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Robust exponential hedging in a Brownian setting

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Abstract

This paper studies the robust exponential hedging in a Brownian factor model, giving a solvable example using a PDE argument. The dual problem is reduced to a standard stochastic control problem, of which the HJB equation admits a classical solution. The optimal strategy will be expressed in terms of the solution to the HJB equation.

Keywords robust utility maximization, stochastic control, duality

Research Activity Group Mathematical Finance

1. Introduction

This paper aims to provide a solvable example for the robust exponential hedging problem studied by [1]:

$$\text{minimize } \sup_{P\in\mathcal{P}} E^P[e^{-\alpha(\theta\cdot S_T-H)}], \quad \text{over } \theta\in\Theta. \tag{1}$$

Here S is a d-dim. càdlàg locally bounded semimartingale on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},R)$, $\mathcal{P}$ is a convex set of probability measures absolutely continuous w.r.t. R, H is a random variable and Θ is a set of admissible integrands for S. The set $\mathcal{P}$ is a mathematical expression of *model uncertainty*, and (1) is equivalent to the maximization of the *robust exponential utility* from the net terminal wealth for the seller of the claim H.

The problem (1) is solved via its dual:

$$\text{minimize } \mathcal{H}(Q|P)-\alpha E^Q[H], \quad \text{over } (Q,P)\in\mathcal{Q}_f\times\mathcal{P}, \tag{2}$$

where $\mathcal{H}(\cdot|\cdot)$ denotes the relative entropy, and $\mathcal{Q}_f$ is the set of R-absolutely continuous local martingale measures for S, having finite relative entropy with some $P\in\mathcal{P}$.

Assume:

(A1) $\{dP/dR : P\in\mathcal{P}\}$ is weakly compact in $L^1(R)$.

(A2) $\mathcal{Q}_f^e(S):=\{Q\in\mathcal{Q}_f : Q\sim R\}\neq\emptyset$.

(A3) $\{e^{\alpha|H|}dP/dR : P\in\mathcal{P}\}$ is uniformly integrable and $\sup_{P\in\mathcal{P}} E^P[e^{(\alpha+\varepsilon)|H|}]<\infty$, for some $\varepsilon>0$.

Under (A1)–(A3), [1] shows that the dual problem (2) of (1) admits a solution $(\widehat{Q}_H,\widehat{P}_H)\in\mathcal{Q}_f\times\mathcal{P}$ which is maximal in that if $(\widetilde{Q},\widetilde{P})\in\mathcal{Q}_f\times\mathcal{P}$ is another solution, then $\widetilde{P}\ll\widehat{P}_H$ and $d\widetilde{Q}/d\widetilde{P}=d\widehat{Q}_H/d\widehat{P}_H$, $\widetilde{P}$-a.s. This solution has a kind of martingale representation:

$$\frac{d\widehat{Q}_H}{d\widehat{P}_H}=\hat{c}\cdot e^{-\alpha(\hat{\theta}\cdot S_T-H)}, \quad \widehat{Q}_H\text{-a.s.}, \tag{3}$$

where $\hat{c}$ is a constant, and $\hat{\theta}$ is a predictable $(S,\widehat{Q}_H)$-integrable process such that $\hat{\theta}\cdot S$ is a $\widehat{Q}_H$-martingale. Finally, if we assume additionally:

(A4) $\widehat{Q}_H\sim R$,

the strategy $\hat{\theta}$ is shown to be optimal for (1) with the admissible class Θ_H defined as the set of all (S,R)-integrable predictable processes θ such that $\theta\cdot S$ is a martingale under all $Q\in\mathcal{Q}_f$ with $\mathcal{H}(Q|\widehat{P}_H)<\infty$.

In the sequel, we investigate this problem in a specific setting for which the optimal strategy $\hat{\theta}$ is explicitely represented, using a standard stochastic control technique.

2. Main results

This section states the main results of this paper. All proofs are collected in Section 4.

