# Weighted local Hardy spaces associated to Schrödinger operators 

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#### Abstract

In this paper, we characterize the weighted local Hardy spaces $h_{\rho}^{p}(\omega)$ related to the critical radius function $\rho$ and weights $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ which locally behave as Muckenhoupt's weights and actually include them, by the local vertical maximal function, the local nontangential maximal function and the atomic decomposition. By the atomic characterization, we also prove the existence of finite atomic decompositions associated with $h_{\rho}^{p}(\omega)$. Furthermore, we establish boundedness in $h_{\rho}^{p}(\omega)$ of quasi- Banach-valued sublinear operators. As their applications, we establish the equivalence of the weighted local Hardy space $h_{\rho}^{1}(\omega)$ and the weighted Hardy space $H_{\mathcal{L}}^{1}(\omega)$ associated to Schrödinger operators $\mathcal{L}$ with $\omega \in A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$.


## 1 Introduction

The theory of classical local Hardy spaces, originally introduced by Goldberg [17], plays an important role in various field of analysis and partial differential equations; see [6, 24, 26, 32, [33, 34] and their references. In particular, pseudo-difference operators are bounded on local Hardy spaces $h^{p}\left(\mathbb{R}^{n}\right)$ for $p \in(0,1]$, but they are not bounded on Hardy spaces $H^{p}\left(\mathbb{R}^{n}\right)$ for $p \in(0,1]$; see [17] (also [33, 34). In [6], Bui studied the weighted local Hardy space $h_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ with $\omega \in A_{\infty}\left(\mathbb{R}^{n}\right)$, where and in what follows, $A_{p}\left(\mathbb{R}^{n}\right)$ for $p \in[1, \infty]$ denotes the class of Muckenhoupt's weights; see [8, 15, 18, 26] for their definition and properties.

In [23], Rychkov introduced and studied some properties of the weighted Besov-Lipschitz spaces and Triebel-Lizorkin spaces with weights that are locally in $A_{p}\left(\mathbb{R}^{n}\right)$ but may grow or decrease exponentially, which contain Hardy spaces. In particular, Rychkov [23] generalized some of theories of weighted local Hardy spaces developed by Bui $[6]$ to $A_{\infty}^{l o c}\left(\mathbb{R}^{n}\right)$ weights, where $A_{\infty}^{\text {loc }}\left(\mathbb{R}^{n}\right)$ weights denote local $A_{\infty}\left(\mathbb{R}^{n}\right)$ weights which are non-doubling weights, and $A_{\infty}^{\text {loc }}\left(\mathbb{R}^{n}\right)$ weights include $A_{\infty}\left(\mathbb{R}^{n}\right)$ weights. Recently, Tang [28] established the weighted atomic decomposition characterization of the weighted local Hardy space $h_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ with $\omega \in A_{\infty}^{l o c}\left(\mathbb{R}^{n}\right)$ via the

[^0]local grand maximal function, and gave some criterions about boundedness of $\mathcal{B}_{\beta}$-sublinear operators on $h_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ which was first introduced in [38]; meanwhile, Tang [28] also proved that pseudo-difference operators are bounded on local Hardy spaces $h_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ for $p \in(0,1]$ by using above criterions and main results in [29]. Furthermore, Yang-Yang [36] extended the main results in [28] to the weighted local Orlicz-Hardy space $h_{\omega}^{\Phi}\left(\mathbb{R}^{n}\right)$ case by applying similar methods in [28].

On the other hand, the study of schrödinger operator $L=-\triangle+V$ recently attracted much attention; see [3, 4, 10, 11, [25, 30, 31, 38, 39, 40, 41, In particularly, J. Dziubański and J. Zienkiewicz [10, 11] studied Hardy space $H_{\mathcal{L}}^{1}$ associated to Schrödinger operators $\mathcal{L}$ with potential satisfying reverse Hölder inequality. Recently, Bongioanni, etc. 33 introduced new classes of weights, related to Schrödinger operators $\mathcal{L}$, that is, $A_{p}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ weight which are in general larger than Muckenhoupt's (see Section 2 for notions of $A_{p}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ weight). Nature, it is a very interesting problem that whether we can give a atomic characterization for weighted Hardy space $H_{\mathcal{L}}^{1}(\omega)$ with $\omega \in A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$.

The purpose of this paper is to give a positive answer. More precisely, we first introduce the weighted local Hardy spaces $h_{\rho}^{p}(\omega)$ with $A_{q}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ weights, and establish the atomic characterization of the weighted local Hardy spaces $h_{\rho}^{p}(\omega)$ with $\omega \in A_{q}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ weights. Then, we establish the equivalence between the weighted local Hardy spaces $h_{\rho}^{1}(\omega)$ and the weighted Hardy space $H_{\mathcal{L}}^{1}(\omega)$ associated to Schrödinger operator $\mathcal{L}$ with $\omega \in A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$. In particular, it should be pointed out that we can not directly obtain the atomic characterization of $H_{\mathcal{L}}^{1}(\omega)$ with $A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ weights by using the methods in [10, 11, 12], which forces us to use the above weighted local Hardy spaces $h_{\rho}^{1}(\omega)$ theory to overcome the difficulty.

The paper is organized as follows. In Section 2, we review some notions and notations concerning the weight classes $A_{p}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ introduced in [3, 30, 31]. In Section 3, we first introduce the weighted local Hardy space $h_{\rho, N}^{p}(\omega)$ via the local grand maximal function, and then the weighted atomic local Hardy space $h_{\rho}^{p, q, s}(\omega)$ for any admissible triplet $(p, q, s)_{\omega}$ (see Definition 3.4 below), furthermore, we establish the local vertical and the local nontangential maximal function characterizations of $h_{\rho, N}^{p}(\omega)$ via a local Calderón reproducing formula and some useful estimates established by Rychkov [23]. In Section 4, we establish the Calderón-Zygmund decomposition associated with the grand maximal function. In Section 5, we prove that for any given admissible triplet $(p, q, s)_{\omega}, h_{\rho, N}^{p}(\omega)=h_{\rho}^{p, q, s}(\omega)$ with equivalent norms. It is worth pointing out that we obtain Theorem 5.1 by a way different from the methods in [17, 6], but close to those in [1, 28, 36]. For simplicity, in the rest of this introduction, we denote by $h_{\rho}^{p}(\omega)$ the weighted local Hardy space $h_{\rho, N}^{p}(\omega)$. In Section 6 , we prove that $\|\cdot\|_{h_{\rho, f i n}^{p, q, s}(\omega)}$ and $\|\cdot\|_{h_{\rho}^{p}(\omega)}$ are equivalent quasi-norms on $h_{\rho, \text { fin }}^{p, q, s}(\omega)$ with $q<\infty$, and we obtain criterions for boundedness of $\mathcal{B}_{\beta}$-sublinear operators in $h_{\rho}^{p}(\omega)$. We remark that this extends both the results of Meda-Sjögren-Vallarino [21] and Yang-Zhou [38] to the setting of weighted local Hardy spaces. In

Section 7 , we apply the atomic characterization of the weighted local Hardy spaces $h_{\rho}^{1}(\omega)$ to establish atomic characterization of weighted Hardy space $H_{\mathcal{L}}^{1}(\omega)$ associated to Schrödinger operator $\mathcal{L}$ with $A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ weights.

Throughout this paper, we let $C$ denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a constant $C>1$ such that $1 / C \leq A / B \leq C$. The symbol $A \lesssim B$ means that $A \leq C B$. The symbol $[s]$ for $s \in \mathbb{R}$ denotes the maximal integer not more than $s$. We also set $\mathbb{N} \equiv\{1,2, \cdots\}$ and $\mathbb{Z}_{+} \equiv \mathbb{N} \cup\{0\}$. The multi-index notation is usual: for $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ and $\partial^{\alpha}=\left(\partial / \partial_{x_{1}}\right)^{\alpha_{1}} \cdots\left(\partial / \partial_{x_{n}}\right)^{\alpha_{n}}$. Given a function $g$ on $\mathbb{R}^{n}$, we let $L_{g} \in \mathbb{Z}_{+}$denote the maximal number such that $g$ has vanishing moments up to the order $L_{g}$, i.e., $\int x^{\alpha} g(x) d x=0$ for all multi-indices $\alpha$ with $|\alpha| \leq L_{g}$. If no vanishing moments of $g$, then we put $L_{g}=-1$.

## 2 Preliminaries

In this section, we review some notions and notations concerning the weight classes $A_{p}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ introduced in [3, 30, 31]. Given $B=B(x, r)$ and $\lambda>0$, we will write $\lambda B$ for the $\lambda$-dilate ball, which is the ball with the same center $x$ and with radius $\lambda r$. Similarly, $Q(x, r)$ denotes the cube centered at $x$ with side length $r$ (here and below only cubes with sides parallel to the axes are considered), and $\lambda Q(x, r)=Q(x, \lambda r)$. Especially, we will denote $2 B$ by $B^{*}$, and $2 Q$ by $Q^{*}$.

Let $\mathcal{L}=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^{n}, n \geq 3$, where $V \not \equiv 0$ is a fixed nonnegative potential. We assume that $V$ belongs to the reverse Hölder class $R H_{s}\left(\mathbb{R}^{n}\right)$ for some $s \geq n / 2$; that is, there exists $C=C(s, V)>0$ such that

$$
\left(\frac{1}{|B|} \int_{B} V(x)^{s} d x\right)^{\frac{1}{s}} \leq C\left(\frac{1}{|B|} \int_{B} V(x) d x\right),
$$

for every ball $B \subset \mathbb{R}^{n}$. Trivially, $R H_{q}\left(\mathbb{R}^{n}\right) \subset R H_{p}\left(\mathbb{R}^{n}\right)$ provided $1<p \leq q<\infty$. It is well known that, if $V \in R H_{q}\left(\mathbb{R}^{n}\right)$ for some $q>1$, then there exists $\varepsilon>0$, which depends only on $d$ and the constant $C$ in above inequality, such that $V \in R H_{q+\varepsilon}\left(\mathbb{R}^{n}\right)$ (see [16]). Moreover, the measure $V(x) d x$ satisfies the doubling condition:

$$
\int_{B(y, 2 r)} V(x) d x \leq C \int_{B(y, r)} V(x) d x
$$

With regard to the Schrödinger operator $\mathcal{L}$, we know that the operators derived from $\mathcal{L}$ behave "locally" quite similar to those corresponding to the Laplacian (see [9, (25]). The notion of locality is given by the critical radius function

$$
\begin{equation*}
\rho(x)=\frac{1}{m_{V}(x)}=\sup _{r>0}\left\{r: \frac{1}{r^{n-2}} \int_{B(x, r)} V(y) d y \leq 1\right\} . \tag{2.1}
\end{equation*}
$$

Throughout the paper we assume that $V \not \equiv 0$, so that $0<\rho(x)<\infty$ (see [25]). In particular, $m_{V}(x)=1$ with $V=1$ and $m_{V}(x) \sim(1+|x|)$ with $V=|x|^{2}$.

Lemma 2.1. (see [25]) There exist $C_{0} \geq 1$ and $k_{0} \geq 1$ so that for all $x, y \in \mathbb{R}^{n}$

$$
\begin{equation*}
C_{0}^{-1} \rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-k_{0}} \leq \rho(y) \leq C_{0} \rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{k_{0}}{k_{0}+1}} . \tag{2.2}
\end{equation*}
$$

In particular, $\rho(x) \sim \rho(y)$ when $y \in B(x, r)$ and $r \leq C \rho(x)$, where $C$ is a positive constant.
A ball of the form $B(x, \rho(x))$ is called critical, and in what follows we will call critical radius function to any positive continuous function $\rho$ that satisfies (2.1), not necessarily coming from a potential $V$. Clearly, if $\rho$ is such a function, so it is $\beta \rho$ for any $\beta>0$. As the consequence of the above lemma we acquire the following result:

Lemma 2.2. (see [10]) There exists a sequence of points $x_{j} \in \mathbb{R}^{n}, j \geq 1$, such that the family $B_{j}=B\left(x_{j}, \rho\left(x_{j}\right)\right), j \geq 1$ satisfies:
(a) $\quad \bigcup_{j} B_{j}=\mathbb{R}^{n}$.
(b) For every $\sigma \geq 1$ there exist constants $C$ and $N_{1}$ such that $\Sigma_{j} \chi_{\sigma B_{j}} \leq C \sigma^{N_{1}}$.

In this paper, we write $\Psi_{\theta}(B)=\left(1+r / \rho\left(x_{0}\right)\right)^{\theta}$, where $\theta \geq 0, x_{0}$ and $r$ denotes the center and radius of $B$ respectively.

A weight always refers to a positive function which is locally integrable. As in [3], we say that a weight $\omega$ belongs to the class $A_{p}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ for $1<p<\infty$, if there is a constant $C$ such that for all balls $B$

$$
\left(\frac{1}{\Psi_{\theta}(B)|B|} \int_{B} \omega(y) d y\right)\left(\frac{1}{\Psi_{\theta}(B)|B|} \int_{B} \omega^{-\frac{1}{p-1}}(y) d y\right)^{p-1} \leq C .
$$

We also say that a nonnegative function $\omega$ satisfies the $A_{1}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ condition if there exists a constant $C$ such that

$$
M_{V, \theta}(\omega)(x) \leq C \omega(x) \text {, a.e. } x \in \mathbb{R}^{n} .
$$

where

$$
M_{V, \theta} f(x) \equiv \sup _{x \in B} \frac{1}{\Psi_{\theta}(B)|B|} \int_{B}|f(y)| d y .
$$

When $V=0$, we denote $M_{0} f(x)$ by $M f(x)$ (the standard Hardy-Littlewood maximal function). It is easy to see that $|f(x)| \leq M_{V, \theta} f(x) \leq M f(x)$ for a.e. $x \in \mathbb{R}^{n}$ and any $\theta \geq 0$.

Clearly, the classes $A_{p}^{\rho, \theta}$ are increasing with $\theta$, and we denote $A_{p}^{\rho, \infty}=\bigcup_{\theta \geq 0} A_{p}^{\rho, \theta}$. By Hölder's inequality, we see that $A_{p_{1}}^{\rho, \theta} \subset A_{p_{2}}^{\rho, \theta}$, if $1 \leq p_{1}<p_{2}<\infty$, and we also denote
$A_{\infty}^{\rho, \infty}=\bigcup_{p \geq 1} A_{p}^{\rho, \infty}$. In addition, for $1 \leq p \leq \infty$, denote by $p^{\prime}$ the adjoint number of $p$, i.e. $1 / p+1 / p^{\prime}=1$.

Since $\Psi_{\theta}(B) \geq 1$ with $\theta \geq 0$, then $A_{p} \subset A_{p}^{\rho, \theta}$ for $1 \leq p<\infty$, where $A_{p}$ denotes the classical Muckenhoupt weights; see [15] and [22]. Moreover, the inclusions are proper. In fact, as the example given in [30], let $\theta>0$ and $0 \leq \gamma \leq \theta$, it is easy to check that $\omega(x)=(1+|x|)^{-(n+\gamma)} \notin$ $A_{\infty}=\bigcup_{p \geq 1} A_{p}$ and $\omega(x) d x$ is not a doubling measure, but $\omega(x)=(1+|x|)^{-(n+\gamma)} \in A_{1}^{\rho, \theta}$ provided that $V=1$ and $\Psi_{\theta}\left(B\left(x_{0}, r\right)\right)=(1+r)^{\theta}$.

In what follows, given a Lebesgue measurable set $E$ and a weight $\omega,|E|$ will denote the Lebesgue measure of $E$ and $\omega(E):=\int_{E} \omega(x) d x$. For any $\omega \in A_{\infty}^{\rho, \infty}$, the space $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ with $p \in(0, \infty)$ denotes the set of all measurable functions $f$ such that

$$
\|f\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \equiv\left(\int_{\mathbb{R}^{n}}|f(x)|^{p} \omega(x) d x\right)^{1 / p}<\infty
$$

and $L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right) \equiv L^{\infty}\left(\mathbb{R}^{n}\right)$. The symbol $L_{\omega}^{1, \infty}\left(\mathbb{R}^{n}\right)$ denotes the set of all measurable functions $f$ such that

$$
\|f\|_{L_{\omega}^{1, \infty}\left(\mathbb{R}^{n}\right)} \equiv \sup _{\lambda>0}\left\{\lambda \omega\left(\left\{x \in \mathbb{R}^{n}:|f(x)|>\lambda\right\}\right)\right\}<\infty
$$

We define the local Hardy-Littlewood maximal operator by

$$
\begin{equation*}
M^{l o c} f(x) \equiv \sup _{\substack{x \in B\left(x_{0}, r\right) \\ r \leq \rho\left(x_{0}\right)}} \frac{1}{|B|} \int_{B}|f(y)| d y \tag{2.3}
\end{equation*}
$$

We remark that balls can be replaced by cubes in definition of $A_{p}^{\rho, \theta}$ and $M_{V, \theta}$, since $\Psi(B) \leq \Psi(2 B) \leq 2^{\theta} \Psi(B)$. In fact, for the cube $Q=Q\left(x_{0}, r\right)$, we can also define $\Psi_{\theta}(Q)=$ $\left(1+r / \rho\left(x_{0}\right)\right)^{\theta}$. Then we give the weighted boundedness of $M_{V, \theta}$.

Lemma 2.3. (see [30]) Let $1<p<\infty, p^{\prime}=p /(p-1)$ and assume that $\omega \in A_{p}^{\rho, \theta}$. There exists a constant $C>0$ such that

$$
\left\|M_{V, p^{\prime} \theta} f\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}
$$

Next, we give some properties of weights class $A_{p}^{\rho, \theta}$ for $p \geq 1$.
Lemma 2.4. Let $\omega \in A_{p}^{\rho, \infty}=\bigcup_{\theta \geq 0} A_{p}^{\rho, \theta}$ for $p \geq 1$. Then
(i) If $1 \leq p_{1}<p_{2}<\infty$, then $A_{p_{1}}^{\rho, \theta} \subset A_{p_{2}}^{\rho, \theta}$.
(ii) $\omega \in A_{p}^{\rho, \theta}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p^{\prime}}^{\rho, \theta}$, where $1 / p+1 / p^{\prime}=1$.
(iii) If $\omega \in A_{p}^{\rho, \infty}, 1<p<\infty$, then there exists $\epsilon>0$ such that $\omega \in A_{p-\epsilon}^{\rho, \infty}$.
(iv) Let $f \in L_{l o c}(\rho), 0<\delta<1$, then $\left(M_{V, \theta} f\right)^{\delta} \in A_{1}^{\rho, \theta}$.
(v) Let $1<p<\infty$, then $\omega \in A_{p}^{\rho, \infty}$ if and only if $\omega=\omega_{1} \omega_{2}^{1-p}$, where $\omega_{1}, \omega_{2} \in A_{1}^{\rho, \infty}$.
(vi) For $\omega \in A_{p}^{\rho, \theta}, Q=Q(x, r)$ and $\lambda>1$, there exists a positive constant $C$ such that

$$
\omega(\lambda Q) \leq C\left(\Psi_{\theta}(\lambda Q)\right)^{p} \lambda^{n p} \omega(Q)
$$

(vii) If $p \in(1, \infty)$ and $\omega \in A_{p}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$, then the local Hardy-Littlewood maximal operator $M^{\text {loc }}$ is bounded on $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$.
(viii) If $\omega \in A_{1}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$, then $M^{\text {loc }}$ is bounded from $L_{\omega}^{1}\left(\mathbb{R}^{n}\right)$ to $L_{\omega}^{1, \infty}\left(\mathbb{R}^{n}\right)$.

Proof. (i)-(viii) have been proved in [3, 31].

For any $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, define the critical index of $\omega$ by

$$
\begin{equation*}
q_{\omega} \equiv \inf \left\{p \in[1, \infty): \omega \in A_{p}^{\rho, \infty}\left(\mathbb{R}^{n}\right)\right\} \tag{2.4}
\end{equation*}
$$

Obviously, $q_{\omega} \in[1, \infty)$. If $q_{\omega} \in(1, \infty)$, then $\omega \notin A_{q_{\omega}}^{\rho, \infty}$.
The symbols $\mathcal{D}\left(\mathbb{R}^{n}\right)=C_{0}^{\infty}\left(\mathbb{R}^{n}\right), \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ is the dual space of $\mathcal{D}\left(\mathbb{R}^{n}\right)$, and for $\mathcal{D}\left(\mathbb{R}^{n}\right), \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$, we have the following conclusions.

Lemma 2.5. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, $q_{\omega}$ be as in (2.4) and $p \in\left(q_{\omega}, \infty\right]$.
(i) If $\frac{1}{p}+\frac{1}{p^{\prime}}=1$, then $\mathcal{D}\left(\mathbb{R}^{n}\right) \subset L_{\omega^{-1 /(p-1)}}^{p^{\prime}}\left(\mathbb{R}^{n}\right)$.
(ii) $L_{\omega}^{p}\left(\mathbb{R}^{n}\right) \subset \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and the inclusion is continuous.

By the same method as the proof of Lemma 2.2 in [28], we can get the Lemma 2.5, and we omit the details here.

For any $\varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$, let $\varphi_{t}(x)=t^{-n} \varphi(x / t)$ for $t>0$ and $\varphi_{j}(x)=2^{j n} \varphi\left(2^{j} x\right)$ for $j \in \mathbb{Z}$. It is easy to see that we have the following results.

Lemma 2.6. (see [28]) Let $\varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ and $\int_{\mathbb{R}^{n}} \varphi(x) d x=1$.
(i) For any $\Phi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ and $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, $\Phi * \varphi_{t} \rightarrow \Phi$ in $\mathcal{D}\left(\mathbb{R}^{n}\right)$ as $t \rightarrow 0$, and $f * \varphi_{t} \rightarrow f$ in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ as $t \rightarrow 0$.
(ii) Let $\omega \in A_{\infty}^{\rho, \infty}$ and $q_{\omega}$ be as in (2.3). If $q \in\left(q_{\omega}, \infty\right)$, then for any $f \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right), f * \varphi_{t} \rightarrow f$ in $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$ as $t \rightarrow 0$.

## 3 Weighted local Hardy spaces and their maximal function characterizations

In this section, we introduce the weighted local Hardy spaces $h_{\rho, N}^{p}(\omega)$ via the local grand maximal function and establish its local vertical and nontangential maximal function characterizations via a local Calderón reproducing formula. We also introduce the weighted atomic local Hardy space $h_{\rho}^{p, q, s}(\omega)$ and give some basic properties of these spaces.

We first introduce some local maximal functions. For $N \in \mathbb{Z}_{+}$and $R \in(0, \infty)$, let

$$
\begin{aligned}
\mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right) \equiv\left\{\varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right):\right. & \operatorname{supp}(\varphi) \subset B(0, R) \\
& \|\varphi\|_{\left.\mathcal{D}_{N}\left(\mathbb{R}^{n}\right) \equiv \sup _{x \in \mathbb{R}^{n}} \sup _{\alpha \in \mathbb{Z}_{+}^{n},|\alpha| \leq N}\left|\partial^{\alpha} \varphi(x)\right| \leq 1\right\}} .
\end{aligned}
$$

Definition 3.1. Let $N \in \mathbb{Z}_{+}$and $R \in(0, \infty)$. For any $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, the local nontangential grand maximal function $\widetilde{\mathcal{M}}_{N, R}(f)$ of $f$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\widetilde{\mathcal{M}}_{N, R}(f)(x) \equiv \sup \left\{\left|\varphi_{l} * f(z)\right|:|x-z|<2^{-l}<\rho(x), \varphi \in \mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right)\right\} \tag{3.1}
\end{equation*}
$$

and the local vertical grand maximal function $\mathcal{M}_{N, R}(f)$ of $f$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\mathcal{M}_{N, R}(f)(x) \equiv \sup \left\{\left|\varphi_{l} * f(x)\right|: 0<2^{-l}<\rho(x), \varphi \in \mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right)\right\} \tag{3.2}
\end{equation*}
$$

For convenience's sake, when $R=1$, we denote $\mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right), \widetilde{\mathcal{M}}_{N, R}(f)$ and $\mathcal{M}_{N, R}(f)$ simply by $\mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right), \widetilde{\mathcal{M}}_{N}^{0}(f)$ and $\mathcal{M}_{N}^{0}(f)$, respectively; when $R=\max \left\{R_{1}, R_{2}, R_{3}\right\}>1$ (in which $R_{1}, R_{2}$ and $R_{3}$ are defined as in Lemma 4.2, 4.4 and 4.8), we denote $\mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right), \widetilde{\mathcal{M}}_{N, R}(f)$ and $\mathcal{M}_{N, R}(f)$ simply by $\mathcal{D}_{N}\left(\mathbb{R}^{n}\right), \widetilde{\mathcal{M}}_{N}(f)$ and $\mathcal{M}_{N}(f)$, respectively. For any $N \in \mathbb{Z}_{+}$and $x \in \mathbb{R}^{n}$, obviously,

$$
\mathcal{M}_{N}^{0}(f)(x) \leq \mathcal{M}_{N}(f)(x) \leq \widetilde{\mathcal{M}}_{N}(f)(x)
$$

For the local grand maximal function $\mathcal{M}_{N}^{0}(f)$, we have the following Proposition 3.1, which can be proved by the same method as in [28, Proposition 2.2]. Here and in what follows, the space $L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ denotes the set of all locally integrable functions on $\mathbb{R}^{n}$.

Proposition 3.1. Let $N \geq 2$. Then
(i) There exists a positive constant $C$ such that for all $f \in L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right) \cap \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and almost every $x \in \mathbb{R}^{n}$,

$$
|f(x)| \leq \mathcal{M}_{N}^{0}(f)(x) \leq C M^{\mathrm{loc}}(f)(x)
$$

(ii) If $\omega \in A_{p}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ with $p \in(1, \infty)$, then $f \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ if and only if $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\mathcal{M}_{N}^{0}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right) ;$ moreover,

$$
\|f\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \sim\left\|\mathcal{M}_{N}^{0}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}
$$

(iii) If $\omega \in A_{1}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$, then $\mathcal{M}_{N}^{0}$ is bounded from $L_{\omega}^{1}\left(\mathbb{R}^{n}\right)$ to $L_{\omega}^{1, \infty}\left(\mathbb{R}^{n}\right)$.

Now we introduce the weighted local Hardy space via the local grand maximal function as follows.

