# AN APPLICATION OF STEFAN'S PROBLEM TO THE FREEZING OF A CYLINDRICAL FOOD-STUFF*

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The analytical solution of the problem involving a change of phase with convection at the surface is very difficult. Previously, heat conduction problems with a change of phase (called Neumann's problem) have been solved. The concept may be used to obtain analytical solutions for a semi-infinite solid and an infinite cylinder in the freezing of foods.

This paper presents an analytical Stefan-type solution for a cylinder with convection at the surface and experimental results of the temperature distribution, fusion front moving and the time required to freeze a cylinder in the freezing of cylindrical food stuff.

## Introduction

Problems involving a change of phase are of practical importance in the technical field. Typical applications of these problems to practical cases include the freezing of foods, ice formation and the solidification of metals in the casting process. Recently two heat conduction problems involving a change of phase (called Neumann's problem) have been solved to obtain analytical solutions for a semi-infinite solid and an infinite cylinder in the freezing of foods[3,4].

Assuming that the temperature in the liquid layer is kept at the melting temperature all the time, Neumann's problem will be mathematically converted into the Stefan type. This paper presents an analytical Stefan-type solution of the temperature distribution and the position of the fusion front in the freezing of a cylindrical ice cream.

## Statement of the Problem

The schematic model used in this problem to be analyzed is shown in **Fig. 1**. In this case, the temperature of the cylinder is at the melting temperature initially and freezing proceeds toward the center of the cylinder. Hence two solutions must be sought, that is, the temperature distribution in the solid phase, which is a function of both time and radius, and the position of the solidification front, which changes with time.

Since the temperature in the liquid phase is kept at the melting temperature all the time, there is no flow of heat.

To solve this problem, the following assumptions are made;

1) heat flows only in the direction of the radius;
2) physical properties are independent of temperature;
3) material is homogeneous;
4) natural convection does not occur in the liquid portion;
5) freezing takes place at a single, unique temperature and only at the solid-liquid interface;
6) dilatation of bulk due to the change of phase is neglected.

Upon these assumptions, the above solutions are obtained by solving a one-dimensional conduction equation, with suitable boundary and initial conditions.

Thus, the differential equation of the cylindrical coordinate for the solid phase may be expressed as:

$$\frac{\partial T}{\partial \theta} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial T}{\partial r}\right) \tag{1}$$

If the crust thickness increases by $d\delta$ in time $d\theta$, as shown in **Fig. 2**, the heat balance at the solidification front is

$$K\frac{\partial T}{\partial r} = -L\rho\frac{d\delta}{d\theta} \quad \text{at} \quad r = R-\delta \tag{2}$$

where $L$ is the latent heat of fusion. The other boundary conditions and initial condition are:

$$-K\frac{\partial T}{\partial r} = hT \quad \text{at} \quad r = R \tag{3}$$

$$T = T_f \quad \text{at} \quad r = R-\delta \tag{4}$$

$$T = T_f \quad \text{at} \quad \theta = 0 \tag{5}$$

where the temperature $T$ is expressed on the basis of the temperature of the cold air, namely

$$T = t - t_a \qquad T_f = t_f - t_a \tag{6}$$

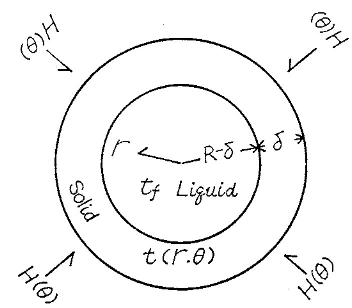



Fig. 1 Schematic model

* Received on December 9, 1968
"Heat Transfer Problem involving a Change of Phase" 6-th Symposium at Nagoya, Nov. 27, 1967

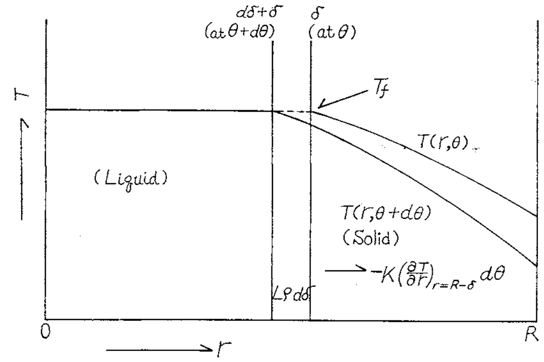



**Fig. 2** A model of fusion front

There is no doubt that the crust thickness is zero at the beginning of freezing.