2.1 Setup

Let $W=(W^1,W^2)$ be a 2-dimensional R-Brownian motion, $(\mathcal{F}_t)_{t\in[0,T]}$ be its augmented natural filtration. Suppose that the price process S is given by the SDE:

$$\begin{cases} dS_t = S_t(b(Y_t)dt+\sigma(Y_t)dW_t^1), \\ dY_t = g(Y_t)dt+\rho dW_t^1+\bar{\rho}dW_t^2, \end{cases} \tag{4}$$

where $\rho\in[-1,1]$ and $\bar{\rho}=\sqrt{1-\rho^2}$. The set $\mathcal{P}$ of candidate models is given as follows. Let C be a convex compact subset of $\mathbb{R}^2$ containing the origin, and $\mathcal{I}_{\mathcal{P}}$ be the set of 2-dimensional predictable C-valued processes. Then we set

$$\mathcal{P}:=\left\{P^\nu\sim R : \frac{dP^\nu}{dR}=\mathcal{E}_T(-\nu\cdot W),\ \nu\in\mathcal{I}_{\mathcal{P}}\right\}, \tag{5}$$

where $\mathcal{E}(M):=\exp(M-\langle M\rangle/2)$ denotes the *Doléans-Dade exponential* of a continuous local martingale M. Finally, the claim H is assumed to be of the form $H=h(Y_T)$ for some measurable function h.

Remark 1 *A typical situation underlying our setup is as follows. A financial institution sells an option written on an untradable index Y, and want to maximize her utility by trading an asset S which is correlated to Y. However, the probabilistic model of assets (S,Y) is uncertain in its expected rate of return (drift, in mathematical language). Actually, the dynamics under the proba-*

bility P^ν is:

$$\begin{cases} dS_t = S_t((b(Y_t) - \nu^1_t \sigma(Y_t))dt + \sigma(Y_t)dW^{1,\nu}_t), \\ dY_t = (g(Y_t) - \rho\nu^1_t - \bar\rho\nu_2)dt + \rho dW^{1,\nu}_t + \bar\rho dW^{2,\nu}_t. \end{cases}$$

In this context, we can know only the range *of the drift through the set C appearing in the definition of $\mathcal{P}$.*

In what follows, we assume

(B1) $b, \sigma, g \in C^2_b(\mathbb{R})$, where $C^2_b(\mathbb{R}) = \{f \in C^2(\mathbb{R}) : f, f', f''$ are bounded$\}$.

(B2) For some $k > 0$, $\sigma(y) \geq k$ for all y.

(B3) $h \in C^2(\mathbb{R})$, h' is bounded and h'' has a polynomial growth.

Our first task is to check that:

Lemma 2 *Under* (B1) – (B3)*, the conditions* (A1) – (A4) *of* [1] *are satisfied.*

Once this lemma is established, an optimal strategy $\hat\theta$ will be derived via (i) solving the dual problem (2), and (ii) finding $\hat\theta$ satisfying (3).

Remark 3

(I) *In this setting, we can show that*

$$\mathcal{H}(Q|P) < \infty \text{ for some } P \in \mathcal{P} \ \Leftrightarrow\ \mathcal{H}(Q|R) < \infty, \tag{6}$$

for all local martingale measures Q. In particular, Θ_H is characterized as the class of predictable (S, R)-integrable processes θ such that $\theta \cdot S$ is a martingale under all absolutely continuous local martingale measures Q with $\mathcal{H}(Q|R) < \infty$. This condition is further reduced to "all equivalent *martingale measures with...". Therefore, the class Θ_H is actually independent of $\widehat{P}_H$, hence of H. This point is conceptually important since the dependence of Θ on $\widehat{P}_H$, which is a part of the* solution *to the dual problem, implies that we can not specify the admissible class for the primal problem until we solve the dual problem.*