Definition 3.2. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), q_{\omega}$ be as in (2.4), $p \in(0,1]$ and $\widetilde{N}_{p, \omega} \equiv\left[n\left(\frac{q_{\omega}}{p}-1\right)\right]+2$. For each $N \in \mathbb{N}$ with $N \geq \widetilde{N}_{p, \omega}$, the weighted local Hardy space is defined by

$$
h_{\rho, N}^{p}(\omega) \equiv\left\{f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right): \mathcal{M}_{N}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)\right\} .
$$

Moreover, let $\|f\|_{h_{\rho, N}^{p}(\omega)} \equiv\left\|\mathcal{M}_{N}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}$.
Obviously, for any integers $N_{1}$ and $N_{2}$ with $N_{1} \geq N_{2} \geq \widetilde{N}_{p, \omega}$,

$$
h_{\rho, \widetilde{N}_{p, \omega}}^{p}(\omega) \subset h_{\rho, N_{2}}^{p}(\omega) \subset h_{\rho, N_{1}}^{p}(\omega),
$$

and the inclusions are continuous.
Next, we introduce the weighted local atoms, via which, we give the definition of the weighted atomic local Hardy space.

Definition 3.3. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, $q_{\omega}$ be as in (2.4). A triplet $(p, q, s)_{\omega}$ is called to be admissible, if $p \in(0,1], q \in\left(q_{\omega}, \infty\right]$ and $s \in \mathbb{N}$ with $s \geq\left[n\left(q_{\omega} / p-1\right)\right]$. A function a on $\mathbb{R}^{n}$ is said to be a $(p, q, s)_{\omega}-$ atom if
(i) $\operatorname{supp} a \subset Q(x, r)$ and $r \leq L_{1} \rho(x)$,
(ii) $\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \leq[\omega(Q)]^{1 / q-1 / p}$, (iii) $\int_{\mathbb{R}^{n}} a(x) x^{\alpha} d x=0$ for all $\alpha \in \mathbb{Z}_{+}^{n}$ with $|\alpha| \leq s$, when $Q=Q(x, r), r<L_{2} \rho(x)$,
where $L_{1} \equiv 4 C_{0}(3 \sqrt{n})^{k_{0}}$, $L_{2} \equiv 1 / C_{0}^{2}(3 \sqrt{n})^{k_{0}+1}$, and $C_{0}, k_{0}$ are constant given in Lemma 2.1. Moreover, a function $a(x)$ on $\mathbb{R}^{n}$ is called a $(p, q)_{\omega}$-single-atom with $q \in\left(q_{\omega}, \infty\right]$, if

$$
\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \leq\left[\omega\left(\mathbb{R}^{n}\right)\right]^{1 / q-1 / p} .
$$

Definition 3.4. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), q_{\omega}$ be as in (2.4), and $(p, q, s)_{\omega}$ be admissible, The weighted atomic local Hardy space $h_{\rho}^{p, q, s}(\omega)$ is defined as the set of all $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfying that

$$
f=\sum_{i=0}^{\infty} \lambda_{i} a_{i}
$$

in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, where $\left\{a_{i}\right\}_{i \in \mathbb{N}}$ are $(p, q, s)_{\omega}$-atoms with $\operatorname{supp}\left(a_{i}\right) \subset Q_{i}$, $a_{0}$ is a $(p, q)_{\omega \text {-single-atom, }}$, $\left\{\lambda_{i}\right\}_{i \in \mathbb{Z}_{+}} \subset \mathbb{C}$. Moreover, the quasi-norm of $f \in h_{\rho}^{p, q, s}(\omega)$ is defined by

$$
\|f\|_{h_{\rho}^{p, q, s}(\omega)} \equiv \inf \left\{\left[\sum_{i=0}^{\infty}\left|\lambda_{i}\right|^{p}\right]^{1 / p}\right\}
$$

where the infimum is taken over all the decompositions of $f$ as above.

It is easy to see that if triplets $(p, q, s)_{\omega}$ and $(p, \bar{q}, \bar{s})_{\omega}$ are admissible and satisfy $\bar{q} \leq q$ and $\bar{s} \leq s$, then $(p, q, s)_{\omega}$-atoms are $(p, \bar{q}, \bar{s})_{\omega}$-atoms, which implies that $h_{\rho}^{p, q, s}(\omega) \subset h_{\rho}^{p, \bar{q}, \bar{s}}(\omega)$ and the inclusion is continuous.

Next, we introduce some local vertical, tangential and nontangential maximal functions, and then we establish the characterizations of the weighted local Hardy space $h_{\rho, N}^{p}(\omega)$ by these local maximal functions.

Definition 3.5. Let

$$
\begin{equation*}
\psi_{0} \in \mathcal{D}\left(\mathbb{R}^{n}\right) \text { with } \int_{\mathbb{R}^{n}} \psi_{0}(x) d x \neq 0 \tag{3.3}
\end{equation*}
$$

For every $x \in \mathbb{R}^{n}$, there exists an integer $j_{x} \in \mathbb{Z}$ satisfying $2^{-j_{x}}<\rho(x) \leq 2^{-j_{x}+1}$, and then for $j \geq j_{x}, A, B \in[0, \infty)$ and $y \in \mathbb{R}^{n}$, let $m_{j, A, B, x}(y) \equiv\left(1+2^{j}|y|\right)^{A} 2^{B|y| / \rho(x)}$.

The local vertical maximal function $\psi_{0}^{+}(f)$ of $f$ associated to $\psi_{0}$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\psi_{0}^{+}(f)(x) \equiv \sup _{j \geq j_{x}}\left|\left(\psi_{0}\right)_{j} * f(x)\right|, \tag{3.4}
\end{equation*}
$$

the local tangential Peetre-type maximal function $\psi_{0, A, B}^{* *}(f)$ of $f$ associated to $\psi_{0}$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\psi_{0, A, B}^{* *}(f)(x) \equiv \sup _{j \geq j_{x}, y \in \mathbb{R}^{n}} \frac{\left|\left(\psi_{0}\right)_{j} * f(x-y)\right|}{m_{j, A, B, x}(y)} \tag{3.5}
\end{equation*}
$$

and the local nontangential maximal function $\left(\psi_{0}\right)_{\underset{\sim}{*}}^{*}(f)$ of $f$ associated to $\psi_{0}$ is defined by setting, for all $x \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\left(\psi_{0}\right)_{\nabla}^{*}(f)(x) \equiv \sup _{|x-y|<2^{-l}<\rho(x)}\left|\left(\psi_{0}\right)_{l} * f(y)\right|, \tag{3.6}
\end{equation*}
$$

where $l \in \mathbb{Z}$.

Obviously, for any $x \in \mathbb{R}^{n}$, we have

$$
\psi_{0}^{+}(f)(x) \leq\left(\psi_{0}\right)_{\nabla}^{*}(f)(x) \lesssim \psi_{0, A, B}^{* *}(f)(x) .
$$

We point out that the local tangential Peetre-type maximal function $\psi_{0, A, B}^{* *}(f)$ was introduced by Rychkov [23].

In order to characterize $h_{\rho, N}^{p}(\omega)$ by the local vertical and the local nontangential maximal function, we need to establish some relations in the norm of $L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ of the local maximal functions $\psi_{0, A, B}^{* *}(f), \psi_{0}^{+}(f)$ and $\widetilde{\mathcal{M}}_{N, R}(f)$, which further imply the desired characterizations.

We begin with a lemma on local reproducing formula, which can be deduced from the Lemma 1.6 in [23], and we omit the details of its proof here.

Lemma 3.1. Let $\psi_{0}$ be as in (3.3) and $\psi(x) \equiv \psi_{0}(x)-\left(1 / 2^{n}\right) \psi_{0}(x / 2)$ for all $x \in \mathbb{R}^{n}$. Then for any given integers $j \in \mathbb{Z}$ and $L \in \mathbb{Z}_{+}$, there exist $\varphi_{0}, \varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ such that $L_{\varphi} \geq L$ and

$$
\begin{equation*}
f=\left(\varphi_{0}\right)_{j} *\left(\psi_{0}\right)_{j} * f+\sum_{k=j+1}^{\infty} \varphi_{k} * \psi_{k} * f \tag{3.7}
\end{equation*}
$$

in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ for all $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$.
Lemma 3.2. Let $0<r<\infty, \psi_{0}$ be as in (3.3) and $\psi(x) \equiv \psi_{0}(x)-\left(1 / 2^{n}\right) \psi_{0}(x / 2)$. Then there exists a positive constant $A_{0}$ depending only on the support of $\psi_{0}$ such that for any $A \in\left(A_{0}, \infty\right)$ and $B \in[0, \infty)$, there exists a positive constant $C$ depending only on $n, r, \psi_{0}, A$ and $B$, such that for all $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right), x, x_{0} \in \mathbb{R}^{n}$ and $j \geq j_{x_{0}}\left(\right.$ where $\left.2^{-j_{x_{0}}}<\rho\left(x_{0}\right) \leq 2^{-j_{x_{0}}+1}\right)$, we have

$$
\begin{equation*}
\left|\psi_{j} * f(x)\right|^{r} \leq C \sum_{k=j}^{\infty} 2^{(j-k) A r} 2^{k n} \int \frac{\left|\psi_{k} * f(x-y)\right|^{r}}{m_{j, A r, B r, x_{0}}(y)} d y \tag{3.8}
\end{equation*}
$$

Proof. By Lemma 3.1, we can find $\varphi_{0}, \varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ so that $L_{\varphi} \geq A$ and (3.7) is true. Hence, we have

$$
\begin{equation*}
\psi_{j} * f=\left(\varphi_{0}\right)_{j} *\left(\psi_{0}\right)_{j} * \psi_{j} * f+\sum_{k=j+1}^{\infty} \psi_{j} * \varphi_{k} * \psi_{k} * f \tag{3.9}
\end{equation*}
$$

The function $\psi_{j} * \varphi_{k}(k \geq j+1)$ have support size $\leq C 2^{-j}$ and enjoy the uniform estimate

$$
\begin{equation*}
\left\|\psi_{j} * \varphi_{k}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \leq C 2^{(j-k) A} 2^{j n} \tag{3.10}
\end{equation*}
$$

which can be easily deduced by the moment condition on $\varphi$ (see [23, (2.13)]). Therefore, we may write

$$
\begin{equation*}
\left|\psi_{j} * \varphi_{k}(y)\right| \leq C \frac{2^{(j-k) A} 2^{k n}}{m_{j, A, B, x_{0}}(y)} \quad\left(y \in \mathbb{R}^{n}\right) \tag{3.11}
\end{equation*}
$$

Putting (3.11) together with the similar estimate for $\left(\varphi_{0}\right)_{j} *\left(\psi_{0}\right)_{j}$ into (3.9) gives (3.8) for $r=1$, and the case $r>1$ follows by Hölder's inequality. To obtain the case $r<1$, we introduce the maximal functions

$$
M_{A, B, x_{0}}(x, j)=\sup _{k \geq j, y \in \mathbb{R}^{n}} 2^{(j-k) A} \frac{\left|\psi_{k} * f(x-y)\right|}{m_{j, A, B, x_{0}}(y)}
$$

The (3.8) with $r=1$ gives

$$
\begin{equation*}
2^{(j-k) A}\left|\psi_{k} * f(x-y)\right| \leq C \sum_{l=k}^{\infty} 2^{(j-l) A} 2^{l n} \int \frac{\left|\psi_{l} * f(x-z)\right|}{m_{k, A, B, x_{0}}(z-y)} d z \tag{3.12}
\end{equation*}
$$

and the right of (3.12) decreases as $k$ increases. Hence, to get the estimate for $M_{A, B, x_{0}}(x, j)$, we may only consider (3.12) with $k=j$. Combing with the elementary inequality

$$
\begin{equation*}
m_{j, A, B, x_{0}}(z) \leq m_{j, A, B, x_{0}}(y) m_{k, A, B, x_{0}}(z-y) . \tag{3.13}
\end{equation*}
$$

we can get

$$
\begin{align*}
M_{A, B, x_{0}}(x, j) & \leq C \sum_{k=j}^{\infty} 2^{(j-k) A} 2^{k n} \int \frac{\left|\psi_{l} * f(x-z)\right|}{m_{j, A, B, x_{0}}(z)} d z \\
& \leq C M_{A, B, x_{0}}(x, j)^{1-r} \sum_{k=j}^{\infty} 2^{(j-k) A r} 2^{k n} \int \frac{\left|\psi_{l} * f(x-z)\right|^{r}}{m_{j, A r, B r, x_{0}}(z)} d z . \tag{3.14}
\end{align*}
$$

Considering $\left|\psi_{j} * f(x)\right| \leq M_{A, B, x_{0}}(x, j)$, (3.14) implies (3.8), if $M_{A, B, x_{0}}(x, j)<\infty$. By [18, Proposition 2.3.4(a)], for any $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, we have $M_{A, B, x_{0}}(x, j)<\infty$ for all $x \in \mathbb{R}^{n}$ and $j \geq j_{x_{0}}$, provided $A>A_{0}$, where $A_{0}$ is a positive constant depending only on the support of $\psi_{0}$. This finishes the proof.

For $f \in L_{\mathrm{loc}}^{1}\left(\mathbb{R}^{n}\right), B \in[0, \infty)$ and $x \in \mathbb{R}^{n}$, let

$$
\begin{equation*}
K_{B} f(x)=\frac{1}{(\rho(x))^{n}} \int_{\mathbb{R}^{n}}|f(y)|^{-B \frac{|x-y|}{\rho(x)}} d y, \tag{3.15}
\end{equation*}
$$

and for the operator $K_{B}$, we have the following lemma:
Lemma 3.3. Let $p \in(1, \infty)$ and $\omega \in A_{p}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$, then there exist constants $C>0$ and $B_{0} \equiv$ $B_{0}(\omega, n)>0$ such that for all $B>B_{0} / p$,

$$
\left\|K_{B} f\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)},
$$

for all $f \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$.
Proof. It is suffice to show that there exists a constant $C>0$ such that for all $B>B_{0}$,

$$
K_{B} f(x) \leq C M_{V, p^{\prime} \theta} f(x),
$$

then combining with Lemma 2.3, we get the boundedness of the operator $K_{B}$.
To control $K_{B} f(x)$, we argue as follows:

$$
\begin{aligned}
& K_{B} f(x)= \frac{1}{(\rho(x))^{n}} \int_{\mathbb{R}^{n}}|f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} d y \\
&= \frac{1}{(\rho(x))^{n}} \int_{|y-x|<\rho(x)}|f(y)|^{-B \frac{|x-y|}{\rho(x)}} d y+\frac{1}{(\rho(x))^{n}} \int_{|y-x| \geq \rho(x)}|f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} d y \\
&= \frac{1}{(\rho(x))^{n}} \int_{|y-x|<\rho(x)}|f(y)|^{-B \frac{|x-y|}{\rho(x)}} d y \\
& \quad \quad \quad \sum_{k=0}^{\infty} \frac{1}{(\rho(x))^{n}} \int_{|y-x| \sim 2^{k} \rho(x)}|f(y)| 2^{-B \frac{|x-y|}{\rho(x)}} d y \\
& \equiv I_{1}+I_{2} .
\end{aligned}
$$

For $I_{1}$, it is easy to get

$$
I_{1} \leq \frac{C}{\Psi_{p^{\prime} \theta}\left(B_{1}\right)\left|B_{1}\right|} \int_{B_{1}}|f(y)| d y \leq C M_{V, p^{\prime} \theta} f(x)
$$

in which $B_{1}=B(x, \rho(x))$ is a critical ball.
For $I_{2}$, we have

$$
\begin{aligned}
I_{2} & \leq C \sum_{k=0}^{\infty} \frac{\left(1+2^{k+1}\right)^{p^{\prime} \theta} 2^{k n}}{2^{B 2^{k}}} \frac{1}{\Psi_{p^{\prime} \theta}\left(2^{k+1} B_{1}\right)\left|2^{k+1} B_{1}\right|} \int_{2^{k+1} B_{1}}|f(y)| d y \\
& \leq C\left(\sum_{k=0}^{\infty} \frac{\left(1+2^{k+1}\right)^{p^{\prime} \theta} 2^{k n}}{2^{B 2^{k}}}\right) M_{V, p^{\prime} \theta} f(x) \\
& \leq C M_{V, p^{\prime} \theta} f(x),
\end{aligned}
$$

where the sum converges when $B>B_{0} / p$.
Lemma 3.4. Let $\psi_{0}$ be as in (3.3) and $r \in(0, \infty)$. Then for any $A \in\left(\max \left\{A_{0}, n / r\right\}, \infty\right)$ (where $A_{0}$ is as in Lemma 3.2) and $B \in[0, \infty)$, there exists a positive constant $C$, depending only on $n, r, \psi_{0}, A$ and $B$, such that for all $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right), x \in \mathbb{R}^{n}$ and $j \geq j_{x}$ (where $2^{-j_{x}}<$ $\left.\rho(x) \leq 2^{-j_{x}+1}\right)$,

$$
\begin{aligned}
{\left[\left(\psi_{0}\right)_{j, A, B}^{*}(f)(x)\right]^{r} } & \leq C \sum_{k=j}^{\infty} 2^{(j-k)(A r-n)}\left\{M^{\mathrm{loc}}\left(\left|\left(\psi_{0}\right)_{k} * f\right|^{r}\right)(x)\right. \\
& \left.+K_{B r}\left(\left|\left(\psi_{0}\right)_{k} * f\right|^{r}\right)(x)\right\}
\end{aligned}
$$

where

$$
\left(\psi_{0}\right)_{j, A, B}^{*}(f)(x) \equiv \sup _{y \in \mathbb{R}^{n}} \frac{\left|\left(\psi_{0}\right)_{j} * f(x-y)\right|}{m_{j, A, B, x}(y)}
$$

for all $x \in \mathbb{R}^{n}$.

Proof. First we can get the stronger version of (3.8) by virtue of (3.13), that is:

$$
\begin{aligned}
{\left[\left(\psi_{0}\right)_{j, A, B}^{*}(f)(x)\right]^{r} \leq } & C \sum_{k=j}^{\infty} 2^{(j-k) A r} 2^{k n} \int_{\mathbb{R}^{n}} \frac{\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r}}{m_{j, A r, B r, x}(x-y)} d y \\
\leq & C \sum_{k=j}^{\infty} 2^{(j-k)(A r-n)}\left\{2^{j n} \int_{|y-x|<2^{-j_{x}}} \frac{\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r}}{\left(1+2^{j}|x-y|\right)^{A r}} d y\right. \\
& \left.\quad+2^{j n} \int_{|y-x| \geq 2^{-j_{x}}} \frac{\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r}}{\left(2^{j}|x-y|\right)^{A r} 2^{B r|x-y| / \rho(x)}} d y\right\} \\
\equiv & C \sum_{k=j}^{\infty} 2^{(j-k)(A r-n)}\{I+I I\} .
\end{aligned}
$$

Since $2^{-j_{x}}<\rho(x) \leq 2^{-j_{x}+1}$ and $j \geq j_{x}$, for $I$ we have

$$
\begin{aligned}
I & =2^{j n} \int_{2^{-j} \leq|y-x|<2^{-j_{x}}} \frac{\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r}}{\left(1+2^{j}|x-y|\right)^{A r}} d y+2^{j n} \int_{|y-x| \leq 2^{-j}} \frac{\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r}}{\left(1+2^{j}|x-y|\right)^{A r}} d y \\
& \equiv I_{1}+I_{2} .
\end{aligned}
$$

According to the definition of $M^{l o c} f(x)$ (see (2.3)), for $I_{2}$ we have

$$
I_{2} \leq 2^{j n} \int_{|y-x| \leq 2^{-j}}\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r} d y \leq C M^{l o c}\left(\left|\left(\psi_{0}\right)_{k} * f\right|^{r}\right)(x),
$$

and for $I_{1}$ we have

$$
\begin{aligned}
I_{1} & \leq 2^{j n} \sum_{l=j_{x}+1}^{j} \int_{2^{-l} \leq|y-x|<2^{-l+1}} \frac{\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r}}{\left(2^{j}|x-y|\right)^{A r}} d y \\
& \leq \sum_{l=j_{x}+1}^{j} \frac{2^{j n}\left(2^{-l+1}\right)^{n}}{\left(2^{j-l}\right)^{A r}} \frac{1}{\left(2^{-l+1}\right)^{n}} \int_{|y-x| \leq 2^{-l+1}}\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r} d y \\
& \leq \sum_{l=j_{x}+1}^{j} \frac{2^{n}}{2^{(A r-n)(j-l)}} M^{l o c}\left(\left|\left(\psi_{0}\right)_{k} * f\right|^{r}\right)(x) \\
& \leq C M^{l o c}\left(\left|\left(\psi_{0}\right)_{k} * f\right|^{r}\right)(x),
\end{aligned}
$$

where $A r>n$. In addition, with regard to $I I$, we have the following estimate,

$$
\begin{aligned}
I I & \leq \frac{2^{j n}(\rho(x))^{n}}{\left(2^{j-j_{x}}\right)^{A r}} \frac{1}{(\rho(x))^{n}} \int_{\mathbb{R}^{n}}\left|\left(\psi_{0}\right)_{k} * f(y)\right|^{r} 2^{-B r \frac{|x-y|}{\rho(x)}} d y \\
& \leq C \frac{2^{j n}\left(2^{-j_{x}}\right)^{n}}{\left(2^{j-j_{x}}\right)^{A r}} K_{B r}\left(\left|\left(\psi_{0}\right)_{k} * f\right|\right)(x) \\
& \leq C \frac{\left(2^{j-j_{x}}\right)^{n}}{\left(2^{j-j_{x}}\right)^{A r}} K_{B r}\left(\left|\left(\psi_{0}\right)_{k} * f\right|\right)(x) \\
& \leq C K_{B r}\left(\left|\left(\psi_{0}\right)_{k} * f\right|\right)(x),
\end{aligned}
$$

where the last inequality is a consequence of the face that $j \geq j_{x}$ and $A r>n$. This finishes the proof.

Now we can establish weighted norm inequalities of $\psi_{0}^{+}(f), \psi_{0, A, B}^{* *}(f)$ and $\widetilde{\mathcal{M}}_{N, R}(f)$.
Theorem 3.1. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), R \in(0, \infty), p \in(0,1], \psi_{0}$ and $q_{\omega}$ be respectively as in (3.3) and (2.4), and let $\psi_{0}^{+}(f), \psi_{0, A, B}^{* *}(f)$ and $\widetilde{\mathcal{M}}_{N, R}(f)$ be respectively as in (3.4), (3.5) and (3.1). Let $A_{1} \equiv \max \left\{A_{0}, n q_{\omega} / p\right\}, B_{1} \equiv B_{0} / p$ and $N_{0} \equiv\left[2 A_{1}\right]+1$, where $A_{0}$ and $B_{0}$ are respectively as in Lemmas 3.2 and 3.3. Then for any $A \in\left(A_{1}, \infty\right), B \in\left(B_{1}, \infty\right)$ and integer $N \geq N_{0}$, there exists a positive constant $C$, depending only on $A, B, N, R, \psi_{0}, \omega$ and $n$, such that for all $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\left\|\psi_{0, A, B}^{* *}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq C\left\|\psi_{0}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \tag{3.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\widetilde{\mathcal{M}}_{N, R}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq C\left\|\psi_{0}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}, \tag{3.17}
\end{equation*}
$$

Proof. Let $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$. First, we prove (3.16). Let $A \in\left(A_{1}, \infty\right)$ and $B \in\left(B_{1}, \infty\right)$. By $A_{1} \equiv \max \left\{A_{0}, n q_{\omega} / p\right\}$ and $B_{1} \equiv B_{0} / p$, we know that there exists $r_{0} \in\left(0, p / q_{\omega}\right)$ such that $A>n / r_{0}$ and $B r_{0}>B_{0} / q_{\omega}$, where $A_{0}$ and $B_{0}$ are respectively as in Lemmas 3.2 and 3.3. Thus, by Lemma 3.4 , for all $x \in \mathbb{R}^{n}$ and $j \geq j_{x}$ we have

$$
\begin{gather*}
{\left[\left(\psi_{0}\right)_{j, A, B}^{*}(f)(x)\right]^{r_{0}} \lesssim \sum_{k=j}^{\infty} 2^{(j-k)\left(A r_{0}-n\right)}\left\{M^{\mathrm{loc}}\left(\left|\left(\psi_{0}\right)_{k} * f\right|^{r_{0}}\right)(x)\right.} \\
\left.+K_{B r_{0}}\left(\left|\left(\psi_{0}\right)_{k} * f\right|^{r_{0}}\right)(x)\right\} . \tag{3.18}
\end{gather*}
$$

Let $\psi_{0}^{+}(f)$ and $\psi_{0, A, B}^{* *}(f)$ be respectively as in (3.4) and (3.5). We notice that for any $x \in \mathbb{R}^{n}$ and $k \geq j_{x}$,

$$
\left|\left(\psi_{0}\right)_{k} * f(x)\right| \leq \psi_{0}^{+}(f)(x)
$$

which together with (3.18) implies that for all $x \in \mathbb{R}^{n}$,

$$
\begin{equation*}
\left[\psi_{0, A, B}^{* *}(f)(x)\right]^{r_{0}} \lesssim M^{\mathrm{loc}}\left(\left[\psi_{0}^{+}(f)\right]^{r_{0}}\right)(x)+K_{B r_{0}}\left(\left[\psi_{0}^{+}(f)\right]^{r_{0}}\right)(x) \tag{3.19}
\end{equation*}
$$

Then by (3.19) we have

$$
\begin{align*}
\int_{\mathbb{R}^{n}}\left|\psi_{0, A, B}^{* *}(f)(x)\right|^{p} \omega(x) d x \lesssim & \int_{\mathbb{R}^{n}}\left|\left\{M^{\mathrm{loc}}\left(\left[\psi_{0}^{+}(f)\right]^{r_{0}}\right)(x)\right\}\right|^{p / r_{0}} \omega(x) d x \\
& +\int_{\mathbb{R}^{n}}\left|\left\{K_{B r_{0}}\left(\left[\psi_{0}^{+}(f)\right]^{r_{0}}\right)(x)\right\}\right|^{p / r_{0}} \omega(x) d x \\
\equiv & \mathrm{I}_{1}+\mathrm{I}_{2} \tag{3.20}
\end{align*}
$$

