UNIFORM K-STABILITY AND PLT BLOWUPS OF LOG FANO PAIRS

KENTO FUJITA

ABSTRACT. We show relationships between uniform K-stability and plt blowups of log Fano pairs. We see that it is enough to evaluate certain invariants defined by volume functions for all plt blowups in order to test uniform K-stability of log Fano pairs. We also discuss the uniform K-stability of two log Fano pairs under crepant finite covers. Moreover, we give another proof of Ksemistability of the projective plane.

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1. INTRODUCTION

In this paper, we work over an arbitrary algebraically closed field \Bbbk of characteristic zero. Let (X, Δ) be a log Fano pair, that is, X is a normal projective variety over \Bbbk and Δ is an effective \mathbb{Q} -divisor such that (X, Δ) is a klt pair and $-(K_X + \Delta)$ is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor. (For the minimal model program, we refer the readers to [KM98] and [BCHM10].) We are interested in the problem whether (X, Δ) is uniformly K-stable (resp., K-semistable) or not (see [Tia97, Don02, Szé06, Szé15, Der15, BHJ15, BBJ15, Fjt16b] and references therein). In [Li16, Theorem 3.7] and [Fjt16b, Theorem 6.5], we have seen that the uniform K-stability (and the K-semistability) of (X, Δ) is equivalent to measure the positivity of certain invariants associated to

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divisorial valuations on X. Before recalling those results, we prepare some definitions.

Definition 1.1. Let (V, Γ) be a *log pair*, that is, V is a normal variety and Γ is an effective \mathbb{R} -divisor on V such that $K_V + \Gamma$ is \mathbb{R} -Cartier. Let F be a prime divisor *over* V, that is, there exists a projective birational morphism $\sigma: W \to V$ with W normal such that F is a prime divisor on W.

- (1) The log discrepancy $A(F) := A_{(V,\Gamma)}(F)$ of (V,Γ) along F is defined to be $A(F) := 1 + \operatorname{ord}_F(K_W \sigma^*(K_V + \Gamma))$. We remark that the value A(F) does not depend on the choice of σ . Moreover, let $c_V(F) \subset V$ be the *center* of F on V, that is, the image of F on V.
- (2) ([Ish04]) The divisor F is said to be primitive over V if there exists a projective birational morphism $\tau: T \to V$ with T normal such that F is a prime divisor on T and -F on T is a τ ample \mathbb{Q} -Cartier divisor. We call the morphism τ the associated prime blowup. (We do not assume that F is exceptional over V. We remark that the associated prime blowup is uniquely determined from F [Ish04, Proposition 2.4].) Moreover, let Γ_T be the \mathbb{R} -divisor on T defined by

$$K_T + \Gamma_T + (1 - A(F))F = \tau^*(K_V + \Gamma).$$

We often denote the associated prime blowup by

$$\tau\colon (T,\Gamma_T+F)\to (V,\Gamma).$$

(3) ([Sho96, 3.1], [Pro00, Definition 2.1]) Assume that F is a primitive prime divisor over V and $\tau: (T, \Gamma_T + F) \to (V, \Gamma)$ is the associated prime blowup. The divisor F is said to be *plt-type* (resp., *lc-type*) over (V, Γ) if the pair $(T, \Gamma_T + F)$ is plt (resp., lc). Under the situation, we call the associated morphism τ the *associated plt blowup* (resp., the *associated lc blowup*).

Definition 1.2 ([Fjt16b, Definition 6.1]). Let (X, Δ) be a log Fano pair of dimension $n, L := -(K_X + \Delta)$, and let F be a prime divisor over X.

(1) For arbitrary $k \in \mathbb{Z}_{\geq 0}$ with kL Cartier and $x \in \mathbb{R}_{\geq 0}$, let $H^0(X, kL - xF)$ be the sub k-vector space of $H^0(X, kL)$ defined by the set of global sections vanishing at the generic point of F at least x times.

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(2) The divisor F is said to be *dreamy* over (X, Δ) if the graded k-algebra

$$\bigoplus_{x,i\in\mathbb{Z}_{\geq 0}} H^0(X, krL - iF)$$

is finitely generated over k for some (hence, for an arbitrary) $r \in \mathbb{Z}_{>0}$ such that rL is Cartier.

(3) For an arbitrary $x \in \mathbb{R}_{\geq 0}$, set

k

$$\operatorname{vol}_X(L-xF) := \limsup_{\substack{k \to \infty \\ kL: \text{ Cartier}}} \frac{\dim_{\mathbb{K}} H^0(X, kL-kxF)}{k^n/n!}.$$

By [Laz04a, Laz04b], the function $\operatorname{vol}_X(L - xF)$ is continuous and non-increasing on $x \in [0, \infty)$. Moreover, the limsup is actually a limit.