(II) *For our purpose, it suffices to consider $\mathcal{Q}^e_f$ for the domain of dual problem since we already know that a solution to the dual problem is obtained in $\mathcal{Q}^e_f \times \mathcal{P}$. Let $\mathcal{I}_M$ be the set of predictable processes η with $E^R[\int_0^T \eta_t^2 dt] < \infty$, and $E^R[\mathcal{E}_T(-(\lambda(Y), \eta)\cdot W)] = 1$, where $\lambda := b/\sigma$, and*

$$\frac{dQ^\eta}{dR} := \mathcal{E}_T(-(\lambda(Y), \eta) \cdot W), \quad \eta \in \mathcal{I}_M. \tag{7}$$

Then $\mathcal{Q}^e_f = \{Q^\eta : \eta \in \mathcal{I}_M\}$.

2.2 Dual problem

Let

$$J^{\eta,\nu}_t := E^\eta \left[\alpha h(Y_T) - \frac{1}{2}\int_t^T \|\nu_s - (\lambda(Y_s), \eta_s)'\|^2 ds | \mathcal{F}_t \right],$$

where $E^\eta[\,\cdot\,]$ denotes the expectation under Q^η, " $'$ " is the transpose, and $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^2$. The dual problem (2) is now reduced to the following stochastic control problems:

$$\text{maximize } J^{\eta,\nu}_0 \quad \text{among } (\eta, \nu) \in \mathcal{I}_M \times \mathcal{I}_\mathcal{P}. \tag{8}$$

For each constant $\eta \in \mathbb{R}$, set

$$\begin{aligned} \mathcal{A}^\eta &:= (g - \rho\lambda - \bar\rho\eta)\partial_y + \frac{1}{2}\partial_{yy} \\ &= \mathcal{A}^0 - \bar\rho\eta\partial_y, \end{aligned} \tag{9}$$

where $\partial_y := \partial/\partial y$ and $\partial_{yy} := \partial^2/\partial y^2$ etc. Then the HJB equation for (8) is formally given by

$$\begin{cases} v_t + \sup_{(\eta,\nu)\in\mathbb{R}\times C}\left(\mathcal{A}^\eta v - \frac{1}{2}\|\nu - (\lambda, \eta)'\|^2\right) = 0, \\ v(T, y) = \alpha h(y). \end{cases} \tag{10}$$

Theorem 4 *The HJB equation* (10) *admits a unique classical solution $v \in C^{1,2}((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ such that $v_y := \partial_y v$ is bounded. Then we can choose measurable functions $\hat\nu : [0, T] \times \mathbb{R} \longrightarrow C$ and $\hat\eta : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ so that*

$$\begin{cases} \hat\nu(t, y) \in \arg\inf_{\nu\in C}\left(\frac{1}{2}(\nu_1 - \lambda(y))^2 + \nu_2\bar\rho v_y(t, y)\right), \\ \hat\eta = \hat\nu_2(t, y) - \bar\rho v_y(t, y), \end{cases}$$

and $(\hat\nu_\cdot, \hat\eta_\cdot) := (\hat\nu(\cdot, Y_\cdot), \hat\eta(\cdot, Y_\cdot))$ is an optimal control for (8)*. In particular, $(Q^{\hat\eta}, P^{\hat\nu})$ is a solution to* (2)*.*

2.3 Optimal strategy

We now give a representation of an optimal strategy $\hat\theta$ via Theorem 4 and the duality result of [1].

Theorem 5 *An optimal strategy for the problem* (1) *is given by*

$$\hat\theta_t = \frac{\rho v_Y(t, Y_t) + \lambda(Y_t) - \hat\nu_1(t, Y_t)}{\alpha\sigma(Y_t)S_t}. \tag{11}$$

Remark 6 *Here we give a brief review of related literature. In the case without uncertainty i.e., $\mathcal{P} = \{R\}$ ($\Leftrightarrow$ $C = \{(0, 0)\}$ in our setup), explicit solutions to exponential hedging through duality are studied by* [2] *using BSDE arguments with the help of Malliavin calculus, and by* [3] *using PDE arguments close to ours.*