## Analytical Solution

A solution of Eq. (1) that satisfies the boundary condition Eq. (3) and initial condition Eq. (5) is

$$\frac{T}{T_f} = \sum_{m=1}^{\infty} V_m J_0(r\alpha_m) \tag{7}$$

where $J_0(x)$ is a zero-order Bessel function of the first kind. To satisfy the boundary condition Eq. (3) $\alpha_m$ must be a root of

$$\alpha_m J_1(R\alpha_m) - HJ_0(R\alpha_m) = 0 \tag{8}$$

This equation has an infinite number of real positive roots. $V_m$ is expressed as:

$$V_m = \frac{2He^{-k\alpha_m^2\theta}}{R(H^2+\alpha_m^2)J_0(R\alpha_m)} \tag{9}$$

Conforming the analysis for this problem to Neumann's solution for the semi-infinite solid[2], it is supposed that the general solution is given by:

$$\frac{T}{T_f} = A\sum_{m=1}^{\infty} V_m J_0(r\alpha_m) + B \tag{10}$$

where $A$ and $B$ are constants, to satisfy Eqs. (1), (3) and (4). On the other hand, as has been shown in previous papers[3,4], the crust thickness is supposed to be given by:

$$\delta = n\sqrt{\theta} \tag{11}$$

In Eqs. (10) and (11), there are three unknown quantity terms $A$, $B$ and $n$. $A$ and $B$ are determined from boundary conditions Eqs. (3) and (4). $n$ is a numerical constant to be determined from the boundary condition, Eq. (2) by the trial and error method[3,4].

Substituting Eq. (10) in boundary conditions, Eqs. (3) and (4), the values of $A$ and $B$ are given by:

$$\left.\begin{aligned} A &= \frac{1}{\sum_{m=1}^{\infty} V_m J_0\{(R-\delta)\alpha_m\}} \\ B &= 0 \end{aligned}\right\} \tag{12}$$

Substituting Eqs. (10) and (11) into Eq. (2) gives

$$\frac{KT_f\sum_{m=1}^{\infty} V_m\alpha_m J_1\{(R-\delta)\alpha_m\}}{\sum_{m=1}^{\infty} V_m J_0\{(R-\delta)\alpha_m\}} = \frac{L\rho n}{2\sqrt{\theta}} \tag{13}$$

To decide a numerical constant $n$ from Eq. (13), the graphical method is used in this analysis by plotting each value of the left and right hand side term in Eq. (13) against $n$ or $\delta$ on the rectangular coordinate. Thus, three unknown quantity terms may be determined from Eqs. (12) and (13). But, to satisfy Eq. (1), the value of these terms must be constant for all values of time. In this case these terms are the function of time, therefore Eq. (10), which is assumed to be the general solution, does not strictly satisfy the differential equation, Eq. (1).

Accordingly, if these terms are approximately constant with respect to time, the analytical solution of the temperature for the solid phase can be written down from Eqs. (10), (11) and (12) as

$$\frac{T}{T_f} = \frac{\sum_{m=1}^{\infty} V_m J_0(r\alpha_m)}{\sum_{m=1}^{\infty} V_m J_0\{(R-\delta)\alpha_m\}} \tag{14}$$

**Rate of crust formation in the vicinity of the center.** With the proceeding of the solidification front toward the center of the cylinder, Eq. (13) becomes gradually unsuitable to decide a numerical constant $n$ or the crust thickness because the rate of heat flow per unit area will be different from a heat balance given by Eq. (2) due to increasing curvature at the solid-liquid interface. In other words, the boundary condition, Eq. (2), will be not satisfied any longer in the center portion of the cylinder.

For a Stefan's or Neumann's problem of the cylindrical coordinate, a convenient approach to the solution for the time required to freeze the center of the cylinder has been presented in the previous work[4]; viz., an approximate mathematical means utilizing the average volume rate of crust formation.

To obtain the average volume rate of crust formation, the following assumptions are made:

1) physical properties are independent of time;
2) quasi-steady state is assumed;
3) heat transferred to the air is entirely used to remove the latent heat of liquid.

