *et al.* [10] have studied Vallis model with local derivatives, Caputo derivatives and Caputo-Fabrizio derivative. They have also drawn phase portraits for commensurate fractional Vallis system with Caputo

¹Department of Mathematics, IIT Bombay, Mumbai - 400076, India.

 $^{^2} Email: \ 2009 as deshpande @gmail.com, \ a meyd @math.iitb.ac.in.$

³Department of Mathematics, Savitribai Phule Pune University, Pune - 411007, India.

⁴Email: vsgejji@gmail.com, vsgejji@unipune.ac.in.

derivative for p = 0.3 and $\alpha = 0.85, 0.55, 0.25$. Merdan [11] have calculated numerical solution for fractional order Vallis system for parameter p = 0 by using multi-step differential transformation method.

In the present article, fractional version of commensurate as well as incommensurate Vallis system has been studied in detail for all values of the parameter p. For $p \neq 0$ this analysis becomes more tedious. Our main objective is to study the effect of fractional orders on the chaotic behavior of the Vallis system.

The present article is organized as follows. Section 2 introduces preliminaries and notations used in the article. Section 3 presents stability analysis of fractional order Vallis system. Section 4 analyzes commensurate fractional Vallis system w.r.t. parameter p and fractional order α . Section 5 deals with incommensurate fractional Vallis system. Section 6 describes synchronization of Vallis system. Section 7 presents conclusions.

2 Preliminaries and Notations

This section introduces the basic definitions and notations used throughout this paper [12, 13, 14].

Definition 1 ([13]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of $f \in C^0$ is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau.$$
 (1)

Definition 2 ([13]). The Caputo derivative of order $\alpha \in (k-1,k], k \in \mathbb{N}$ of $f \in C^k$ is defined as

$$D^{\alpha}f(t) = I^{k-\alpha}f^{(k)}(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-\tau)^{k-\alpha-1} f^{(k)}(\tau) \, d\tau, \quad \alpha \in (k-1,k),$$
(2)

$$D^{\alpha}f(t) = f^{(k)}(t), \ \alpha = k.$$
 (3)

Hereafter in the article, we always assume Caputo derivative of order α , $0 < \alpha \le 1$, unless stated otherwise.

Definition 3. [14] For $0 < \alpha_i \le 1$, and $f_i \in C^1$, (i = 1, 2, 3), fractional dynamical

system along with initial conditions is given as

$$D^{\alpha_1} x(t) = f_1(x(t), y(t), z(t)),$$

$$D^{\alpha_2} y(t) = f_2(x(t), y(t), z(t)),$$

$$D^{\alpha_3} z(t) = f_3(x(t), y(t), z(t)),$$

$$(x(0), y(0), z(0)) = (x_0, y_0, z_0).$$

(4)

If $\alpha_1 = \alpha_2 = \alpha_3$, then the system (4) is called as *commensurate* system and this common fractional order is denoted as α . System is called as *incommensurate* if it is not commensurate[15]. For $\alpha_1 = \alpha_2 = \alpha_3 = 1$, the system (4) reduces to integer order dynamical system.

Denote $\Sigma = \alpha_1 + \alpha_2 + \alpha_3$.

Definition 4. The critical value (Σ_{cr}) is defined as the largest value of Σ , such that for the $\Sigma \leq \Sigma_{cr}$, the system (4) is not chaotic.

Definition 5. The point $(x^*, y^*, z^*) \in \mathbb{R}^3$ is called as equilibrium point of the system (4), if $f_i(x^*, y^*, z^*) = 0$, i = 1, 2, 3.

2.1 Stability analysis of the commensurate fractional systems

Consider the system (4), with $\alpha_1 = \alpha_2 = \alpha_3 \in (0, 1]$. Let $f = (f_1, f_2, f_3)$ and $X = (x, y, z), X^* = (x^*, y^*, z^*) \in \mathbb{R}^3$.