For $I_{1}$, as $r_{0}<p / q_{\omega}$, we have $q \equiv p / r_{0}>q_{\omega}$ and $\omega \in A_{q}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, therefore by Lemma 2.4(vii) we get

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}\left|M^{\mathrm{loc}}\left(\left[\psi_{0}^{+}(f)\right]^{r_{0}}\right)(x)\right|^{p / r_{0}} \omega(x) d x \lesssim \int_{\mathbb{R}^{n}}\left|\psi_{0}^{+}(f)\right|^{p} \omega(x) d x \tag{3.21}
\end{equation*}
$$

and for $I_{2}$ by Lemma 3.3 we get

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}\left|K_{B r_{0}}\left(\left[\psi_{0}^{+}(f)\right]^{r_{0}}\right)(x)\right|^{p / r_{0}} \omega(x) d x \lesssim \int_{\mathbb{R}^{n}}\left|\psi_{0}^{+}(f)\right|^{p} \omega(x) d x \tag{3.22}
\end{equation*}
$$

which together with (3.21) implies (3.16).
Now we prove (3.17). By $N_{0} \equiv\left[2 A_{1}\right]+1$, we know that there exists $A \in\left(A_{1}, \infty\right)$ such that $2 A<N_{0}$. In the rest of this proof, we fix $A \in\left(A_{1}, \infty\right)$ satisfying $2 A<N_{0}$ and $B \in\left(B_{1}, \infty\right)$. Take an integer $N \geq N_{0}$ and $R \in(0, \infty)$. For any $\gamma \in \mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right), x \in \mathbb{R}^{n}, l \in \mathbb{Z}$ (where $l$
satisfies $\left.2^{-l} \in(0, \rho(x))\right)$ and $j \geq j_{x}\left(\right.$ where $\left.2^{-j_{x}}<\rho(x) \leq 2^{-j_{x}+1}\right)$, from Lemma 3.1, it follows that

$$
\begin{equation*}
\gamma_{l} * f=\gamma_{l} *\left(\varphi_{0}\right)_{j} *\left(\psi_{0}\right)_{j} * f+\sum_{k=j+1}^{\infty} \gamma_{l} * \varphi_{k} * \psi_{k} * f \tag{3.23}
\end{equation*}
$$

where $\varphi_{0}, \varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ with $L_{\varphi} \geq N$ and $\psi$ is as in Lemma 3.1.
For any given $l_{0} \in \mathbb{Z}$ which satisfies $2^{-l_{0}} \in(0, \rho(x))$, and $z \in \mathbb{R}^{n}$ which satisfies $|z-x|<$ $2^{-l_{0}}$, by (3.23) we have

$$
\begin{align*}
\left|\gamma_{l_{0}} * f(z)\right| \leq & \left|\gamma_{l_{0}} *\left(\varphi_{0}\right)_{l_{0}} *\left(\psi_{0}\right)_{l_{0}} * f(z)\right|+\sum_{k=l_{0}+1}^{\infty}\left|\gamma_{l_{0}} * \varphi_{k} * \psi_{k} * f(z)\right| \\
\leq & \int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}} *\left(\varphi_{0}\right)_{l_{0}}(y)\right|\left|\left(\psi_{0}\right)_{l_{0}} * f(z-y)\right| d y \\
& +\sum_{k=l_{0}+1}^{\infty} \int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}} * \varphi_{k}(y)\right|\left|\psi_{k} * f(z-y)\right| d y \equiv I_{3}+I_{4} . \tag{3.24}
\end{align*}
$$

To estimate $I_{3}$, from

$$
\begin{aligned}
\psi_{0, A, B}^{* *}(f)(x) & =\sup _{j \geq j_{x}, y \in \mathbb{R}^{n}} \frac{\left|\left(\psi_{0}\right)_{j} * f(x-y)\right|}{m_{j, A, B, x}(y)} \\
& =\sup _{j \geq j_{x}, y \in \mathbb{R}^{n}} \frac{\left|\left(\psi_{0}\right)_{j} * f(x-(y+x-z))\right|}{m_{j, A, B, x}(y+x-z)} \\
& =\sup _{j \geq j_{x}, y \in \mathbb{R}^{n}} \frac{\left|\left(\psi_{0}\right)_{j} * f(z-y)\right|}{m_{j, A, B, x}(y+x-z)},
\end{aligned}
$$

we infer that

$$
\left|\left(\psi_{0}\right)_{l_{0}} * f(z-y)\right| \leq \psi_{0, A, B}^{* *}(f)(x) m_{l_{0}, A, B, x}(y+x-z),
$$

which together with the facts that

$$
m_{l_{0}, A, B, x}(y+x-z) \leq m_{l_{0}, A, B, x}(x-z) m_{l_{0}, A, B, x}(y)
$$

and

$$
m_{l_{0}, A, B, x}(x-z)=\left(1+2^{l_{0}}|x-z|\right)^{A} 2^{B \frac{|x-z|}{\rho(x)}} \lesssim 2^{A}
$$

implies that

$$
\left|\left(\psi_{0}\right)_{l_{0}} * f(z-y)\right| \lesssim 2^{A} \psi_{0, A, B}^{* *}(f)(x) m_{l_{0}, A, B, x}(y)
$$

Thus, we have

$$
I_{3} \lesssim 2^{A}\left\{\int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}} *\left(\varphi_{0}\right)_{l_{0}}(y)\right| m_{l_{0}, A, B, x}(y) d y\right\} \psi_{0, A, B}^{* *}(f)(x)
$$

To estimate $\mathrm{I}_{4}$, by the definition of $\psi$, it is easy to know that for any $k \in \mathbb{Z}$,

$$
\left|\psi_{k} * f(z-y)\right| \leq\left|\left(\psi_{0}\right)_{k} * f(z-y)\right|+\left|\left(\psi_{0}\right)_{k-1} * f(z-y)\right| .
$$

By the definition of $\psi_{0, A, B}^{* *}(f)$ and the facts that

$$
m_{k, A, B, x}(y+x-z) \leq m_{k, A, B, x}(x-z) m_{k, A, B, x}(y),
$$

for any $k \in \mathbb{Z}$ and $m_{k, A, B, x}(x-z) \lesssim 2^{\left(k-l_{0}\right) A}$, we conclude that

$$
\begin{aligned}
\left|\left(\psi_{0}\right)_{k} * f(z-y)\right| & \leq \psi_{0, A, B}^{* *}(f)(x) m_{k, A, B, x}(y+x-z) \\
& \leq \psi_{0, A, B}^{* *}(f)(x) m_{k, A, B, x}(x-z) m_{k, A, B, x}(y) \\
& \lesssim 2^{\left(k-l_{0}\right) A} m_{k, A, B, x}(y) \psi_{0, A, B}^{* *}(f)(x) .
\end{aligned}
$$

Similarly, we also have

$$
\left|\left(\psi_{0}\right)_{k-1} * f(z-y)\right| \lesssim 2^{\left(k-l_{0}\right) A} m_{k, A, B, x}(y) \psi_{0, A, B}^{* *}(f)(x) .
$$

Thus,

$$
I_{4} \lesssim \sum_{k=l_{0}+1}^{\infty} 2^{\left(k-l_{0}\right) A}\left\{\int_{\mathbb{R}^{n}}\left|\gamma_{t} * \varphi_{k}(y)\right| m_{k, A, B, x}(y) d y\right\} \psi_{0, A, B}^{* *}(f)(x) .
$$

From (3.24) and the above estimates of $\mathrm{I}_{3}$ and $\mathrm{I}_{4}$, it follows that

$$
\begin{align*}
\left|\gamma_{l_{0}} * f(z)\right| \lesssim & \left\{\int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}} *\left(\varphi_{0}\right) l_{0}(y)\right| m_{l_{0}, A, B, x}(y) d y\right. \\
& \left.+\sum_{k=l_{0}+1}^{\infty} 2^{\left(k-l_{0}\right) A} \int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}} * \varphi_{k}(y)\right| m_{k, A, B, x}(y) d y\right\} \psi_{0, A, B}^{* *}(f)(x) \tag{3.25}
\end{align*}
$$

Assume that $\operatorname{supp}\left(\varphi_{0}\right) \subset B\left(0, R_{0}\right)$. Then $\operatorname{supp}\left(\left(\varphi_{0}\right)_{j}\right) \subset B\left(0,2^{-j} R_{0}\right)$ for all $j \geq j_{x}$. Moreover, by $\operatorname{supp}(\gamma) \subset B(0, R)$, we see that $\operatorname{supp}\left(\gamma_{l_{0}}\right) \subset B\left(0,2^{-l_{0}} R\right)$. From this, we further deduce that $\operatorname{supp}\left(\gamma_{l_{0}} *\left(\varphi_{0}\right)_{l_{0}}\right) \subset B\left(0,2^{-l_{0}}\left(R_{0}+R\right)\right)$ and

$$
\left|\gamma_{l_{0}} *\left(\varphi_{0}\right)_{l_{0}}(y)\right| \lesssim \int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}}(s)\right|\left|\left(\varphi_{0}\right)_{l_{0}}(y-s)\right| d s \lesssim 2^{l_{0} n} \int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}}(s)\right| d s \sim 2^{l_{0} n}
$$

which implies that

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}} *\left(\varphi_{0}\right) l_{0}(y)\right| m_{l_{0}, A, B, x}(y) d y \lesssim 2^{l_{0} n} \int_{B\left(0,2^{\left.-l_{0}\left(R_{0}+R\right)\right)}\right.}\left(1+2^{l_{0}}|y|\right)^{A} 2^{\frac{B|y|}{\rho(x)}} d y \lesssim 1 . \tag{3.26}
\end{equation*}
$$

Moreover, since $\varphi$ has vanishing moments up to order $N$, it was proved in [23, (2.13)] that

$$
\left\|\gamma_{l_{0}} * \varphi_{k}\right\|_{L^{\infty}\left(\mathbb{R}^{n}\right)} \lesssim 2^{\left(l_{0}-k\right) N} 2^{l_{0} n}
$$

for all $k \in \mathbb{Z}$ with $k \geq l_{0}+1$, which, together with the facts that $l_{0} \geq j_{x}, N>2 A$ and

$$
\operatorname{supp}\left(\gamma_{l_{0}} * \varphi_{k}\right) \subset B\left(0,2^{-l_{0}} R_{0}+2^{-k} R\right),
$$

implies that

$$
\begin{align*}
\sum_{k=l_{0}+1}^{\infty} 2^{\left(k-l_{0}\right) A} & \int_{\mathbb{R}^{n}}\left|\gamma_{l_{0}} * \varphi_{k}(y)\right| m_{k, A, B, x}(y) d y \\
& \lesssim \sum_{k=l_{0}+1}^{\infty} 2^{\left(k-l_{0}\right) A} 2^{\left(l_{0}-k\right) N} 2^{l_{0} n}\left(2^{-l_{0}} R_{0}+2^{-k} R\right)^{n} \\
& \times\left[1+2^{k}\left(2^{-l_{0}} R_{0}+2^{-k} R\right)\right]^{A} 2^{B\left(2^{-l_{0}} R_{0}+2^{-k} R\right) / \rho(x)} \\
& \lesssim \sum_{k=l_{0}+1}^{\infty} 2^{\left(l_{0}-k\right)(N-2 A)} \lesssim 1 . \tag{3.27}
\end{align*}
$$

Thus, from (3.25), (3.26) and (3.27), we deduce that $\left|\gamma_{l_{0}} * f(z)\right| \lesssim \psi_{0, A, B}^{* *}(f)(x)$. Then, by the arbitrariness of $l_{0} \geq j_{x}$ and $z \in B\left(x, 2^{-l_{0}}\right)$, we know that

$$
\begin{equation*}
\widetilde{\mathcal{M}}_{N, R}(f)(x) \lesssim \psi_{0, A, B}^{* *}(f)(x), \tag{3.28}
\end{equation*}
$$

which deduces the (3.17) and completes the proof of this theorem.

As a corollary of Theorem 3.1 we immediately obtain the local vertical and the local nontangential maximal function characterizations of $h_{\rho, N}^{p}(\omega)$ with $N \geq N_{p, \omega}$ as follows. Here and in what follows,

$$
\begin{equation*}
N_{p, \omega} \equiv \max \left\{\widetilde{N}_{p, \omega}, N_{0}\right\}, \tag{3.29}
\end{equation*}
$$

where $\widetilde{N}_{p, \omega}$ and $N_{0}$ are respectively as in Definition 3.2 and Theorem 3.1.
Theorem 3.2. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), \psi_{0}$ and $N_{p, \omega}$ be respectively as in (3.3) and (3.29). Then for any integer $N \geq N_{p, \omega}$, the following are equivalent:
(i) $f \in h_{\rho, N}^{p}(\omega)$;
(ii) $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\psi_{0}^{+}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$;
(iii) $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\left(\psi_{0}\right)_{\nabla}^{*}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$;
(iv) $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\widetilde{\mathcal{M}}_{N}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$;
(v) $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\widetilde{\mathcal{M}}_{N}^{0}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$;
(vi) $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\mathcal{M}_{N}^{0}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$.

Moreover, for all $f \in h_{\rho, N}^{p}(\omega)$

$$
\begin{align*}
\|f\|_{h_{p, N}^{p}(\omega)} & \sim\left\|\psi_{0}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \sim\left\|\left(\psi_{0}\right)_{\nabla}^{*}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \\
& \sim\left\|\widetilde{\mathcal{M}}_{N}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \sim\left\|\widetilde{\mathcal{M}}_{N}^{0}(f)\right\|_{L_{\omega\left(\mathbb{R}^{n}\right)}^{p}} \sim\left\|\mathcal{M}_{N}^{0}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}, \tag{3.30}
\end{align*}
$$

where the implicit constants are independent of $f$.

Proof. (i) $\Rightarrow$ (ii). Pick an integer $N \geq N_{p, \omega}$ and $f \in h_{\rho, N}^{p}(\omega)$. Let $\widetilde{\psi}_{0}$ satisfy (3.3) and $\widetilde{\psi}_{0} \in \mathcal{D}_{N}\left(\mathbb{R}^{n}\right)$. Then from the definition of $\mathcal{M}_{N}(f)$, we infer that $\widetilde{\psi}_{0}^{+}(f) \leq \mathcal{M}_{N}(f)$ and hence $\widetilde{\psi}_{0}^{+}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$. For any $\psi_{0}$ satisfying (3.3), assume that $\operatorname{supp}\left(\psi_{0}\right) \subset B(0, R)$. Then, by (3.17) and the above argument, we have

$$
\left\|\widetilde{\mathcal{M}}_{N, R}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\widetilde{\psi}_{0}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{h_{\rho, N}^{p}(\omega)}
$$

which together with $\psi_{0}^{+}(f) \lesssim \widetilde{\mathcal{M}}_{N, R}(f)$ implies that $\psi_{0}^{+}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ and

$$
\left\|\psi_{0}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\|f\|_{h_{\rho, N}^{p}(\omega)}
$$

(ii) $\Rightarrow$ (iii). Let $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfy $\psi_{0}^{+}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$, where $\psi_{0}$ is as in (3.3). Then from the fact that

$$
\psi_{0}^{+}(f) \leq\left(\psi_{0}\right)_{\nabla}^{*}(f) \lesssim \psi_{0, A, B}^{* *}(f)
$$

and (3.16), we deduce that $\left(\psi_{0}\right)_{\nabla}^{*}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ and

$$
\left\|\left(\psi_{0}\right)_{\vec{\gamma}}^{*}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\psi_{0}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} .
$$

(iii) $\Rightarrow$ (iv). Let $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfy $\left(\psi_{0}\right)_{\nabla}^{*}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$, where $\psi_{0}$ is as in (3.3). By (3.17),

$$
\left\|\widetilde{\mathcal{M}}_{N}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\psi_{0}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}
$$

which together with the fact that

$$
\psi_{0}^{+}(f) \leq\left(\psi_{0}\right)_{\nabla}^{*}(f)
$$

and the assumption that $\left(\psi_{0}\right)_{\nabla}^{*}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ implies $\widetilde{\mathcal{M}}_{N}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$ and

$$
\left\|\widetilde{\mathcal{M}}_{N}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\left(\psi_{0}\right)_{\nabla}^{*}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}
$$

$(\mathrm{iv}) \Rightarrow(\mathrm{v}) \Rightarrow(\mathrm{vi})$. By the facts that $\mathcal{M}_{N}^{0}(f) \leq \widetilde{\mathcal{M}}_{N}^{0}(f) \leq \widetilde{\mathcal{M}}_{N}(f)$ for any $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, we see that all the conclusions hold. Moreover, it is obvious that

$$
\left\|\mathcal{M}_{N}^{0}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq\left\|\widetilde{\mathcal{M}}_{N}^{0}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq\left\|\widetilde{\mathcal{M}}_{N}(f)\right\|_{L_{\omega\left(\mathbb{R}^{n}\right)}^{p}}
$$

$(\mathrm{vi}) \Rightarrow(\mathrm{i})$. Let $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfy $\mathcal{M}_{N}^{0}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$. Let $\psi_{1}$ satisfy (3.3) and $\psi_{1} \in$ $\mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right)$. Then by (3.17), we have that

$$
\left\|\widetilde{\mathcal{M}}_{N}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\psi_{1}^{+}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}
$$

which together with the facts that $\psi_{1}^{+}(f) \leq \mathcal{M}_{N}^{0}(f)$ and $\mathcal{M}_{N}(f) \leq \widetilde{\mathcal{M}}_{N}(f)$ implies that

$$
\left\|\mathcal{M}_{N}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \lesssim\left\|\mathcal{M}_{N}^{0}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}
$$

Thus, by the definition of $h_{\rho, N}^{p}(\omega)$, we know that $f \in h_{\rho, N}^{p}(\omega)$ and

$$
\|f\|_{h_{\rho, N}^{p}(\omega)} \lesssim\left\|\mathcal{M}_{N}^{0}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}
$$

which completes the proof of Theorem 3.2,

By the Theorems 3.1 and 3.2, we also have the following corollary about local tangential maximal function characterization of $h_{\rho, N}^{p}(\omega)$, and we omit the details here.

Corollary 3.1. Let $\psi_{0}$ be as in (3.3), $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), N_{p, \omega}$ be as in (3.29), $A$ and $B$ be as in Theorem 3.1. Then for integer $N \geq N_{p, \omega}, f \in h_{\rho, N}^{p}(\omega)$ if and only if $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\psi_{0, A, B}^{* *}(f) \in L_{\omega}^{p}\left(\mathbb{R}^{n}\right)$; moreover,

$$
\|f\|_{h_{\rho, N}^{p}(\omega)} \sim\left\|\psi_{0, A, B}^{* *}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} .
$$

Next we give some basic properties of $h_{\rho, N}^{p}(\omega)$ and $h_{\rho}^{p, q, s}(\omega)$.
Proposition 3.2. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), p \in(0,1]$ and $N_{p, \omega}$ be as in (3.29). For any integer $N \geq N_{p, \omega}$, the inclusion $h_{\rho, N}^{p}(\omega) \hookrightarrow \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ is continuous.

Proof. First, for any $x \in B(0, \rho(0))$, by Lemma 2.1, there exist $C_{0} \geq 1$ and $k_{0} \geq 1$, such that

$$
\rho(0) \leq C_{0}\left(1+\frac{|x|}{\rho(0)}\right)^{k_{0}} \rho(x) \leq C_{0} 2^{k_{0}} \rho(x) .
$$

We take $r_{1} \equiv \rho(0) / C_{0} 2^{k_{0}+1}<\min \{\rho(x), \rho(0)\}$, then we have $B\left(0, r_{1}\right) \subset B(0, \rho(0))$. In addition, for any $x \in B\left(0, r_{1}\right)$, we also have $|x|<r_{1}<\rho(x)$.

Next, let $f \in h_{\rho, N}^{p}(\omega)$. For any given $\varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$, assume that $\operatorname{supp}(\varphi) \subset B(0, R)$ with $R \in(0, \infty)$. Then by Theorem 3.1 and 3.2, we have

$$
\begin{aligned}
|\langle f, \varphi\rangle|=|f * \widetilde{\varphi}(0)| & \leq\|\widetilde{\varphi}\|_{\mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right)} \inf _{x \in B\left(0, r_{1}\right)} \widetilde{\mathcal{M}}_{N, R}(f)(x) \\
& \leq\|\widetilde{\varphi}\|_{\mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right)}\left[\omega\left(B\left(0, r_{1}\right)\right)\right]^{-1 / p}\left\|\widetilde{\mathcal{M}}_{N, R}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\|\widetilde{\varphi}\|_{\mathcal{D}_{N, R}\left(\mathbb{R}^{n}\right)}\left[\omega\left(B\left(0, r_{1}\right)\right)\right]^{-1 / p}\|f\|_{h_{p, N}^{p}(\omega)},
\end{aligned}
$$

where $\widetilde{\mathcal{M}}_{N, R}(f)$ is as in (3.1) and $\widetilde{\varphi}(x) \equiv \varphi(-x)$ for all $x \in \mathbb{R}^{n}$. This implies $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and the inclusion is continuous. The proof is finished.

Proposition 3.3. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), p \in(0,1]$ and $N_{p, \omega}$ be as in (3.29). For any integer $N \geq N_{p, \omega}$, the space $h_{\rho, N}^{p}(\omega)$ is complete.

Proof. For any $\psi \in \mathcal{D}_{N}\left(\mathbb{R}^{n}\right)$ and $\left\{f_{i}\right\}_{i \in \mathbb{N}} \subset \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ such that $\left\{\sum_{i=1}^{j} f_{i}\right\}_{j \in \mathbb{N}}$ converges in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ to a distribution $f$ as $j \rightarrow \infty$, and the series $\sum_{i} f_{i} * \psi(x)$ converges pointwise to $f * \psi(x)$ for each $x \in \mathbb{R}^{n}$. Therefore,

$$
\left(\mathcal{M}_{N}(f)(x)\right)^{p} \leq\left(\sum_{i=1}^{\infty} \mathcal{M}_{N}\left(f_{i}\right)(x)\right)^{p} \leq \sum_{i=1}^{\infty}\left(\mathcal{M}_{N}\left(f_{i}\right)(x)\right)^{p} \quad \text { for all } x \in \mathbb{R}^{n},
$$

and hence $\|f\|_{h_{\rho, N}^{p}(\omega)} \leq \sum_{i}^{\infty}\left\|f_{i}\right\|_{h_{\rho, N}^{p}(\omega)}$.
To prove that $h_{\rho, N}^{p}(\omega)$ is complete, it suffices to show that for every sequence $\left\{f_{j}\right\}_{j \in \mathbb{N}}$ with $\left\|f_{j}\right\|_{h_{\rho, N}^{p}(\omega)}<2^{-j}$ for any $j \in \mathbb{N}$, the series $\sum_{j \in \mathbb{N}} f_{j}$ convergence in $h_{\rho, N}^{p}(\omega)$. Since $\left\{\sum_{i=1}^{j} f_{i}\right\}_{j \in \mathbb{N}}$ is a Cauchy sequence in $h_{\rho, N}^{p}(\omega)$, by Proposition 3.2 and the completeness of $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, $\left\{\sum_{i=1}^{j} f_{i}\right\}_{j \in \mathbb{N}}$ is also a Cauchy sequence in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and thus converges to some $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$. Therefore,

$$
\left\|f-\sum_{i=1}^{j} f_{i}\right\|_{h_{p, N}^{p}(\omega)}^{p}=\left\|\sum_{i=j+1}^{\infty} f_{i}\right\|_{h_{p, N}^{p}(\omega)}^{p} \leq \sum_{i=j+1}^{\infty} 2^{-i p} \rightarrow 0
$$

as $j \rightarrow \infty$. This finishes the proof.
Theorem 3.3. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ and $N_{p, \omega}$ be as in (3.29). If $(p, q, s)_{\omega}$ is an admissible triplet (see Definition 3.3) and integer $N \geq N_{p, \omega}$, then

$$
h_{\rho}^{p, q, s}(\omega) \subset h_{\rho, N_{p, \omega}}^{p}(\omega) \subset h_{\rho, N}^{p}(\omega),
$$

and moreover, there exists a positive constant $C$ such that for all $f \in h_{\rho}^{p, q, s}(\omega)$,

$$
\|f\|_{h_{\rho, N}^{p}(\omega)} \leq\|f\|_{h_{\rho, N_{p, \omega}}^{p}(\omega)} \leq C\|f\|_{h_{\rho}^{p, q, s}(\omega)} .
$$

Proof. Obviously, we only need to prove $h_{\rho}^{p, q, s}(\omega) \subset h_{\rho, N_{p, \omega}}^{p}(\omega)$. For all $f \in h_{\rho}^{p, q, s}(\omega)$,

$$
\|f\|_{h_{p, N_{p, \omega}}^{p}(\omega)} \lesssim\|f\|_{h_{p}^{p, q, s}(\omega)} .
$$

By Definition 3.4 and Theorem 3.2, it suffices to prove that there exists a positive constant $C$ such that

$$
\begin{equation*}
\left\|\mathcal{M}_{N_{p, \omega}}^{0}(a)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq C, \quad \text { for all }(p, q)_{\omega}-\text { single }- \text { atoms } a, \tag{3.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\mathcal{M}_{N_{p, \omega}}^{0}(a)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)} \leq C, \quad \text { for all }(p, q, s)_{\omega}-\text { atoms } a . \tag{3.32}
\end{equation*}
$$

We first prove (3.31). Since $q \in\left(q_{\omega}, \infty\right]$, so $\omega \in A_{q}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$. Let $a$ be a $(p, q)_{\omega}$-single-atom. When $\omega\left(\mathbb{R}^{n}\right)=\infty$, by the definition of the single atom, we know that $a=0$ for almost every
$x \in \mathbb{R}^{n}$. In this case, it is easy to see that (3.31) holds. When $\omega\left(\mathbb{R}^{n}\right)<\infty$, from Hölder's inequality, $\omega \in A_{q}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ and Proposition 3.1(i), we deduce that

$$
\begin{aligned}
\left\|\mathcal{M}_{N_{p, \omega}}^{0}(a)\right\|_{\left.L_{\omega(\mathbb{R}}^{p}\right)}^{p} & =\int_{\mathbb{R}^{n}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{p} \omega(x) d x \\
& \leq\left(\int_{\mathbb{R}^{n}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{q} \omega(x) d x\right)^{p / q}\left(\int_{\mathbb{R}^{n}} \omega(x) d x\right)^{1-p / q} \\
& \leq C\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)}^{p}\left[\omega\left(\mathbb{R}^{n}\right)\right]^{1-p / q} \leq C .
\end{aligned}
$$

Next, we prove (3.32). Let $a$ be a $(p, q, s)_{\omega}$-atom supported in the cube $Q \equiv Q\left(x_{0}, r\right)$. We consider the following two cases for $Q$.