The rate of heat flow per unit area through the resistances due to the crust and the air film is equal to the latent heat of fusion necessary for freezing at the interface $r=r_\delta$ $(=R-\delta)$, that is,

$$\frac{q}{S} = \frac{T_f}{1/h + (R/K)\ln(R/r_\delta)} \tag{15}$$

and

$$\frac{q}{S} = L\rho\cdot\frac{1}{S}\cdot\frac{dV}{d\theta} = L\rho u_v \tag{16}$$

where $\frac{1}{S}\frac{dV}{d\theta}$ and $u_v$ are the volume rate of crust formation per unit area at the moving solidification front in $m^3/hr\cdot m^2$.

Combining Eqs. (15) and (16) gives

$$u_v = \frac{T_f}{L\rho}\cdot\frac{1}{1/h + (R/K)\ln(R/r_\delta)} \tag{17}$$

But since the volume rate of crust formation $u_v$ is equal to zero at $r_\delta=0$, it takes an infinite time to freeze the cylinder, so the time required to freeze the center, as well as the position of the solid-liquid interface in the center portion cannot be obtained from Eq. (17) directly. Therefore, an approximate calculation method will be applied to the center portion of the cylinder inside the critical position of the solid-liquid interface, $r_{c\delta}$, where Eq. (2) is not satisfied. The average volume rate of crust formation in this portion is calculated as

$$\bar{u}_v = \frac{1}{r_{c\delta}} \int_0^{r_{c\delta}} u_v dr_\delta \tag{18}$$

Denoting the time corresponding to the critical position of interface $r_{c\delta}$ with $\theta_c$, the crust thickness for a time $\theta > \theta_c$ is calculated by the following equation

$$\delta = \bar{u}_v \theta_q + n\sqrt{\theta_c} \tag{19}$$

where $\theta = \theta_c + \theta_q$, and the time required to freeze the center portion is

$$\theta_{qp} = \frac{r_{c\delta}}{\bar{u}_v} + \theta_c \tag{20}$$

where the critical position of interface $r_{c\delta}$ at the time $\theta_c$ is decided graphically by plotting $d\delta/d\theta$ in Eq. (13) and $u_v$ in Eq. (17) against $r_\delta/R$ on the rectangular coordinate.

## Description of Equipment and Experiment

**Fig. 3** shows details of a cylindrical container used in this work. The containers are made of a vinyl chloride plate of 0.5mm in thickness. One container, 12.8cm in diam. and 10cm long, has a capacity of 1287cm$^3$ and the other, 15.0cm in diam. (10cm long), has a capacity of 1766cm$^3$.

The food used is a soft ice cream projected out of an ice-cream freezer at the melting temperature. In the containers, Cu-Co thermocouples are inserted in space along the $r$-direction only. The temperature distribution of the crust of the ice cream was measured by mV-meter. In this case, since thickness of crust cannot be found directly from the surface of the cylinder, it was succesively found by measuring individually every half hour the position of the solidification front of several similar containers frozen simultaneously under the same conditions by means of insertion of a fine vinyl chloride stick into the container. To find thickness of crust by this method, the temperature of crust in the respective containers must be a similar profile for all times*.

Exact physical properties for ice cream have not been reported, so data obtained empirically or employed in a confectionery were used. They are tabulated in **Table 1**. Further, the heat transfer coefficient at the surface of the cylinder was found in a similar manner to that described in the previous papers[3,4].

## Discussion and Experimental Results

**Unknown quantity terms $1/A$ and $n$.** It is found that, as far as the values of these terms are independent of time, the analytical solution satisfies completely the fundamental differential equation.

* Temperature difference at discretionary points in the crust of every container was not over a range of 0.3 to 2.1 °C, so temperature profiles of crust in the respective containers were assumed to be similar.