An equilibrium point X^* of the system (4) is called as a hyperbolic equilibrium point if $|\arg(\lambda)| \neq \frac{\pi\alpha}{2}$, for every eigenvalue λ of the matrix $J = Df(X^*)$ [16]. Assume X^* is an equilibrium point of the system (4). Let $\xi = X - X^* \in \mathbb{R}^3$, then [14]

$$D^{\alpha}\xi = D^{\alpha}(X - X^*)$$

= $D^{\alpha}X$
= $f(\xi + X^*)$
= $f(X^*) + Df(X^*)\xi + \cdots$

Thus we get,

$$D^{\alpha}\xi = J\xi, \ \alpha = (\alpha_1, \alpha_2, \alpha_3), \ \alpha_1 = \alpha_2 = \alpha_3 \in (0, 1].$$
 (5)

The eigenvalues of the matrix J determine the stability properties of the system about an equilibrium point X^* [14]. The stability criteria is due to Matignon [16] which states: If X^* is a hyperbolic equilibrium point then the trajectories of the system (5) are asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha \pi}{2}$, for every eigenvalue λ of J.

2.2 Numerical Method

Various numerical methods exist in the literature for solving nonlinear fractional differential equations. For simulations pertaining to fractional order dynamical systems, predictor-corrector type methods such as Fractional Adams Method (FAM) or New Predictor Corrector Method (NPCM) are more suitable and are extensively used in the literature.

In the present article, we have used New Predictor Corrector Method (NPCM) which is proposed by Daftardar-Gejji *et al.* [17]. This method is more time efficient, accurate and has better stability properties.

We briefly describe NPCM below. Consider the following fractional initial value problem:

$$D^{\alpha}y(t) = f(t, y(t)), \ \frac{d^{k}y(0)}{dt^{k}} = y_{0}^{k}, \ k = 0, 1, \cdots, m - 1,$$
(6)

where D^{α} denotes Caputo derivative, with order $\alpha, m-1 < \alpha < m, m \in \mathbb{N}$. Then eqn. (6) is equivalent to the integral equation

$$y(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} y_0^k + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) \, d\tau.$$
(7)

In this method we discretize integral eqn. (7) on a uniform grid $t_j = jh$, $j = 0, 1, \dots, N$, $h = \frac{T}{N}$, on [0, T]. Assume y_1, y_2, \dots, y_k , values at first k points, are known. To find y_{k+1} , we define predictors

$$y_{k+1}^{P} = \sum_{j=0}^{m-1} \frac{t_{k+1}^{j}}{j!} y_{0}^{j} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{k} a_{j,k+1} f(t_{j}, y_{j}),$$
(8)

$$z_{k+1}^{P} = \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{k+1}, y_{k+1}^{P}),$$
(9)

where $a_{j, k+1}$ are defined as

$$a_{j,k+1} = \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}, & j = 0, \\ (k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}, & 1 \le j \le k, \\ 1, & j = k+1. \end{cases}$$
(10)

Using predicted values, we get the corrected value of y_{k+1} as

$$y_{k+1} = y_{k+1}^P + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_{k+1}, y_{k+1}^P + z_{k+1}^P).$$
(11)

Equations eqn. (8) to eqn. (11), constitute the new predictor corrector method [17].

Theorem 1 (Error estimate [17]). Assume that $D^{\alpha}y$ is C^2 over interval [0,T] and $y(t_j)$ denote exact values of y at point t_j then

$$\max_{0 \le j \le N} |y(t_j) - y_j| = O(h^2), \quad \alpha > 0.$$
(12)

3 Analysis of fractional Order Vallis system

The fractional version of Vallis system is given below.

$$D^{\alpha_1}x = By - C(x+p)$$

$$D^{\alpha_2}y = xz - y$$

$$D^{\alpha_3}z = -xy - z + 1$$
(13)

where $0 < \alpha_i \leq 1$, i = 1, 2, 3. The system (13), is called as commensurate if $\alpha_1 = \alpha_2 = \alpha_3$ and incommensurate otherwise. The system reduces to fractional Lorenz system whenever p = 0, B = 1.

