# Symmetries, Quotients <br> and <br> Kähler-Einstein metrics 

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#### Abstract

We consider Fano manifolds $M$ that admit a collection of finite automorphism groups $G_{1}, \ldots, G_{k}$, such that the quotients $M / G_{i}$ are smooth Fano manifolds possessing a Kähler-Einstein metric. Under some numerical and smoothness assumptions on the ramification divisors, we prove that $M$ admits a Kähler-Einstein metric too.


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## 1 Introduction

The aim of this paper is to provide new examples of Kähler-Einstein metrics of positive scalar curvature. The existence of such a metric on a Fano manifold is a subtle problem, due to the presence of obstructions, that have been discovered during the years, beginning with Matsushima's theorem in 1957, Futaki invariants in 1982, Tian's theorem stating that Kähler-Einstein manifolds of positive scalar curvature are semistable (see [24, Theorem 8.1]), up to Donaldson's result
[11. Corollary 4], which shows that the existence of Kähler-Einstein metrics (even more generally of a Kähler constant scalar curvature metric) forces the algebraic underlying manifolds to be asympotically stable (see also [1]).

Existence theorems on the other hand are always very hard. The only necessary and sufficient condition, established by Tian, is of a truly analytic character. It says that a Fano manifold $M$ admits a Kähler-Einstein metric, if and only if an integral functional $F$ defined on Kähler metrics in the class $\mathrm{c}_{1}(M)$ is proper (see Theorem 2.1 below). The equivalence of properness of $F$ with the algebraic stability of the underlying manifold, in an appropriate sense, would represent the final solution of the problem, but is still unknown. (This has been suggested by Yau, and made precise by Tian, who has also proved that properness implies stability.) Work in progress by Paul and Tian [18] indicates a new stability condition as a candidate for the equivalence with the existence of a Kähler-Einstein metric.

Although by now there is a good deal of examples, the only broad class of manifolds for which the problem is solved is the one of toric Fano manifolds, thanks to a recent theorem of Xujia Wang and Xiaohua Zhu ([27], see also Donaldson's work [12] for related results for extremal metrics). Otherwise, even for manifolds that are deceptively simple from the algebro-geometric point of view, one has often no clue on how to check the properness of $F$, and finding the metric. The case of Del Pezzo surfaces is quite eloquent from this point of view, as the reader of 23 might verify. Another striking example of the difficulties on which one suddenly runs, is the hypersurface case. Indeed, it is expected that any smooth Fano hypersurface has a Kähler-Einstein metric, nonetheless the only ones for which this is known are the ones lying in a suitable small analytic neighbourhood of the Fermat's hypersurfaces (see [25, p. 85-87]). In fact a standard implicit function theorem argument shows that the Kähler-Einstein condition is open in the moduli space in the analytic topology, provided the automorphism group is finite. This remark can be applied also to some of the examples discussed below.

In trying to construct explicit examples a good help comes from having many holomorphic symmetries to work with. This has been crucial for example in estimating the so called $\alpha$-invariant for some Del Pezzo surfaces with reductive automorphisms group. This has been the heart of the work of Tian-Yau [26, Proposition 2.2].

The aim of this paper is to use in a different way the symmetries
of the manifold to prove existence of Kähler-Einstein metrics, inspired by Tian's work on Fermat hypersurfaces. In Section 20 we study the behaviour of properness of $F_{\omega}$ (see p. (2) in presence of a Galois covering and find conditions under which the existence of a Kähler-Einstein metric on the base allows one to prove a version of properness, and thus existence, on the covering space. We find algebraic conditions on the covering maps (Theorems 2.3 and (2.6) ensuring that the desired inequalities hold on the covering space. In 3 we show how this can be used to prove the existence of Kähler-Einstein metrics on some classes of Fano manifolds, chosen from the lists of Del Pezzo manifolds, and Fano threefolds with Pic $=\mathbb{Z}$ (see [13, p. 214-215]). Our examples include:
a) hypersurfaces of the form $\left\{x_{0}^{d}+\ldots+x_{k-1}^{d}+f\left(x_{k}, \ldots, x_{n+1}\right)=\right.$ $0\} \subset \mathbb{P}^{n+1}$ where $f$ is a homogeneous polynomial of degree $d$, and $k>n+2-d$;
b) $n$-dimensional intersections of hypersurfaces of the same form as above, all of the same degree $d$ and with $k>n+2-d$;
c) arbitrary intersections of two (hyper)quadrics;
d) double covers of $\mathbb{P}^{n}$ ramified along a smooth hypersurface of degree $2 d$ with $\frac{n+1}{2}<d \leq n$;
e) double covers of the $n$-dimensional quadric $Q_{n} \subset \mathbb{P}^{n+1}$ with smooth branching locus cut out by a hypersurface of degree $2 d$ with $\frac{n}{2}<d<n$.
(See section 3) Example (a) generalises Tian's theorem about Fermat's hypersurfaces. Examples (a), (b), (d) and (e) give positivedimensional algebraic families of Kähler-Einstein manifolds. This becomes even more striking in example (c) since every element in the moduli of such manifolds has a Kähler-Einstein metric. A particular case of (c) (the intersection of two specific quadrics in $\mathbb{P}^{5}$ ) had been previously studied by Alan Nadel (see [17, p. 589]).

Some interesting questions arise naturally from these results. In the first place, when a finite group $G$ acts on an algebraic manifold $M$, the quotient $M / G$ can always be endowed with the structure of a complex analytic orbifold. We believe that our theorems can be generalised to cover this case, provided the quotient admits a Kähler-Einstein orbifold metric. Nevertheless there are few examples of Kähler-Einstein orbifolds (see e.g. [9], [14], 4]), and it is probably hard to apply our results to coverings with orbifold base.

From a different perspective, in light of our results (c)-(f), one could study the Weil-Petersson geometry of the moduli spaces of these new families, or one can try to generalise Mabuchi and Mukai's results ([16]) on compactification of moduli spaces. A situation which seems geometrically appealing is the one of the intersection of two quadrics (which is in fact Mabuchi-Mukai's case in dimension 2). A classical result says the moduli space of the intersection of two quadrics in $\mathbb{P}^{2 n+3}$ is isomorphic to the moduli space of hyperelliptic curves of genus $n$ (see [2] and reference therein). Therefore this moduli space inherits two Weil-Petersson geometries, one coming from the Kähler-Einstein metrics on the intersection of quadrics, the other from Poincare metrics on curves. It would be interesting to compare them.

We wish to thank Gang Tian for many helpful conversations and for his interest in this work., and the referees for useful suggestions.

## 2 Existence theorems on covering spaces

Let $M$ be a compact $n$-dimensional Kähler manifold and $\omega$ a smooth closed (1,1)-form on $M$ such that

$$
\left\langle[\omega]^{n},[M]\right\rangle=\int_{M} \omega^{n}>0
$$

When $\varphi \in C^{\infty}(M)$ put $\omega_{\varphi}=\omega+\mathrm{i} \partial \bar{\partial} \varphi$. Define the following functionals on $C^{\infty}(M)$ :

$$
\begin{align*}
& I_{\omega}(\varphi)=\frac{1}{\left\langle[\omega]^{n},[M]\right\rangle} \int \varphi\left(\omega^{n}-\omega_{\varphi}^{n}\right)  \tag{1}\\
& J_{\omega}(\varphi)=\int_{0}^{1} \frac{I_{\omega}(s \varphi)}{s} d s  \tag{2}\\
& F_{\omega}^{0}(\varphi)=J_{\omega}(\varphi)-\frac{1}{\left\langle[\omega]^{n},[M]\right\rangle} \int \varphi \omega^{n} . \tag{3}
\end{align*}
$$

When no confusion is possible, we will write $V=\left\langle[\omega]^{n},[M]\right\rangle$. For the reader's convenience we recall the following equivalent definitions of these functionals.

Lemma 2.1 If $M$ and $\omega$ are as above, and $\varphi \in C^{\infty}(M)$, then

$$
\begin{align*}
J_{\omega}(\varphi) & =-\frac{n!}{\left\langle[\omega]^{n},[M]\right\rangle} \sum_{p=1}^{n} \frac{1}{(n-p)!(p+1)!} \int_{M} \varphi \omega^{n-p}(\mathrm{i} \partial \bar{\partial} \varphi)^{p}=  \tag{4}\\
& =-\frac{1}{\left\langle[\omega]^{n},[M]\right\rangle} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_{M} \varphi \mathrm{i} \partial \bar{\partial} \varphi \wedge \omega^{k} \wedge \omega_{\varphi}^{n-k-1}=  \tag{5}\\
& =\frac{1}{\left\langle[\omega]^{n},[M]\right\rangle} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_{M} \mathrm{i} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{k} \wedge \omega_{\varphi}^{n-k-1} \tag{6}
\end{align*}
$$

and

$$
\begin{equation*}
F_{\omega}^{0}(\varphi)=-\frac{n!}{\left\langle[\omega]^{n},[M]\right\rangle} \sum_{p=0}^{n} \frac{1}{(n-p)!(p+1)!} \int_{M} \varphi \omega^{n-p}(\mathrm{i} \partial \bar{\partial} \varphi)^{p} . \tag{7}
\end{equation*}
$$

Proof (sketch). To prove (4) expand $\omega_{s \varphi}^{n}=(\omega+s \mathrm{i} \partial \bar{\partial} \varphi)^{n}$ in powers of $s$ and use the result to compute $I_{\omega}(s \varphi)$ in (2). As for (5) compute $I_{\omega}(s \varphi)$ in (2) using the fact that

$$
\omega^{n}-\omega_{s \varphi}^{n}=\left(\omega-\omega_{s \varphi}\right) \sum_{q=0}^{n-1} \omega^{n-q-1} \wedge \omega_{s \varphi}^{q}
$$