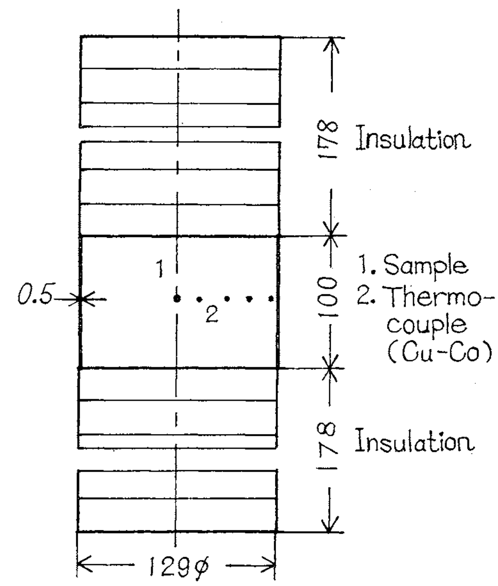
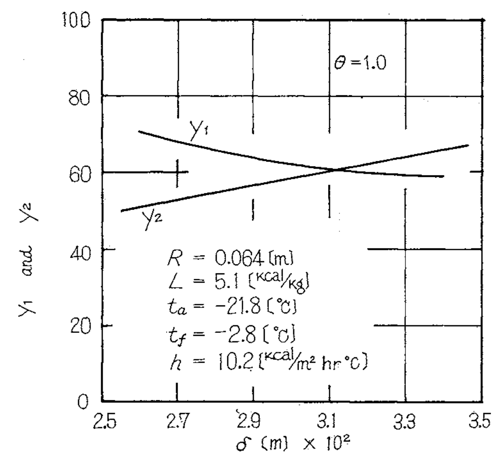



Fig. 3 Details of a container




(where $y_1 = KT_f \sum_{m=1}^{\infty} V_m\alpha_m J_1\{(R-\delta)\alpha_m\} / \sum_{m=1}^{\infty} V_m J_0\{(R-\delta)\alpha_m\}$ and $y_2 = L\rho n/2\sqrt{\theta} = L\rho\delta/2\theta$)

Fig. 4 An example of numerical plot to decide the crust thickness

Table 1 Physical properties of ice cream

| | |
|---|---|
| $C_p$ [kcal/kg·°C] | 0.40 |
| $K$ [kal/m·hr·°C] | 0.24 |
| $\rho$ [kg/m$^3$]] | 767 |
| $L$ [kcal/kg] | 5.1 (R=0.064m)<br>3.0 (R=0.075m) |

However, the values of these terms could not be exactly constant with respect to time except in special cases. Hence the relationship between these values and time must be confirmed in this analysis. **Fig. 4** shows an example of crust thickness determined in the way described above. In **Fig. 5**, the values of $n$ obtained in this manner are plotted against time and Fig. 5 shows simultaneously the relationship between time and several values of $n$ of the semi-infinite solid[3] and an infinite cylinder[4] obtained in previous works.

As shown in Fig. 5, the value of $n$ increases considerably at the beginning and becomes constant as time proceeds. Moreover, the absolute values of $n$ depend on physical properties, particularly the latent heat of fusion of materials.

The values of $1/A$ obtained by Eq. (12) are plotted against time in **Fig. 6**. A variable constant, $1/A$ varies rapidly with time at the beginning, but finally it is consistent with what is indicated in Fig. 5. Accordingly, as these variable constants $1/A$ and $n$ are approximately independent of time except at the beginning of the freezing process, it is considered that the analytical solution, Eq. (14), would be applicable to this problem.