This model is developed by treating equatorial ocean as a box of fluid characterized by temperatures in the east and west. Variable x represents current generated by the temperature gradient, co-ordinate y represents half of the difference of east-west temperatures and z represents the average of the east and west temperatures. The parameter B governs strength of air-sea interactions and the vertical temperature difference, while parameter C represents the ratio of time scales of decay of sea-surface temperature anomalies to a frictional time scale. The parameter p measures the average effect of equatorial winds on the sea [5, 9].

The equilibrium points of the system (13) are $F_1 \equiv (x_1, y_1, z_1)$, $F_2 \equiv (x_2, y_2, z_2)$, $F_3 \equiv (x_3, y_3, z_3)$, where x_1, x_2, x_3 are roots of the polynomial

$$C\xi^{3} + Cp\xi^{2} + (C - B)\xi + Cp,$$
(14)

and y_i , z_i are given as

$$y_i = \frac{x_i}{1 + x_i^2}, \quad z_i = \frac{1}{1 + x_i^2}, \quad i = 1, 2, 3.$$
 (15)

The Jacobian matrix J and characteristic polynomial for J are given as

$$J = \begin{pmatrix} -C & B & 0\\ z & -1 & x\\ -x & -y & -1 \end{pmatrix},$$
 (16)

$$\phi(\lambda) = \lambda^3 + (2+C)\lambda^2 + (1+2C+xy-Bz)\lambda + (C+Bx^2+Cxy-Bz).$$
(17)

For the values B = 150, p = 0.35 and C = 4, the equilibrium points are $F_1 \equiv (0.0095, 0.0095, 0.9999)$, $F_2 \equiv (5.86412, 0.16571, 0.02825)$ and $F_3 \equiv (-6.22371, -0.156632, 0.025167)$. The corresponding eigenvalues of the Jacobian matrix are given in Table (1). Further the phase portrait of commensurate system (13) with fractional order $\alpha = 0.98$ is drawn in fig. (1). A positive value of the largest Lyapunov exponent (LLE) indicates a chaotic behavior. In this case,

the LLE of system (13), calculated by using Wolf algorithm [18], is found out to be LLE = 0.4181.



Figure 1: Chaotic attractor of system (13) for $\alpha = 0.98$, and p = 0.35, B = 150, C = 4.

	λ_1	λ_2	λ_3
F_1	-14.8384	-0.9999	9.83835
F_2	-19.4082	6.704 - 14.866i	6.704 + 14.866i
F_3	-20.093	7.0464 - 15.4828i	7.0464 + 15.4828i

Table 1: Eigenvalues of Jacobian matrix $\lambda_1, \lambda_2, \lambda_3$, around equilibrium points F_1, F_2, F_3 .

4 Commensurate Vallis System

4.1 Chaos and Bifurcation w.r.t. parameter p

We analyze system (13) by varying the system parameter p. Let B = 150, C = 4. We analyze stability of the system (13) by varying parameter p. The eqn. (14) has at least one real root. Using discriminant Δ of the cubic polynomial we see that eqn. (14) will have all three real roots whenever $|p| \leq 17.7754$. Thus we see that for |p| = 17.7754 system (13) undergoes a pitchfork bifurcation.

Using eqn. (17), we numerically calculate eigenvalues around each equilibrium point. From these calculations we observe that, for $|p| \leq 17.35$, equilibrium point F_1 is index 1 saddle with all three real eigenvalues while F_2 , F_3 are index 2 saddles with one real and two complex conjugate eigenvalues with positive real part. Thus for $|p| \leq 17.35$ system (13) satisfies necessary condition for the existence of chaos. Note that for fractional order $0 < \alpha \leq 1$, the values of p for which system may be chaotic is given as |p| < M where $0 < M \leq 17.35$. This is due to the stability criteria of fractional systems [16].