Substituting $\omega_{s \varphi}=s \omega_{\varphi}+(1-s) \omega$ and expanding $\omega_{s \varphi}^{q}$ yields

$$
J_{\omega}(\varphi)=-\frac{1}{\left\langle[\omega]^{n},[M]\right\rangle} \sum_{p=0}^{n-1} C_{p} \int_{M} \varphi \mathrm{i} \partial \bar{\partial} \varphi \wedge \omega^{n-1-p} \wedge \omega_{\varphi}^{p}
$$

where

$$
C_{p}=\sum_{q=p}^{n-1}\binom{q}{p} \int_{0}^{1} s^{p+1}(1-s)^{q-p} d s
$$

This can be computed using the combinatorial identities

$$
\begin{gathered}
\int_{0}^{1} s^{p+1}(1-s)^{k} d s=\frac{(p+1)!k!}{(p+k+2)!} \\
\sum_{k=0}^{n-p-1} \frac{p+1}{(p+k+1)(p+k+2)}=\frac{n-p}{n+1}
\end{gathered}
$$

and gives the desired result. To get (6) it is enough to integrate by parts, using that $\omega$ is closed and $M$ is Kähler . Finally (7) is an immediate consequence of (4).
Q.D.E.

Formula (7) says that $F^{0}$ coincides (up to a constant factor) with the functional called $I$ by other authors. Compare with eq. (25) in 10 where Donaldson gives a nice geometric interpretation of $F^{0}$.

Lemma 2.2 Let $M$ and $\omega$ be as above. If $\lambda$ is a positive constant then

$$
\begin{equation*}
F_{\lambda \omega}^{0}(\lambda \varphi)=\lambda F_{\omega}^{0}(\varphi) \tag{8}
\end{equation*}
$$

Let $\omega_{0}$ be a closed (1,1)-form such that $\left\langle\left[\omega_{0}\right],[M]\right\rangle>0$. Given $\varphi_{01}$, $\varphi_{12} \in C^{\infty}(M)$ put $\omega_{1}=\omega_{0}+\mathrm{i} \partial \bar{\partial} \varphi_{01}, \varphi_{02}=\varphi_{01}+\varphi_{12}$. Then

$$
\begin{equation*}
F_{\omega_{0}}^{0}\left(\varphi_{02}\right)=F_{\omega_{0}}^{0}\left(\varphi_{01}\right)+F_{\omega_{1}}^{0}\left(\varphi_{12}\right) . \tag{9}
\end{equation*}
$$

For the proof see [25, pp. 60f].
Assume from now on that $M$ is a Fano manifold and $\omega$ is a Kähler metric in the class $2 \pi c_{1}(M)$. Then $V=\left\langle[\omega]^{n},[M]\right\rangle=n!\operatorname{vol}(M)$. Let $f=f(\omega)$ be the unique function on $M$ satisfying

$$
\begin{equation*}
\operatorname{Ric}(\omega)=\omega+\mathrm{i} \partial \bar{\partial} f(\omega), \quad \int_{M} e^{f(\omega)} \omega^{n}=V \tag{10}
\end{equation*}
$$

Define $A_{\omega}, F_{\omega}: C^{\infty}(M) \rightarrow \mathbb{R}$ by

$$
\begin{aligned}
& A_{\omega}(\varphi)=\log \left[\frac{1}{V} \int_{M} e^{f(\omega)-\varphi} \omega^{n}\right] \\
& F_{\omega}(\varphi)=F_{\omega}^{0}(\varphi)-A_{\omega}(\varphi) .
\end{aligned}
$$

Although these functionals (as well as the ones defined before) are defined on the whole of $C^{\infty}(M)$, their interest for Kähler-Einstein metrics lies in their behaviour on a smaller space, whose definition we now recall.

Let $G$ be a compact group of isometries of $(M, \omega)$. Put

$$
\begin{equation*}
P_{G}(M, \omega)=\left\{\varphi \in C^{\infty}(M): \omega_{\varphi}>0, \text { and } \varphi \text { is } G \text {-invariant }\right\} . \tag{11}
\end{equation*}
$$

By $\omega_{\varphi}>0$ we mean that $\omega_{\varphi}$ is a Kähler metric. If $G=\{1\}$ we simply write $P(M, \omega)$. We say that $F_{\omega}$ is proper on $P_{G}(M, \omega)$ if there is a proper increasing function $\mu: \mathbb{R} \rightarrow \mathbb{R}$, such that the inequality

$$
F_{\omega}(\varphi) \geq \mu\left(J_{\omega}(\varphi)\right)
$$

holds for any $\varphi \in P_{G}(M, \omega)$. The importance of this notion is mainly due to the following theorem (see [24, Theorem 1.6] and [25, Chapter 7]).

Theorem 2.1 (Tian) Let $M$ be a Fano manifold, $G$ a maximal compact subgroup of $\operatorname{Aut}(M)$ and $\omega$ a G-invariant Kähler metric in the class $2 \pi \mathrm{c}_{1}(M)$. Then $M$ admits a Kähler-Einstein metric if and only if $F_{\omega}$ is proper on $P_{G}(M, \omega)$. Moreover, in this case $F_{\omega}$ is bounded from below on all $P(M, \omega)$.

The elements of $P_{G}(M, \omega)$ parametrise metrics only up to a constant, because $\omega_{\varphi}$ does not change by adding a constant to $\varphi$, and the functional $F_{\omega}$ depends on $\varphi \in P_{G}(M, \omega)$ only up to a constant. Therefore we can normalise the elements of $P_{G}(M, \omega)$ one way or another. The following normalisation is useful in this context:

$$
\begin{equation*}
Q_{G}(M, \omega)=\left\{\varphi \in P_{G}(M, \omega): A_{\omega}(\varphi)=0\right\} . \tag{12}
\end{equation*}
$$

For any $\varphi \in P_{G}(M, \omega), \varphi+A_{\omega}(\varphi) \in Q_{G}(M, \omega)$ is the corresponding normalised potential.

The following proposition gives a sufficient condition for the existence of Kähler-Einstein metrics on Fano manifolds.

Proposition 2.1 Let $M$ be a Fano manifold, $\omega$ a Kähler metric in the class $2 \pi c_{1}(M)$ and $G$ a compact group of isometries of $(M, \omega)$. If there are constants $C_{1}, C_{2}>0$ such that

$$
\begin{equation*}
F_{\omega}(\varphi) \geq C_{1} \sup _{M} \varphi-C_{2} \tag{13}
\end{equation*}
$$

for any $\varphi \in Q_{G}(M, \omega)$, then $M$ admits a Kähler-Einstein metric.
Proof. One exploits the same estimates used in the proof of Theorem 2.1 (compare [25, Chapter 7].) Indeed, let $\varphi_{t}, t \in[0, T)$ be the curve of potentials obtained by applying the continuity method:

$$
\begin{equation*}
\left(\omega+\mathrm{i} \partial \bar{\partial} \varphi_{t}\right)^{n}=e^{f-t \varphi_{t}} \omega^{n} \tag{14}
\end{equation*}
$$

Then it is known that for some constants $C_{3}, C_{4}>0$

$$
\begin{gather*}
F_{\omega}^{0}\left(\varphi_{t}\right) \leq 0  \tag{15}\\
F_{\omega}\left(\varphi_{t}\right) \leq-A_{\omega}\left(\varphi_{t}\right) \leq \frac{1-t}{V} \int_{M} \varphi_{t} \omega_{t}^{n}  \tag{16}\\
0 \leq-\inf _{M} \varphi_{t} \leq C_{3}\left(\frac{1}{V} \int_{M}\left(-\varphi_{t}\right) \omega_{t}^{n}+C_{4}\right)  \tag{17}\\
\frac{1}{V} \int_{M} \varphi_{t} \omega_{t}^{n} \leq C_{4}  \tag{18}\\
F_{\omega}\left(\varphi_{t}\right) \leq-A_{\omega}\left(\varphi_{t}\right) \leq C_{4}(1-t) \leq C_{4} \tag{19}
\end{gather*}
$$

(see [25] p. 72]). Since $\varphi_{t}+A_{\omega}\left(\varphi_{t}\right) \in Q_{G}(M, \omega)$, and $F_{\omega}\left(\varphi_{t}\right)$ does not change by adding a constant to $\varphi_{t}$, an application of (13) yields

$$
\begin{gather*}
F_{\omega}\left(\varphi_{t}\right)=F_{\omega}\left(\varphi_{t}+A_{\omega}\left(\varphi_{t}\right)\right) \geq \\
\geq C_{1} \sup _{M}\left(\varphi_{t}+A_{\omega}\left(\varphi_{t}\right)\right)-C_{2}=C_{1} \sup _{M} \varphi_{t}+C_{1} A_{\omega}\left(\varphi_{t}\right)-C_{2} \tag{20}
\end{gather*}
$$

Therefore using (19)

$$
C_{1} \sup _{M} \varphi_{t} \leq F_{\omega}\left(\varphi_{t}\right)-C_{1} A_{\omega}\left(\varphi_{t}\right)+C_{2} \leq C_{4}+C_{2}+C_{1} C_{4}
$$

Hence $\sup _{M} \varphi_{t}$ is uniformly bounded. But from (15)

$$
J_{\omega}\left(\varphi_{t}\right) \leq \frac{1}{V} \int_{M} \varphi_{t} \omega^{n} \leq \sup _{M} \varphi_{t}
$$

So $J_{\omega}\left(\varphi_{t}\right)$ is bounded and this is enough to bound the $C^{0}$ norm (see [25] p. 67]). Therefore, by Yau's estimates, one can solve equations (14) up to $t=1$, and $\omega+\mathrm{i} \partial \bar{\partial} \varphi_{1}$ is the Kähler-Einstein metric. Q.D.E.