There are also a few recent works deriving explicit form of optimal strategies for robust utility maximization. Our setup and idea for the proof of Theorem 4 are due to [4]*, where robust power utility maximization is considered. See also* [5] *for the case of logarithmic utility.*

3. Explicit examples

This section provides two explicit examples which may be reduced to *linear* PDEs, hence can be computed via both elementary numerical schemes and the Feynman-Kac formula. Recall that our model is characterized by the compact set C, and the HJB equation takes the form:

$$\begin{cases} v_t + \mathcal{A}^0 v + \frac{\bar\rho^2 v_y^2}{2} - l(y, v_y) = 0, \\ v(T, y) = \alpha h(y), \end{cases}$$

where

$$l(y, p) := \inf_{\nu\in C}\left(\frac{1}{2}(\nu_1 - \lambda(y))^2 + \bar\rho\nu_2 p\right).$$

Thus, if $l(y, p)$ can be explicitly calculated, then we can expect an explicit solution.

3.1 The case of disk

We first consider the case where the set C is a disk in $\mathbb{R}^2$ with radius r:

$$C = \{x \in \mathbb{R}^2 : \|x\| \le r\}. \tag{12}$$

But due to a technical difficulty, we assume the drift b of S under R is identically zero, or equivalently, λ is identically zero. In this case,

$$\begin{aligned} l(y,p) &= \inf_{\|\nu\|\le r}\left(\frac{\nu_1^2}{2} + \bar{\rho}\nu_2 p\right) \\ &= -r\bar{\rho}|p|, \end{aligned}$$

and $\hat{\nu}(y,p) = (0, -r\cdot\mathrm{sgn}(p))$ is a minimizer. Then the HJB equation is written as:

$$v_t + \mathcal{A}^0 v + \frac{\bar{\rho}^2 v_y^2}{2} + r\bar{\rho}|v_y| = 0.$$

Now suppose that the payoff function h is non-increasing. Then noting that the 1-dimensional stochastic flow associated to Y is order-preserving under (B1) and (B2), the value function is also decreasing in y variable, hence $v_y \le 0$. Therefore the term $r\bar{\rho}|v_y|$ in the equation is replaced by $-r\bar{\rho}v_y$. Moreover, *changing the drift*, the equation becomes:

$$v_t + \mathcal{A}^{r\bar{\rho}} v + \frac{\bar{\rho}^2 v_y^2}{2} = 0.$$

Here $\mathcal{A}^{r\bar{\rho}}$ is the generator of Y under $Q^{r\rho}$. Note that a simple calculation using the Itô formula yields:

$$de^{\bar{\rho}^2 v(t,Y_t)} = \bar{\rho}^2 e^{\bar{\rho}^2 v(t,Y_t)} v_y(t,Y_t) d\bar{W}_t^{r\bar{\rho}}.$$

Thus $e^{\bar{\rho}^2 v(t,Y_t)}$ is a martingale, and since $v(T,y) = \alpha h(y)$,

$$\begin{aligned} v(t,y) &= \frac{1}{\bar{\rho}^2}\log E^{r\bar{\rho}}\left[e^{\alpha\bar{\rho}^2 h(Y_T)}\middle| Y_t = y\right] \\ &=: \frac{1}{\bar{\rho}^2}\tilde{v}(t,y). \end{aligned}$$

Now the Feynman-Kac formula yields:

Corollary 7 *Suppose that C is given by (12), $\lambda \equiv 0$ and h is non-increasing. Then the value function is represented as*

$$v(t,y) = \frac{1}{\bar{\rho}^2}\log\tilde{v}(t,y),$$

where $\tilde{v}$ is the solution to the Cauchy problem:

$$\begin{cases} \tilde{v}_t + \mathcal{A}^{r\bar{\rho}}\tilde{v} = 0, \\ \tilde{v}(T,y) = e^{\alpha\bar{\rho}^2 h(y)}. \end{cases} \tag{13}$$