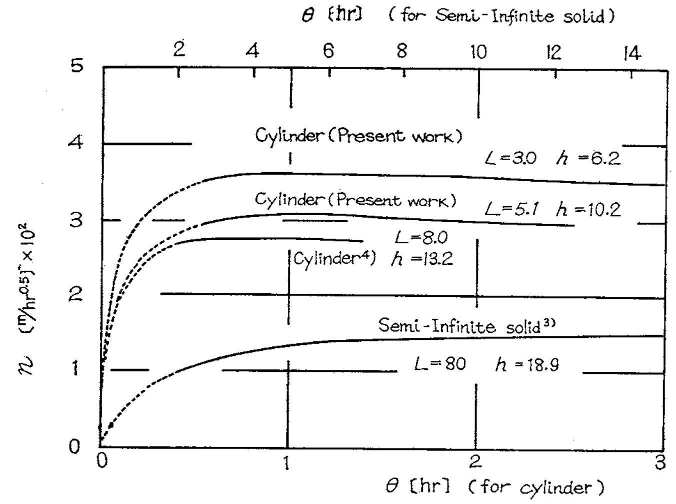



**Fig. 5** Relationship between variable constant $n$ and time

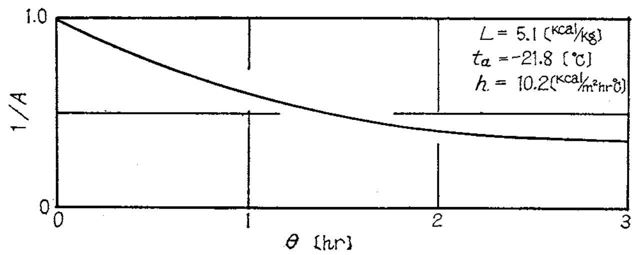



**Fig. 6** Relationship between variable constant $1/A$ and time

**Crust thickness.** The critical position of interface $r_{c\delta}$ is determined in **Fig. 7** where the values of $d\delta/d\theta$ in Eq. (13) and $u_v$ in Eq. (17) are plotted against $r_\delta/R$. In **Fig. 8**, the values of crust thickness calculated from Eqs. (11) and (19) using the value of $r_{c\delta}$ obtained in Fig. 7 are plotted against time. Previously, several investigators[1,5,6,8] have reported analytical or numerical solutions for problems involving the solidification or melting of materials.