Lemma 2.3 Let $M$ be a Fano manifold, and $\omega$ a Kähler metric in the class $2 \pi \mathrm{c}_{1}(M)$. Then for any $\beta>0$ there are constants $C_{1}, C_{2}>0$ such that for any $\varphi \in Q(M, \omega)$

$$
\begin{equation*}
\log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \omega^{n}\right] \geq C_{1} \sup _{M} \varphi-C_{2} \tag{21}
\end{equation*}
$$

Proof. According to one of the basic results of Tian's theory of the $\alpha$-invariant (see [22, Prop. 2.1]), there are $\alpha \in(0,1)$ and $C_{3}>0$, such that for any $\varphi \in P_{G}(M, \omega)$

$$
\begin{equation*}
\frac{1}{V} \int_{M} e^{-\alpha(\varphi-\sup \varphi)} \omega^{n} \leq C_{3} \tag{22}
\end{equation*}
$$

Let $p$ be such that

$$
\begin{equation*}
\frac{p-\alpha}{p-1}=1+\beta . \tag{23}
\end{equation*}
$$

Then

$$
\begin{equation*}
p=1+\frac{1-\alpha}{\beta} \tag{24}
\end{equation*}
$$

so $p \in(1,+\infty)$, because $\alpha<1$. Let $d \mu$ denote the measure $(1 / V) \omega^{n}$ on $M$. By definition, if $\varphi \in Q_{G}(M, \omega)$

$$
\begin{equation*}
\int e^{f-\varphi} d \mu=1 \tag{25}
\end{equation*}
$$

so

$$
e^{-\sup f} \leq \int e^{-\varphi} d \mu
$$

Since

$$
\begin{aligned}
& -\varphi=\frac{\alpha}{p}(\sup \varphi-\varphi)-\frac{\alpha}{p} \sup \varphi+\left(1-\frac{\alpha}{p}\right)(-\varphi) \\
& e^{-\sup f} \leq e^{-\frac{\alpha}{p} \sup \varphi} \int e^{\frac{\alpha}{p}(\sup \varphi-\varphi)} \cdot e^{\frac{p-\alpha}{p}(-\varphi)} d \mu .
\end{aligned}
$$

Therefore applying Hölder inequality with exponent $p$ yields

$$
\begin{equation*}
e^{\frac{\alpha}{p} \sup \varphi-\sup f} \leq\left[\int e^{\alpha(\sup \varphi-\varphi)} d \mu\right]^{1 / p}\left[\int e^{\frac{p^{\prime}}{p}(p-\alpha)(-\varphi)} d \mu\right]^{1 / p^{\prime}} . \tag{26}
\end{equation*}
$$

Using (22) and observing that

$$
\frac{p^{\prime}}{p}(p-\alpha)=1+\beta,
$$

we get

$$
e^{\frac{\alpha}{p} \sup \varphi-\sup f} \leq C_{3}^{1 / p}\left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \omega^{n}\right]^{1 / p^{\prime}}
$$

Taking logarithms

$$
\frac{p^{\prime} \alpha}{p} \sup \varphi-p^{\prime} \sup f-\frac{p^{\prime}}{p} C_{3} \leq \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \omega^{n}\right]
$$

that is (21) with

$$
C_{1}=\frac{p^{\prime} \alpha}{p}=\frac{\alpha \beta}{1-\alpha}>0, \quad C_{2}=p^{\prime} \sup f+\frac{C_{3}}{p-1} .
$$

Q.D.E.

Corollary 2.1 If there are constants $C_{1}, C_{2}>0$ and $\beta>0$ such that

$$
\begin{equation*}
F_{\omega}(\varphi) \geq C_{1} \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \omega^{n}\right]-C_{2} \tag{27}
\end{equation*}
$$

for any $\varphi \in Q_{G}(M, \omega)$, then $M$ admits a Kähler-Einstein metric.
This is an immediate consequence of the previous lemma and Proposition 2.1.

In the proof of the existence theorems below we will need a slight extension of the integral functionals defined above. Let $M$ be a compact complex manifold and $\gamma$ a continuous hermitian form on $M$. A closed positive current $T$ of bidegree $(1,1)$ is called a Kähler current if for some constant $c>0$ one has $T \geq c \gamma$ in the sense of currents. The definition does not depend on the choice of $\gamma$, since $M$ is compact. If $M$ is a Fano manifold, $G \subset \operatorname{Aut}(M)$ is a compact subgroup, and $\omega$ is a $G$-invariant Kähler form in the class $2 \pi \mathrm{c}_{1}(M)$, we put

$$
P_{G}^{0}(M, \omega)=\left\{\psi \in C^{0}(M): \omega+\mathrm{i} \partial \bar{\partial} \psi \text { is a Kähler current }\right\} .
$$

This means that $\psi$ belongs to $P_{G}^{0}(M, \omega)$ if and only if $\omega+\mathrm{i} \partial \bar{\partial} \psi \geq c \omega$ in the sense of currents for some $c>0$.

Lemma 2.4 The map $\varphi \mapsto(\omega+\mathrm{i} \partial \bar{\partial} \varphi)^{n}$ can be extended to a map

$$
P_{G}^{0}(M, \omega) \longrightarrow\{\text { positive Borel measures on } M\} .
$$

The extension is continuous with respect to the $C^{0}$-topology on the domain and the weak convergence of measures on the target.

Proof. This follows from basic results on the complex Monge-Ampère operator. Consider a covering $\left\{U_{k}\right\}$ of $M$ with contractible open subsets. On $U_{k}$ we have $\omega=\mathrm{i} \partial \bar{\partial} u_{k}$ for some smooth strictly plurisubharmonic function $u_{k}$. If $\varphi \in P^{0}$ then $u_{k}+\varphi$ is plurisubharmonic and continuous on $U_{k}$. Although in general currents cannot be multiplied, Bedford and Taylor showed how to define consistently $\left(\mathrm{i} \partial \bar{\partial}\left(u_{k}+\varphi\right)\right)^{n}$ as a positive measure on $U_{k}$. Moreover, it follows from the Chern-Levine-Nirenberg inequality that this measure depends continuously on $\varphi$ (see e.g. [7, Corollary 2.6]). As $\left(\mathrm{i} \partial \bar{\partial}\left(u_{k}+\varphi\right)\right)^{n}=\left(\mathrm{i} \partial \bar{\partial}\left(u_{j}+\varphi\right)\right)^{n}$ on $U_{k} \cap U_{j}$, these local measures glue together, and the resulting measure on $M$, denoted by $(\omega+\mathrm{i} \partial \bar{\partial} \varphi)^{n}$ depends continuously on $\varphi$. Q.D.E.

Proposition 2.2 The functionals $I_{\omega}, J_{\omega}, F_{\omega}^{0}$ and $F_{\omega}$ can be extended to $P_{G}^{0}(M, \omega)$. The extensions are continuous with respect to the $C^{0}$ topology.

Proof. It follows from the previous lemma that we can extend continuously $I_{\omega}$. Using formula (2) we can extend continuously $J_{\omega}$, and therefore $F_{\omega}^{0}$. $A_{\omega}(\varphi)$ can be clearly extended continuously to $P_{G}^{0}(M, \omega)$. Q.D.E.

In the proof of the next Theorem we will need the following density result.

Proposition 2.3 Any $\psi \in P_{G}^{0}(M, \omega)$ is the $C^{0}$-limit of a sequence $\varphi_{n} \in P_{G}(M, \omega)$.

This is a straightforward application of a result due to Richberg ([20]) that we quote in the version given by Demailly ([6, Lemma 2.15]).

Lemma 2.5 (Richberg) Let $\psi \in C^{0}(M)$ be such that i $\partial \bar{\partial} \psi \geq \alpha$ for some continuous (1,1)-form $\alpha$. Then given any hermitian form $\gamma$ and any $\varepsilon>0$, there is a function $\psi^{\prime} \in C^{\infty}(M)$ such that $\psi \leq \psi^{\prime}<\psi+\varepsilon$ and $\mathrm{i} \partial \bar{\partial} \psi^{\prime} \geq \alpha-\varepsilon \gamma$.

The following two lemmata deal specifically with coverings.
Lemma 2.6 If $\pi: M \rightarrow N$ is a finite holomorphic map of compact complex manifolds, the direct image via $\pi$ of a Kähler current on $M$ is a Kähler current on $N$.