Figure 2: Bifurcation diagram versus p for $\alpha = 1$



Figure 3: Phase and time-series for p = -0.5 for $\alpha = 1$

The bifurcation diagram versus parameter p for fractional order $\alpha = 1$ (*c.f.* fig. (2)) shows that for $-2.45 \le p \le 2.45$, system is chaotic. The system will break into El-Niño effect, if the difference between the eastern and western sea temperatures is high. Fig. (3) shows that p = -0.5, the chances of system breaking into El-Niño are higher than for p = 0, while when |p| > 2.45, system stabilizes to one of the equilibrium points and hence does not admit El-Niño effect.

The bifurcation diagrams w.r.t. parameter p corresponding to orders $\alpha = 0.985, 0.975, 0.965$ are shown in figure (4). For $\alpha = 0.985$, the chaotic region shrinks to $-1.5 \leq p \leq 1.5$, while for $\alpha = 0.975$, region further shrinks to $-0.5 \leq p < 0$. This reduction in chaotic region reduces the chances of system breaking into El-Niño. For $\alpha = 0.965$, system (13) completely loses chaos for all values of p.

Table (2) shows the calculated values of LLE for various values of p and $\alpha = 1, 0.985, 0.975, 0.965$. A positive value of LLE indicates existence of chaos while negative or zero value confirms non-existence of chaotic behavior.



Figure 4: Bifurcation diagrams versus parameter p for fractional orders (a) $\alpha = 0.985$ (b) $\alpha = 0.975$ (c) $\alpha = 0.965$ 9

p	$\alpha = 1$	$\alpha=0.985$	$\alpha=0.975$	$\alpha=0.965$
-3.5	-0.1002	-0.0229	-0.0142	-0.00072
-2.4	0.2656	0.00018	-0.0160	-0.00010
-1.5	0.5115	-0.0694	-0.015	-0.0196
-0.5	0.4735	0.4970	-0.076	-0.06522
0	0.5069	0.49427	-0.0055	-0.9
0.35	0.52828	0.2133	-0.010	-0.9
1.5	0.5350	0.0793	-0.9	-0.0332
2.4	0.2317	0.04	-0.9	-0.0245
3.5	-0.1270	-0.032	-0.9	-0.0168

Table 2: Largest Lyapunov exponent (LLE) values of system (13) for various values of parameter p and fractional orders $\alpha = 1$, 0.985, 0.975 and $\alpha = 0.965$.

4.2 Chaos and Bifurcation w.r.t. fractional order α

Let B = 105, C = 4 and p = 0.35. We analyze eqn. (13), in particular for the case $\alpha = \alpha_1 = \alpha_2 = \alpha_3$. Figure (5) shows bifurcation of system (13) versus parameter α . Table (3) shows LLE values for some values of α .



Figure 5: Bifurcation diagram versus fractional order α

α	LLE
0.970	-0.0246
0.976	0.31576
0.985	0.47899
1.000	0.3683

Table 3: Largest Lyapunov exponents (LLE) of commensurate Vallis system for various fractional orders α .

From fig. (5) we observe that the integer order system is chaotic. As the fractional order reduces, chaotic nature of the system remains till $\alpha \approx 0.975$, below which chaos disappears. Thus the lowest order for which the commensurate system (13) shows chaos is $\alpha = 0.975$.

5 Incommensurate Vallis System

Consider incommensurate order system (13), for parameter values B = 105, C = 4and p = 0.35. We consider $\alpha_2 = \alpha_3 = 1$. The bifurcation diagram w.r.t. α_1 has been plotted in fig. (6). Table (4) shows values of LLE for some values of fractional order α_1 .



Figure 6: Bifurcation diagram versus fractional order α_1 , $\alpha_2 = \alpha_3 = 1$

α_1	LLE
0.670	-0.00386
0.720	0.2113
0.850	0.4913
1.000	0.41214

Table 4: Largest Lyapunov exponents (LLE) of incommensurate Vallis system for various values of α_1 , $\alpha_2 = \alpha_3 = 1$.

The fractional incommensurate system (13) remains chaotic until $\alpha_1 = 0.71$ and below this value the system stabilizes. Thus critical order below which the system loses chaos, is 2.71.