The first case is when $r<L_{2} \rho\left(x_{0}\right)$. Let $\widetilde{Q} \equiv 2 \sqrt{n} Q$, then we have

$$
\begin{align*}
\int_{\mathbb{R}^{n}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{p} \omega(x) d x & =\int_{\widetilde{Q}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{p} \omega(x) d x+\int_{\widetilde{Q}^{0}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{p} \omega(x) d x \\
& \equiv \mathrm{I}_{1}+\mathrm{I}_{2} \tag{3.33}
\end{align*}
$$

For $I_{1}$, by Hölder's inequality and the properties of $A_{q}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ (see Lemma 2.4(vi)), we have

$$
\begin{align*}
I_{1} & \leq\left(\int_{\mathbb{R}^{n}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{q} \omega(x) d x\right)^{p / q}\left(\int_{\widetilde{Q}} \omega(x) d x\right)^{1-p / q} \\
& \leq C\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)}^{p}[\omega(\widetilde{Q})]^{1-p / q} \leq C . \tag{3.34}
\end{align*}
$$

To estimate $I_{2}$, we claim that for $x \in \widetilde{Q}^{\complement}$

$$
\begin{equation*}
\mathcal{M}_{N_{p, \omega}}^{0}(a)(x) \leq C|Q|^{\left(s_{0}+n+1\right) / n}[\omega(Q)]^{-1 / p}\left|x-x_{0}\right|^{-\left(s_{0}+n+1\right)} \chi_{B\left(x_{0}, c_{1} \rho\left(x_{0}\right)\right)}(x), \tag{3.35}
\end{equation*}
$$

where $s_{0} \equiv\left[n\left(q_{\omega} / p-1\right)\right]$ and $c_{1}>2 \sqrt{n}$ is an constant independent of the atom $a$. Indeed, for any $\psi \in \mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right)$ and $2^{-l} \in(0, \rho(x))$, let $P$ be the Taylor expansion of $\psi$ about $\left(x-x_{0}\right) / 2^{-l}$ with degree $s_{0}$. By Taylor's remainder theorem, for any $y \in \mathbb{R}^{n}$, we have

$$
\begin{aligned}
& \left|\psi\left(\frac{x-y}{2^{-l}}\right)-P\left(\frac{x-x_{0}}{2^{-l}}\right)\right| \\
& \leq C \sum_{\substack{\alpha \in \mathbb{Z}^{n} \\
|\alpha|=s_{0}+1}}\left|\left(\partial^{\alpha} \psi\right)\left(\frac{\theta(x-y)+(1-\theta)\left(x-x_{0}\right)}{2^{-l}}\right)\right|\left|\frac{x_{0}-y}{2^{-l}}\right|^{s_{0}+1},
\end{aligned}
$$

where $\theta \in(0,1)$. By $2^{-l} \in(0, \rho(x))$ and $x \in \widetilde{Q}^{\complement}$, we see that $\operatorname{supp}\left(a * \psi_{l}\right) \subset B\left(x_{0}, c_{1} \rho\left(x_{0}\right)\right)$, and $a * \psi_{l}(x) \neq 0$ implies that $2^{-l}>\left|x-x_{0}\right| / 2$. Thus, from the above facts and Definition
3.3, we get that for all $x \in \widetilde{Q}^{\complement}$,

$$
\begin{aligned}
\left|a * \psi_{l}(x)\right| & \leq \frac{1}{2^{-l n}}\left\{\int_{Q}|a(y)|\left|\psi\left(\frac{x-y}{2^{-l}}\right)-P\left(\frac{x-x_{0}}{2^{-l}}\right)\right| d y\right\} \chi_{B\left(x_{0}, c_{1} \rho\left(x_{0}\right)\right)}(x) \\
& \leq C\left|x-x_{0}\right|^{-\left(s_{0}+n+1\right)}\left\{\int_{Q}\left|a(y) \| x_{0}-y\right|^{s_{0}+1} d y\right\} \chi_{B\left(x_{0}, c_{1} \rho\left(x_{0}\right)\right)}(x) \\
& \left.\leq C|Q|^{\left(s_{0}+1\right) / n}\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)}\left(\int_{Q}[\omega(y)]^{-q^{\prime} / q} d y\right)^{1 / q^{\prime}}\left|x-x_{0}\right|^{-\left(s_{0}+n+1\right)} \chi_{B\left(x_{0}, c_{1} \rho\left(x_{0}\right)\right)(x)}\right) \\
& \leq C|Q|^{\left(s_{0}+n+1\right) / n}[\omega(Q)]^{-1 / p}\left|x-x_{0}\right|^{-\left(s_{0}+n+1\right)} \chi_{B\left(x_{0}, c_{1} \rho\left(x_{0}\right)\right)}(x),
\end{aligned}
$$

which together with the arbitrariness of $\psi \in \mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right)$ implies (3.35). Thus, the claim holds.
Let $Q_{i} \equiv 2^{i} \sqrt{n} Q$ for all $i \in \mathbb{N}$ and $i_{0} \in \mathbb{N}$ satisfying $2^{i_{0}} r \leq c_{1} \rho\left(x_{0}\right)<2^{i_{0}+1} r$. As $s_{0}=\left[n\left(q_{\omega} / p-1\right)\right]$, we know that there exists $q_{0} \in\left(q_{\omega}, \infty\right)$ such that $p\left(s_{0}+n+1\right)>n q_{0}$. Then from the Lemma [2.4] we conclude that

$$
\begin{aligned}
I_{2} & \leq \int_{\sqrt{n} r \leq\left|x-x_{0}\right|<c_{1} \rho\left(x_{0}\right)}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{p} \omega(x) d x \\
& \leq C|Q|^{p\left(s_{0}+n+1\right) / n}[\omega(Q)]^{-1} \int_{\sqrt{n} r \leq\left|x-x_{0}\right|<c_{1} \rho\left(x_{0}\right)}\left|x-x_{0}\right|^{-p\left(s_{0}+n+1\right)} \omega(x) d x \\
& \leq C r^{p\left(s_{0}+n+1\right)}[\omega(Q)]^{-1} \sum_{i=0}^{i_{0}} \int_{Q_{i+1} \backslash Q_{i}}\left|x-x_{0}\right|^{-p\left(s_{0}+n+1\right)} \omega(x) d x \\
& \leq C[\omega(Q)]^{-1} \sum_{i=0}^{i_{0}} 2^{-i p\left(s_{0}+n+1\right)} \omega\left(Q_{i+1}\right) \\
& \leq C[\omega(Q)]^{-1} \sum_{i=0}^{i_{0}} 2^{-i\left[p\left(s_{0}+n+1\right)-n q_{0}\right]} \omega(Q) \leq C,
\end{aligned}
$$

which together with (3.33) and (3.34) implies (3.32) in the first case.
Now we consider the case $L_{2} \rho\left(x_{0}\right) \leq r \leq L_{1} \rho\left(x_{0}\right)$, let $Q^{*} \equiv Q\left(x_{0}, c_{2} r\right)$, in which $c_{2}>1$ is an constant independent of atom $a$. Thus, from $\operatorname{supp}\left(\mathcal{M}_{N_{p, w}}^{0}(a)\right) \subset Q^{*}$, Hölder's inequality and Lemma 2.4. we get

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{p} \omega(x) d x & =\int_{Q^{*}}\left|\mathcal{M}_{N_{p, \omega}}^{0}(a)(x)\right|^{p} \omega(x) d x \\
& \leq C\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)}^{p}\left[\omega\left(Q^{*}\right)\right]^{1-p / q} \\
& \leq C\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)}^{p}[\omega(Q)]^{1-p / q} \\
& \leq C .
\end{aligned}
$$

This finishes the proof of Theorem 3.3.

## 4 Calderón-Zygmund decompositions

In this section, we establish the Calderón-Zygmund decompositions associated with local grand maximal functions on weighted Euclidean space $\mathbb{R}^{n}$. We follow the constructions in [26], 1] and [5.

Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ and $q_{\omega}$ be as in (2.4). For integer $N \geq 2$, let $\mathcal{M}_{N}(f)$ and $\mathcal{M}_{N}^{0}(f)$ be as in (3.2). Throughout this section, we consider a distribution $f$ so that for all $\lambda>0$,

$$
\omega\left(\left\{x \in \mathbb{R}^{n}: \mathcal{M}_{N}(f)(x)>\lambda\right\}\right)<\infty
$$

For a given $\lambda>\inf _{x \in \mathbb{R}^{n}} \mathcal{M}_{N}(f)(x)$, we set

$$
\Omega_{\lambda} \equiv\left\{x \in \mathbb{R}^{n}: \mathcal{M}_{N}(f)(x)>\lambda\right\} .
$$

It is obvious that $\Omega_{\lambda}$ is a proper open subset of $\mathbb{R}^{n}$. As in [26], we give the usual Whitney decomposition of $\Omega_{\lambda}$. Thus we can find closed cubes $Q_{k}$ whose interiors distance from $\Omega_{\lambda}^{\complement}$, with $\Omega_{\lambda}=\bigcup_{k} Q_{k}$ and

$$
\operatorname{diam}\left(Q_{k}\right) \leq 2^{-(6+n)} \operatorname{dist}\left(Q_{k}, \Omega_{\lambda}^{\complement}\right) \leq 4 \operatorname{diam}\left(Q_{k}\right)
$$

In what follows, fix $a \equiv 1+2^{-(11+n)}$ and $b \equiv 1+2^{-(10+n)}$, and if we denote $\bar{Q}_{k}=a Q_{k}, Q_{k}^{*}=$ $b Q_{k}$, we have $Q_{k} \subset \bar{Q}_{k} \subset Q_{k}^{*}$. Moveover, $\Omega_{\lambda}=\bigcup_{k} Q_{k}^{*}$, and $\left\{Q_{k}^{*}\right\}_{k}$ have the bounded interior property, namely, every point in $\Omega_{\lambda}$ is contained in at most a fixed number of $\left\{Q_{k}^{*}\right\}_{k}$.

Now we take a function $\xi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ such that $0 \leq \xi \leq 1, \operatorname{supp}(\xi) \subset a Q(0,1)$ and $\xi \equiv 1$ on $Q(0,1)$. For $x \in \mathbb{R}^{n}$, set $\xi_{k}(x) \equiv \xi\left(\left(x-x_{k}\right) / l_{k}\right)$, where and in what follows, $x_{k}$ is the center of the cube $Q_{k}$ and $l_{k}$ is its sidelength. Obviously, by the construction of $\left\{Q_{k}^{*}\right\}_{k}$ and $\left\{\xi_{k}\right\}_{k}$, for any $x \in \mathbb{R}^{n}$, we have $1 \leq \sum_{k} \xi_{k}(x) \leq M$, where $M$ is a fixed positive integer independent of $x$. Let $\eta_{k} \equiv \xi_{k} /\left(\sum_{j} \xi_{j}\right)$. Then $\left\{\eta_{k}\right\}_{k}$ form a smooth partition of unity for $\Omega_{\lambda}$ subordinate to the locally finite covering $\left\{Q_{k}^{*}\right\}_{k}$ of $\Omega_{\lambda}$, namely, $\chi_{\Omega_{\lambda}}=\sum_{k} \eta_{k}$ with each $\eta_{k} \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ supported in $\bar{Q}_{k}$.

Let $s \in \mathbb{Z}_{+}$be some fixed integer and $\mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$ denote the linear space of polynomials in $n$ variables of degrees no more than $s$. For each $i \in \mathbb{N}$ and $P \in \mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$, set

$$
\begin{equation*}
\|P\|_{i} \equiv\left[\frac{1}{\int_{\mathbb{R}^{n}} \eta_{i}(y) d y} \int_{\mathbb{R}^{n}}|P(x)|^{2} \eta_{i}(x) d x\right]^{1 / 2} \tag{4.1}
\end{equation*}
$$

Then it is easy to see that $\left(\mathcal{P}_{s}\left(\mathbb{R}^{n}\right),\|\cdot\|_{i}\right)$ is a finite dimensional Hilbert space. Let $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$. Since $f$ induces a linear functional on $\mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$ via

$$
P \mapsto \frac{1}{\int_{\mathbb{R}^{n}} \eta_{i}(y) d y}\left\langle f, P \eta_{i}\right\rangle,
$$

by the Riesz represent theorem, there exists a unique polynomial $P_{i} \in \mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$ for each $i$ such that for all $Q \in \mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$,

$$
\left\langle f, Q \eta_{i}\right\rangle=\left\langle P_{i}, Q \eta_{i}\right\rangle=\int_{\mathbb{R}^{n}} P_{i}(x) Q(x) \eta_{i}(x) d x
$$

For each $i$, define the distribution $b_{i} \equiv\left(f-P_{i}\right) \eta_{i}$ when $l_{i} \in\left(0, L_{3} \rho\left(x_{i}\right)\right)$ (where $L_{3}=2^{k_{0}} C_{0}$, $x_{i}$ is the center of the cube $\left.Q_{i}\right)$ and $b_{i} \equiv f \eta_{i}$ when $l_{i} \in\left[L_{3} \rho\left(x_{i}\right), \infty\right)$.

We will show that for suitable choices of $s$ and $N$, the series $\sum_{i} b_{i}$ converge in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, and in this case, we define $g \equiv f-\sum_{i} b_{i}$ in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$. We point out that the represent $f=g+\sum_{i} b_{i}$, where $g$ and $b_{i}$ are as above, is called a Calderón-Zygmund decomposition of $f$ of degree $s$ and height $\lambda$ associated with $\mathcal{M}_{N}(f)$.

The rest of this section consists of a series of lemmas. In Lemma 4.1 and Lemma 4.2, we give some properties of the smooth partition of unity $\left\{\eta_{i}\right\}_{i}$. In Lemmas 4.3 through 4.6, we derive some estimates for the bad parts $\left\{b_{i}\right\}_{i}$. Lemma 4.7 and Lemma 4.8 give controls over the good part $g$. Finally, Corollary 4.1 shows the density of $L_{\omega}^{q}\left(\mathbb{R}^{n}\right) \bigcap h_{\rho, N}^{p}(\omega)$ in $h_{\rho, N}^{p}(\omega)$, where $q \in\left(q_{\omega}, \infty\right)$.

Lemma 4.1. There exists a positive constant $C_{1}$ depending only on $N$, such that for all $i$ and $l \leq l_{i}$,

$$
\sup _{|\alpha| \leq N} \sup _{x \in \mathbb{R}^{n}}\left|\partial^{\alpha} \eta_{i}(l x)\right| \leq C_{1} .
$$

Lemma 4.1 is essentially Lemma 5.2 in [1.
Lemma 4.2. If $l_{i}<L_{3} \rho\left(x_{i}\right)$, then there exists a constant $C_{2}>0$ independent of $f \in \mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, $l_{i}$ and $\lambda>0$ so that

$$
\sup _{y \in \mathbb{R}^{n}}\left|P_{i}(y) \eta_{i}(y)\right| \leq C_{2} \lambda .
$$

Proof. As in the proof of Lemma 5.3 in [1]. Let $\pi_{1}, \cdots, \pi_{m}\left(m=\operatorname{dim} \mathcal{P}_{s}\right)$ be an orthonormal basis of $\mathcal{P}_{s}$ with respect to the norm (4.1). we have

$$
\begin{equation*}
P_{i}=\sum_{k=1}^{m}\left(\frac{1}{\int \eta_{i}} \int f(x) \pi_{k}(x) \eta_{i}(x) d x\right) \bar{\pi}_{k}, \tag{4.2}
\end{equation*}
$$

where the integral is understood as $\left\langle f, \pi_{k} \eta_{i}\right\rangle$. Therefore,

$$
\begin{align*}
1 & =\frac{1}{\int \eta_{i}} \int_{\bar{Q}_{i}}\left|\pi_{k}(x)\right|^{2} \eta_{i}(x) d x \geq \frac{2^{-n}}{\left|Q_{i}\right|} \int_{\bar{Q}_{i}}\left|\pi_{k}(x)\right|^{2} \eta_{i}(x) d x \\
& \geq \frac{2^{-n}}{\left|Q_{i}\right|} \int_{Q_{i}}\left|\pi_{k}(x)\right|^{2} d x=2^{-n} \int_{Q^{0}}\left|\widetilde{\pi}_{k}(x)\right|^{2} d x, \tag{4.3}
\end{align*}
$$

where $\widetilde{\pi}_{k}(x)=\pi_{k}\left(x_{i}+l_{i} x\right)$ and $Q^{0}$ denotes the cube of side length 1 centered at the origin.

Since $\mathcal{P}_{s}$ is finite dimensional, all norms on $\mathcal{P}_{s}$ are equivalent, then there exists $A_{1}>0$ such that for all $P \in \mathcal{P}_{s}$

$$
\sup _{|\alpha| \leq s} \sup _{z \in b Q^{0}}\left|\partial^{\alpha} P(z)\right| \leq A_{1}\left(\int_{Q^{0}}|P(z)|^{2} d z\right)^{1 / 2}
$$

From this and (4.3), for $k=1, \cdots, m$, we have

$$
\begin{equation*}
\sup _{|\alpha| \leq s} \sup _{z \in b Q^{0}}\left|\partial^{\alpha} \widetilde{\pi}_{k}(z)\right| \leq A_{1} 2^{n / 2} \tag{4.4}
\end{equation*}
$$

If $z \in 2^{8+n} n Q_{i} \cap \Omega^{\complement}$, by Lemma 2.1, we have $\rho\left(x_{i}\right) \leq C_{0}\left(1+2^{8+n} n^{2} L_{3}\right)^{k_{0}} \rho(z)$, then we let $\widetilde{L} \equiv 1 / C_{0}\left(1+2^{8+n} n^{2} L_{3}\right)^{k_{0}} L_{3}$. For $k=1, \cdots, m$, we define

$$
\Phi_{k}(y)=\frac{2^{-k_{i} n}}{\int \eta_{i}} \pi_{k}\left(z-2^{-k_{i}} y\right) \eta_{i}\left(z-2^{-k_{i}} y\right)
$$

where $z \in 2^{8+n} n Q_{i} \cap \Omega^{\complement}$ and $2^{-k_{i}} \leq \widetilde{L} l_{i}<2^{-k_{i}+1}$. It is easy to see that $\operatorname{supp} \Phi_{k} \subset B\left(0, R_{1}\right)$ where $R_{1} \equiv 2^{9+n} n^{2} / \widetilde{L}$, and $\left\|\Phi_{k}\right\|_{\mathcal{D}_{N}} \leq A_{2}$ by Lemma 4.1.

Note that

$$
\frac{1}{\int \eta_{i}} \int f(x) \pi_{k}(x) \eta_{i}(x) d x=\left(f *\left(\Phi_{k}\right)_{k_{i}}\right)(z)
$$

since $2^{-k_{i}} \leq \widetilde{L} l_{i}<\widetilde{L} L_{3} \rho\left(x_{i}\right) \leq \rho(z)$, then we have

$$
\left|\frac{1}{\int \eta_{i}} \int f(x) \pi_{k}(x) \eta_{i}(x) d x\right| \leq \mathcal{M}_{N} f(z)\left\|\Phi_{k}\right\|_{\mathcal{D}_{N}} \leq A_{2} \lambda
$$

By (4.2), (4.4) and above estimate, we have

$$
\sup _{z \in Q_{i}^{*}}\left|P_{i}(z)\right| \leq m 2^{n / 2} A_{1} A_{2} \lambda
$$

Thus,

$$
\sup _{z \in \mathbb{R}^{n}}\left|P_{i}(z) \eta_{i}(z)\right| \leq C_{2} \lambda
$$

The proof is complete.

By the same method, we can get the following lemma as the Lemma 4.3 in [28], and we omit the details here.

Lemma 4.3. There exists a constant $C_{3}>0$ such that

$$
\begin{equation*}
\mathcal{M}_{N}^{0} b_{i}(x) \leq C_{3} \mathcal{M}_{N} f(x) \quad \text { for } x \in Q_{i}^{*} \tag{4.5}
\end{equation*}
$$

Lemma 4.4. Suppose $Q \subset \mathbb{R}^{n}$ is bounded, convex, and $0 \in Q$, and $N$ is a positive integer. Then there is a constant $C$ depending only on $Q$ and $N$ such that for every $\phi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ and every integer $s, 0 \leq s<N$ we have

$$
\sup _{z \in Q} \sup _{|\alpha| \leq N}\left|\partial^{\alpha} R_{y}(z)\right| \leq C \sup _{z \in y+Q} \sup _{s+1 \leq|\alpha| \leq N}\left|\partial^{\alpha} \phi(z)\right|
$$

where $R_{y}$ is the remainder of the Taylor expansion of $\phi$ of order $s$ at the point $y \in \mathbb{R}^{n}$.

Lemma 4.4 is Lemma 5.5 in [1].
Lemma 4.5. Suppose $0 \leq s<N$. Then there exist positive constants $C_{4}, C_{5}$ so that for $i \in \mathbb{N}$,

$$
\begin{equation*}
\mathcal{M}_{N}^{0}\left(b_{i}\right)(x) \leq C \frac{\lambda l_{i}^{n+s+1}}{\left(l_{i}+\left|x-x_{i}\right|\right)^{n+s+1}} \chi_{\left\{\left|x-x_{i}\right|<C_{4} \rho(x)\right\}}(x) \quad \text { if } x \notin Q_{i}^{*} \tag{4.6}
\end{equation*}
$$

Moreover,

$$
\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)=0, \text { if } x \notin Q_{i}^{*} \text { and } l_{i} \geq C_{5} \rho(x)
$$

Proof. Take $\varphi \in \mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right)$. Recall that $\eta_{i}$ is supported in the cube $\bar{Q}_{i}$, and we have taken $\bar{Q}_{i}$ to be strictly contained in $Q_{i}^{*}$. Thus if $x \notin Q_{i}^{*}$ and $\eta_{i}(y) \neq 0$, then there exists a positive constant $C_{4}$ such that $|x-y| \leq\left|x-x_{i}\right| \leq C_{4}|x-y|$. On the other hand, the support property of $\varphi$ requires that $\rho(x)>2^{-l} \geq|x-y| \geq 2^{-11-n} l_{i}$. Hence, $\left|x-x_{i}\right| \leq C_{4} 2^{-l}$, $l_{i}<2^{11+n} \rho(x) \equiv C_{5} \rho(x)$ and $l_{i}<C_{5} 2^{-l}$. Pick some $w \in\left(2^{8+n} n Q_{i}\right) \cap \Omega^{\complement}$, and we discuss the following two cases.