Proof. Let $R \subset M$ and $B \subset N$ denote ramification and branching locus of $\pi$, and $d$ its degree. Let $\gamma_{M}$ and $\gamma_{N}$ be continuous hermitian forms on $M$ and $N$ respectively. Since $\pi^{*} \gamma_{N}$ is continuous and $\gamma_{M}$ is positive definite, there is $c_{1}>0$ such that $\gamma_{M} \geq c_{1} \pi^{*} \gamma_{N}$. If $T$ is a Kähler current on $M$, by definition $T \geq c_{2} \gamma_{M}$ for some $c_{2}>0$, so that $T \geq c \pi^{*} \gamma_{N}$ with $c=c_{1} c_{2}>0$. Given a positive form $\eta \in \wedge^{n-1, n-1}(N)$ we have

$$
\begin{aligned}
& \left\langle\pi_{*} T, \eta\right\rangle=\left\langle T, \pi^{*} \eta\right\rangle \geq c\left\langle\pi^{*} \gamma_{N}, \pi^{*} \eta\right\rangle=c \int_{M} \pi^{*}\left(\gamma_{N} \wedge \eta\right)= \\
= & c \int_{M \backslash R} \pi^{*}\left(\gamma_{N} \wedge \eta\right)=c \cdot d \int_{N \backslash B} \gamma_{N} \wedge \eta=c \cdot d \int_{N} \gamma_{N} \wedge \eta=c \cdot d\left\langle\gamma_{N}, \eta\right\rangle
\end{aligned}
$$

so that $T \geq c \cdot d \gamma_{N}$. This proves the lemma.
Q.D.E.

Lemma 2.7 Let $\pi: M \rightarrow N$ be a degree $d$ covering between $n$ dimensional Kähler manifolds. Let $\omega_{N}$ be a Kähler metric on $N$, and $\psi \in P^{0}\left(N, \omega_{N}\right)$ a continuous potential such that $\pi^{*} \psi$ be a smooth function on $M$. Then

$$
\begin{equation*}
F_{\pi^{*} \omega_{N}}^{0}\left(\pi^{*} \psi\right)=F_{\omega_{N}}^{0}(\psi) \tag{28}
\end{equation*}
$$

Proof. Put $V_{N}=\left\langle N,\left[\omega_{N}\right]^{n}\right\rangle$. Then $\left\langle M,\left[\pi^{*} \omega_{N}\right]^{n}\right\rangle=d \cdot V_{N}$.

$$
\begin{align*}
I_{\omega_{N}}(s \psi) & =\frac{1}{V_{N}} \int_{N} s \psi\left[\omega_{N}^{n}-\left(\omega_{N}+s \mathrm{i} \partial \bar{\partial} \psi\right)^{n}\right]= \\
& =\frac{1}{d V_{N}} \int_{M} s \pi^{*} \psi\left[\left(\pi^{*} \omega_{N}\right)^{n}-\left(\pi^{*} \omega_{N}+s \mathrm{i} \partial \bar{\partial} \pi^{*} \psi\right)^{n}\right]= \\
& =I_{\pi^{*} \omega_{N}}\left(s \pi^{*} \psi\right) \\
J_{\omega_{N}}(\psi) & =\int_{0}^{1} \frac{I_{\omega_{N}}(s \psi)}{s} d s=\int_{0}^{1} \frac{I_{\pi^{*} \omega_{N}}\left(s \pi^{*} \psi\right)}{s} d s=  \tag{29}\\
& =J_{\pi^{*} \omega_{N}}\left(\pi^{*} \psi\right) \\
\frac{1}{V_{N}} \int_{N} \psi \omega_{N}^{n} & =\frac{1}{d V_{N}} \int_{M} \pi^{*} \psi\left(\pi^{*} \omega_{N}\right)^{n} \tag{30}
\end{align*}
$$

Plugging (29) and (30) in the definition of $F^{0}$ we get finally

$$
\begin{equation*}
F_{\omega_{N}}^{0}(\psi)=F_{\pi^{*} \omega_{N}}^{0}\left(\pi^{*} \psi\right) \tag{31}
\end{equation*}
$$

Mark that the functionals $I_{\omega_{N}}(\psi), J_{\omega_{N}}(\psi)$ and $F_{\omega_{N}}^{0}(\psi)$ are well-defined because $\omega_{N}$ is a Kähler metric and $\psi \in P^{0}\left(N, \omega_{N}\right)$. On the other hand, $\pi^{*} \omega_{N}$ degenerates along the ramification, so it is not a Kähler metric. Nevertheless it is a smooth closed (1,1)-form and $\pi^{*} \psi$ is a smooth function, so the functionals $I_{\pi^{*} \omega_{N}}\left(\pi^{*} \psi\right), J_{\pi^{*} \omega_{N}}\left(\pi^{*} \psi\right)$ and $F_{\pi^{*} \omega_{N}}^{0}\left(\pi^{*} \psi\right)$ are well-defined too, thanks to the discussion at p. 4 Q.D.E.

Theorem 2.2 Let $M$ and $N$ be Fano manifolds, $\pi: M \rightarrow N$ a ramified Galois covering of degree $d$ with structure group $G, \omega_{N}$ a KählerEinstein metric on $N$ and $\omega \in 2 \pi \mathrm{c}_{1}(M)$ a $G$-invariant Kähler metric. Denote by $R(\pi)$ be the ramification divisor of $\pi$ (with multiplicities), and assume that numerically (i.e. in homology)

$$
R(\pi)=-\beta K_{M}
$$

for some $\beta \in \mathbb{Q}$. (Since $R(\pi)$ is effective and $-K_{M}$ is ample, $\beta>0$.) Then there is a constant constant $C$ such that for any $\varphi \in P_{G}\left(M, \omega_{M}\right)$

$$
\begin{equation*}
F_{\omega}^{0}(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \pi^{*} \omega_{N}^{n}\right]-C \tag{32}
\end{equation*}
$$

(Here $V=\left\langle[\omega]^{n},[M]\right\rangle$.)
Proof. The classical Hurwitz formula for the canonical bundle of a ramified covering, $\pi^{*} K_{N}=K_{M}-R(\pi)$, yields that

$$
\pi^{*}\left[\omega_{N}\right]=(1+\beta)[\omega] .
$$

Choose a $G$-invariant $u \in C^{\infty}(M)$ such that

$$
\begin{equation*}
\pi^{*} \omega_{N}=(1+\beta) \omega+\mathrm{i} \partial \bar{\partial} u \tag{33}
\end{equation*}
$$

We claim that any $\varphi \in P_{G}(M, \omega)$ is of the form

$$
\varphi=\frac{u+\pi^{*} \psi}{1+\beta}
$$

for some $\psi \in P^{0}\left(N, \omega_{N}\right)$. Indeed $(1+\beta) \varphi-u$ is $G$-invariant, so $(1+\beta) \varphi-u=\pi^{*} \psi$ for some continuous function $\psi$, because $N=M / G$ has the quotient topology. Since (as currents)

$$
\omega_{N}+\mathrm{i} \partial \bar{\partial} \psi=\frac{1+\beta}{d} \pi_{*}(\omega+\mathrm{i} \partial \bar{\partial} \varphi)
$$

Lemma 2.6 implies that $\omega_{N}+\mathrm{i} \partial \bar{\partial} \psi$ is a Kähler current, i.e. that $\psi \in P^{0}\left(N, \omega_{N}\right)$. We have shown that to any potential $\varphi \in P_{G}(M, \omega)$ corresponds a continuous potential $\psi \in P^{0}\left(N, \omega_{N}\right)$ such that

$$
\begin{equation*}
\pi^{*}\left(\omega_{N}+\mathrm{i} \partial \bar{\partial} \psi\right)=(1+\beta)(\omega+\mathrm{i} \partial \bar{\partial} \varphi) . \tag{34}
\end{equation*}
$$

Since $N$ is Kähler-Einstein by hypothesis, Tian's Theorem [2.1]implies that there is a constant $C_{3}$ such that $F_{\omega_{N}}(\eta) \geq-C_{3}$ for any $\eta \in$ $P\left(N, \omega_{N}\right)$. By Proposition 2.2 the functional $F_{\omega_{N}}$ can be extended continuously to $P^{0}\left(N, \omega_{N}\right)$, and by Proposition [2.3] $P\left(N, \omega_{N}\right)$ is dense in $P^{0}\left(N, \omega_{N}\right)$, so we can conclude that

$$
\begin{equation*}
F_{\omega_{N}}(\psi) \geq-C_{3} \tag{35}
\end{equation*}
$$

for $\psi$ as in (34). To finish the proof we need to "lift" this inequality from $N$ to $M$. From (33) and (9) of Lemma 2.2, applied to the forms $(1+\beta) \omega$ and $\pi^{*} \omega_{N}$, it follows that

$$
F_{(1+\beta) \omega}^{0}((1+\beta) \varphi)=F_{(1+\beta) \omega}^{0}(u)+F_{\pi^{*} \omega_{N}}^{0}\left(\pi^{*} \psi\right) .
$$

Since $u$ does not depend on $\varphi, F_{(1+\beta) \omega}^{0}(u)$ is a constant. Next (8) in Lemma 2.2 implies that

$$
F_{(1+\beta) \omega}^{0}((1+\beta) \varphi)=(1+\beta) F_{\omega}^{0}(\varphi) .
$$

So

$$
F_{\omega}^{0}(\varphi)=\frac{1}{1+\beta} F_{\pi^{*} \omega_{N}}^{0}\left(\pi^{*} \psi\right)-C_{4} .
$$

Using Lemma 2.7 and (35) we get

$$
\begin{aligned}
& F_{\omega}^{0}(\varphi)=\frac{1}{1+\beta} F_{\omega_{N}}^{0}(\psi)-C_{4} \geq \\
\geq & \frac{1}{1+\beta} \log \left[\frac{1}{V_{N}} \int_{N} e^{-\psi} \omega_{N}^{n}\right]-C_{5} .
\end{aligned}
$$

Now

$$
\begin{aligned}
\frac{1}{V_{N}} \int_{N} e^{-\psi} \omega_{N}^{n} & =\frac{1}{d V_{N}} \int_{M} e^{-\pi^{*} \psi}\left(\pi^{*} \omega_{N}\right)^{n} \\
d V_{N} & =\left\langle\pi^{*}\left[\omega_{N}^{n}\right],[M]\right\rangle=(1+\beta)^{n} V \\
\frac{1}{V_{N}} \int_{N} e^{-\psi} \omega_{N}^{n} & =\frac{1}{(1+\beta)^{n} V} \int e^{-(1+\beta) \varphi} e^{u}\left(\pi^{*} \omega_{N}\right)^{n} \geq \\
& \geq \frac{e^{\inf u}}{(1+\beta)^{n} V} \int e^{-(1+\beta) \varphi}\left(\pi^{*} \omega_{N}\right)^{n}
\end{aligned}
$$

Therefore

$$
F_{\omega}^{0}(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi}\left(\pi^{*} \omega_{N}\right)^{n}\right]-C_{6}
$$

Q.D.E.