Figure 7: Bifurcation diagram versus fractional order α_2 , $\alpha_1 = \alpha_3 = 1$

α_2	LLE
0.930	-0.13101
0.960	0.4852
0.980	0.54821
1.000	0.412

Table 5: Largest Lyapunov exponents (LLE) of incommensurate Vallis system for various values of α_2 , $\alpha_1 = \alpha_3 = 1$.

Fig. (7) shows bifurcation diagram versus order α_2 (with $\alpha_1 = \alpha_3 = 1$). Further Table (5) shows values of LLE for some values of fractional order α_2 . The chaos in

this case, disappears below $\alpha_2 = 0.942$. Thus effective critical order for the system (13) is $\Sigma = 2.942$.



Figure 8: Bifurcation diagram versus fractional order α_3 , $\alpha_1 = \alpha_2 = 1$

α_3	LLE
0.970	-0.0246
0.976	0.31576
0.985	0.47899
1.000	0.3683

Table 6: Largest Lyapunov exponents (LLE) of incommensurate Vallis system for various values of α_3 , $\alpha_1 = \alpha_2 = 1$.

Let $\alpha_1 = \alpha_2 = 1$. The bifurcation with respect to the order α_3 is plotted in fig. (8). Further Table (6) shows the values of LLE for some values of fractional order α_3 . It is clear that the minimum order for existence of chaos is $\alpha_3 = 0.948$. Hence the critical order is $\Sigma = 2.948$.

6 Synchronization of Vallis system and Bhalekar-Gejji system

Bhalekar and Daftardar-Gejji introduced a new chaotic system [19], which is referred as BG system. The fractional version of BG system is given as follows.

$$D^{\alpha}x_{r} = \omega x_{r} - y_{r}^{2},$$

$$D^{\alpha}y_{r} = \mu(z - y),$$

$$D^{\alpha}z_{r} = Ay_{r} - bz_{r} + x_{r}y_{r},$$
(18)

where $\omega < 0, \mu > 0$ and A, b are parameters and $0 < \alpha \leq 1$. BG system was analyzed by Deshpande and Daftardar-Gejji [20] and shown to be chaotic for $\alpha_c < \alpha \leq 1$, where α_c denotes the critical value for α below which the system loses chaos. Moreover the authors have shown that $\alpha_c > \frac{2}{3}$. Figure (9) shows phase portrait of Bhalekar-Gejji system for fractional order $\alpha = 0.98$ with parameter values $\mu =$ $10, \omega = -2.667, A = 26, b = 1$ with initial condition $(x_0, y_0, z_0) = (4, 0.5, 0.1)$. The LLE of the BG system under this case is calculated and found out to be LLE = 0.8806. Thus the BG system shows chaotic behavior.



Figure 9: Chaotic behavior of BG system with $\alpha = 0.98$, $\mu = 10$, $\omega = -2.667$, A = 26, b = 1. Initial condition is $(x_0, y_0, z_0) = (4, 0.5, 0.1)$.

In this section, we synchronize fractional order commensurate Vallis system (13) as a Drive system with the fractional order BG system (18) as a Response system using active control functions $u_1(t), u_2(t), u_3(t)$.

Let $e_1 := x_r - x$, $e_2 := y_r - y$ and $e_3 := z_r - z$ be the error variables. Then (13) and (18) along with active control functions gives the error system

$$D^{\alpha}e_{1} = (\omega - C)e_{1} + Be_{2} + (Cp - y_{r}^{2} + \omega x + Cx_{r} - By_{r}) + u_{1}(t),$$

$$D^{\alpha}e_{2} = \mu e_{3} - \mu e_{2} + (\mu z - xz - (\mu - 1)y) + u_{2}(t),$$

$$D^{\alpha}e_{3} = Ae_{2} - be_{3} + (x_{r}y_{r} + xy - 1 + Ay - (b - 1)z) + u_{3}(t).$$
(19)