Case I. If $L_{3} \rho\left(x_{i}\right) \leq l_{i}<C_{5} 2^{-l}<C_{5} \rho(x)$, then according to the Lemma 2.1 we have $l_{i}<$ $C_{5} C_{0}\left(1+C_{4}\right)^{k_{0}} \rho\left(x_{i}\right)$ and

$$
\rho(\omega) \geq C_{0}^{-1}\left(1+\frac{\left|\omega-x_{i}\right|}{\rho\left(x_{i}\right)}\right)^{-k_{0}} \rho\left(x_{i}\right) \geq C_{0}^{-1}\left(1+2^{8+n} n \sqrt{n} L_{2}\right)^{-k_{0}} \rho\left(x_{i}\right)
$$

therefore, $l_{i}<a_{1} \rho(w)$, where $a_{1}>1$ is a constant.
Now we define $\bar{l}_{i}=l_{i} / a_{1}<\rho(w)$ and take $k_{i} \in \mathbb{Z}$ such that $2^{-k_{i}} \leq \bar{l}_{i}<2^{-k_{i}+1}$, then for $\varphi \in \mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right), \phi(z) \equiv \varphi\left(2^{-k_{i}} z / 2^{-l}\right)$ and $2^{-l}<\rho(x)$ we have

$$
\begin{aligned}
\left(b_{i} * \varphi_{l}\right)(x) & =2^{l n} \int b_{i}(z) \varphi\left(2^{l}(x-z)\right) d z \\
& =2^{l n} \int b_{i}(z) \phi\left(2^{k_{i}}(x-z)\right) d z \\
& =2^{l n} \int b_{i}(z) \phi_{2^{k_{i}}(x-w)}\left(2^{k_{i}}(w-z)\right) d z \\
& =\frac{2^{l n}}{2^{k_{i} n}}\left(f * \Phi_{k_{i}}\right)(w)
\end{aligned}
$$

where

$$
\Phi(z) \equiv \phi_{2^{k_{i}}(x-w)}(z) \eta_{i}\left(w-2^{-k_{i}} z\right), \phi_{2^{k_{i}}(x-w)}(z) \equiv \phi\left(z+2^{k_{i}}(x-w)\right)
$$

Obviously, $\operatorname{supp} \Phi \subset B\left(0, R_{2}\right)$, where $R_{2} \equiv 2^{9+n} n^{2} a_{1}$. Notice that $l_{i}<C_{5} 2^{-l}$ and $\left|x-x_{i}\right| \leq$ $C_{4} 2^{-l}$, we obtain

$$
\begin{equation*}
\left|\left(b_{i} * \varphi_{l}\right)(x)\right| \leq C \frac{2^{l n}}{2^{k_{i} n}} \mathcal{M}_{N} f(w) \leq C \lambda \frac{2^{l n}}{2^{k_{i} n}} \leq C \lambda \frac{l_{i}^{n+s+1}}{\left(l_{i}+\left|x-x_{i}\right|\right)^{n+s+1}} \tag{4.7}
\end{equation*}
$$

Case II. If $l_{i}<L_{3} \rho\left(x_{i}\right)$ and $\varphi \in \mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right)$, taking $j_{i} \in \mathbb{Z}$ such that $2^{-j_{i}} \leq l_{i}<2^{-j_{i}+1}$, then we define $\phi(z)=\varphi\left(2^{-j_{i}} z / 2^{-l}\right)$ and consider the Taylor expansion of $\phi$ of order $s$ at the point $y=2^{j_{i}}(x-w)$ :

$$
\phi(y+z)=\sum_{|\alpha| \leq s} \frac{\partial^{\alpha} \phi(y)}{\alpha!} z^{\alpha}+R_{y}(z)
$$

where $R_{y}$ denotes the remainder. Thus we get

$$
\begin{align*}
\left(b_{i} * \varphi_{l}\right)(x) & =2^{l n} \int b_{i}(z) \varphi\left(2^{l n}(x-z)\right) d z \\
& =2^{l n} \int b_{i}(z) \phi\left(2^{j_{i} n}(x-z)\right) d z \\
& =2^{l n} \int b_{i}(z) R_{2^{j_{i}}(x-w)}\left(2^{j_{i}}(w-z)\right) d z \\
& =\frac{2^{l n}}{2^{j_{i} n}}\left(f * \Phi_{j_{i}}\right)(w)-2^{l n} \int P_{i}(z) \eta_{i}(z) R_{2^{j_{i}}(x-w)}\left(2^{j_{i}}(w-z)\right) d z \tag{4.8}
\end{align*}
$$

where

$$
\Phi(z) \equiv R_{2^{j_{i}}(x-w)}(z) \eta_{i}\left(\omega-2^{-j_{i}} z\right)
$$

Obviously, $\operatorname{supp} \Phi \subset B_{n} \equiv B\left(0, R_{2}\right)$. Applying Lemma 4.4 to $\phi(z)=\varphi\left(2^{-j_{i}} z / 2^{-l}\right), y=$ $2^{j_{i}}(x-w)$ and $B_{n}$, we have

$$
\begin{aligned}
\sup _{z \in B_{n}} \sup _{|\alpha| \leq N}\left|\partial^{\alpha} R_{y}(z)\right| & \leq C \sup _{z \in y+B_{n}} \sup _{s+1 \leq|\alpha| \leq N}\left|\partial^{\alpha} \phi(z)\right| \\
& \leq C \sup _{z \in y+B_{n}}\left(\frac{2^{-j_{i}}}{2^{-l}}\right)^{s+1} \sup _{s+1 \leq|\alpha| \leq N}\left|\partial^{\alpha} \varphi\left(2^{-j_{i}} z / 2^{-l}\right)\right| \\
& \leq C\left(\frac{2^{-j_{i}}}{2^{-l}}\right)^{s+1}
\end{aligned}
$$

Notice that $l_{i}<C_{5} 2^{-l}$ and $\left|x-x_{i}\right| \leq C_{4} 2^{-l}$, therefore by (4.8), we obtain

$$
\begin{align*}
\left(b_{i} * \varphi_{l}\right)(x) & \leq \frac{2^{l n}}{2^{j_{i} n}}\left|\left(f * \Phi_{j_{i}}\right)(w)\right|+2^{l n} \int\left|P_{i}(z) \eta_{i}(z) R_{2^{j_{i}}(x-w)}\left(2^{j_{i}}(w-z)\right)\right| d z \\
& \leq C \frac{2^{l n}}{2^{j_{i} n}}\left(\mathcal{M}_{N} f(w)\|\Phi\|_{\mathcal{D}_{N}}+\lambda \sup _{z \in B_{n}} \sup _{\operatorname{lup}^{\prime} \mid \leq N}\left|\partial^{\alpha} R_{y}(z)\right|\right) \\
& \leq C \lambda \frac{l_{i}^{n+s+1}}{\left(l_{i}+\left|x-x_{i}\right|\right)^{n+s+1}} . \tag{4.9}
\end{align*}
$$

By combining both cases, we obtain (4.6).

Lemma 4.6. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ and $q_{\omega}$ be as in (2.4). If $p \in(0,1], s \geq\left[n\left(q_{\omega} / p-1\right)\right], N>s$ and $N \geq N_{p, \omega}$, where $N_{p, \omega}$ is as in (3.29), then there exists a positive constant $C_{6}$ such that for all $f \in h_{\rho, N}^{p}(\omega), \lambda>\inf _{x \in \mathbb{R}^{n}} \mathcal{M}_{N} f(x)$ and $i \in \mathbb{N}$,

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}\left(\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)\right)^{p} \omega(x) d x \leq C_{6} \int_{Q_{i}^{*}}\left(\mathcal{M}_{N}(f)(x)\right)^{p} \omega(x) d x \tag{4.10}
\end{equation*}
$$

Moreover the series $\sum_{i} b_{i}$ converges in $h_{\rho, N}^{p}(\omega)$ and

$$
\begin{equation*}
\int_{\mathbb{R}^{n}}\left(\mathcal{M}_{N}^{0}\left(\sum_{i} b_{i}\right)(x)\right)^{p} \omega(x) d x \leq C_{6} \int_{\Omega}\left(\mathcal{M}_{N}(f)(x)\right)^{p} \omega(x) d x \tag{4.11}
\end{equation*}
$$

Proof. By the proof of Lemma4.5, we know $\left|x-x_{i}\right|<C_{4} \rho(x), l_{i}<C_{5} \rho(x)$ and $\rho(x) \leq C_{0}(1+$ $\left.C_{4}\right)^{k_{0}} \rho\left(x_{i}\right)$, thus $Q_{i}^{*} \subset a_{2} \rho\left(x_{i}\right) Q_{i}^{0}$, where $a_{2} \equiv 2 C_{0}\left(1+C_{4}\right)^{k_{0}} \max \left\{C_{4}, C_{5}\right\}$ and $Q_{i}^{0} \equiv Q\left(x_{i}, 1\right)$. Furthermore, we have

$$
\begin{align*}
\int_{\mathbb{R}^{n}}\left(\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)\right)^{p} \omega(x) d x \leq & \int_{Q_{i}^{*}}\left(\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)\right)^{p} \omega(x) d x \\
& +\int_{a_{2} \rho\left(x_{i}\right) Q_{i}^{0} \backslash Q_{i}^{*}}\left(\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)\right)^{p} \omega(x) d x \tag{4.12}
\end{align*}
$$

Notice that $s \geq\left[n\left(q_{\omega} / p-1\right)\right]$ implies $2^{-n\left(q_{\omega}+\eta\right)} 2^{(s+n+1) p}>1$ for sufficient small $\eta>0$. By using Lemma 2.1 (iii) with $\omega \in A_{q_{\omega}+\eta}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, Lemma 4.5 and the fact that $\mathcal{M}_{N}(f)(x)>\lambda$ for all $x \in Q_{i}^{*}$, we have

$$
\begin{align*}
\int_{a_{2} \rho\left(x_{i}\right) Q_{i}^{0} \backslash Q_{i}^{*}}\left(\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)\right)^{p} \omega(x) d x & \leq \sum_{k=0}^{k_{0}} \int_{2^{k} Q_{i}^{*} \backslash 2^{k-1} Q_{i}^{*}}\left(\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)\right)^{p} \omega(x) d x \\
& \leq \lambda^{p} \omega\left(Q_{i}^{*}\right) \sum_{k=0}^{k_{0}}\left[2^{-n\left(q_{\omega}+\eta\right)+(s+n+1) p}\right]^{-k}  \tag{4.13}\\
& \leq C \int_{Q_{i}^{*}}\left(\mathcal{M}_{N} f(x)\right)^{p} \omega(x) d x
\end{align*}
$$

where $b=1+2^{-(10+n)}, k_{0} \in \mathbb{Z}$ such that $2^{k_{0}-1} b l_{i} \leq a_{2} \rho\left(x_{i}\right)<2^{k_{0}} b l_{i}$.
Combining the last two estimates we obtain (4.10), furthermore, we have

$$
\sum_{i} \int_{\mathbb{R}^{n}}\left(\mathcal{M}_{N}^{0}\left(b_{i}\right)(x)\right)^{p} \omega(x) d x \leq C \sum_{i} \int_{Q_{i}^{*}}\left(\mathcal{M}_{N} f(x)\right)^{p} \omega(x) d x \leq C \int_{\Omega}\left(\mathcal{M}_{N}(f)(x)\right)^{p} \omega(x) d x
$$

which together with complete of $h_{\rho, N}^{p}(\omega)$ (see Proposition 3.3)implies that $\sum_{i} b_{i}$ converges in $h_{\rho, N}^{p}(\omega)$. Therefore, the series $\sum_{i} b_{i}$ converges in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $\mathcal{M}_{N}^{0}\left(\sum_{i} b_{i}\right)(x) \leq \sum_{i} \mathcal{M}_{N}^{0}\left(b_{i}\right)(x)$ by Proposition 3.2, which gives (4.11). This finishes the proof of Lemma 4.6.

Lemma 4.7. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ and $q_{\omega}$ be as in (2.4), $s \in \mathbb{Z}_{+}$, and integer $N \geq 2$. If $q \in\left(q_{\omega}, \infty\right]$ and $f \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$, then the series $\sum_{i} b_{i}$ converges in $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$ and there exists $a$ positive constant $C_{7}$, independent of $f$ and $\lambda$, such that

$$
\left\|\sum_{i} \mid b_{i}\right\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \leq C_{7}\|f\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} .
$$

Proof. The proof for $q=\infty$ is similar to that for $q \in\left(q_{\omega}, \infty\right)$. So we only give the proof for $q \in\left(q_{\omega}, \infty\right)$. Set $F_{1}=\left\{i \in \mathbb{N}: l_{i} \geq L_{3} \rho\left(x_{i}\right)\right\}$ and $F_{2}=\left\{i \in \mathbb{N}: l_{i}<L_{3} \rho\left(x_{i}\right)\right\}$. By Lemma 4.2, for $i \in F_{2}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}\left|b_{i}(x)\right|^{q} \omega(x) d x & \leq \int_{Q_{i}^{*}}|f(x)|^{q} \omega(x) d x+\int_{Q_{i}^{*}}\left|P_{i}(x) \eta_{i}(x)\right|^{q} \omega(x) d x \\
& \leq \int_{Q_{i}^{*}}|f(x)|^{q} \omega(x) d x+\lambda^{q} \omega\left(Q_{i}^{*}\right)
\end{aligned}
$$

For $i \in F_{1}$, we have

$$
\int_{\mathbb{R}^{n}}\left|b_{i}(x)\right|^{q} \omega(x) d x \leq \int_{Q_{i}^{*}}|f(x)|^{q} \omega(x) d x
$$

From these, we obtain

$$
\begin{aligned}
\sum_{i} \int_{\mathbb{R}^{n}}\left|b_{i}(x)\right|^{q} \omega(x) d x & =\sum_{i \in F_{1}} \int_{\mathbb{R}^{n}}\left|b_{i}(x)\right|^{q} \omega(x) d x+\sum_{i \in F_{2}} \int_{\mathbb{R}^{n}}\left|b_{i}(x)\right|^{q} \omega(x) d x \\
& \leq \sum_{i} \int_{Q_{i}^{*}}|f(x)|^{q} \omega(x) d x+C \sum_{i \in F_{2}} \lambda^{q} \omega\left(Q_{i}^{*}\right) \\
& \leq \sum_{i} \int_{Q_{i}^{*}}|f(x)|^{q} \omega(x) d x+C \lambda^{q} \omega(\Omega) \\
& \leq C \int_{\mathbb{R}^{n}}|f(x)|^{q} \omega(x) d x
\end{aligned}
$$

Combining above estimates with the fact that $\left\{b_{i}\right\}_{i}$ have finite covering, we obtain

$$
\left\|\sum_{i}\left|b_{i}\right|\right\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \leq C_{7}\|f\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)}
$$

This finishes the proof.
Lemma 4.8. If $N>s \geq 0$ and $\sum_{i} b_{i}$ converges in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$, then there exists a positive constant $C_{8}$, independent of $f$ and $\lambda$, such that for all $x \in \mathbb{R}^{n}$,
$\mathcal{M}_{N}^{0}(g)(x) \leq \mathcal{M}_{N}^{0}(f)(x) \chi_{\Omega^{\mathrm{c}}}(x)+C_{8} \lambda \sum_{i} \frac{l_{i}^{n+s+1}}{\left(l_{i}+\left|x-x_{i}\right|\right)^{n+s+1}} \chi_{\left\{\left|x-x_{i}\right|<C_{4} \rho(x)\right\}}(x)+C_{8} \lambda \chi_{\Omega}(x)$,
where $x_{i}$ is the center of $Q_{i}$ and $C_{4}$ is as in Lemma 4.5.

Proof. If $x \notin \Omega$, since

$$
\mathcal{M}_{N}^{0}(g)(x) \leq \mathcal{M}_{N}^{0}(f)(x)+\sum_{i} \mathcal{M}_{N}^{0}\left(b_{i}\right)(x)
$$

by Lemma 4.5, we obtain

$$
\mathcal{M}_{N}^{0}(g)(x) \leq \mathcal{M}_{N}^{0}(f)(x) \chi_{\Omega^{0}}(x)+C \lambda \sum_{i} \frac{l_{i}^{n+s+1}}{\left(l_{i}+\left|x-x_{i}\right|\right)^{n+s+1}} \chi_{\left\{\left|x-x_{i}\right|<C_{4} \rho(x)\right\}}(x) .
$$

If $x \in \Omega$, take $k \in \mathbb{N}$ such that $x \in Q_{k}^{*}$. Let $J \equiv\left\{i \in \mathbb{N}: Q_{i}^{*} \cap Q_{k}^{*} \neq \emptyset\right\}$. Then the cardinality of $J$ is bounded by $L$. By Lemma 4.5, we have

$$
\sum_{i \notin J} \mathcal{M}_{N}^{0}\left(b_{i}\right)(x) \leq C \lambda \sum_{i \notin J} \frac{l_{i}^{n+s+1}}{\left(l_{i}+\left|x-x_{i}\right|\right)^{n+s+1}} \chi_{\left\{\left|x-x_{i}\right|<C_{4} \rho(x)\right\}}(x) .
$$

It suffices to estimate the grand maximal function of $g+\sum_{i \notin J} b_{i}=f-\sum_{i \in J} b_{i}$. Take $\varphi \in \mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right)$ and $l \in \mathbb{Z}$ such that $0<2^{-l}<\rho(x)$, then we write

$$
\begin{align*}
\left(f-\sum_{i \in J} b_{i}\right) * \varphi_{i}(x) & =(f \xi) * \varphi_{l}(x)+\left(\sum_{i \in J} P_{i} \eta_{i}\right) * \varphi_{l}(x) \\
& =f * \widetilde{\Phi}_{l}(w)+\left(\sum_{i \in J} P_{i} \eta_{i}\right) * \varphi_{l}(x) \tag{4.14}
\end{align*}
$$

where $w \in\left(2^{8+n} n Q_{k}\right) \bigcap \Omega^{\complement}, \xi=1-\sum_{i \in J} \eta_{i}$ and

$$
\widetilde{\Phi}(z) \equiv \varphi\left(z+2^{l}(x-w)\right) \xi\left(w-2^{-l} z\right)
$$

Since for $N \geq 2$ there is a constant $C>0$ such that $\|\varphi\|_{L^{1}\left(\mathbb{R}^{n}\right)} \leq C$ for all $\varphi \in \mathcal{D}_{N}^{0}\left(\mathbb{R}^{n}\right)$ and Lemma 4.1, we have

$$
\left|\left(\sum_{i \in J} P_{i} \eta_{i}\right) * \varphi_{l}(x)\right| \leq C \lambda
$$

Finally, we estimate $f * \Phi_{l}(w)$. There are two cases: If $2^{-l} \leq 2^{-(11+n)} l_{k}$, then $f * \Phi_{l}(w)=0$, because $\xi$ vanishes in $Q_{k}^{*}$ and $\varphi_{l}$ is supported in $B\left(0,2^{-l}\right)$. On the other hand, if $2^{-l} \geq$ $2^{-(11+n)} l_{k}$, then there exists a positive constant $a_{3}>1$ such that $2^{-l}<\rho(x)<a_{3} \rho(w)$. Take $\Phi(x) \equiv \widetilde{\Phi}\left(x / 2^{m_{1}}\right)$ and $m_{1} \in \mathbb{N}$ satisfying $2^{m_{1}-1} \leq a_{3}<2^{m_{1}}$, then $\operatorname{supp} \Phi \subset B\left(0, R_{3}\right)$ where $R_{3} \equiv 2^{3(11+n)} a_{3}$, and $\|\Phi\|_{\mathcal{D}_{N}} \leq C$. Therefore, $2^{-l-m_{1}}<\rho(x) / a_{3}<\rho(w)$ and

$$
\left|\left(f * \widetilde{\Phi}_{l}\right)(w)\right|=2^{-m_{1} n}\left|\left(f * \Phi_{l+m_{1}}\right)(w)\right| \leq C \mathcal{M}_{N} f(w)\|\Phi\|_{\mathcal{D}_{N}} \leq C \lambda .
$$

According to above estimates, we have

$$
\left|\left(f-\sum_{i \in J} b_{i}\right) * \varphi_{l}\right| \leq C \lambda,
$$

then we can get

$$
\mathcal{M}_{N}^{0}\left(\left(f-\sum_{i \in J} b_{i}\right)\right)(x) \leq C \lambda .
$$

This finishes the proof of the lemma.

Lemma 4.9. Let $\omega \in A_{\infty}^{\rho \infty}\left(\mathbb{R}^{n}\right)$, $q_{\omega}$ be as in (2.4), $q \in\left(q_{\omega}, \infty\right), p \in(0,1]$ and $N \geq N_{p, \omega}$, where $N_{p, \omega}$ is as in (3.29).
(i) If $N>s \geq\left[n\left(q_{\omega} / p-1\right)\right]$ and $f \in h_{\rho, N}^{p}(\omega)$, then $\mathcal{M}_{N}^{0}(g) \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$ and there exists a positive constant $C_{9}$, independent of $f$ and $\lambda$, such that

$$
\int_{\mathbb{R}^{n}}\left[\mathcal{M}_{N}^{0}(g)(x)\right]^{q} \omega(x) d x \leq C_{9} \lambda^{q-p} \int_{\mathbb{R}^{n}}\left[\mathcal{M}_{N}(f)(x)\right]^{p} \omega(x) d x
$$

(ii) If $N \geq 2$ and $f \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$, then $g \in L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)$ and there exists a positive constant $C_{10}$, independent of $f$ and $\lambda$, such that $\|g\|_{L_{\omega}^{\infty}} \leq C_{10} \lambda$.

Proof. We first prove (i). Since $f \in h_{\rho, N}^{p}(\omega)$, by Lemma 4.6 and Proposition 3.2, $\sum_{i} b_{i}$ converges in both $h_{\rho, N}^{p}(\omega)$ and $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$. Notice that $s \geq\left[n\left(q_{\omega} / p-1\right)\right]$, by Lemma 4.8 and the proof of Lemma 4.6, we get

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}\left(\mathcal{M}_{N}^{0}(g)(x)\right)^{q} \omega(x) d x \leq & C \lambda^{q} \sum_{i} \int_{\mathbb{R}^{n}}\left[\frac{l_{i}^{(n+s+1)}}{\left(l_{i}+\left|x-x_{i}\right|\right)^{(n+s+1)}} \chi_{B\left(x_{i}, a_{2} \rho\left(x_{i}\right)\right)}(x)\right]^{q} \omega(x) d x \\
& +C \lambda^{q} \int_{\mathbb{R}^{n}} \chi_{\Omega}(x) \omega(x) d x+\int_{\Omega^{\mathrm{C}}}\left(\mathcal{M}_{N}(f)(x)\right)^{q} \omega(x) d x \\
\leq & C \lambda^{q} \sum_{i} \omega\left(Q_{i}^{*}\right)+C \lambda^{q} \omega(\Omega)+\int_{\Omega^{\mathrm{C}}}\left(\mathcal{M}_{N}(f)(x)\right)^{q} \omega(x) d x \\
\leq & C \lambda^{q} \omega(\Omega)+C \lambda^{q-p} \int_{\Omega^{\mathrm{C}}}\left(\mathcal{M}_{N}(f)(x)\right)^{p} \omega(x) d x \\
\leq & C_{9} \lambda^{q-p} \int_{\mathbb{R}^{n}}\left(\mathcal{M}_{N}(f)(x)\right)^{p} \omega(x) d x
\end{aligned}
$$

Thus, (i) holds.
Next, we prove (ii). If $f \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$, then $g$ and $\left\{b_{i}\right\}_{i}$ are functions. By Lemma 4.7, we know that $\sum_{i} b_{i}$ converges in $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$ and hence in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ by Lemma 2.5(ii). Write

$$
g=f-\sum_{i} b_{i}=f\left(1-\sum_{i} \eta_{i}\right)+\sum_{i \in F_{2}} P_{i} \eta_{i}=f \chi_{\Omega^{\complement}}+\sum_{i \in F_{2}} P_{i} \eta_{i}
$$

By Lemma 4.3, we have $|g(x)| \leq C \lambda$ for all $x \in \Omega$, and by Proposition 3.1(i), we also have $|g(x)|=|f(x)| \leq \mathcal{M}_{N} f(x) \leq \lambda$ for almost everywhere $x \in \Omega^{\complement}$. Therefore, $\|g\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{10} \lambda$ which yields (ii).

Corollary 4.1. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ and $q_{\omega}$ be as in (2.4). If $q \in\left(q_{\omega}, \infty\right), p \in(0,1]$ and $N \geq N_{p, \omega}$, where $N_{p, \omega}$ is as in (3.29), then $h_{\rho, N}^{p}(\omega) \cap L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$ is dense in $h_{\rho, N}^{p}(\omega)$.

Proof. Let $f \in h_{\rho, N}^{p}(\omega)$. For any $\lambda>\inf _{x \in \mathbb{R}^{n}} \mathcal{M}_{N} f(x)$, let $f=g^{\lambda}+\sum_{i} b_{i}^{\lambda}$ be the CalderónZygmund decomposition of $f$ of degree $s$ with $\left[n\left(q_{\omega} / p-1\right)\right] \leq s<N$ and height $\lambda$ associated
to $\mathcal{M}_{N} f$. By Lemma 4.6, we have

$$
\left\|\sum_{i} b_{i}^{\lambda}\right\|_{h_{\rho, N}^{p}(\omega)} \leq C \int_{\left\{x \in \mathbb{R}^{n}: \mathcal{M}_{N} f(x)>\lambda\right\}}\left(\mathcal{M}_{N} f(x)\right)^{p} \omega(x) d x .
$$

Therefore, $g^{\lambda} \rightarrow f$ in $h_{\rho, N}^{p}(\omega)$ as $\lambda \rightarrow \infty$. Moreover, by Lemma 4.9, we have $\mathcal{M}_{N}^{0}\left(g^{\lambda}\right) \in$ $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$, which combined with Proposition 3.1(ii) implies $g^{\lambda} \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$. Thus, Corollary 4.1 is proved.

## 5 Weighted atomic decompositions of $h_{\rho, N}^{p}(\omega)$

In this section, we establish the equivalence between $h_{\rho, N}^{p}(\omega)$ and $h_{\rho}^{p, q, s}(\omega)$ by using the Calderón-Zygmund decomposition associated to the local grand maximal function stated in Section 4.

Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), q_{\omega}$ be as in (2.4), $p \in(0,1], N \geq N_{p, \omega}, s \equiv\left[n\left(q_{\omega} / p-1\right)\right]$ and $f \in$ $h_{\rho, N}^{p}(\omega)$. Take $m_{0} \in \mathbb{Z}$ such that $2^{m_{0}-1} \leq \inf _{x \in \mathbb{R}^{n}} \mathcal{M}_{N} f(x)<2^{m_{0}}, \operatorname{if~}_{\inf }^{x \in \mathbb{R}^{n}} \mathcal{M}_{N} f(x)=0$, write $m_{0}=-\infty$. For each integer $m \geq m_{0}$ consider the Calderón-Zygmund decomposition of $f$ of degree $s$ and height $\lambda=2^{m}$ associated to $\mathcal{M}_{N} f$, namely

$$
\begin{equation*}
f=g^{m}+\sum_{i \in \mathbb{N}} b_{i}^{m}, \tag{5.1}
\end{equation*}
$$

and

$$
\Omega_{m} \equiv\left\{x \in \mathbb{R}^{n}: \mathcal{M}_{N} f(x)>2^{m}\right\}, Q_{i}^{m} \equiv Q_{l_{i}^{m}} .
$$

In this section, we write $\left\{Q_{i}\right\}_{i},\left\{\eta_{i}\right\}_{i},\left\{P_{i}\right\}_{i}$ and $\left\{b_{i}\right\}_{i}$, respectively, as $\left\{Q_{i}^{m}\right\}_{i},\left\{\eta_{i}^{m}\right\}_{i},\left\{P_{i}^{m}\right\}_{i}$ and $\left\{b_{i}^{m}\right\}_{i}$. The center and the sidelength of $Q_{i}^{m}$ are respectively denoted by $x_{i}^{m}$ and $l_{i}^{m}$. Recall that for all $i$ and $m$,

$$
\begin{equation*}
\sum_{i} \eta_{i}^{m}=\chi_{\Omega_{m}}, \quad \operatorname{supp}\left(b_{i}^{m}\right) \subset \operatorname{supp}\left(\eta_{i}^{m}\right) \subset Q_{i}^{m *} \tag{5.2}
\end{equation*}
$$

$\left\{Q_{i}^{m *}\right\}_{i}$ has the bounded interior property, and for all $P \in \mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\left\langle f, P \eta_{i}^{m}\right\rangle=\left\langle P_{i}^{m}, P \eta_{i}^{m}\right\rangle . \tag{5.3}
\end{equation*}
$$

For each integer $m \geq m_{0}$ and $i, j \in \mathbb{N}$, let $P_{i, j}^{m+1}$ be the orthogonal projection of $\left(f-P_{j}^{m+1}\right) \eta_{i}^{m}$ on $\mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$ with respect to the norm

$$
\|P\|_{j}^{2} \equiv \frac{1}{\int_{\mathbb{R}^{n}} \eta_{j}^{m+1}(y) d y} \int_{\mathbb{R}^{n}}|P(x)|^{2} \eta_{j}^{m+1}(x) d x
$$

namely, $P_{i, j}^{m+1}$ is the unique polynomial of $\mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$ such that for any $P \in \mathcal{P}_{s}\left(\mathbb{R}^{n}\right)$,

$$
\begin{equation*}
\left\langle\left(f-P_{j}^{m+1}\right) \eta_{i}^{k}, P \eta_{j}^{m+1}\right\rangle=\int_{\mathbb{R}^{n}} P_{i, j}^{m+1}(x) P(x) \eta_{j}^{m+1}(x) d x \tag{5.4}
\end{equation*}
$$

In what follows, we denote $Q_{i}^{m *}=\left(1+2^{-(10+n)}\right) Q_{i}^{m}$,

$$
\begin{array}{rll}
E_{1}^{m} \equiv\left\{i \in \mathbb{N}: l_{i}^{m} \geq \rho\left(x_{i}^{m}\right) /\left(2^{5} n\right)\right\}, & & E_{2}^{k} \equiv\left\{i \in \mathbb{N}: l_{i}^{m}<\rho\left(x_{i}^{m}\right) /\left(2^{5} n\right)\right\}, \\
F_{1}^{k} \equiv\left\{i \in \mathbb{N}: l_{i}^{m} \geq L_{3} \rho\left(x_{i}^{m}\right)\right\}, & & F_{2}^{k} \equiv\left\{i \in \mathbb{N}: l_{i}^{m}<L_{3} \rho\left(x_{i}^{m}\right)\right\},
\end{array}
$$

where $L_{3}=2^{k_{0}} C_{0}$ is as in Section 4.
Observe that

$$
\begin{equation*}
P_{i, j}^{m+1} \neq 0 \quad \text { if and only if } \quad Q_{i}^{m *} \cap Q_{j}^{(m+1) *} \neq \emptyset \tag{5.5}
\end{equation*}
$$

Indeed, this follows directly from the definition of $P_{i, j}^{m+1}$. The following Lemmas 5.155.3 can be proved by similar methods of Lemmas 5.1-5.3 in [28].