The first criterion for the existence of Kähler-Einstein metrics is the following

Theorem 2.3 Let $M$ be an $n$-dimensional Fano manifold. Assume that ramified coverings $\pi_{i}: M \rightarrow M_{i}$ are given for $i=1, \ldots, k$, satisfying the following assumptions:

1. $M_{i}$ is a Fano manifold and admits a Kähler-Einstein metric;
2. the coverings are Galois, i.e. $M_{i}=M / G_{i}$ for some finite group $G_{i}$,
3. the groups $G_{i}$ are contained in some compact subgroup $G \subset$ Aut $(M)$;
4. if $R\left(\pi_{1}\right), \ldots, R\left(\pi_{k}\right)$ are the ramification divisors, then

$$
\bigcap_{i=1}^{k} R\left(\pi_{i}\right)=\emptyset ;
$$

5. the divisors $R\left(\pi_{i}\right)$ are all proportional to the anticanonical divisor of $M$, i.e. there are some (necessarily positive) rational numbers $\beta_{i}$ such that numerically (i.e. in homology)

$$
R\left(\pi_{i}\right)=-\beta_{i} K_{M}
$$

Then $M$ has a Kähler-Einstein metric.
Proof. Fix a $G$-invariant Kähler form $\omega \in 2 \pi \mathrm{c}_{1}(M)$ and KählerEinstein metrics $\omega_{i}$ on $M_{i}$. As $G_{i} \subset G$ we have that

$$
\begin{equation*}
P_{G}(M, \omega) \subset \bigcap_{i=1}^{k} P_{G_{i}}(M, \omega) . \tag{36}
\end{equation*}
$$

From Theorem 2.2 it follows that for some constants $C_{1 i} \in \mathbb{R}$ we have

$$
\begin{equation*}
F_{\omega}^{0}(\varphi) \geq \frac{1}{1+\beta_{i}} \log \left[\frac{1}{V} \int_{M} e^{-\left(1+\beta_{i}\right) \varphi}\left(\pi_{i}^{*} \omega_{i}\right)^{n}\right]-C_{1 i} \tag{37}
\end{equation*}
$$

for all $\varphi \in P_{G}(M, \omega)$. Put

$$
\begin{aligned}
& C_{1}=\max C_{1 i} \quad p_{i}=1+\beta_{i} \quad \psi=e^{-\varphi} \\
& \beta=\min \beta_{i} \quad p=\min p_{i}=1+\beta \quad d \mu=\frac{1}{V} \omega^{n}
\end{aligned}
$$

and define $\eta_{i} \in C^{\infty}(M)$ by

$$
\pi_{i}^{*} \omega_{i}^{n}=\eta_{i} \omega^{n}
$$

Clearly $\beta>0, p>1$ and $\eta_{i} \geq 0$. Then (37) becomes

$$
F_{\omega}^{0}(\varphi)+C_{1 i} \geq \frac{1}{p_{i}} \log \left[\int_{M} \psi^{p_{i}} \eta_{i} d \mu\right]
$$

so

$$
F_{\omega}^{0}(\varphi)+C_{1} \geq \log \left\|\psi \eta_{i}^{1 / p_{i}}\right\|_{p_{i}}
$$

where $\left\|\|_{s}\right.$ denotes the norm of $L^{s}(M, \mu)$. By construction, $p_{i} \geq p$, for $i=1, \ldots, k$. If $p_{i}=p$, then clearly

$$
\left\|\psi \eta_{i}^{1 / p_{i}}\right\|_{p_{i}} \geq C_{2 i}\left\|\psi \eta_{i}^{1 / p}\right\|_{p}
$$

with $C_{2 i}=1$. If $p_{i}>p$, then

$$
\frac{1}{p}=\frac{1}{p_{i}}+\frac{1}{q}
$$

for some $q>p>1$. By Hölder inequality

$$
\left\|\psi \eta_{i}^{1 / p}\right\|_{p} \leq\left\|\psi \eta_{i}^{1 / p_{i}}\right\|_{p_{i}} \cdot\left\|\eta_{i}^{1 / q}\right\|_{q}
$$

so

$$
\left\|\psi \eta_{i}^{1 / p_{i}}\right\|_{p_{i}} \geq C_{2 i}\left\|\psi \eta_{i}^{1 / p}\right\|_{p}
$$

with

$$
C_{2 i}=\frac{1}{\left\|\eta_{i}^{1 / q}\right\|_{q}}>0
$$

Actually $\left\|\eta_{i}^{1 / q}\right\|_{q}=p_{i}^{n / q}$ so $C_{2 i}=p_{i}^{-n / q}$. Put $C_{2}=\min C_{2 i}>0$. Then

$$
\begin{gathered}
F_{\omega}^{0}(\varphi)+C_{1}-\log C_{2} \geq \log \left\|\psi \eta_{i}^{1 / p}\right\|_{p}= \\
=\frac{1}{p} \log \left[\int_{M} \psi^{p} \eta_{i} d \mu\right] \\
\exp \left(p F_{\omega}^{0}(\varphi)+C_{3}\right) \geq \int_{M} \psi^{p} \eta_{i} d \mu
\end{gathered}
$$

Taking the average over $i$

$$
\exp \left(p F_{\omega}^{0}(\varphi)+C_{3}\right) \geq \frac{1}{k} \sum_{i=1}^{k} \int_{M} \psi^{p} \eta_{i} d \mu
$$

Taking the logarithm we get

$$
\begin{gather*}
F_{\omega}^{0}(\varphi) \geq \frac{1}{p} \log \left[\frac{1}{k} \sum_{i=1}^{k} \int_{M} \psi^{p} \eta_{i} d \mu\right]-C_{4} \\
F_{\omega}^{0}(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi}\left(\frac{1}{k} \sum_{i=1}^{k} \eta_{i}\right) \omega^{n}\right]-C_{4} . \tag{38}
\end{gather*}
$$

It follows from assumption (4) that for some constant $C_{5}>0$

$$
\begin{equation*}
\frac{1}{k} \sum_{i=1}^{k} \eta_{i} \geq C_{5} \tag{39}
\end{equation*}
$$

Therefore

$$
F_{\omega}^{0}(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \omega^{n}\right]-C_{6}
$$

This holds for any $\varphi \in P_{G}(M \omega)$. If $\varphi \in Q_{G}(M, \omega)$, then $F_{\omega}(\varphi)=$ $F_{\omega}^{0}(\varphi)$, so we can apply Corollary 2.1 thus proving the existence of a Kähler-Einstein metric on $M$.
Q.D.E.

The reader will notice that assumption (4) on the ramification divisors is used only to ensure that (39) holds for some constant $C_{2}>$ 0 . This allows to bound

$$
\frac{1}{V} \int_{M} e^{-(1+\beta)} \omega^{n} \quad \text { with } \quad \frac{1}{V} \int_{M} e^{-(1+\beta)} \sum_{i=1}^{k}\left(\pi_{i}^{*} \omega_{i}\right)^{n}
$$

If the intersection of the ramification divisors is non-vacuous, the sum of the pull-back measures is degenerate along it. Nevertheless, under some numerical assumptions, it is still possible to bound the integral on the left with the one on the right.

Proposition 2.4 Let $M$ be an n-dimensional Fano manifold. Assume that ramified coverings $\pi_{i}: M \rightarrow M_{i}$ are given for $i=1, \ldots, k$, satisfying the following assumptions:

1. $M_{i}$ is a Fano manifold and admits a Kähler-Einstein metric;
2. the coverings are Galois, i.e. $M_{i}=M / G_{i}$ for some finite group $G_{i}$;
3. the groups $G_{i}$ are contained in some compact subgroup $G \subset$ $\operatorname{Aut}(M)$;
4. there are (positive) rational numbers $\beta_{i}$ such that numerically

$$
R\left(\pi_{i}\right)=-\beta_{i} K_{M}
$$

Define $\eta \in C^{\infty}(M)$ by

$$
\begin{equation*}
\frac{1}{k} \sum_{i=1}^{k} \pi_{i}^{*} \omega_{i}^{n}=\eta \omega^{n}, \tag{40}
\end{equation*}
$$

and put

$$
\begin{equation*}
c:=\sup \left\{\lambda \geq 0: \eta^{-\lambda} \in L^{1}\left(M, \omega^{n}\right)\right\} \tag{41}
\end{equation*}
$$

and $\beta:=\min \beta_{i}$. If

$$
\begin{equation*}
\frac{1}{c}<\beta \tag{42}
\end{equation*}
$$

then $M$ admits a Kähler-Einstein metric.
Proof of Proposition 2.4. Proceeding as in the proof of Theorem 2.3 one shows that for any $\varphi \in P_{G}(M, \omega)$

$$
\begin{equation*}
F_{\omega}^{0}(\varphi) \geq \frac{1}{1+\beta} \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \eta \omega^{n}\right]-C_{1} . \tag{43}
\end{equation*}
$$