Furthermore, $(\hat{\eta},\hat{\nu}) = (r - \bar{\rho}(\tilde{v}_y/\tilde{v})(\cdot,Y), 0, r)$ is an optimal control, and an optimal portfolio strategy is given by

$$\hat{\theta}_t = \frac{\rho}{\alpha\bar{\rho}^2}\frac{\tilde{v}_y(t,Y_t)}{\tilde{v}(t,Y_t)\sigma(Y_t)S_t}. \tag{14}$$

Remark 8 *The case of* non-decreasing h *can be treated in a symmetric way.*

3.2 The case of rectangle

Let C be a rectangle in $\mathbb{R}^2$, that is:

$$C = \{x \in \mathbb{R}^2 : |x_1| \le m_1,\ |x_2| \le m_2\}. \tag{15}$$

In this case,

$$\begin{aligned} l(y,p) &= \frac{1}{2}(\hat{\nu}_1(y) - \lambda(y))^2 + \bar{\rho}\hat{\nu}_2(p)p \\ &= \frac{k(y;m_1)}{2} - \bar{\rho}m_2|p|, \end{aligned}$$

where

$$\begin{aligned} &\hat{\nu}_1(y) = \mathrm{sgn}(\lambda(y))(|\lambda(y)| \wedge m_1), \\ &\hat{\nu}_2(p) = -m_2\mathrm{sgn}(p), \quad k(y;m_1) := \{(|\lambda(y)| - m_1)^+\}^2. \end{aligned}$$

Therefore, the HJB equation is written as:

$$v_t + \mathcal{A}^0 v + \frac{\bar{\rho}^2 v_y^2}{2} + \bar{\rho}m_2|v_y| - \frac{k(y;m_1)}{2} = 0. \tag{16}$$

As in the case of disk, if the value function is monotone (e.g., h is non-increasing and λ is constant), the *linearization procedure* as in the previous subsection yields a linear PDE and a Feynman-Kac representation.

4. Proofs

Proof of Lemma 2 (A1) is guaranteed by [4, Lemma 3.1] and [6, Lemma 3.2]. The function $b/\sigma =: \lambda$ is bounded by the assumptions (B1) and (B2). Therefore $dQ^0/dR := \mathcal{E}_T(-(\lambda(Y),0)\cdot W)$ defines an equivalent local martingale measure. Since $R \in \mathcal{P}$ and $\mathcal{H}(Q^0|R) = E^R[\int_0^T \lambda(Y_s)^2 ds]/2 < \infty$, (A2) is satisfied. Also, (B3) implies that h is globally Lipschitz continuous, hence admits a constant K_h such that $|h(y)| \le K_h(1+|y|)$ for all $y \in \mathbb{R}$. Then (A3) will be verified by checking that $\{e^{\gamma|h(Y_T)|}\mathcal{E}_T(-\nu\cdot W) : \nu \in \mathcal{I}_{\mathcal{P}}\}$ is bounded in $L^2(R)$ for any $\gamma > \alpha$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} &E^R\left[\left(e^{\gamma|h(Y_T)|}\mathcal{E}_T(-\nu\cdot W)\right)^2\right] \\ &\quad \le E^R\left[e^{4\gamma|h(Y_T)|}\right]^{\frac{1}{2}} E^R\left[e^{-4\nu\cdot W_T}\right]^{\frac{1}{2}}. \end{aligned} \tag{17}$$

Introducing another R-Brownian motion $\bar{W} = \rho W^1 + \bar{\rho}W^2$,

$$\begin{aligned} e^{4\gamma|h(Y_T)|} &\le e^{4\gamma K_h(1+|Y_T|)} \\ &\le e^{4\gamma K_h(1+|Y_0|+\|g\|_\infty T + |\bar{W}_T|)}. \end{aligned}$$

Therefore, the first component in the RHS of (17) is bounded by $\sqrt{2}e^{2\gamma K_h(1+|Y_0|+(\|g\|_\infty + 2\gamma K_h)T)}$. For the second, we can apply [7, Th. III 39] to get an upper bound $e^{8T(\mathrm{diam}C)^2}$. Thus (A3) is verified, and the dual problem admits a maximal solution $(\widehat{Q}_H, \widehat{P}_H)$. Finally, (A4) is trivially satisfied since all $P \in \mathcal{P}$ are equivalent.