Lemma 5.1. Notice that $\Omega_{m+1} \subset \Omega_{m}$, then
(i) If $Q_{i}^{m *} \cap Q_{j}^{(m+1) *} \neq \emptyset$, then $l_{j}^{m+1} \leq 2^{4} \sqrt{n} l_{i}^{m}$ and $Q_{j}^{(m+1) *} \subset 2^{6} n Q_{i}^{k *} \subset \Omega_{m}$.
(ii) There exists a positive integer $L$ such that for each $i \in \mathbb{N}$, the cardinality of $\{j \in \mathbb{N}$ : $\left.Q_{i}^{m *} \cap Q_{j}^{(m+1) *} \neq \emptyset\right\}$ is bounded by $L$.
Lemma 5.2. There exists a positive constant $C$ such that for all $i, j \in \mathbb{N}$ and integer $m \geq m_{0}$ with $l_{j}^{m+1}<L_{3} \rho\left(x_{j}^{m+1}\right)$,

$$
\begin{equation*}
\sup _{y \in \mathbb{R}^{n}}\left|P_{i, j}^{m+1}(y) \eta_{j}^{m+1}(y)\right| \leq C 2^{m+1} \tag{5.6}
\end{equation*}
$$

Lemma 5.3. For any $k \in \mathbb{Z}$ with $m \geq m_{0}$,

$$
\sum_{i \in \mathbb{N}}\left(\sum_{j \in F_{2}^{m+1}} P_{i, j}^{m+1} \eta_{j}^{m+1}\right)=0,
$$

where the series converges both in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and pointwise.
The following lemma gives the weighted atomic decomposition for a dense subspace of $h_{\rho, N}^{p}(\omega)$.

Lemma 5.4. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), q_{\omega}$ and $N_{p, \omega}$ be respectively as in (2.4) and (3.29). If $p \in(0,1], s \geq\left[n\left(q_{\omega} / p-1\right)\right], N>s$ and $N \geq N_{p, \omega}$, then for any $f \in\left(L_{\omega}^{q}\left(\mathbb{R}^{n}\right) \cap h_{\rho, N}^{p}(\omega)\right)$, there exists numbers $\lambda_{0} \in \mathbb{C}$ and $\left\{\lambda_{i}^{m}\right\}_{m \geq k_{0}, i} \subset \mathbb{C},(p, \infty, s)_{\omega^{-}}$-atoms $\left\{a_{i}^{m}\right\}_{m \geq m_{0}, i}$ and a single atom $a_{0}$ such that

$$
\begin{equation*}
f=\sum_{m \geq m_{0}} \sum_{i} \lambda_{i}^{m} a_{i}^{m}+\lambda_{0} a_{0}, \tag{5.7}
\end{equation*}
$$

where the series converges almost everywhere and in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$. Moreover, there exists a positive constant $C$, independent of $f$, such that

$$
\begin{equation*}
\sum_{m \geq m_{0}, i}\left|\lambda_{i}^{m}\right|^{p}+\left|\lambda_{0}\right|^{p} \leq C\|f\|_{h_{\rho, N}^{p}(\omega)} . \tag{5.8}
\end{equation*}
$$

Proof. Let $f \in\left(L_{\omega}^{q}\left(\mathbb{R}^{n}\right) \cap h_{\rho, N}^{p}(\omega)\right)$. We first consider the case $m_{0}=-\infty$. As above, for each $m \in \mathbb{Z}, f$ has a Calderón-Zygmund decomposition of degree $s$ and height $\lambda=2^{m}$ associated to $\mathcal{M}_{N}(f)$ as in (5.1), namely, $f=g^{m}+\sum_{i} b_{i}^{m}$. By Corollary 4.1 and Proposition 3.1, $g^{m} \rightarrow f$ in both $h_{\rho, N}^{p}(\omega)$ and $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ as $m \rightarrow \infty$. By Lemma 4.9 (i), $\left\|g^{m}\right\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \rightarrow 0$ as $m \rightarrow-\infty$, and moreover, by Lemma 2.5 (ii), $g^{m} \rightarrow 0$ in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ as $m \rightarrow-\infty$. Therefore,

$$
\begin{equation*}
f=\sum_{m=-\infty}^{\infty}\left(g^{m+1}-g^{m}\right) \tag{5.9}
\end{equation*}
$$

in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$. Moreover, since $\operatorname{supp}\left(\sum_{i} b_{i}^{m}\right) \subset \Omega_{m}$ and $\omega\left(\Omega_{m}\right) \rightarrow 0$ as $m \rightarrow \infty$, then $g^{m} \rightarrow f$ almost everywhere as $m \rightarrow \infty$. Thus, (5.9) also holds almost everywhere. By Lemma 5.3 and $\sum_{i} \eta_{i}^{m} b_{j}^{m+1}=\chi_{\Omega_{m}} b_{j}^{m+1}=b_{j}^{m+1}$ for all $j$, then we have

$$
\begin{align*}
g^{m+1}-g^{m} & =\left(f-\sum_{j} b_{j}^{m+1}\right)-\left(f-\sum_{i} b_{i}^{m}\right) \\
& =\sum_{i} b_{i}^{m}-\sum_{j} b_{j}^{m+1}+\sum_{i}\left(\sum_{j \in F_{2}^{m+1}} P_{i, j}^{m+1} \eta_{j}^{m+1}\right) \\
& =\sum_{i}\left[b_{i}^{m}-\sum_{j} b_{j}^{m+1} \eta_{i}^{m}+\sum_{j \in F_{2}^{m+1}} P_{i, j}^{m+1} \eta_{j}^{m+1}\right] \equiv \sum_{i} h_{i}^{m}, \tag{5.10}
\end{align*}
$$

where all the series converge in both $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and almost everywhere. Furthermore, from the definitions of $b_{j}^{m}$ and $b_{j}^{m+1}$, we infer that when $l_{i}^{m}<L_{3} \rho\left(x_{i}^{m}\right)$,

$$
\begin{equation*}
h_{i}^{m}=f \chi_{\Omega_{m+1}^{\complement}} \eta_{i}^{m}-P_{i}^{m} \eta_{i}^{m}+\sum_{j \in F_{2}^{m+1}} P_{j}^{m+1} \eta_{i}^{m} \eta_{j}^{m+1}+\sum_{j \in F_{2}^{m+1}} P_{i, j}^{m+1} \eta_{j}^{m+1}, \tag{5.11}
\end{equation*}
$$

and when $l_{i}^{m} \geq L_{3} \rho\left(x_{i}^{m}\right)$,

$$
\begin{equation*}
h_{i}^{m}=f \chi_{\Omega_{m+1}^{\complement}} \eta_{i}^{m}+\sum_{j \in F_{2}^{m+1}} P_{j}^{m+1} \eta_{i}^{m} \eta_{j}^{m+1}+\sum_{j \in F_{2}^{m+1}} P_{i, j}^{m+1} \eta_{j}^{m+1} \tag{5.12}
\end{equation*}
$$

By Proposition 3.1(i), we know that for almost every $x \in \Omega_{m+1}^{\complement}$,

$$
|f(x)| \leq \mathcal{M}_{N}(f)(x) \leq 2^{m+1}
$$

which combined with Lemma 4.2, Lemma 5.1(ii), Lemma 5.2, (5.11) and (5.12) implies that there exists a positive constant $C_{11}$ such that for all $i \in \mathbb{N}$,

$$
\begin{equation*}
\left\|h_{i}^{m}\right\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{11} 2^{m} . \tag{5.13}
\end{equation*}
$$

Next, we show that for each $i$ and $m, h_{i}^{m}$ is either a multiple of a $(p, \infty, s)_{\omega}$-atom or a finite linear combination of $(p, \infty, s)_{\omega}$-atom by considering the following two cases for $i$.
Case I. For $i \in E_{1}^{m}, l_{i}^{m} \geq \rho\left(x_{i}^{m}\right) / 2^{5} n$. Clearly, $h_{i}^{m}$ is supported in a cube $\widetilde{Q}_{i}^{m}$ that contains $Q_{i}^{m *}$ as well as all the $Q_{j}^{(m+1) *}$ that intersect $Q_{i}^{m *}$. In fact, observe that if $Q_{i}^{m *} \cap Q_{j}^{(m+1) *} \neq$
$\emptyset$, by Lemma 5.1, we have $Q_{j}^{(m+1) *} \subset 2^{6} n Q_{i}^{m *} \subset \Omega_{m}$, thus, we set $\widetilde{Q}_{i}^{m} \equiv 2^{6} n Q_{i}^{m *}$. Since $l\left(\widetilde{Q}_{i}^{m}\right) \geq 2 \rho\left(x_{i}^{m}\right)$, by the same method of Lemma 3.1 in 30], $\widetilde{Q}_{i}^{m}$ can be decomposed into finite disjoint cubes $\left\{Q_{i, k}^{m}\right\}_{k}$ such that $\widetilde{Q}_{i}^{m}=\bigcup_{k=1}^{n_{i}} Q_{i, k}^{m}$ and $l_{i, k}^{m} / 4<\rho(x) \leq C_{0}(3 \sqrt{n})^{k_{0}} l_{i, k}^{m}$ for some $x \in Q_{i, k}^{m}=Q\left(x_{i, k}^{m}, l_{i, k}^{m}\right)$, where $C_{0}, k_{0}$ are constants given in Lemma 2.1. Moreover, by Lemma 2.1, we also have $l_{i, k}^{m} \leq L_{1} \rho\left(x_{i, k}^{m}\right)$ and $l_{i, k}^{m}>L_{2} \rho\left(x_{i, k}^{m}\right)$. Therefore, let

$$
\lambda_{i, k}^{m} \equiv C_{11} 2^{m}\left[\omega\left(Q_{i, k}^{m}\right)\right]^{1 / p} \quad \text { and } \quad a_{i, k}^{m} \equiv\left(\lambda_{i, k}^{m}\right)^{-1} \frac{h_{i}^{m} \chi_{i, k}^{m}}{\sum_{k=1}^{n_{i}} \chi_{Q_{i, k}^{m}}^{m}},
$$

then supp $a_{i, k}^{m} \subset Q_{i, k}^{m}$ and $\left\|a_{i, k}^{m}\right\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \leq\left[\omega\left(Q_{i, k}^{m}\right)\right]^{-1 / p}$, hence each $a_{i, k}^{m}$ is a $(p, \infty, s)_{\omega \text {-atom }}$ and $h_{i}^{m}=\sum_{k=1}^{n_{i}} \lambda_{i, k}^{m} a_{i, k}^{m}$.
Case II. For $i \in E_{2}^{m}$, if $j \in F_{1}^{m+1}$, we claim that $Q_{i}^{m *} \cap Q_{j}^{(m+1) *}=\emptyset$. In fact, if $Q_{i}^{m *} \cap Q_{j}^{(m+1) *} \neq$ $\emptyset$, by Lemma 5.1 (i), we know $l_{j}^{m+1} \leq 2^{4} \sqrt{n} l_{i}^{m}$ then we can deduce that $l_{i}^{m}<l_{i}^{m} / 2 \sqrt{n}$ which is a contradiction, hence the claim is true. Thus, we have

$$
\begin{align*}
h_{i}^{m}= & \left(f-P_{i}^{m}\right) \eta_{i}^{m}-\sum_{j \in F_{1}^{m+1}} f \eta_{j}^{m+1} \eta_{i}^{m}-\sum_{j \in F_{2}^{m+1}}\left(f-P_{j}^{m+1}\right) \eta_{j}^{m+1} \eta_{i}^{m} \\
& +\sum_{j \in F_{2}^{m+1}} P_{i, j}^{m+1} \eta_{j}^{m+1} \\
= & \left(f-P_{i}^{m}\right) \eta_{i}^{m}-\sum_{j \in F_{2}^{m+1}}\left\{\left(f-P_{j}^{m+1}\right) \eta_{j}^{m+1} \eta_{i}^{m}-P_{i, j}^{m+1} \eta_{j}^{m+1}\right\} . \tag{5.14}
\end{align*}
$$

Let $\widetilde{Q}_{i}^{m} \equiv 2^{6} n Q_{i}^{m *}$, then $l\left(\widetilde{Q}_{i}^{m}\right)<L_{1} \rho\left(x_{i}^{m}\right)$ and $\operatorname{supp} h_{i}^{m} \subset \widetilde{Q}_{i}^{m}$. Moveover, $h_{i}^{m}$ satisfies the desired moment conditions, which are deduced from the moment conditions of $\left(f-P_{i}^{m}\right) \eta_{i}^{m}$ (see (5.3)) and $\left(f-P_{j}^{m+1}\right) \eta_{j}^{m+1} \eta_{i}^{m}-P_{i, j}^{m+1} \eta_{j}^{m+1}$ (see (5.4)). Let $\lambda_{i}^{m} \equiv C_{11} 2^{m}\left[\omega\left(\widetilde{Q}_{i}^{m}\right)\right]^{1 / p}$ and $a_{i}^{m} \equiv\left(\lambda_{i}^{m}\right)^{-1} h_{i}^{m}$, then $a_{i}^{m}$ is a $(p, \infty, s)_{\omega}$-atom.

Thus, from (5.9), (5.10), Case I and Case II, we infer that

$$
f=\sum_{m \in \mathbb{Z}}\left(\sum_{i \in E_{1}^{m}}\left(\sum_{k=1}^{n_{i}} \lambda_{i, k}^{m} a_{i, k}^{m}\right)+\sum_{i \in E_{2}^{m}} \lambda_{i}^{m} a_{i}^{m}\right)
$$

holds in both $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and almost everywhere. Moreover, by Lemma [2.4, we get

$$
\begin{aligned}
\sum_{k \in \mathbb{Z}}\left[\sum_{i \in E_{1}^{m}}\left[\sum_{k=1}^{n_{i}}\left|\lambda_{i, k}^{m}\right|^{p}\right]+\sum_{i \in E_{2}^{m}}\left|\lambda_{i}^{m}\right|^{p}\right] & \leq C \sum_{k \in \mathbb{Z}} 2^{m p}\left[\sum_{i \in E_{1}^{m}}\left[\sum_{k=1}^{n_{i}} \omega\left(Q_{i, k}^{m}\right)\right]+\sum_{i \in E_{2}^{m}} \omega\left(\widetilde{Q}_{i}^{m}\right)\right] \\
& \leq C \sum_{k \in \mathbb{Z}} 2^{m p}\left[\sum_{i \in E_{1}^{m}} \omega\left(\widetilde{Q}_{i}^{m}\right)+\sum_{i \in E_{2}^{m}} \omega\left(\widetilde{Q}_{i}^{m}\right)\right] \\
& \leq C \sum_{m \in \mathbb{Z}} \sum_{i \in \mathbb{N}} 2^{m p} \omega\left(\widetilde{Q}_{i}^{m}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \leq C \sum_{m \in \mathbb{Z}} \sum_{i \in \mathbb{N}} 2^{m p} \omega\left(Q_{i}^{m *}\right) \\
& \leq C \sum_{m \in \mathbb{Z}} 2^{m p} \omega\left(\Omega_{m}\right) \\
& \leq C\left\|\mathcal{M}_{N}(f)\right\|_{L_{\omega}^{p}\left(\mathbb{R}^{n}\right)}^{p}=C\|f\|_{h_{p, N}^{p}(\omega)}^{p},
\end{aligned}
$$

which implies (5.8) in the case that $m_{0}=-\infty$.
Finally, we consider the case that $k_{0}>-\infty$. In this case, by $f \in h_{\rho, N}^{p}(\omega)$, we see that $\omega\left(\mathbb{R}^{n}\right)<\infty$. Adapting the previous arguments, we conclude that

$$
\begin{equation*}
f=\sum_{m=m_{0}}^{\infty}\left(g^{m+1}-g^{m}\right)+g^{m_{0}} \equiv \tilde{f}+g^{m_{0}} . \tag{5.15}
\end{equation*}
$$

For the function $\widetilde{f}$, we have the same $(p, \infty, s)_{\omega}$ atomic decomposition as above

$$
\begin{equation*}
\tilde{f}=\sum_{m \geq m_{0}, i} \lambda_{i}^{m} a_{i}^{m}, \tag{5.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{m \geq m_{0}} \sum_{i \in \mathbb{N}}\left|\lambda_{i}^{m}\right|^{p} \leq C\|f\|_{h_{p, N}(\omega)}^{p} . \tag{5.17}
\end{equation*}
$$

For the function $g^{m_{0}}$, by Lemma 4.9 (ii), we have

$$
\begin{equation*}
\left\|g^{m_{0}}\right\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \leq C_{10} 2^{m_{0}} \leq 2 C_{10} \inf _{x \in \mathbb{R}^{n}} \mathcal{M}_{N} f(x), \tag{5.18}
\end{equation*}
$$

where $C_{10}$ is the same as in Lemma 4.9 (ii). Let $\lambda_{0} \equiv C_{10} 2^{m_{0}}\left[\omega\left(\mathbb{R}^{n}\right)\right]^{1 / p}$ and $a_{0} \equiv \lambda_{0}^{-1} g^{m_{0}}$, then

$$
\begin{equation*}
\left\|a_{0}\right\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \leq\left[\omega\left(\mathbb{R}^{n}\right)\right]^{-1 / p} \quad \text { and } \quad\left|\lambda_{0}\right|^{p} \leq\left(2 C_{10}\right)^{p}\|f\|_{h_{p, N}}^{p}(\omega) . \tag{5.19}
\end{equation*}
$$

Thus, $a_{0}$ is a $(p, \infty)_{\omega}$-single-atom and $g^{m_{0}}=\lambda_{0} a_{0}$, which together with (5.15) and (5.16) implies (5.7) in the case that $k_{0}>-\infty$. Furthermore, by combining (5.17) with (5.19), we obtain

$$
\sum_{m \geq m_{0}} \sum_{i \in \mathbb{N}}\left|\lambda_{i}^{m}\right|^{p}+\left|\lambda_{0}\right|^{p} \leq C\|f\|_{h_{\rho, N}^{p}(\omega)}^{p} .
$$

The proof of the lemma is complete.

Now we state the weighted atomic decompositions of $h_{\rho, N}^{p}(\omega)$.
Theorem 5.1. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), q_{\omega}$ and $N_{p, \omega}$ be respectively as in (2.4) and (3.29). If $q \in\left(q_{\omega}, \infty\right], p \in(0,1]$, and integers $s$ and $N$ satisfy $N \geq N_{p, \omega}$ and $N>s \geq\left[n\left(q_{\omega} / p-1\right)\right]$, then $h_{\rho}^{p, q, s}(\omega)=h_{\rho, N}^{p}(\omega)=h_{\rho, N_{p, \omega}}^{p}(\omega)$ with equivalent norms.

Proof. It is easy to see that

$$
h_{\rho}^{p, \infty, \bar{s}}(\omega) \subset h_{\rho}^{p, q, s}(\omega) \subset h_{\rho, N_{p, \omega}}^{p}(\omega) \subset h_{\rho, N}^{p}(\omega) \subset h_{\rho, \bar{N}}^{p}(\omega)
$$

where $\bar{s}$ is an integer no less than $s$ and $\bar{N}$ is an integer larger than $N$, and the inclusions are continuous. Thus, to prove Theorem 5.1, it suffices to prove that for any $N>s \geq\left[n\left(q_{\omega} / p-1\right)\right]$, $h_{\rho, N}^{p}(\omega) \subset h_{\rho}^{p, \infty, s}(\omega)$, and for all $f \in h_{\rho, N}^{p}(\omega),\|f\|_{h_{\rho}^{p, \infty, s}(\omega)} \leq C\|f\|_{h_{\rho, N}^{p}(\omega)}$.

Let $f \in h_{\rho, N}^{p}(\omega)$. By Corollary 4.1, there exists a sequence of functions $\left\{f_{m}\right\}_{m \in \mathbb{N}} \subset$ $\left(h_{\rho, N}^{p}(\omega) \cap L_{\omega}^{q}\left(\mathbb{R}^{n}\right)\right)$ such that for all $m \in \mathbb{N}$,

$$
\begin{equation*}
\left\|f_{m}\right\|_{h_{\rho, N}^{p}(\omega)} \leq 2^{-m}\|f\|_{h_{\rho, N}^{p}(\omega)} \tag{5.20}
\end{equation*}
$$

and $f=\sum_{m \in \mathbb{N}} f_{m}$ in $h_{\rho, N}^{p}(\omega)$. By Lemma 5.4, for each $m \in \mathbb{N}, f_{m}$ has an atomic decomposition

$$
f_{m}=\sum_{i \in \mathbb{Z}_{+}} \lambda_{i}^{m} a_{i}^{m}
$$

in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ with

$$
\sum_{i \in \mathbb{Z}_{+}}\left|\lambda_{i}^{m}\right|^{p} \leq C\left\|f_{m}\right\|_{h_{\rho, N}^{p}(\omega)}^{p}
$$

where $\left\{\lambda_{i}^{m}\right\}_{i \in \mathbb{Z}_{+}} \subset \mathbb{C},\left\{a_{i}^{m}\right\}_{i \in \mathbb{N}}$ are $(p, \infty, s)_{\omega}$-atoms and $a_{0}^{m}$ is a $(p, \infty)_{\omega \text {-single-atom. }}$.
Let

$$
\widetilde{\lambda}_{0} \equiv\left[\omega\left(\mathbb{R}^{n}\right)\right]^{1 / p} \sum_{m=1}^{\infty}\left|\lambda_{0}^{m}\right|\left\|a_{0}^{m}\right\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \quad \text { and } \quad \widetilde{a}_{0} \equiv\left(\widetilde{\lambda}_{0}\right)^{-1} \sum_{m=1}^{\infty} \lambda_{0}^{m} a_{0}^{m}
$$

Then

$$
\widetilde{\lambda}_{0} \widetilde{a}_{0}=\sum_{m=1}^{\infty} \lambda_{0}^{m} a_{0}^{m}
$$

It is easy to see that

$$
\left\|\widetilde{a}_{0}\right\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \leq\left[\omega\left(\mathbb{R}^{n}\right)\right]^{-1 / p}
$$

which implies that $\widetilde{a}_{0}$ is a $(\rho, \infty)_{\omega}$-single-atom. Since $\left\|a_{0}^{m}\right\|_{L_{\omega}^{\infty}\left(\mathbb{R}^{n}\right)} \leq\left(\omega\left(\mathbb{R}^{n}\right)\right)^{-1 / p}$ and

$$
\left|\lambda_{0}^{m}\right| \leq C\left\|f_{m}\right\|_{h_{\rho, N}^{p}(\omega)} \leq C 2^{-m}\|f\|_{h_{\rho, N}^{p}(\omega)}
$$

we have

$$
\left|\widetilde{\lambda}_{0}\right| \leq C\left(\sum_{m=1}^{\infty} 2^{-m}\right)\|f\|_{h_{\rho, N}^{p}(\omega)} \leq C\|f\|_{h_{\rho, N}^{p}(\omega)}
$$

moreover, we also have

$$
\sum_{m \in \mathbb{N}} \sum_{i \in \mathbb{N}}\left|\lambda_{i}^{m}\right|^{p}+\left|\tilde{\lambda}_{0}\right|^{p} \leq C\left(\sum_{m \in \mathbb{N}}\left\|f_{m}\right\|_{h_{\rho, N}^{p}(\omega)}^{p}+\|f\|_{h_{\rho, N}^{p}(\omega)}^{p}\right) \leq C\|f\|_{h_{\rho, N}^{p}(\omega)}^{p}
$$

Finally, we obtain

$$
f=\sum_{m \in \mathbb{N}} \sum_{i \in \mathbb{N}} \lambda_{i}^{m} a_{i}^{m}+\widetilde{\lambda}_{0} \widetilde{a}_{0} \in h_{\rho}^{p, \infty, s}(\omega)
$$

and

$$
\|f\|_{h_{\rho}^{p, \infty, s}(\omega)} \leq C\|f\|_{h_{\rho, N}^{p}(\omega)} .
$$

The theorem is proved.

For simplicity, from now on, we denote by $h_{\rho}^{p}(\omega)$ the weighted local Hardy space $h_{\rho, N}^{p}(\omega)$ when $N \geq N_{p, \omega}$.

## 6 Finite atomic decompositions

In this section, we prove that for any given finite linear combination of weighted atoms when $q<\infty$, its norm in $h_{\rho, N}^{p}(\omega)$ can be achieved via all its finite weighted atomic decompositions. This extends the main results in [21 to the setting of weighted local Hardy spaces.