But few solutions presented by them have been con-

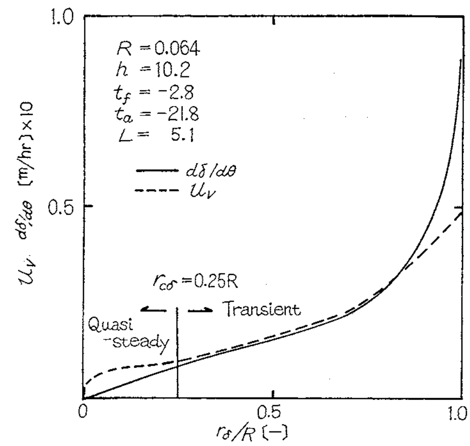



**Fig. 7** Relationship between the rate of crust formation and the position of the solid-liquid interface.

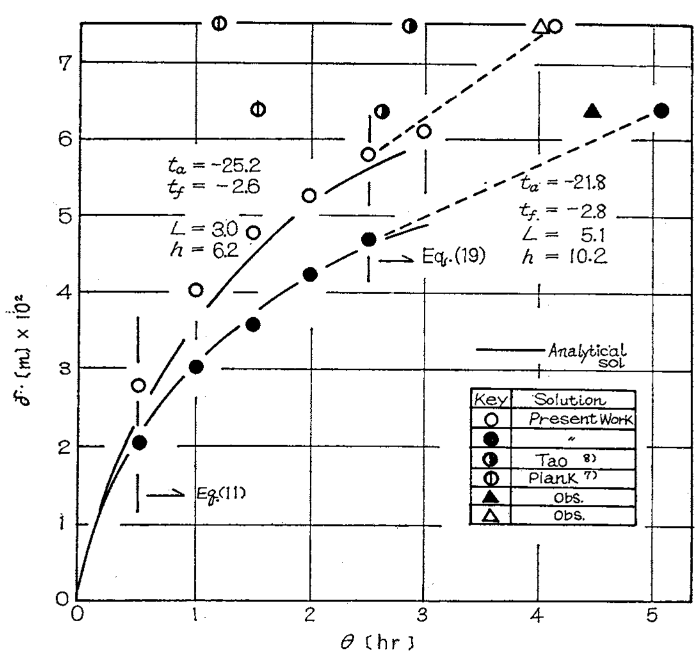



**Fig. 8** Increment of solidified thickness

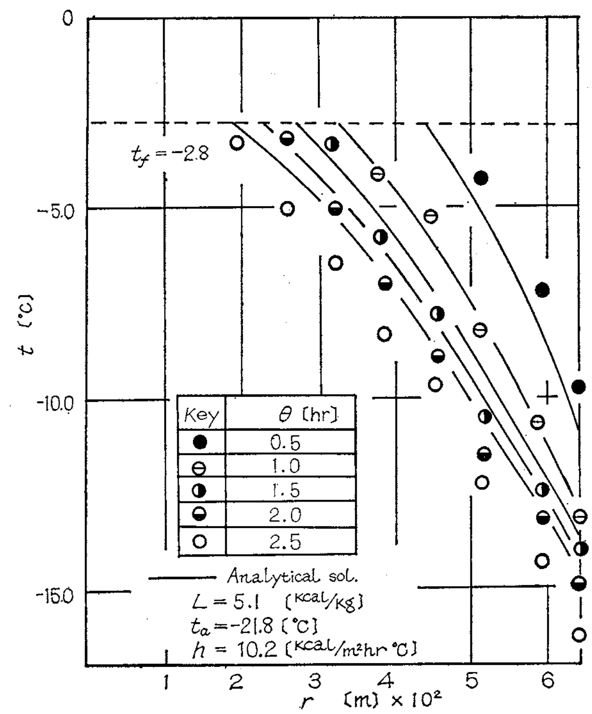



Fig. 9 Temperature distribution in solidified crust

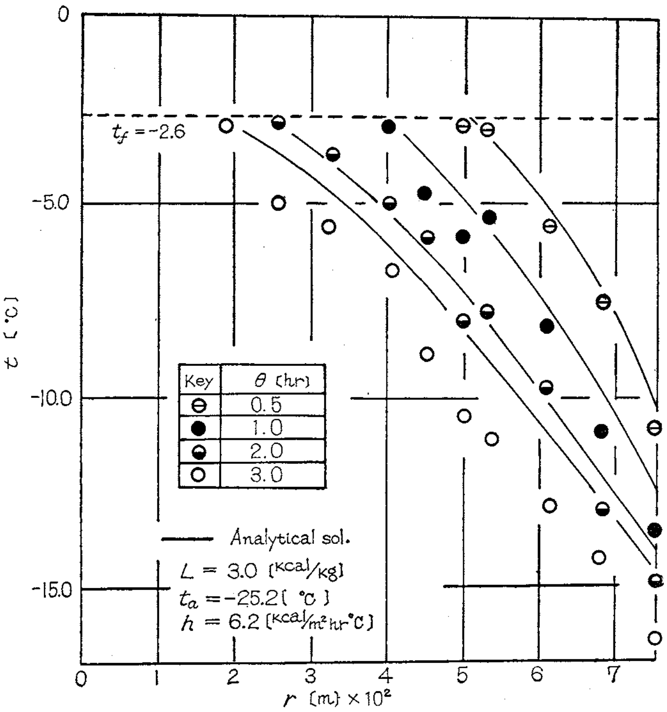



Fig. 10 Temperature distribution in solidified crust

firmed empirically. Hence, the time required to freeze the center was calculated from several of them by using the data of physical properties for ice cream. These results were plotted in Fig. 8 together with the experimental results. The authors' predictions are in the best agreement with the experimental results.

**Temperature distribution.** The temperature distribution** in the solid phase at various times is shown in **Figs. 9** and **10**. The solid lines in these figures are analytical lines obtained from Eq. (14). It can be observed that the experimental results of temperature in the solid phase are fairly consistent with the analytical solution. However, the experimental data are going to deviate from the analytical line due to deficient insulation of the container as time proceeds.

## Conclusion

An analytical solution of the heat transfer problem involving a change of phase for the cylindrical system has been presented and applied to the freezing of ice cream. For a Stefan's problem of the cylindrical coordinate, if the variable constants $A$ and $n$ are approximately independent of time, the crust thickness is given by Eq. (11) and the temperature distribution in the crust is given by Eq. (14).

But since the rate of heat flow in the vicinity of the center cannot satisfy the boundary condition, Eq. (2), on account of increasing curvature at the fusion front, the solution of the crust thickness, Eq. (11), must be restricted to the transient state in which the heat balance satisfies the boundary condition, Eq. (2), at the solid-liquid interface.

Then, after the critical time, assuming that a freezing process is a quasi-steady state, the crust thickness for a time is obtained from Eq. (19) and the time required to freeze the center may be calculated from Eq. (20) with the average volume rate of crust formation expressed in Eq. (18).