Let us choose u_1, u_2, u_3 as

$$u_{1}(t) = -Cp + y_{r}^{2} - \omega x - Cx_{r} + By_{r} + V_{1}(t),$$

$$u_{2}(t) = -\mu z + xz + (\mu - 1)y + V_{2}(t),$$

$$u_{3}(t) = -x_{r}y_{r} - xy + 1 - Ay + (b - 1)z + V_{3}(t),$$
(20)

where V_1, V_2, V_3 are linear functions of e_1, e_2, e_3 , whose choice is not necessarily unique. Substituting (20) into (19) we get

$$D^{\alpha}e_{1} = (\omega - C)e_{1} + Be_{2} + V_{1}(t),$$

$$D^{\alpha}e_{2} = \mu e_{3} - \mu e_{2} + V_{2}(t),$$

$$D^{\alpha}e_{3} = Ae_{2} - be_{3} + V_{3}(t).$$
(21)

The two systems (13) and (18) will be synchronized if and only if the error system (21) is asymptotically stable. We choose values of V_1, V_2, V_3 such that the eigenvalues λ_i , i = 1, 2, 3 of system (21) satisfy $|\arg(\lambda_i)| > \frac{\pi \alpha}{2}$. This will ensure asymptotic stability of the system (21).

In particular let

$$V_{1}(t) = -(\omega - C + K_{1})e_{1},$$

$$V_{2}(t) = (\mu - K_{2})e_{2},$$

$$V_{3}(t) = -Ae_{2} + (b - K_{3})e_{3},$$

(22)

where K_1, K_2, K_3 are positive real numbers called as control parameters. In view of (22), system (21) becomes

$$D^{\alpha} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{bmatrix} -K_1 & B & 0 \\ 0 & -K_2 & \mu \\ 0 & 0 & -K_3 \end{bmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$
(23)

having eigenvalues as $-K_1, -K_2, -K_3$. For positive values of K_i , the system is asymptotically stable. In particular, for $K_1 = K_2 = K_3 = 4$, system (21) is asymptotically stable and system (13) and (18) will synchronize.

We simulate the Drive system (13) for the parameter values B = 150, C = 4, p = 0.35 and initial condition (5.86, 0.165, 0.028) and for the Response system (18), we consider $\omega = -2.667, \mu = 10, A = 26, b = 1$ and initial condition (4, 0.5, 0.1). Fractional order α is taken as 0.98.

Both the systems are allowed to evolve independently till t = 75, after which the active control is switched on. Figure (10) shows timeline of x, y, z variables of Vallis system (Blue) and BG sytem (Orange and dashed). It is clear that after t = 75, both system synchronize with each other. Time evolution of corresponding error variables is shown in fig. (11).



Figure 10: Evolution of time-series of x, y, z variables of Vallis system (13) (shown by Blue continuous line) and BG system (18) (Orange, dashed line) for $\alpha = 0.98$. The control is triggered at t = 75.



Figure 11: Evolution of error e_1, e_2, e_3 for $\alpha = 0.98$. The control is triggered at t = 75.

7 Conclusions

Vallis proposed a model for El-Niño phenomenon and showed that El-Niño effect is related to the chaotic behavior of the Vallis system. Hence it is important to study the chaos in Vallis system.

In the present article, fractional version of commensurate as well as incommensurate Vallis system has been studied in detail. The effect of variation of system parameter p and the fractional order, on the chaotic behavior of the Vallis system is investigated using bifurcation diagrams and largest Lyapunov exponents. In particular it is observed that the range of the parameter p for which the system is chaotic; reduces rapidly as the fractional order is reduced. For $\alpha \leq 0.965$, the system is no longer chaotic for any value of parameter p.

The critical fractional order for which fractional Vallis system is no longer chaotic has been computed in case of B = 105, C = 4, and p = 0.35. This critical order is found to be $0.975 \times 3 = 2.925$ for the commensurate Vallis system, while the lowest critical order for the incommensurate fractional system, is found to be 2.71. Further we have synchronized fractional Vallis system with fractional Bhalekar-Gejji sytem.

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