(Compare with equation (38).) It follows from (42) that we can choose a real number $s$ such that

$$
\begin{equation*}
1+\frac{1}{c}<s<1+\beta . \tag{44}
\end{equation*}
$$

Put

$$
\gamma=\frac{1}{s}(1+\beta)-1 .
$$

It follows that $s>1$ and $\gamma>0$. Applying Hölder inequality with exponent $s$ we see that

$$
\begin{align*}
& \frac{1}{V} \int_{M} e^{-(1+\gamma) \varphi} \omega^{n}=\frac{1}{V} \int_{M} e^{-(1+\gamma) \varphi} \eta^{1 / s} \eta^{-1 / s} \omega^{n} \leq \\
& \quad \leq\left[\frac{1}{V} \int_{M} e^{-s(1+\gamma) \varphi} \eta \omega^{n}\right]^{\frac{1}{s}} \cdot\left[\frac{1}{V} \int_{M} \eta^{-\frac{s^{\prime}}{s}} \omega^{n}\right]^{\frac{1}{s^{s}}} \tag{45}
\end{align*}
$$

But (44)

$$
\frac{s^{\prime}}{s}=\frac{1}{s-1}<c
$$

so by the definition of $c$

$$
C_{2}=\left[\frac{1}{V} \int_{M} \eta^{-\frac{s^{\prime}}{s}} \omega^{n}\right]^{\frac{1}{s^{\prime}}}<+\infty
$$

On the other hand, $s(1+\gamma)=1+\beta$, so taking the logarithm on both sides of (45) we get

$$
\log \left[\frac{1}{V} \int_{M} e^{-(1+\gamma) \varphi} \omega^{n}\right] \leq \frac{1}{s} \log \left[\frac{1}{V} \int_{M} e^{-(1+\beta) \varphi} \eta \omega^{n}\right]+\log C_{2}
$$

and applying (43)

$$
F_{\omega}^{0}(\varphi) \geq \frac{s}{1+\beta} \log \left[\frac{1}{V} \int_{M} e^{-(1+\gamma) \varphi} \omega^{n}\right]-C_{3} .
$$

Since $\gamma>0$ we can still apply Corollary 2.1 to get the existence of a Kähler-Einstein metric.
Q.D.E.

It is clear that the last proposition is of some use only if $c$ can be computed or at least bounded from below. This number is an instance of an interesting invariant of a singularity studied - among others - by Demailly and Kollár (see 9] and [15). Indeed, in the situation of Proposition [2.4] let $\mathcal{I}$ be the ideal sheaf on $M$ that on any coordinate chart $U$ is given by $\mathcal{I}=\left(f_{1}, \ldots, f_{k}\right)$, where $f_{1}, \ldots, f_{k} \in \mathcal{O}_{M}(U)$ are local defining equations for the divisors $R\left(\pi_{1}\right), \ldots, R\left(\pi_{k}\right)$. The complex singularity exponent of $\mathcal{I}$ at a point $x \in U$ is defined as

$$
\begin{equation*}
c_{x}(\mathcal{I})=\sup \left\{\lambda \geq 0: e^{-2 \lambda \varphi} \text { is } L^{1} \text { on a neighbourhood of } x\right\}, \tag{46}
\end{equation*}
$$

where

$$
\begin{equation*}
\varphi=\log \left(\left|f_{1}\right|+\ldots+\left|f_{k}\right|\right) \tag{47}
\end{equation*}
$$

(See 9, p. 528].) Put

$$
\begin{equation*}
c_{M}(\mathcal{I})=\inf _{x \in M} c_{x}(\mathcal{I}) \tag{48}
\end{equation*}
$$

Lemma 2.8 If $c$ is defined by (41) and $\mathcal{I}$ is the ideal defined above, then $c=c_{M}(\mathcal{I})$.

Proof. Let $\left(U, z^{1}, \ldots, z^{n}\right)$ and $\left(V, w^{1}, \ldots, w^{n}\right)$ be coordinate charts on $M$ and $M_{i}$ respectively, such that $\pi_{i}(U) \subset V$. Let $w^{s}=\pi_{i}^{s}(z)$ be the local representation of $\pi_{i}$. Then the ramification divisor $R\left(\pi_{i}\right)$ is defined by $f_{i}=\operatorname{det}\left(\partial \pi_{i}^{s} / \partial z^{t}\right)$. On the other hand let

$$
\begin{aligned}
\omega & =\mathrm{i} g_{s t} d z^{s} \wedge d \bar{z}^{t} \\
\omega_{i} & =\mathrm{i} h_{s t} d w^{s} \wedge d \bar{w}^{t}
\end{aligned}
$$

be the local representations of $\omega$ and $\omega_{i}$ on $M$ and $M_{i}$ respectively. It is easy to check that $\pi_{i}^{*} \omega_{i}^{n}=\left|f_{i}\right|^{2} \psi_{i} \omega^{n}$, where

$$
\psi_{i}=\frac{\left|\operatorname{det}\left(h_{s t}\right)\right|^{2}}{\left|\operatorname{det}\left(g_{s t}\right)\right|^{2}}
$$

This is a smooth positive function, and by restricting $U$ we can assume that it be bounded and uniformly bounded away from 0 . Cover $M$ with a finite collection of open sets $U_{\alpha}$ such that this holds for all coverings $\pi_{1}, \ldots, \pi_{k}$. On each such $U_{\alpha}$ we have

$$
\eta=\frac{1}{k}\left(\left|f_{1}\right|^{2} \psi_{1}+\ldots+\left|f_{k}\right|^{2} \psi_{k}\right),
$$

so for some $C>0$

$$
\frac{1}{C} \eta \leq\left|f_{1}\right|^{2}+\ldots+\left|f_{k}\right|^{2} \leq C \eta
$$

Since $\left|f_{1}\right|+\ldots+\left|f_{k}\right| \leq \sqrt{k} \sqrt{\left|f_{1}\right|^{2}+\ldots+\left|f_{k}\right|^{2}} \leq \sqrt{k}\left(\left|f_{1}\right|+\ldots+\left|f_{k}\right|\right)$, the local integrability of $\eta^{-\lambda}$ is equivalent to the local integrability of $e^{-2 \lambda \varphi}$ (where $\varphi$ is defined by (47)). Taking the minimum over $\alpha$ we get the result.
Q.D.E.

The complex singularity exponent is in general quite difficult to compute, even for reasonably simple singularities (see [15, §8]). We present below two cases in which the computation is very simple. Although in many other explicit examples it is possible to compute $c$ and to successfully apply Proposition [2.4] a general computation of $c$ seems to be hard, although the singularities of the ramification divisors are relatively mild compared to other kinds of singularities.

We first recall some results on the ramification divisor of a Galois covering.

Lemma 2.9 (Cartan, [5, p. 97]) Given a finite group $G$ acting holomorphically on a complex manifold $M$ and leaving a point $x \in M$ fixed, there is a biholomorphism between a neighbourhood of $x$ and a neighbourhood of the origin in $T_{x} M$, that intertwines the action of $G$ and the tangent representation.

Definition $2.1 A$ (pseudo)reflection is a linear map $g \in \operatorname{Gl}(n, \mathbb{C})$ that is diagonalisable and has exactly $n-1$ eigenvalues equal to 1. A reflection group is a finite subgroup $G \subset \operatorname{Gl}(n, \mathbb{C})$ that is generated by reflections.

The eigenvalues of a reflection $g$ of finite order (i.e. such that $g^{m}=1$ ) are an $m$-th root of unity (with multiplicity 1 ) and (with multiplicity $n-1$ ). (When $m=2, g$ is indeed the reflection across its 1-eigenspace.)

Theorem 2.4 (Chevalley-Shephard-Todd) A finite subgroup $G \subset$ $\mathrm{Gl}(n, \mathbb{C})$ is a reflection group if and only if the affine variety $\mathbb{C}^{n} / G$ is smooth.

For the proof of this Theorem we refer to [21, p. 76]. Let now $\pi: M \rightarrow N=M / G$ be a Galois covering and $x$ a point in $M$. Denote by $G_{x}$ the stabiliser. Since the action is properly discontinuous, we can find a neighbourhood $U_{x}$ of $x$ that is $G_{x}$-stable and such that $g U_{x} \cap U_{x}=\emptyset$ if $g \notin G_{x}$. By Cartan's lemma we can assume that $U_{x}$ be isomorphic to some neighbourhood $V$ of the origin in $T_{x} M$ with the tangent representation. But $U_{x} / G_{x}$ is isomorphic to a neighbourhood of $\pi(x)$ in $N$, and therefore is smooth. Hence, Chevalley-Shephard-Todd's theorem implies that $G_{x}$ acts on $T_{x} M$ as a reflection subgroup. Moreover the invariant theory of finite groups provides a nice model for the map $U_{x} \rightarrow \pi\left(U_{x}\right)$, and in particular ensures that the projection $\pi$ can be written locally using invariant polynomials: $\pi(z)=\left(F_{1}(z), \ldots, F_{n}(z)\right)$. Here $F_{j}$ is a $G_{x}$-invariant polynomial on $T_{x} M \cong \mathbb{C}^{n}$ of degree $d_{i}$. The polynomial $f=\operatorname{det}\left(\partial F_{i} / \partial z^{j}\right)$ is a local defining equation for $R(\pi)$. It has degree $\left(d_{1}-1\right)+\ldots+\left(d_{n}-1\right)$. On the other hand the (local) degree of the covering $\pi$ is of course $d_{1} \ldots d_{n}$. The inequality $d_{1}+\ldots+d_{n}-n \leq d_{1} \ldots d_{n}-1$ implies that in these coordinates the ramification divisor is given by a homogeneous polynomial $f$ whose degree is strictly smaller than the local degree of $\pi$, hence a fortiori smaller than the global degree of $\pi$. Thus we have proved the following.