** Temperature difference at two points separated by 4.8 cm in axial direction was not over 2.5°C.

## Nomenclature

| | | |
|---|---|---|
| $C_p$ | = Specific heat | [kcal/kg·°C] |
| $H$ | = $h/K$ | [1/m] |
| $K$ | = thermal conductivity | [kcal/m·hr·°C] |
| $h$ | = heat transfer coefficient | [kcal/m²·hr·°C] |
| $k$ | = thermal diffusivity | [m²/hr] |
| $L$ | = latent heat of fusion | [kcal/kg] |
| $q$ | = rate of heat flow | [kcal/hr] |
| $r$, $R$ | = radius | [m] |
| $S$ | = solid-liquid interfacial area | [m²] |
| $t$, $T$ | = temperature | [°C] |
| $u_v$ | = volume rate of crust formation | [m³/hr·m²] |
| $V$ | = bulk of solidified crust | [m³] |
| $\alpha_m$ | = $m$-th positive root given by Eq. (8) | |
| $\delta$ | = crust thickness | [m] |
| $\theta$ | = time | [hr] |
| $\rho$ | = density | [kg/m³] |

Subscript

| | |
|---|---|
| $a$ | = air |
| $c$ | = critical |
| $\delta$ | = position of interface |
| $f$ | = melting temperature |
| $q$ | = quasi-steady |

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# HEAT TRANSFER BETWEEN FLUIDIZED BEDS AND HEATED SURFACES*

## ——EFFECT OF PARTICLE SIZE——

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**Heat transfer coefficients between fluidized beds and heating surfaces were experimentally determined by unsteady state method. An equation of the heat transfer coefficient was proposed by considering a simple model. For high velocity of solid particles, experimental results, especially the effect of particle size, were found to be easily explained by the equation.**

## Introduction

Heat transfer between fluidized beds and heating surfaces has been the subject of many experimental and theoretical studies.

Wicke and Fetting[8] have proposed a model, assuming that the heating surface is covered with a gas film and a region in which solid particles circulate. They also back-calculated thickness from their experimental results.

Ziegler et al.[9] have proposed a mechanism of unsteady state heat transfer for a single particle, assuming that the surface is covered with a relatively thick gas film, and that the temperature of the heating surface is equal to that of gas in the films. And they have obtained the information from their experimental results that the heat transfer coefficient depends upon the solid heat capacity and is independent of the solid thermal conductivity. From theoretical equation, they have also predicted that the effect of solid heat capacity is very small when the residence time of particle is short, and that the maximum Nusselt number is about 7.2.

Mickley and Fairbanks[7] have treated these phenomena, considering the mechanism of unsteady state heat transfer for semi-infinite solids. They considered that the medium contributing to heat transfer is not a single particle of the first row on the heating surface but a small group or assembly of particles. However, their model may imply that as residence time approaches zero, the heat transfer coefficient becomes infinite. This is contrary to experimental results. Besides, they have not clearly described the effect of particle size on the heat transfer coefficient.

Botterill et al.[3,4,5] have also considered unsteady state heat transfer for a single particle which contacts with a heating surface, and computed temperature distribution within a particle by the relaxation method. They obtained the heat transfer coefficient as a function of residence time on the heating surface. However, they obtained experimental values smaller than calculated values. They considered that this inconsistency is due to the gas gap between a heated surface and solid particles and predicted that the gap is about one tenth of the particle size.

Baskakov[2] has adopted a quasi-steady state heat transfer model between a heating surface and a particle, assuming that the heat conduction is controlled only by the thermal resistance between the heating surface and a particle of the first row. From these assumptions, heat transfer coefficients are proportional to effective thermal conductivity between a surface and a particle and inversely proportional to particle size. Since the effective thermal conductivity depends on the thermal conductivity of the solid particle, the use of a particle with considerably different thermal conductivity is expected to have an effect. However, this assumption is contrary to the experimental results of Ziegler[9], and of Botterill[3,4,5].

This report also deals with unsteady state heat transfer for a single particle with relatively short residence time on a heating surface and with rather a high velocity. In case of short residence time, a particle of the first row is not too much heated up, so that heat conduction from the heating surface to a particle is controlled only by the thermal resistance between them. On the contrary, the particle of the first row is heated

* Received on June 21, 1969
Presentet at 34th Annual Meeting of the Society of Chemical Engineers, Japan, at Tokyo, April 2, 1969