Definition 6.1. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$ and $(p, q, s)_{\omega}$ be admissible as in Definition 3.3. Then $h_{\rho, \text { fin }}^{p, q, s}(\omega)$ is defined to be the vector space of all finite linear combinations of $(p, q, s)_{\omega}$-atoms and $a(p, q)_{\omega \text {-single-atom, and the norm of } f \text { in } h_{\rho, \text { fin }}^{p, q, s}(\omega) \text { is defined by }}$

$$
\begin{aligned}
\|f\|_{h_{\rho, \text { in }}^{p, q, s}(\omega)} \equiv \inf \{ & {\left[\sum_{i=0}^{k}\left|\lambda_{i}\right|^{p}\right]^{1 / p}: f=\sum_{i=0}^{k} \lambda_{i} a_{i}, k \in \mathbb{Z}_{+},\left\{\lambda_{i}\right\}_{i=0}^{k} \subset \mathbb{C},\left\{a_{i}\right\}_{i=1}^{k} \text { are } } \\
& \left.(p, q, s)_{\omega} \text { atoms, and } a_{0} \text { is a }(p, q)_{\omega} \text { single atom }\right\} .
\end{aligned}
$$

Obviously, for any admissible triplet $(p, q, s)_{\omega}$ atom and $(p, q)_{\omega}$ single atom, $h_{\rho, \text { fin }}^{p, q, s}(\omega)$ is dense in $h_{\rho}^{p, q, s}(\omega)$ with respect to the quasi-norm $\|\cdot\|_{h_{\rho}^{p, q, s}(\omega)}$.

Theorem 6.1. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), q_{\omega}$ be as in (2.4) and $(p, q, s)_{\omega}$ be admissible as in Definition 53.3. If $q \in\left(q_{\omega}, \infty\right)$, then $\|\cdot\|_{h_{\rho, \text { in }}^{p, q, s}(\omega)}$ and $\|\cdot\|_{h_{\rho}^{p}(\omega)}$ are equivalent quasi-norms on $h_{\rho, \text { fin }}^{p, q, s}(\omega)$.

Proof. Obviously, by Theorem [5.1, we infer that $h_{\rho, \text { fin }}^{p, q, s}(\omega) \subset h_{\rho}^{p, q, s}(\omega)=h_{\rho}^{p}(\omega)$ and for all $f \in h_{\rho, \text { fin }}^{p, q, s}(\omega)$, we have

$$
\|f\|_{h_{\rho}^{p}(\omega)} \leq C\|f\|_{h_{\rho, f i n}^{p, q, s}(\omega)} .
$$

Thus, it suffices to show that for every $q \in\left(q_{\omega}, \infty\right)$ there exists a constant $C$ such that for all $f \in h_{\rho, \operatorname{fin}}^{p, q, s}(\omega)$,

$$
\begin{equation*}
\|f\|_{h_{\rho}^{p, q, i n}\left(, \operatorname{li}^{s}(\omega)\right.} \leq C\|f\|_{h_{p}^{p}(\omega)} . \tag{6.1}
\end{equation*}
$$

Suppose that $f$ is in $h_{\rho, \text { fin }}^{p, q, s}(\omega)$ with $\|f\|_{h_{\rho}^{p}(\omega)}=1$. In this section, as in Section 5 , we take $m_{0} \in \mathbb{Z}$ such that $2^{m_{0}-1} \leq \inf _{x \in \mathbb{R}^{n}} \mathcal{M}_{N} f(x)<2^{m_{0}}$, and for $\inf _{x \in \mathbb{R}^{n}} \mathcal{M}_{N} f(x)=0$, write $m_{0}=-\infty$. For each integer $m \geq m_{0}$, set

$$
\Omega_{m} \equiv\left\{x \in \mathbb{R}^{n}: \mathcal{M}_{N} f(x)>2^{m}\right\}
$$

where and in what follows $N=N_{p, \omega}$. Since $f \in\left(h_{\rho, N}^{p}(\omega) \cap L_{\omega}^{q}\left(\mathbb{R}^{n}\right)\right)$, by Lemma 5.4, there exist $\lambda_{0} \in \mathbb{C},\left\{\lambda_{i}^{m}\right\}_{m \geq k_{0}, i} \subset \mathbb{C}$, a $(p, \infty)_{\omega}$-single-atom $a_{0}$ and $(p, \infty, s)_{\omega}$-atoms $\left\{a_{i}^{m}\right\}_{m \geq m_{0}, i}$, such that

$$
\begin{equation*}
f=\sum_{m \geq m_{0}} \sum_{i} \lambda_{i}^{m} a_{i}^{m}+\lambda_{0} a_{0} \tag{6.2}
\end{equation*}
$$

holds both in $\mathcal{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and almost everywhere. First, we claim that (6.2) also holds in $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$. For any $x \in \mathbb{R}^{n}$, by $\mathbb{R}^{n}=\cup_{m \geq m_{0}}\left(\Omega_{2^{m}} \backslash \Omega_{2^{k+1}}\right)$, we see that there exists $j \in \mathbb{Z}$ such that $x \in\left(\Omega_{2^{j}} \backslash \Omega_{2^{j+1}}\right)$. By the proof of Lemma 5.4, we know that for all $m>j, \operatorname{supp}\left(a_{i}^{m}\right) \subset \widetilde{Q}_{i}^{m} \subset$ $\Omega_{m} \subset \Omega_{j+1}$; then from (5.13) and (5.18), we conclude that

$$
\left|\sum_{m \geq m_{0}} \sum_{i} \lambda_{i}^{m} a_{i}^{m}(x)\right|+\left|\lambda_{0} a_{0}(x)\right| \leq C \sum_{k_{0} \leq k \leq j} 2^{k}+2^{k_{0}} \leq C 2^{j} \leq C \mathcal{M}_{N}(f)(x)
$$

Since $f \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$, from Proposition 3.1 (ii), we infer that $\mathcal{M}_{N}(f)(x) \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$. This combined with the Lebesgue dominated convergence theorem implies that

$$
\sum_{m \geq m_{0}} \sum_{i} \lambda_{i}^{m} a_{i}^{m}+\lambda_{0} a_{0}
$$

converges to $f$ in $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$, which deduce the claim.
Next, we show (6.1) by considering the following two cases for $\omega$.
Case I: $\omega\left(\mathbb{R}^{n}\right)=\infty$. In this case, as $f \in L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$, we know that $m_{0}=-\infty$ and $a_{0}(x)=0$ for almost every $x \in \mathbb{R}^{n}$ in (6.2). Thus, in this case, (6.2) can be written as

$$
f=\sum_{m \in \mathbb{Z}} \sum_{i} \lambda_{i}^{m} a_{i}^{m}
$$

Since, when $\omega\left(\mathbb{R}^{n}\right)=\infty$, all $(\rho, q)_{\omega}$-single-atoms are 0 , which implies that $f$ has compact support for $f \in h_{\rho, \text { fin }}^{p, q, s}(\omega)$. Assume that $\operatorname{supp}(f) \subset Q_{0} \equiv Q\left(x_{0}, r_{0}\right)$ and $\widetilde{Q}_{0} \equiv Q\left(x_{0}, r_{1}\right)$, in which $r_{1}=\sqrt{n} r_{0}+C_{0}^{2}(1+R)^{k_{0}+1}\left(1+\sqrt{n} r_{0} / \rho\left(x_{0}\right)\right) \rho\left(x_{0}\right)$. Then for any $\psi \in \mathcal{D}_{N}\left(\mathbb{R}^{n}\right)$, $x \in \mathbb{R}^{n} \backslash \widetilde{Q}_{0}$ and $2^{-l} \in(0, \rho(x))$, we have

$$
\psi_{l} * f(x)=\int_{Q\left(x_{0}, r_{0}\right)} \psi_{l}(x-y) f(y) d y=\int_{B(x, R \rho(x)) \cap Q\left(x_{0}, r_{0}\right)} \psi_{l}(x-y) f(y) d y=0
$$

Thus, for any $m \in \mathbb{Z}, \Omega_{m} \subset \widetilde{Q}_{0}$, which implies that $\operatorname{supp}\left(\sum_{m \in \mathbb{Z}} \sum_{i} \lambda_{i}^{m} a_{i}^{m}\right) \subset \widetilde{Q}_{0}$. For each positive integer $K$, let

$$
F_{K} \equiv\left\{(m, i): m \in \mathbb{Z}, m \geq m_{0}, i \in \mathbb{N},|m|+i \leq K\right\}
$$

and

$$
f_{K} \equiv \sum_{(m, i) \in F_{K}} \lambda_{i}^{m} a_{i}^{m} .
$$

Then, by the above claim, we know that $f_{K}$ converges to $f$ in $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$. Hence, for any given $\varepsilon \in(0,1)$, there exists a $K_{0} \in \mathbb{N}$ large enough such that $\operatorname{supp}\left(f-f_{K_{0}}\right) / \varepsilon \subset \widetilde{Q}_{0}$ and

$$
\left\|\left(f-f_{K_{0}}\right) / \varepsilon\right\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \leq\left[\omega\left(\widetilde{Q}_{0}\right)\right]^{1 / q-1 / p}
$$

For $\widetilde{Q}_{0}$, since $l\left(\widetilde{Q}_{0}\right)=r_{1}>2 \rho\left(x_{0}\right)$, we can decompose it into finite disjoint cubes $\left\{Q_{j}\right\}_{j}$ such that $\widetilde{Q}_{0}=\bigcup_{j=1}^{N_{0}} Q_{j}$ and $l_{j} / 4<\rho(x) \leq C_{0}(3 \sqrt{n})^{k_{0}} l_{j}$ for some $x \in Q_{j}=Q\left(x_{j}, l_{j}\right)$. Moreover, each $l_{j}$ satisfies $L_{2} \rho\left(x_{j}\right)<l_{j}<L_{1} \rho\left(x_{j}\right)$. It is clear that for $q \in\left(q_{\omega}, \infty\right)$ and $p \in(0,1]$ we have

$$
\left\|\left(f-f_{K_{0}}\right) \chi_{Q_{i}} / \varepsilon\right\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \leq\left[\omega\left(\widetilde{Q}_{0}\right)\right]^{1 / q-1 / p} \leq\left[\omega\left(Q_{j}\right)\right]^{1 / q-1 / p},
$$

which together with $\operatorname{supp}\left(\left(f-f_{K_{0}}\right) \chi_{Q_{j}} / \varepsilon\right) \subset Q_{j}$ implies that $\left(f-f_{K_{0}}\right) \chi_{Q_{j}} / \varepsilon$ is a $(p, q, s)_{\omega^{-}}$ atom for $j=1,2, \cdots, N_{0}$. Therefore,

$$
f=f_{K_{0}}+\sum_{j=1}^{N_{0}} \varepsilon \frac{\left(f-f_{K_{0}}\right) \chi_{Q_{j}}}{\varepsilon}
$$

is a finite weighted atom linear combination of $f$ almost everywhere. Then take $\varepsilon \equiv N_{0}^{-1 / p}$, we obtain

$$
\|f\|_{h_{\rho, \text { fin }}^{p, q, s}(\omega)}^{p} \leq \sum_{(m, i) \in F_{K}}\left|\lambda_{i}^{m}\right|^{p}+N_{0} \varepsilon^{p} \leq C,
$$

which implies the Case I.
Case II: $\omega\left(\mathbb{R}^{n}\right)<\infty$. In this case, $f$ may not have compact support. Similarly to Case I, for any positive integer $K$, let

$$
f_{K} \equiv \sum_{(m, i) \in F_{K}} \lambda_{i}^{m} a_{i}^{m}+\lambda_{0} a_{0}
$$

and $b_{K} \equiv f-f_{K}$, where $F_{K}$ is as in Case I. From the above claim, we deduce that $f_{K}$ converges to $f$ in $L_{\omega}^{q}\left(\mathbb{R}^{n}\right)$. Thus, there exists a positive integer $K_{1} \in \mathbb{N}$ large enough such that

$$
\left\|b_{K_{1}}\right\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \leq\left[\omega\left(\mathbb{R}^{n}\right)\right]^{1 / q-1 / p} .
$$

Thus, $b_{K_{1}}$ is a $(p, q)_{\omega}$-single-atom and $f=f_{K_{1}}+b_{K_{1}}$ is a finite weighted atom linear combination of $f$. By Lemma 5.4, we have

$$
\|f\|_{h_{\rho, \text { fin }}^{p, q, s}(\omega)}^{p} \leq C\left(\sum_{(m, i) \in F_{K}}\left|\lambda_{i}^{m}\right|^{p}+\lambda_{0}^{p}\right) \leq C
$$

Thus, (6.1) holds, and the theorem is now proved.

As an application of finite atomic decompositions, we establish boundedness in $h_{\rho}^{p}(\omega)$ of quasi- Banach-valued sublinear operators.

As in [5], a quasi-Banach space space $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate (i.e., $\|f\|_{\mathcal{B}}=0$ if and only if $f=0$ ), homogeneous, and obeys the quasi-triangle inequality, i.e., there exists a positive constant $K$ no less than 1 such that for all $f, g \in \mathcal{B},\|f+g\|_{\mathcal{B}} \leq K\left(\|f\|_{\mathcal{B}}+\|g\|_{\mathcal{B}}\right)$.

Let $\beta \in(0,1]$. A quasi-Banach space $\mathcal{B}_{\beta}$ with the quasi-norm $\|\cdot\|_{\mathcal{B}_{\beta}}$ is called a $\beta$-quasiBanach space if $\|f+g\|_{\mathcal{B}_{\beta}}^{\beta} \leq\|f\|_{\mathcal{B}_{\beta}}^{\beta}+\|g\|_{\mathcal{B}_{\beta}}^{\beta}$ for all $f, g \in \mathcal{B}_{\beta}$.

Notice that any Banach space is a 1-quasi-Banach space, and the quasi-Banach space $l^{\beta}, L_{\omega}^{\beta}\left(\mathbb{R}^{n}\right)$ and $h_{\omega}^{\beta}\left(\mathbb{R}^{n}\right)$ with $\beta \in(0,1)$ are typical $\beta$-quasi-Banach spaces.

For any given $\beta$-quasi-Banach space $\mathcal{B}_{\beta}$ with $\beta \in(0,1]$ and a linear space $\mathcal{Y}$, an operator $T$ from $\mathcal{Y}$ to $\mathcal{B}_{\beta}$ is said to be $\mathcal{B}_{\beta}$-sublinear if for any $f, g \in \mathcal{B}_{\beta}$ and $\lambda, \nu \in \mathbb{C}$,

$$
\|T(\lambda f+\nu g)\|_{\mathcal{B}_{\beta}} \leq\left(|\lambda|^{\beta}\|T(f)\|_{\mathcal{B}_{\beta}}^{\beta}+|\nu|^{\beta}\|T(g)\|_{\mathcal{B}_{\beta}}^{\beta}\right)^{1 / \beta}
$$

and $\|T(f)-T(g)\|_{\mathcal{B}_{\beta}} \leq\|T(f-g)\|_{\mathcal{B}_{\beta}}$.
We remark that if $T$ is linear, then it is $\mathcal{B}_{\beta}$-sublinear. Moreover, if $\mathcal{B}_{\beta}$ is a space of functions, and $T$ is nonnegative and sublinear in the classical sense, then $T$ is also $\mathcal{B}_{\beta}$-sublinear.

Theorem 6.2. Let $\omega \in A_{\infty}^{\rho, \infty}\left(\mathbb{R}^{n}\right), 0<p \leq \beta \leq 1$, and $\mathcal{B}_{\beta}$ be a $\beta$-quasi-Banach space. Suppose $q \in\left(q_{\omega}, \infty\right)$ and $T: h_{\rho, \text { fin }}^{p, q, s}(\omega) \rightarrow \mathcal{B}_{\beta}$ is a $\mathcal{B}_{\beta}$-sublinear operator such that

$$
S \equiv \sup \left\{\|T(a)\|_{\mathcal{B}_{\beta}}: a \text { is a }(p, q, s)_{\omega} \text { atom or }(p, q)_{\omega} \text { single atom }\right\}<\infty
$$

Then there exists a unique bounded $\mathcal{B}_{\beta}$-sublinear operator $\widetilde{T}$ from $h_{\rho}^{p}(\omega)$ to $\mathcal{B}_{\beta}$ which extends $T$.

Proof. For any $f \in h_{\rho, \text { fin }}^{p, q, s}(\omega)$, by Theorem 6.1, there exist a set of numbers $\left\{\lambda_{j}\right\}_{j=0}^{l} \subset \mathbb{C}$,


$$
\sum_{j=0}^{l}\left|\lambda_{j}\right|^{p} \leq C\|f\|_{h_{\rho}^{p}(\omega)}^{p}
$$

Then by the assumption, we have

$$
\|T(f)\|_{\mathcal{B}_{\beta}} \leq C\left[\sum_{j=0}^{l}\left|\lambda_{j}\right|^{p}\right]^{1 / p} \leq C\|f\|_{h_{\rho}^{p}(\omega)}
$$

Since $h_{\rho, \text { fin }}^{p, q, s}(\omega)$ is dense in $h_{\rho}^{p}(\omega)$, a density argument gives the desired results.

## 7 Atomic characterization of $H_{\mathcal{L}}^{1}(\omega)$

In this section, we apply the atomic characterization of the weighted local Hardy spaces $h_{\rho}^{1}(\omega)$ with $A_{1}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ weights to establish atomic characterization of weighted Hardy space $H_{\mathcal{L}}^{1}(\omega)$ associated to Schrödinger operator with $A_{1}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ weights.

Let $\mathcal{L}=-\Delta+V$ be a Schrödinger operator on $\mathbb{R}^{n}, n \geq 3$, where $V \in R H_{n / 2}$ is a fixed non-negative potential.

Let $\left\{T_{t}\right\}_{t>0}$ be the semigroup of linear operators generated by $\mathcal{L}$ and $T_{t}(x, y)$ be their kernels, that is,

$$
\begin{equation*}
T_{t} f(x)=e^{-t \mathcal{L}} f(x)=\int_{\mathbb{R}^{n}} T_{t}(x, y) f(y) d y, \quad \text { for } t>0 \text { and } f \in L^{2}\left(\mathbb{R}^{n}\right) \tag{7.1}
\end{equation*}
$$

Since $V$ is non-negative the Feynman-Kac formula implies that

$$
\begin{equation*}
0 \leq T_{t}(x, y) \leq \widetilde{T}_{t}(x, y) \equiv(4 \pi t)^{-\frac{n}{2}} \exp \left(-\frac{|x-y|^{2}}{4 t}\right) \tag{7.2}
\end{equation*}
$$

Obviously, by (1.2) the maximal operator

$$
\mathcal{T}^{*} f(x)=\sup _{t>0}\left|T_{t} f(x)\right|
$$

is of weak-type (1,1). A weighted Hardy-type space related to $\mathcal{L}$ with $A_{1}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ weights is naturally defined by:

$$
\begin{equation*}
H_{\mathcal{L}}^{1}(\omega) \equiv\left\{f \in L_{\omega}^{1}\left(\mathbb{R}^{n}\right): \mathcal{T}^{*} f(x) \in L_{\omega}^{1}\left(\mathbb{R}^{n}\right)\right\}, \quad \text { with } \quad\|f\|_{H_{\mathcal{L}}^{1}(\omega)} \equiv\left\|\mathcal{T}^{*} f\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \tag{7.3}
\end{equation*}
$$

The $H_{\mathcal{L}}^{1}(\omega)$ with $\omega \in A_{1}\left(\mathbb{R}^{n}\right)$ has been studied in [19, 41]
Now let us recall some basic properties of kernels $T_{t}(x, y)$ and the operator $\mathcal{T}^{*}$
Lemma 7.1. (see [10]) For every $l>0$ there is a constant $C_{l}$ such that

$$
\begin{equation*}
T_{t}(x, y) \leq C_{l}(1+|x-y| / \rho(x))^{-l}|x-y|^{-n} \tag{7.4}
\end{equation*}
$$

for $x, y \in \mathbb{R}^{n}$. Moreover, there is an $\varepsilon>0$ such that for every $C^{\prime}>0$, there exists $C$ so that

$$
\begin{equation*}
\left|T_{t}(x, y)-\widetilde{T}_{t}(x, y)\right| \leq C \frac{(|x-y| / \rho(x))^{\varepsilon}}{|x-y|^{n}} \tag{7.5}
\end{equation*}
$$

for $|x-y| \leq C^{\prime} \rho(x)$.

Since $T_{t}(x, y)$ is a symmetric function, we also have

$$
\begin{equation*}
T_{t}(x, y) \leq C_{l}(1+|x-y| / \rho(y))^{-l}|x-y|^{-n}, \quad \text { for } x, y \in \mathbb{R}^{n} . \tag{7.6}
\end{equation*}
$$

Lemma 7.2. (see [11]) There exist a rapidly decaying function $w \geq 0$ and $a \delta>0$ such that

$$
\begin{equation*}
\left|T_{t}(x, y)-\widetilde{T}_{t}(x, y)\right| \leq\left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} w_{\sqrt{t}}(x-y) \tag{7.7}
\end{equation*}
$$

Lemma 7.3. (see [12]) If $V \in R H_{s}\left(\mathbb{R}^{n}\right), s>n / 2$, then there exist $\delta=\delta(s)>0$ and $c>0$ such that for every $N>0$, there is a constant $C_{N}$ so that, for all $|h|<\sqrt{t}$

$$
\begin{equation*}
\left|T_{t}(x+h, y)-T_{t}(x, y)\right| \leq C_{N}\left(\frac{|h|}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N} \exp \left(-\frac{c|x-y|^{2}}{t}\right) \tag{7.8}
\end{equation*}
$$

Lemma 7.4. (see [3]) For $1<p<\infty$ the operator $\mathcal{T}^{*}$ is bounded on $L^{p}(\omega)$ when $\omega \in$ $A_{p}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, and of weak type $(1,1)$ when $\omega \in A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$.

In order to achieve the desired conclusions, we need the following estimates.
Lemma 7.5. Let $\omega \in A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, then there exists a positive constant $C$ such that for all $f \in h_{\rho}^{1}(\omega)$,

$$
\begin{equation*}
\|f\|_{h_{\rho}^{1}(\omega)} \leq C\left\|\widetilde{T}_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \tag{7.9}
\end{equation*}
$$

where

$$
\widetilde{T}_{\rho}^{+}(f)(x) \equiv \sup _{0<t<\rho(x)}\left|\widetilde{T}_{t^{2}}(f)(x)\right|
$$

and

$$
\widetilde{T}_{t}(f)(x) \equiv \int_{\mathbb{R}^{n}} \widetilde{T}_{t}(x, y) f(y) d y
$$

Proof. Let $h(x)=(4 \pi)^{-n / 2} e^{-|x|^{2} / 4}$, then it is easy to find that $h_{t}(x-y)=\widetilde{T}_{t^{2}}(x, y)$. Now we take a nonnegative function $\varphi \in \mathcal{D}\left(\mathbb{R}^{n}\right)$ such that $\varphi(x)=h(x)$ on $B(0,2)$, and we define $\varphi_{\rho}^{+}(f)(x)$ as follows:

$$
\varphi_{\rho}^{+}(f)(x) \equiv \sup _{0<t<\rho(x)}\left|\varphi_{t} * f(x)\right|
$$

Clearly, for any $x \in \mathbb{R}^{n}$, we have

$$
\begin{equation*}
\varphi^{+}(f)(x) \leq \varphi_{\rho}^{+}(f)(x) \tag{7.10}
\end{equation*}
$$

see (3.4) for the definition of $\varphi^{+}(f)(x)$.
Let $f \in h_{\rho}^{1}(\omega)$, for every $N>0$ we have:

$$
\begin{aligned}
& \left\|\varphi_{\rho}^{+}(f)-\widetilde{T}_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \\
& \leq \int_{\mathbb{R}^{n}} \sup _{0<t<\rho(x)}\left|\varphi_{t} * f(x)-h_{t} * f(x)\right| \omega(x) d x
\end{aligned}
$$

$$
\begin{aligned}
& \leq \int_{\mathbb{R}^{n}}\left(\sup _{0<t<\rho(x)} t^{-n} \int_{\mathbb{R}^{n}}|f(y)|\left|\varphi\left(\frac{x-y}{t}\right)-h\left(\frac{x-y}{t}\right)\right| d y\right) \omega(x) d x \\
& \leq \int_{\mathbb{R}^{n}}\left(\sup _{0<t<\rho(x)} t^{-n} \int_{\mathbb{R}^{n}}|f(y)|\left|\varphi\left(\frac{x-y}{t}\right)-h\left(\frac{x-y}{t}\right)\right| \chi_{\{|y-x|>t\}}(y) d y\right) \omega(x) d x \\
& \leq C \int_{\mathbb{R}^{n}}\left(\int_{\mathbb{R}^{n}}|f(y)| \sup _{0<t<\rho(x)} t^{-n}\left(1+\frac{|x-y|}{t}\right)^{-N} \chi_{\{|y-x|>t\}}(y) d y\right) \omega(x) d x \\
& \leq C \int_{\mathbb{R}^{n}}|f(y)|\left(\int_{\mathbb{R}^{n}}(\rho(x))^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \omega(x) d x\right) d y
\end{aligned}
$$

In the last inequality, we used the following facts that

$$
\sup _{0<t<\rho(x)} t^{-n}\left(1+\frac{|x-y|}{t}\right)^{-N} \leq(\rho(x))^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}
$$

provided that $|x-y|>t$ and $N>2 n$.
We now estimate the inner integral in the last inequality. In fact,

$$
\begin{aligned}
& \int_{\mathbb{R}^{n}}(\rho(x))^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \omega(x) d x \\
& =\int_{|x-y|<\rho(y)}(\rho(x))^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \omega(x) d x \\
& \quad+\int_{|x-y| \geq \rho(y)}(\rho(x))^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \omega(x) d x \\
& \equiv I+I I .
\end{aligned}
$$