Lemma 2.10 If $\pi: M \rightarrow N$ is a Galois covering between smooth complex manifolds with structure group $G$, then in appropriate coordinate charts around an arbitrary point the local defining equation of the ramification divisor is a homogeneous polynomial of degree less than \#G.

The description of the ramification divisor can be made more precise (see [21, Exercise 4.3 .5 p. 85]). Let $H$ be a hyperplane in $\mathbb{C}$. The reflections in $G_{x}$ that fix $H$ form a cyclic group. Denote by $e(H)$ its order, and denote by $\ell_{H}$ a linear function on $\mathbb{C}^{n}$ such that $H=$ $\left\{\ell_{H}=0\right\}$. Since there are a finite number of reflections there are a finite number of hyperplanes, say $H_{1}, \ldots, H_{N}$, that are fixed by some reflection in $G_{x}$. Then on $U_{x}$ the ramification divisor has the following local defining equation:

$$
\begin{equation*}
f=\prod_{i=1}^{N} \ell_{H_{i}}^{e\left(H_{i}\right)-1}=0 \tag{49}
\end{equation*}
$$

If the (reduced) ramification is smooth there is only one hyperplane. Since $e(H) \leq \# G_{x}$, we have proved the following.

Lemma 2.11 Let $\pi: M \rightarrow N$ be a Galois covering between smooth complex manifolds with structure group $G$. If the reduced divisor associated to the ramification divisor is smooth at $x \in M$, then there is a holomorphic function $\ell$ defined on some neighbourhood of $x$, such that $d \ell(x) \neq 0$, and $R(\pi)=\left\{\ell^{m}=0\right\}$, with $m \leq \# G-1$.

We can now give two simple applications of Proposition 2.4
Theorem 2.5 Let $M$ be an n-dimensional Fano manifold, and let $\pi: M \rightarrow N$ be a Galois covering with group $G$ onto a Kähler-Einstein manifold $N$. Assume that homologically $R(\pi)=-\beta K_{M}$, and that

$$
\begin{equation*}
d-1<\beta \tag{50}
\end{equation*}
$$

where $d=\# G=\operatorname{deg}(\pi)$. Then $M$ has a Kähler-Einstein metric.
Proof. Take $x \in M$. If $x$ does not lie in the support of $R=R(\pi)$ then $\eta^{-\lambda}$ is clearly $L_{l o c}^{1}$ for any positive $\lambda$. If $x$ lies in the support of $R$, Lemma 2.10 implies that in appropriate coordinates centered at $x$ the divisor $R$ has a local defining equation that is a homogeneous polynomial $f$ of degree $m$, with $m \leq d-1$. In particular $\operatorname{ord}_{x} f=m$. The following general result gives a lower bound for $c_{x}(f)$ (for the proof see [8] Lemma 8.2, p. 438]).

Lemma 2.12 Let $f$ be a holomorphic function on an open set $U \subset$ $\mathbb{C}^{n}$. If $x \in U$, then $c_{x}(f) \geq 1 / \operatorname{ord}_{x}(f)$.

From this it follows that $c_{x}(\mathcal{I})=c_{x}(f) \geq 1 / m \geq 1 /(d-1)$ for any point of $M$. Hence $c=c_{M}(\mathcal{I}) \geq 1 /(d-1)$, and an application of Proposition 2.4 concludes the proof.
Q.D.E.

Theorem 2.6 Let $M$ be an n-dimensional Fano manifold. Assume that ramified coverings $\pi_{i}: M \rightarrow M_{i}$ are given for $i=1, \ldots, k$, satisfying the following assumptions:

1. $M_{i}$ is a Fano manifold and admits a Kähler-Einstein metric;
2. the coverings are Galois, i.e. $M_{i}=M / G_{i}$;
3. the groups $G_{i}$ are all contained in some fixed compact subgroup $G \subset \operatorname{Aut}(M)$;
4. if $V_{i}$ denotes the reduced divisor of $M$ associated to the ramification divisor of $\pi_{i}$, then the $V_{i}$ 's are smooth hypersurfaces, that intersect transversally in a smooth submanifold $V$;
5. there are (positive) rational numbers $\beta_{i}$ such that

$$
R\left(\pi_{i}\right)=-\beta_{i} K_{M},
$$

and they satisfy

$$
\begin{equation*}
\frac{1}{d_{1}-1}+\ldots+\frac{1}{d_{k}-1}>\frac{1}{\beta} \tag{51}
\end{equation*}
$$

where $\beta:=\min \beta_{i}$ and $d_{i}=\# G_{i}$.
Then $M$ has a Kähler-Einstein metric.
Proof. In order to apply Proposition [2.4 it is necessary to show that

$$
\begin{equation*}
c \geq \frac{1}{d_{1}-1}+\ldots+\frac{1}{d_{k}-1} . \tag{52}
\end{equation*}
$$

By definition $V=V_{1} \cap \ldots \cap V_{k}=V(\mathcal{I})_{\text {red }}=V(\sqrt{\mathcal{I}})$. Let $x$ be a point in $M$. If $x \notin V$ then $e^{-2 \varphi(x)}$ is finite (see (47)), and clearly $c_{x}(\mathcal{I})=+\infty$. Let $x \in V$. Using Lemma 2.11 we find a neighbourhood $U$ of $x$ and holomorphic functions $\ell_{1}, \ldots, \ell_{k}$ such that $R\left(\pi_{i}\right)=\left\{\ell_{i}^{m_{i}}=0\right\}$. Since the $V_{i}$ 's cross normally, $d \ell_{1}, \ldots, d \ell_{k}$ are linearly independent, hence
we can find a coordinate system on a neighbourhood $U$ of $x$ such that $\ell_{i}=z_{i}$ for $1 \leq i \leq k$. Since $m_{i} \leq d_{i}-1$, in order to prove (52) it is enough to show that the integral

$$
I(\lambda)=\int_{U}\left(\left|z_{1}^{m_{1}}\right|+\ldots+\left|z_{k}^{m_{k}}\right|\right)^{-2 \lambda} \omega^{n}
$$

converges for any positive $\lambda<1 / m_{1}+\ldots+1 / m_{k}$, i.e. that $c_{x}(\mathcal{I}) \geq$ $1 / m_{1}+\ldots+1 / m_{k}$. Assuming that the coordinate chart maps $U$ into a polydisk $\Delta^{n}$ (where $\Delta=\{z \in \mathbb{C}:|z|<1\}$ ), we get the estimate

$$
\begin{aligned}
I(\lambda) & \leq C_{1} \int_{\Delta^{n}} \frac{1}{\left(\left|z_{1}\right|^{m_{1}}+\ldots+\left|z_{k}\right|^{m_{k}}\right)^{2 \lambda}} d \mathcal{L}^{n}= \\
& =C_{2} \int_{\Delta^{k}} \frac{1}{\left(\left|z_{1}\right|^{m_{1}}+\ldots+\left|z_{k}\right|^{m_{k}}\right)^{2 \lambda}} d \mathcal{L}^{k}
\end{aligned}
$$

$\mathcal{L}^{n}$ being $2 n$-dimensional Lebesgue measure. Using polar coordinates in each disk $\Delta$ with $t_{i}=\left|z_{i}\right|$, we get

$$
I(\lambda) \leq C_{3} \int_{0}^{1} d t_{1} \ldots \int_{0}^{1} d t_{k} \frac{t_{1} \ldots t_{k}}{\left(t_{1}^{m_{1}}+\ldots+t_{k}^{m_{k}}\right)^{2 \lambda}}
$$

With the substitution $t_{i}=s_{i}^{1 / m_{i}}$

$$
I(\lambda) \leq C_{4} \int_{0}^{1} d s_{1} \ldots \int_{0}^{1} d s_{k} \frac{s_{1}^{\frac{2}{m_{1}}-1} \ldots s_{k}^{\frac{2}{m_{k}}-1}}{\left(s_{1}+\ldots+s_{k}\right)^{2 \lambda}}
$$

If $\lambda<1 / m_{1}+\ldots+1 / m_{k}$, we can choose $\lambda_{1}, \ldots, \lambda_{k}$ such that $0<\lambda_{i}<$ $1 / m_{i}$ and $\lambda=\lambda_{1}+\ldots+\lambda_{k}$. Since $s_{1}+\ldots+s_{k} \geq s_{i}$ we get
$I(\lambda) \leq C_{4} \int_{0}^{1} d s_{1} \ldots \int_{0}^{1} d s_{k} \prod_{i=1}^{k}\left[\frac{s_{i}^{\frac{2}{m_{i}}-1}}{\left(s_{1}+\ldots+s_{k}\right)^{2 \lambda_{i}}}\right] \leq C_{4} \prod_{i=1}^{k} \int_{0}^{1} s_{i}^{2\left(\frac{1}{m_{i}}-\lambda_{i}\right)-1} d s_{i}$.
And this converges since $\frac{1}{m_{i}}-\lambda_{i}>0$ for every $i$. Q.D.E.