(QED)

For the proof of Theorem 4, we first consider a family of auxiliary control problems, restricting the domain of η. For each closed interval $I \subset \mathbb{R}$, set $\mathcal{I}_M^I := \{\eta \in \mathcal{I}_M : \eta_t \in I\,\forall t,\ \text{a.s.}\}$, and consider the equation:

$$\begin{cases} \partial_t v^I + \sup\limits_{\eta\in I,\nu\in C}\left\{\mathcal{A}^\eta v^I - \frac{1}{2}\|\nu - (\lambda(y),\eta)'\|^2\right\} = 0, \\ v^I(T,y) = \alpha h(y). \end{cases} \tag{18}$$

If I is compact, then so is $I \times C$, hence we can apply Theorem VI.4.1 and VI.6.2 of [8] to get:

Lemma 9 *For each compact $I \subset \mathbb{R}$, (18) admits a unique classical solution $v^I \in C_p^{1,2}((0,T)\times\mathbb{R})\cap C([0,T]\times\mathbb{R})$. Then taking $(\eta^I(t,y),\nu^I(t,y)) \in \arg\sup_{\eta\in I,\nu\in C}\{\mathcal{A}^\eta v^I - \|\nu-(\lambda(y),\eta)'\|^2/2\}$, we have*

$$v^I(t,Y_t) = \operatorname*{ess\,sup}_{\eta\in\mathcal{I}_M^I,\nu\in\mathcal{I}_\mathcal{P}} J_t^{\eta,\nu} = J_t^{\eta^I(\cdot,Y),\nu^I(\cdot,Y)}.$$

Lemma 10 *There exists a constant K_v such that $|v_y^I| \le K_v$ for all compact I.*

Proof Let $J_t^{\eta,\nu}(y) := E^\eta[\alpha h(Y_{t,T}(y)) - (1/2)\int_t^T \|\nu_s - (\lambda(Y_{t,s}(y),\eta_s)'\|^2 ds]$, where $Y_{t,T}$ denotes the stochastic flow associated to Y. Then noting that $|\sup_x f(x) - \sup_x g(x)| \le \sup_x |f(x)-g(x)|$, it suffices to show the existence of a constant K_v such that $|J_t^{\eta,\nu}(y) - J_t^{\eta,\nu}(y')| \le K_v|y-y'|$ for all $t\in[0,T]$, $y,y'\in\mathbb{R}$ and $(\eta,\nu)\in\mathcal{I}_M\times\mathcal{I}_\mathcal{P}$.

Since h, g, $\lambda \in C_b^2$, a simple computation yields that

$$\begin{aligned}
&|J_t^{\eta,\nu}(y) - J_t^{\eta,\nu}(y')| \\
&\quad\le \alpha K_h E^\eta\left[|Y_{t,T}(y) - Y_{t,T}(y')|\right] \\
&\qquad + \widetilde{K}K_\lambda \int_t^T E^\eta[|Y_{t,s}(y) - Y_{t,s}(y')|]ds,
\end{aligned}$$

where K_h, K_λ are Lipschitz constants for h,λ, respectively, and $\widetilde{K} = \operatorname{diam}(C) + \max\lambda$. Also, $\forall s\in[t,T]$,

$$\begin{aligned}
&E^\eta\left[|Y_{t,s}(y) - Y_{t,s}(y')|\right] \\
&\quad\le |y-y'| + E^\eta\left[\int_t^s |g(Y_{t,u}(y) - g(Y_{t,u}(y'))|du\right] \\
&\quad\le |y-y'| + K_g\int_t^s E^\eta[|Y_{t,u}(y) - Y_{t,u}(y')|]du,
\end{aligned}$$