For $I$, since $N$ is large enough and (2.2), we have

$$
I \leq \frac{C}{(\rho(y))^{n}} \int_{|x-y|<\rho(y)} \omega(x) d x \leq C \Psi_{\theta}\left(\widetilde{B}_{0}\right) M_{V, \theta}(\omega)(y) \leq C \omega(y)
$$

where $\widetilde{B}_{0}=B(y, \rho(y))$.
For $I I$, by the same reason as above, we have

$$
\begin{aligned}
I I & \leq C \sum_{i=1}^{\infty} \int_{|x-y| \sim 2^{i} \rho(y)}(\rho(x))^{N-n}|x-y|^{-N} \omega(x) d x \\
& \leq C \sum_{i=1}^{\infty} \int_{|x-y| \sim 2^{i} \rho(y)}(\rho(y))^{N-n}\left(1+\frac{|x-y|}{\rho(y)}\right)^{\frac{k_{0}(N-n)}{k_{0}+1}}|x-y|^{-N} \omega(x) d x \\
& \leq C \sum_{i=1}^{\infty} \int_{|x-y| \sim 2^{i} \rho(y)}(\rho(y))^{N-n}\left(1+2^{i}\right)^{\frac{k_{0}(N-n)}{k_{0}+1}}\left(2^{i} \rho(y)\right)^{-N} \omega(x) d x
\end{aligned}
$$

$$
\begin{aligned}
& \leq C \sum_{i=1}^{\infty}\left(2^{-i}\right)^{\frac{N+n k_{0}}{k_{0}+1}} \frac{1}{(\rho(y))^{n}} \int_{|x-y|<2^{i} \rho(y)} \omega(x) d x \\
& \leq C \sum_{i=1}^{\infty}\left(2^{-i}\right)^{\frac{N+n k_{0}}{k_{0}+1}}\left(1+2^{i}\right)^{\theta} M_{V, \theta}(\omega)(y) \\
& \leq C \sum_{i=1}^{\infty}\left(2^{-i}\right)^{\frac{N+n k_{0}}{k_{0}+1}-\theta} \omega(y) \leq C \omega(y),
\end{aligned}
$$

and the last inequality holds because the real number $N$ is large enough.
Combining the above two estimates, we get

$$
\begin{equation*}
\left\|\varphi_{\rho}^{+}(f)-\widetilde{T}_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq C \int_{\mathbb{R}^{n}}|f(y)| \omega(y) d y=C\|f\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \tag{7.11}
\end{equation*}
$$

In addition, it is easy to get $\|f\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq\left\|\widetilde{T}_{\rho}^{+} f\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)}$. Therefore, we obtain

$$
\begin{equation*}
\left\|\varphi_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq\left\|\widetilde{T}_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)}+C\|f\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq C\left\|\widetilde{T}_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \tag{7.12}
\end{equation*}
$$

Finally, from Theorem 3.2, (7.10) and (7.12), it follows that

$$
\|f\|_{h_{\rho}^{1}(\omega)} \leq C\left\|\varphi^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq C\left\|\varphi_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq C\left\|\widetilde{T}_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)},
$$

which finishes the proof.

$$
\begin{aligned}
& \text { For } x, y \in \mathbb{R}^{n} \text {, set } E_{t}(x, y)=T_{t^{2}}(x, y)-\widetilde{T}_{t^{2}}(x, y) \\
& \qquad T_{\rho}^{+}(f)(x) \equiv \sup _{0<t<\rho(x)}\left|T_{t^{2}}(f)(x)\right| \quad \text { and } \quad E_{\rho}^{+}(f)(x) \equiv \sup _{0<t<\rho(x)}\left|E_{t}(f)(x)\right|
\end{aligned}
$$

Lemma 7.6. Let $\omega \in A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, then there exists a positive constant $C$ such that for all $f \in L_{\omega}^{1}\left(\mathbb{R}^{n}\right)$,

$$
\left\|E_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq C\|f\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)}
$$

Proof. By Lemma 2.2, it suffices to prove that for all $j$,

$$
\begin{equation*}
\left\|E_{\rho}^{+}\left(\chi_{B_{j}^{*}} f\right)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq C\left\|\chi_{B_{j}^{*}} f\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \tag{7.13}
\end{equation*}
$$

in which $B_{j}=B\left(x_{j}, \rho\left(x_{j}\right)\right)$. For any $x \in B_{j}^{* *}$ and $y \in B_{j}^{*}$, since $\rho(y) \sim \rho\left(x_{j}\right) \sim \rho(x)$ via Lemma 2.1, by (7.5) we have

$$
\left|E_{t}(x, y)\right| \leq C \frac{(|x-y| / \rho(x))^{\varepsilon}}{|x-y|^{n}} \leq \frac{C}{|x-y|^{n-\varepsilon}\left(\rho\left(x_{j}\right)\right)^{\varepsilon}}
$$

which implies that

$$
\begin{aligned}
\int_{B_{j}^{* *}} & \sup _{0<t<\rho(x)}\left|E_{t}\left(\chi_{B_{j}^{*}} f\right)\right| \omega(x) d x \\
& \leq C \int_{B_{j}^{* *}}\left(\int_{B_{j}^{*}} \frac{|f(y)|}{\left.|x-y|\right|^{n-\varepsilon}\left(\rho\left(x_{j}\right)\right)^{\varepsilon}} d y\right) \omega(x) d x \\
& \leq C \int_{B_{j}^{*}}\left(\int_{B_{j}^{* *}} \frac{\omega(x)}{|x-y|^{n-\varepsilon}\left(\rho\left(x_{j}\right)\right)^{\varepsilon}} d x\right)|f(y)| d y \\
& \leq C \int_{B_{j}^{*}}\left(\sum_{k=-2}^{\infty} \int_{|x-y| \sim 2^{-k} \rho\left(x_{j}\right)} \frac{\omega(x)}{|x-y|^{n-\varepsilon}\left(\rho\left(x_{j}\right)\right)^{\varepsilon}} d x\right)|f(y)| d y \\
& \leq C \int_{B_{j}^{*}}\left(\sum_{k=-2}^{\infty} \frac{\omega\left(B\left(y, 2^{-k} \rho\left(x_{j}\right)\right)\right)}{\left(2^{-k} \rho\left(x_{j}\right)\right)^{n-\varepsilon}\left(\rho\left(x_{j}\right)\right)^{\varepsilon}} d x\right)|f(y)| d y \\
& \leq C \int_{B_{j}^{*}}\left(\sum_{k=-2}^{\infty} \frac{1}{2^{k \varepsilon}}\left(1+C_{0} 2^{k_{0}-k}\right)^{\theta} \omega(y)\right)|f(y)| d y \\
& \leq C \int_{B_{j}^{*}}|f(y)| \omega(y) d y=C\left\|\chi_{B_{j}^{*}} f\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} .
\end{aligned}
$$

For any $x \in\left(B_{j}^{* *}\right)^{\complement}$ and $y \in B_{j}^{*}$, it is easy to see that $\rho\left(x_{j}\right) \lesssim\left|x-x_{j}\right| \sim|x-y|$; in addition, by (2.2) and (7.7), we have $0<t<\rho(x) \lesssim\left|x-x_{j}\right|^{k_{0} /\left(k_{0}+1\right)}\left(\rho\left(x_{j}\right)\right)^{1 /\left(k_{0}+1\right)}$ and $E_{t}(x, y) \lesssim t^{N} /|x-y|^{N+n} \sim t^{N} /\left|x-x_{j}\right|^{N+n}$ for any $N>0$. Therefore, taking $N>\left(k_{0}+1\right) \theta$, we have

$$
\begin{aligned}
& \int_{\left(B_{j}^{* *}\right)^{\mathrm{C}}} \sup _{0<t<\rho(x)}\left|E_{t}\left(\chi_{B_{j}^{*}} f\right)\right| \omega(x) d x \\
& \leq C \int_{\left(B_{j}^{* *}\right)^{\mathrm{c}}}\left(\int_{B_{j}^{*}} \frac{\left(\rho\left(x_{j}\right)\right)^{\frac{N}{k_{0}+1}}|f(y)|}{\left|x-x_{j}\right|^{n+\frac{N}{k_{0}+1}}} d y\right) \omega(x) d x \\
& \leq C \int_{B_{j}^{*}}\left(\int_{\left(B_{j}^{* *}\right)^{\complement}} \frac{\left(\rho\left(x_{j}\right)\right)^{\frac{N}{k_{0}+1}} \omega(x)}{\left|x-x_{j}\right|^{n+\frac{N}{k_{0}+1}}} d x\right)|f(y)| d y \\
& \quad \leq C \int_{B_{j}^{*}}\left(\sum_{i=2}^{\infty} \int_{\left|x-x_{j}\right| \sim^{i} \rho\left(x_{j}\right)} \frac{\left(\rho\left(x_{j}\right)\right)^{\frac{N}{k_{0}+1}} \omega(x)}{\left|x-x_{j}\right|^{n+\frac{N}{k_{0}+1}}} d x\right)|f(y)| d y \\
& \quad \leq C \int_{B_{j}^{*}}\left(\sum_{i=2}^{\infty} \frac{\left(\rho\left(x_{j}\right)\right)^{\frac{N}{k_{0}+1}} \omega\left(B\left(x_{j}, 2^{i} \rho\left(x_{j}\right)\right)\right)}{\left(2^{i} \rho\left(x_{j}\right)\right)^{n+\frac{N}{k_{0}+1}}} d x\right)|f(y)| d y \\
& \quad \leq C \int_{B_{j}^{*}}\left(\sum_{i=2}^{\infty} \frac{\left(1+2^{i}\right)^{\theta}}{\left(2^{i}\right)^{\frac{N}{k_{0}+1}}} \omega(y)\right)|f(y)| d y \\
& \quad \leq C \int_{B_{j}^{*}}|f(y)| \omega(y) d y=C\left\|\chi_{B_{j}^{*}} f\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right),}
\end{aligned}
$$

which completes the proof of (7.13) and hence the proof of lemma.

Next we give several estimates about $(p, q, s)_{\omega}$-atoms and $(p, q)_{\omega}$-single-atom, which are important for our conclusion.

Lemma 7.7. Let $a$ be $a(p, q, s)_{\omega}$-atom, and $\operatorname{supp} a \subset Q\left(x_{0}, r\right)$, then for any $x \in(4 Q)^{\complement}$, we have following estimates:
(i) If $L_{2} \rho\left(x_{0}\right) \leq r \leq L_{1} \rho\left(x_{0}\right)$, then for any $M>0$,

$$
\mathcal{T}^{*} a(x) \lesssim\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \frac{r^{M}}{\left|x-x_{0}\right|^{n+M}}
$$

(ii) If $r<L_{2} \rho\left(x_{0}\right)$ and $\left|x-x_{0}\right| \leq 2 \rho\left(x_{0}\right)$, then there exists $\delta>0$ such that

$$
\mathcal{T}^{*} a(x) \lesssim\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \frac{r^{\delta}}{\left|x-x_{0}\right|^{n+\delta}}
$$

(iii) If $r<L_{2} \rho\left(x_{0}\right)$ and $\left|x-x_{0}\right| \geq \rho\left(x_{0}\right) / \sqrt{n}$, then there exists $\delta>0$ such that for any $M>0$,

$$
\mathcal{T}^{*} a(x) \lesssim\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \frac{r^{\delta}}{\left|x-x_{0}\right|^{n+\delta}}\left(\frac{\rho\left(x_{0}\right)}{\left|x-x_{0}\right|}\right)^{M}
$$

Proof. If $L_{2} \rho\left(x_{0}\right) \leq r \leq L_{1} \rho\left(x_{0}\right)$, since $|x-y| \sim\left|x-x_{0}\right|$ and $\rho(y) \sim \rho\left(x_{0}\right)$ for $x \in(4 Q)^{\complement}$ and $y \in Q$, by Lemma 7.1, for any $M>0$, we have

$$
\begin{aligned}
T_{t} a(x) & \leq \int_{\mathbb{R}^{n}}\left|T_{t}(x, y)\right||a(y)| d y \\
& \lesssim \int_{Q}\left(1+\frac{|x-y|}{\rho(y)}\right)^{-M}|x-y|^{-n}|a(y)| d y \\
& \lesssim \int_{Q}\left(1+\frac{\left|x-x_{0}\right|}{\rho\left(x_{0}\right)}\right)^{-M}\left|x-x_{0}\right|^{-n}|a(y)| d y \\
& \lesssim\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \frac{\rho\left(x_{0}\right)^{M}}{\left|x-x_{0}\right|^{n+M}} \lesssim\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \frac{r^{M}}{\left|x-x_{0}\right|^{n+M}}
\end{aligned}
$$

and then we obtain (i).
If $r<L_{2} \rho\left(x_{0}\right)$, by the moment condition of $a$ and Lemma 7.3 , for any $M>0$ and $y^{\prime} \in Q$ which satisfies $\left|y-y^{\prime}\right|<\sqrt{t}$, we have

$$
\begin{aligned}
T_{t} a(x) & =\int_{\mathbb{R}^{n}} T_{t}(x, y) a(y) d y \\
& =\int_{Q}\left(T_{t}(x, y)-T_{t}\left(x, y^{\prime}\right)\right) a(y) d y \\
& \lesssim \int_{Q}\left(\frac{\left|y-y^{\prime}\right|}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}}\left(1+\frac{\sqrt{t}}{\rho(y)}\right)^{-M} \exp \left(-\frac{c|x-y|^{2}}{t}\right)|a(y)| d y \\
& \lesssim \int_{Q}\left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}}\left(1+\frac{\sqrt{t}}{\rho\left(x_{0}\right)}\right)^{-M}\left(\frac{t}{\left|x-x_{0}\right|^{2}}\right)^{K}|a(y)| d y
\end{aligned}
$$

where $K>0$ is any real number.
For $\left|x-x_{0}\right| \leq 2 \rho\left(x_{0}\right)$, taking $K=(n+\delta) / 2$, we obtain

$$
\begin{aligned}
T_{t} a(x) & \lesssim \int_{Q}\left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}}\left(1+\frac{\sqrt{t}}{\rho\left(x_{0}\right)}\right)^{-M}\left(\frac{t}{\left|x-x_{0}\right|^{2}}\right)^{K}|a(y)| d y \\
& \lesssim\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)}\left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}}\left(\frac{t}{\left|x-x_{0}\right|^{2}}\right)^{K} \\
& =\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \frac{r^{\delta}}{\left|x-x_{0}\right|^{n+\delta}}
\end{aligned}
$$

which implies (ii).
For $\left|x-x_{0}\right| \geq \rho\left(x_{0}\right) / \sqrt{n}$, taking $K=(n+M+\delta) / 2$, we obtain

$$
\begin{aligned}
T_{t} a(x) & \lesssim \int_{Q}\left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}}\left(1+\frac{\sqrt{t}}{\rho\left(x_{0}\right)}\right)^{-M}\left(\frac{t}{\left|x-x_{0}\right|^{2}}\right)^{K}|a(y)| d y \\
& \lesssim\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)}\left(\frac{r}{\sqrt{t}}\right)^{\delta} t^{-\frac{n}{2}}\left(\frac{\rho\left(x_{0}\right)}{\sqrt{t}}\right)^{M}\left(\frac{t}{\left|x-x_{0}\right|^{2}}\right)^{K} \\
& =\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \frac{r^{\delta}}{\left|x-x_{0}\right|^{n+\delta}}\left(\frac{\rho\left(x_{0}\right)}{\left|x-x_{0}\right|}\right)^{M}
\end{aligned}
$$

which finishes the proof of lemma.
Lemma 7.8. Let $\omega \in A_{q}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ and a be $a(p, q, s)_{\omega}$-atom, which satisfies $\operatorname{supp} a \subset Q\left(x_{0}, r\right)$, then there exists a constant $C$ such that:

$$
\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \leq C|Q| \omega(Q)^{-1 / p} \Psi_{\theta}(Q)
$$

Proof. If $q>1$, by Hölder inequality and the definition of $A_{q}^{\rho, \theta}\left(\mathbb{R}^{n}\right)$ weights, we have

$$
\begin{aligned}
\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} & =\int_{Q}|a(x)| \omega(x)^{1 / q} \omega(x)^{-1 / q} d x \\
& \leq\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)}\left(\int_{Q} \omega(x)^{-q^{\prime} / q} d x\right)^{1 / q^{\prime}} \\
& \leq \omega(Q)^{1 / q-1 / p}\left(\int_{Q} \omega(x)^{-q^{\prime} / q} d x\right)^{1 / q^{\prime}}\left(\int_{Q} \omega(x) d x\right)^{1 / q} \omega(Q)^{-1 / q} \\
& \leq C|Q| \omega(Q)^{-1 / p} \Psi_{\theta}(Q)
\end{aligned}
$$

If $q=1$, we have

$$
\omega(Q) \leq C|Q| \Psi_{\theta}(Q) \inf _{x \in Q} \omega(x)
$$

which implies

$$
\left\|\omega^{-1}\right\|_{L^{\infty}(Q)} \leq C|Q| \omega(Q)^{-1} \Psi_{\theta}(Q)
$$

Therefore, we get

$$
\|a\|_{L^{1}\left(\mathbb{R}^{n}\right)} \leq\|a\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)}\left\|\omega^{-1}\right\|_{L^{\infty}(Q)} \leq C|Q| \omega(Q)^{-1 / p} \Psi_{\theta}(Q),
$$

which finishes the proof.

Combining above two lemmas with $\Psi_{\theta}(Q) \lesssim 1$, we can get the following corollary.
Corollary 7.1. Let $a$ be $a(p, q, s)_{\omega^{-}}$atom, and $\operatorname{supp} a \subset Q\left(x_{0}, r\right)$, then for any $x \in(4 Q)^{\complement}$, we have following estimates:
(i) If $L_{2} \rho\left(x_{0}\right) \leq r \leq L_{1} \rho\left(x_{0}\right)$, then for any $M>0$,

$$
\mathcal{T}^{*} a(x) \lesssim \omega(Q)^{-1 / p}\left(\frac{r}{\left|x-x_{0}\right|}\right)^{n+M}
$$

(ii) If $r<L_{2} \rho\left(x_{0}\right)$ and $\left|x-x_{0}\right| \leq 2 \rho\left(x_{0}\right)$, then there exists $\delta>0$ such that

$$
\mathcal{T}^{*} a(x) \lesssim \omega(Q)^{-1 / p}\left(\frac{r}{\left|x-x_{0}\right|}\right)^{n+\delta}
$$

(iii) If $r<L_{2} \rho\left(x_{0}\right)$ and $\left|x-x_{0}\right| \geq \rho\left(x_{0}\right) / \sqrt{n}$, then there exists $\delta>0$ such that for any $M>0$,

$$
\mathcal{T}^{*} a(x) \lesssim \omega(Q)^{-1 / p}\left(\frac{r}{\left|x-x_{0}\right|}\right)^{n+\delta}\left(\frac{\rho\left(x_{0}\right)}{\left|x-x_{0}\right|}\right)^{M} .
$$

Next we give the main theorem of this section.
Theorem 7.1. Let $0 \not \equiv V \in R H_{n / 2}$ and $\omega \in A_{1}^{\rho, \infty}\left(\mathbb{R}^{n}\right)$, then $h_{\rho}^{1}(\omega)=H_{\mathcal{L}}^{1}(\omega)$ with equivalent norms, that is

$$
\|f\|_{h_{\rho}^{1}(\omega)} \sim\|f\|_{H_{\mathcal{L}}^{1}(\omega)}
$$

Proof. Assume that $f \in H_{\mathcal{L}}^{1}(\omega)$, by (7.7), we have

$$
\begin{equation*}
|f(x)|=\lim _{t<\rho(x), t \rightarrow 0}\left|\widetilde{T}_{t}(f)(x)\right| \leq T_{\rho}^{+}(f)(x)+C \lim _{t \rightarrow 0}\left(\frac{t}{\rho(x)}\right)^{\delta} M(f)(x) \leq T_{\rho}^{+}(f)(x) \tag{7.14}
\end{equation*}
$$

Then according to (7.14), Lemma 7.5 and Lemma 7.6, we get $f \in h_{\rho}^{1}(\omega)$ and

$$
\begin{aligned}
\|f\|_{h_{\rho}^{1}(\omega)} & \lesssim\left\|\widetilde{T}_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \lesssim\left\|T_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)}+\left\|E_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|T_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)}+\|f\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \lesssim\left\|T_{\rho}^{+}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \\
& \lesssim\left\|\mathcal{T}^{*}(f)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)}=\|f\|_{H_{\mathcal{L}}^{1}(\omega)} .
\end{aligned}
$$

Conversely, we need to prove that $\mathcal{T}^{*}$ is bounded from $h_{\rho}^{1}(\omega)$ to $L_{\omega}^{1}\left(\mathbb{R}^{n}\right)$. To end this, by Lemma 2.4 and Theorem 5.1, it suffices to prove that for any $(1, q, s)_{\omega \text {-atom }}$ or $(1, q)_{\omega}$-singleatom $a$,

$$
\begin{equation*}
\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \lesssim 1, \tag{7.15}
\end{equation*}
$$

where $1<q \leq 1+\delta / n$.
If $a$ is a $(1, q)_{\omega}$-single-atom, by Hölder inequality and Lemma 7.4, we have

$$
\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \omega\left(\mathbb{R}^{n}\right)^{1-1 / q} \leq C\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \omega\left(\mathbb{R}^{n}\right)^{1-1 / q} \lesssim 1 .
$$

If $a$ is a $(1, q, s)_{\omega}$-atom and $\operatorname{supp} a \subset Q\left(x_{0}, r\right)$ with $r \leq L_{1} \rho\left(x_{0}\right)$, then we have

$$
\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}\left(\mathbb{R}^{n}\right)} \leq\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}(4 Q)}+\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}\left((4 Q)^{c}\right)} \equiv I+I I .
$$

For $I$, by Hölder inequality, Lemma 2.4 and Lemma 7.4, we get

$$
\begin{aligned}
\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}(4 Q)} & \leq\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{q}(4 Q)} \omega(4 Q)^{1-1 / q} \leq C\|a\|_{L_{\omega}^{q}\left(\mathbb{R}^{n}\right)} \omega(4 Q)^{1-1 / q} \\
& \leq C(\omega(4 Q) / \omega(Q))^{1-1 / q} \lesssim 1 .
\end{aligned}
$$

For $I I$, if $L_{2} \rho\left(x_{0}\right) \leq r \leq L_{1} \rho\left(x_{0}\right)$, by Lemma 2.4 and Corollary 7.1, taking $M>q(n+\theta)-n$, we have

$$
\begin{aligned}
\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}\left((4 Q)^{\natural}\right)} & =\sum_{j=3}^{\infty} \int_{2^{j} Q \backslash 2^{j-1} Q} \mathcal{T}^{*}(a)(x) \omega(x) d x \\
& \lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{\infty} \int_{2^{j} Q \backslash 2^{j-1} Q}\left(\frac{r}{\left|x-x_{0}\right|}\right)^{n+M} \omega(x) d x \\
& \lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{\infty} 2^{-j(n+M)} \omega\left(2^{j} Q\right) \\
& \lesssim \sum_{j=3}^{\infty} 2^{-j(n+M)} 2^{j n q}\left(1+\frac{2^{j} r}{\rho\left(x_{0}\right)}\right)^{q \theta} \\
& \lesssim \sum_{j=3}^{\infty} 2^{-j[n+M-n q-q \theta]} \lesssim 1
\end{aligned}
$$

if $r<L_{2} \rho\left(x_{0}\right)$, then there exists $N_{0} \in \mathbb{Z}$ such that $2^{N_{0}-1} \sqrt{n} r \leq \rho\left(x_{0}\right)<2^{N_{0}} \sqrt{n} r$. Let us assume that $N_{0} \geq 3$, otherwise, we just need to consider the $I_{2}$ in the following decomposition:

$$
\left\|\mathcal{T}^{*}(a)\right\|_{L_{\omega}^{1}\left((4 Q)^{\mathfrak{c}}\right)}=\left(\sum_{j=3}^{N_{0}}+\sum_{j=N_{0}+1}^{\infty}\right) \int_{2^{j} Q \backslash 2^{j-1} Q} \mathcal{T}^{*}(a)(x) \omega(x) d x \equiv I_{1}+I_{2},
$$

for $I_{1}$, since $\left|x-x_{0}\right|<2^{j} \sqrt{n} r \leq 2^{N_{0}} \sqrt{n} r \leq 2 \rho\left(x_{0}\right), \Psi_{\theta}\left(2^{j} Q\right) \leq 3^{\theta}$ and $q<1+\delta / n$, by Lemma 2.4 and Corollary 7.1, we get

$$
\begin{aligned}
I_{1} & =\sum_{j=3}^{N_{0}} \int_{2^{j} Q \backslash 2^{j-1} Q} \mathcal{T}^{*}(a)(x) \omega(x) d x \\
& \lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{N_{0}} \int_{2^{j} Q \backslash 2^{j-1} Q}\left(\frac{r}{\left|x-x_{0}\right|}\right)^{n+\delta} \omega(x) d x \\
& \lesssim \frac{1}{\omega(Q)} \sum_{j=3}^{N_{0}} 2^{-j(n+\delta)} \omega\left(2^{j} Q\right) \\
& \lesssim \sum_{j=3}^{N_{0}} 2^{-j[n+\delta-n q]} \lesssim 1
\end{aligned}
$$

for $I_{2}$, since $\left|x-x_{0}\right| \geq 2^{j-1} r \geq 2^{N_{0}} r \geq \rho\left(x_{0}\right) / \sqrt{n}$, then $\Psi_{\theta}\left(2^{j} Q\right) \leq\left(2^{j+1} \sqrt{n} r / \rho\left(x_{0}\right)\right)^{\theta}$, thus, taking $M=q \theta$, by $q<1+\delta / n$, Lemma 2.4 and Corollary 7.1, we obtain

$$
\begin{aligned}
I_{2} & =\sum_{j=N_{0}+1}^{\infty} \int_{2^{j} Q \backslash 2^{j-1} Q} \mathcal{T}^{*}(a)(x) \omega(x) d x \\
& \lesssim \frac{1}{\omega(Q)} \sum_{j=N_{0}+1}^{\infty} \int_{2^{j} Q \backslash 2^{j-1} Q}\left(\frac{r}{\left|x-x_{0}\right|}\right)^{n+\delta}\left(\frac{\rho\left(x_{0}\right)}{\left|x-x_{0}\right|}\right)^{M} \omega(x) d x \\
& \lesssim \frac{1}{\omega(Q)} \sum_{j=N_{0}+1}^{\infty} 2^{-j(n+\delta)} \omega\left(2^{j} Q\right)\left(\frac{\rho\left(x_{0}\right)}{2^{j} r}\right)^{M} \\
& \lesssim \sum_{j=N_{0}+1}^{\infty} 2^{-j[n+\delta-n q]}\left(\Psi_{\theta}\left(2^{j} Q\right)\right)^{q}\left(\frac{\rho\left(x_{0}\right)}{2^{j} r}\right)^{M} \lesssim 1
\end{aligned}
$$

which finally implies the (7.15) and finishes the proof.

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