## 3 Examples

Consider the hypersurface

$$
M=\left\{x_{0}^{d}+\ldots+x_{k-1}^{d}+f\left(x_{k}, \ldots, x_{n+1}\right)=0\right\} \subset \mathbb{P}^{n+1}
$$

where $f$ is any homogeneous polynomial of degree $d$ such that $M$ is smooth. Note that this is equivalent to saying that

$$
V=M \cap\left\{x_{0}=\ldots=x_{k-1}=0\right\} \cong\{f=0\} \subset \mathbb{P}^{n+1-k}
$$

be smooth.
Proposition 3.1 If $k>n+2-d$ then $M$ admits a Kähler-Einstein metric.

Proof. $M$ admits $k$ Galois $\mathbb{Z}_{d}$-coverings $\pi_{i}: M \rightarrow \mathbb{P}^{n}$ obtained by deleting the $i$-th coordinate, $\pi\left(x_{0}, \ldots, x_{n+1}\right)=\left(x_{0}, \ldots, \widehat{x_{i}}, \ldots, x_{n+1}\right)$. $G_{i}=\mathbb{Z}_{d}$ acts by multiplication by roots of unity on the $i$-th coordinate of $\mathbb{P}^{n+1} . R\left(\pi_{i}\right)=\left\{x_{i}^{d-1}=0\right\}=\mathcal{O}(d-1)=-\beta K_{M}$ with

$$
\beta=\frac{d-1}{n+2-d} .
$$

Since the groups $G_{i}$ commute, they generate a subgroup of $\operatorname{Aut}(M)$ which is isomorphic to $G_{0} \times \ldots \times G_{k-1}$. Therefore they all lie inside this (finite) compact subgroup of $\operatorname{Aut}(M)$. The ramifications are smooth hyperplane sections, and their intersection is the submanifold $V$ above. Therefore a straightforward application of Theorem 2.6 yields the existence of the Kähler-Einstein metric.
Q.D.E.

Proposition 3.2 Let $M \subset \mathbb{P}^{n+m}$ be a complete intersection of $m$ hypersurfaces of degree $d$, given by equations of the form

$$
\begin{array}{r}
F_{j}\left(x_{0}, \ldots, x_{n+m}\right)=a_{0}^{j} x_{0}^{d}+\ldots a_{k-1}^{j} x_{k-1}^{d}+f_{j}\left(x_{k}, \ldots, x_{n+m}\right)=0, \\
j=1, \ldots, m .
\end{array}
$$

I.e. the equations are diagonal in the first $k$ coordinates. If $n+2-d<$ $k$, then $M$ admits a Kähler-Einstein metric.

Proof. We proceed by induction over $m$. For $m=1$ it is the last Proposition. Let $m>1$, and assume that the result is true for intersections of $m-1$ hypersurfaces. If we delete one of the first $k$ coordinates,
for example $x_{0}$, we get a degree $d$ covering $\pi_{0}: M \rightarrow M_{0} \subset \mathbb{P}^{n+m-1}$ over a manifold with equations

$$
\begin{aligned}
& a_{0}^{1} F_{j}-a_{0}^{j} F_{1}=b_{1}^{j} x_{1}^{d}+\ldots+b_{k-1}^{j} x_{k-1}^{d}+h_{j}\left(x_{k}, \ldots, x_{k+m}\right)=0, \\
& \text { where } \quad\left\{\begin{array}{l}
b_{s}^{j}=a_{0}^{1} a_{s}^{j}-a_{0}^{j} a_{s}^{1} \\
h_{j}=a_{0}^{1} f_{j}-a_{0}^{j} f_{1}
\end{array}\right.
\end{aligned}
$$

Therefore the base of the covering has equations of the same form, but in smaller number. By induction it has a Kähler-Einstein metric. Moreover we can do the same with any other coordinate $x_{1}, \ldots, x_{k-1}$, so we get $k$ coverings over Kähler-Einstein manifolds. The ramifications are smooth, as well as their intersection, and

$$
\beta=\frac{d-1}{n+m+1-m d} .
$$

Since $n+1+m(1-d) \leq n+2-d<k$, we see that $\beta>(d-1) / k$, and we can apply Theorem [2.6 to get the Kähler-Einstein metric. Q.D.E.

When $d=2$, i.e. when we are intersecting quadrics, one needs $k=n+1$, which means that all the quadrics are in diagonal form. If $m=2$, the following result says that in this way we get all the intersection of two quadrics.

Theorem 3.1 If $Q_{1}, Q_{2}$ are quadrics in $\mathbb{P}^{n+2}$, such that their intersection $M=Q_{1} \cap Q_{2}$ is smooth and $n$-dimensional, then there is a system of homogeneous coordinates $\left(x_{0}: \ldots: x_{n+2}\right)$ such that

$$
\begin{align*}
Q_{1} & =\left\{x_{0}^{2}+\ldots+x_{n+2}^{2}=0\right\} \\
Q_{2} & =\left\{\lambda_{0} x_{0}^{2}+\ldots+\lambda_{n+2} x_{n+2}^{2}=0\right\} \tag{53}
\end{align*}
$$

with $\lambda_{i} \neq \lambda_{j}$ for $i \neq j$.
For this classical result we refer the reader to the detailed proof given by Miles Reid in his PhD thesis [19, p. 36].

Corollary 3.1 Any smooth intersection of two quadrics $M=Q_{1} \cap Q_{2}$ in $\mathbb{P}^{n+2}$ has a Kähler-Einstein metric.

Note that this gives the whole moduli space of such manifolds. In fact a result of Fujita says that these manifolds are characterised by simple numerical invariants (see [13, p. 54, Theorem 3.2 .5 (iv)]).

Browsing through the list of Fano 3-folds with $\rho=h^{1,1}=1$ (see e.g. [13, p. 215]) we see that some of them are already defined as coverings. These are the manifolds that Iskovskikh called hyperelliptic because the anticanonical linear system $\left|-K_{M}\right|$ determines a morphism that is a double cover onto its image $M^{\prime}$. The branching divisor $B \subset M^{\prime}$ is smooth, and the pairs $\left(M^{\prime}, B\right)$ can be classified. The possibilities are the following ones (see [13, p. 33-34]):
a) $M^{\prime}=\mathbb{P}^{3}$ and $B$ is a sextic surface;
b) $M^{\prime}=Q^{3}$ is the 3 -dimensional quadric, and $B$ is cut out by a quartic surface;
c) $M^{\prime} \subset \mathbb{P}^{6}$ is a cone over the Veronese surface, and $B$ is cut out by a cubic hypersurface.

Using Theorem [2.6 we will show that the manifolds in (a) and (b) admit a Kähler-Einstein metric. Actually the same holds for analogous coverings in arbitrary dimension. Whether (c) can be dealt with these methods is not clear.

Theorem 3.2 Let $M$ be an $n$-dimensional Fano manifold that admits a double covering $\pi$ over $\mathbb{P}^{n}$ with branching divisor a smooth hypersurface of degree $2 d$, with $\frac{n+1}{2}<d \leq n$. Then $M$ admits a Kähler-Einstein metric.

Proof. That these coverings are smooth depends on the fact that the branching divisor is smooth, see [3, p. 42]. Recall that given for a double cover $\pi: M \rightarrow N$, the ramification and branching divisors are related by

$$
\begin{equation*}
R=\frac{1}{2} \pi^{*} B \tag{54}
\end{equation*}
$$

Since $B=\mathcal{O}(2 d), R=\pi^{*} \mathcal{O}(d)$ and it follows from Hurwitz formula that $-K_{M}=\pi^{*} \mathcal{O}(n+1-d)$. Therefore $R=-\beta K_{M}$ with

$$
\begin{equation*}
\beta=\frac{d}{n+1-d} . \tag{55}
\end{equation*}
$$

As the pull-back of an ample line bundle by a finite map is ample, these manifolds are Fano for $1 \leq d \leq n$. In order to apply Theorem 2.5 we only need to check that (50) holds, with $d=2$, i.e. $1<\beta$. And this holds if and only if $d>(n+1) / 2$.
Q.D.E.

Theorem 3.3 Let $M$ be an n-dimensional Fano manifold that is a double cover of the quadric $Q_{n} \subset \mathbb{P}^{n}$ ramified along a smooth divisor cut out by a hypersurface of degree $2 d$, with $\frac{n}{2}<d<n$. Then $M$ admits a Kähler-Einstein metric.

Proof. Denote by $\pi: M \rightarrow Q_{n}$ the covering and by $i: Q_{n} \hookrightarrow \mathbb{P}^{n+1}$ the inclusion. Put $\varphi=i \pi$. Then $B=i^{*} \mathcal{O}(2 d)$, and $R=\frac{1}{2} \pi^{*} B=\varphi^{*} \mathcal{O}(d)$, while $-K_{M}=\varphi^{*} \mathcal{O}(n)-\varphi^{*} \mathcal{O}(d)=\varphi^{*} \mathcal{O}(n-d)$. So for $d<n M$ is Fano and

$$
\beta=\frac{d}{n-d} .
$$

When $2 d>n, \beta>1$, and Theorem 2.5 yields the result. Q.D.E.
In case $n=3, d$ has to be equal to 2 , i.e. $B$ is cut out by a quartic, and the branching divisor is an octic hypersurface contained in $Q_{3}$. When $d=1$ (and $n$ arbitrary), it is not possible to apply Theorem [2.6] but in this case the manifold $M$ is simply the intersection of two quadrics.

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