where K_g is a Lipschitz constant for g. Then the Gronwall inequality shows that $E^\eta[|Y_{t,s}(y) - Y_{t,s}(y')|] \le e^{K_g(s-t)}|y-y'| \le e^{K_gT}|y-y'|$ for any $t\le s\le T$. Hence we get the result with $K_v = e^{K_gT}(\alpha K_h + \widetilde{K}K_\lambda T)$.

(QED)

Proof of Theorem 4 The inside of the bracket in (18) is written as:

$$\begin{aligned}
&\mathcal{A}^0v^I + \bar\rho(v_y^I)^2 - \frac{1}{2}\left\{\eta - (\nu_2 - \bar\rho v_y^I)\right\}^2 \\
&\quad - \left\{\frac{1}{2}(\lambda(y)-\nu_1)^2 + \nu_2\bar\rho v_y^I\right\}.
\end{aligned}$$

Here the third term attains the global maximum at $\eta^I = \nu_2 - \bar\rho v_y^I$, which is bounded by $\operatorname{diam}(C) + K_v$ independently of I. Thus taking $I_0 := [-\operatorname{diam}(C) - K_v, \operatorname{diam}(C)+K_v]$, we have

$$\begin{aligned}
-\partial_t v^{I_0} &= \sup_{\eta\in I_0,\nu\in C}\left\{\mathcal{A}^\eta v^{I_0} - \frac{1}{2}\|\nu - (\lambda(y),\eta)'\|^2\right\} \\
&= \sup_{\eta\in\mathbb{R},\nu\in C}\left\{\mathcal{A}^\eta v^{I_0} - \frac{1}{2}\|\nu - (\lambda(y),\eta)'\|^2\right\}.
\end{aligned}$$

Hence $v := v^{I_0}$ is a desired classical solution to (10).

The rest of the proof is a standard verification argument, and we omit this.

(QED)

Proof of Theorem 5 By the duality, it suffices to show that $\hat\theta\in\Theta$ and

$$\frac{dQ^{\hat\eta}}{dP^{\hat\nu}} = \frac{e^{-\alpha(\hat\theta\cdot S_T - h(Y_T))}}{E^{P^{\hat\nu}}\left[e^{-\alpha(\hat\theta\cdot S_T - h(Y_T))}\right]}.$$

Since v satisfies the HJB equation (10), the Itô formula yields:

$$\begin{aligned}
\alpha h(Y_T) &= v(0,Y_0) + \int_0^T \left(\partial_t + \mathcal{A}^{\hat\eta}\right)v(s,Y_s)ds \\
&\qquad + \int_0^T v_y(s,Y_s)d\bar W_s^{\hat\eta} \\
&= v(0,Y_0) + \log\frac{dQ^{\hat\eta}}{dP^{\hat\nu}} \\
&\qquad + \int_0^T (\rho v_y + \lambda - \hat\nu_1)(s,Y_s)dW_s^{1,\hat\eta} \\
&= v(0,Y_0) + \log\frac{dQ^{\hat\eta}}{dP^{\hat\nu}} + \alpha\hat\theta\cdot S_T.
\end{aligned}$$

Therefore we get $dQ^{\hat\eta}/dP^{\hat\nu} = e^{v(0,Y_0)}e^{-\alpha(\hat\theta\cdot S_T - h(Y_T))}$. Finally,

$$\int_0^T \hat\theta_s^2 d\langle S\rangle_s = \frac{1}{\alpha^2}\int_0^T\{(\rho v_y + \lambda - \nu_1)(s,Y_s)\}^2ds$$

is bounded, hence $\hat\theta\cdot S$ is a martingale under every $Q\in\mathcal{Q}_f^e$. This concludes the proof.

(QED)

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