(4) We define the *pseudo-effective threshold* $\tau(F)$ of L along F by

 $\tau(F) := \tau_{(X,\Delta)}(F) := \sup\{x \in \mathbb{R}_{>0} \mid \operatorname{vol}_X(L - xF) > 0\}.$

Note that $\tau(F) \in \mathbb{R}_{>0}$.

(5) We set

$$\beta(F) := \beta_{(X,\Delta)}(F) := A(F) \cdot (L^{\cdot n}) - \int_0^\infty \operatorname{vol}_X(L - xF) dx.$$

Moreover, we set

$$\hat{\beta}(F) := \frac{\beta(F)}{A(F) \cdot (L^{\cdot n})}.$$

(6) We set

$$j(F) := j_{(X,\Delta)}(F) := \int_0^{\tau(F)} \left((L^{\cdot n}) - \operatorname{vol}_X(L - xF) \right) dx.$$

Obviously, we have the inequality j(F) > 0.

- **Remark 1.3.** (1) ([Fjt16c, Lemma 3.1 (2)]) If F is a dreamy prime divisor over (X, Δ) , then F is primitive over X.
 - (2) ([Xu14]) If F is a plt-type prime divisor over a klt pair (V, Γ) and $c_V(F) = \{o\}$, then the divisor F is said to be a Kollár component of the singularity $o \in (V, \Gamma)$.

The following is the valuative criterion for uniform K-stability (and K-semistability) of log Fano pairs introduced in [Li16] and [Fjt16b].

Theorem 1.4. Let (X, Δ) be a log Fano pair.

(1) ([Li16, Theorem 3.7], [Fjt16b, Theorem 6.6]) The following are equivalent:

- (i) (X, Δ) is K-semistable (see for example [Fjt16b, Definition 6.4]).
- (ii) For any prime divisor F over X, the inequality $\beta(F) \ge 0$ holds.
- (iii) For any dreamy prime divisor F over (X, Δ) , the inequality $\beta(F) \ge 0$ holds.
- (2) ([Fjt16b, Theorem 6.6]) The following are equivalent:
 - (i) (X, Δ) is uniformly K-stable (see for example [Fjt16b, Definition 6.4]).
 - (ii) There exists $\delta \in (0, 1)$ such that for any prime divisor F over X, the inequality $\beta(F) \ge \delta \cdot j(F)$ holds.
 - (iii) There exists $\delta \in (0,1)$ such that for any dreamy prime divisor F over (X, Δ) , the inequality $\beta(F) \ge \delta \cdot j(F)$ holds.

The purpose of this paper is to simplify the above theorem. More precisely, we see relationships between the above criterion and plt blowups of log Fano pairs. The following is the main theorem in this paper.

Theorem 1.5 (Main Theorem). Let (X, Δ) be a log Fano pair.

- (1) The following are equivalent:
 - (i) (X, Δ) is K-semistable.
 - (ii) For any plt-type prime divisor F over (X, Δ) , the inequality $\hat{\beta}(F) \geq 0$ holds.
- (2) The following are equivalent:
 - (i) (X, Δ) is uniformly K-stable.
 - (ii) There exists ε ∈ (0,1) such that for any prime divisor F over X, the inequality β(F) ≥ ε holds.
 - (iii) There exists $\varepsilon \in (0,1)$ such that for any dreamy prime divisor F over (X, Δ) , the inequality $\hat{\beta}(F) \geq \varepsilon$ holds.
 - (iv) There exists $\varepsilon \in (0,1)$ such that for any plt-type prime divisor F over (X, Δ) , the inequality $\hat{\beta}(F) \geq \varepsilon$ holds.
- **Remark 1.6.** (1) In Theorem 1.4 (2), we need to evaluate j(F) in order to test uniform K-stability. It seems relatively difficult to evaluate j(F) than to evaluate $\hat{\beta}(F)$ since the value $\tau(F)$ is not easy to treat. It is one of the remarkable point that we do not need to evaluate j(F) in Theorem 1.5 (2).
 - (2) Theorem 1.5 claims that we can check uniform K-stability and K-semistability by evaluating $\hat{\beta}(F)$ for plt-type prime divisors F over (X, Δ) . The theory of plt blowups is important for the theory of minimal model program and singularity theory (see [Pro00, Pro01]). It is interesting that such theories will relate

K-stability via Theorem 1.5. Moreover, Theorem 1.5 seems to relate with [LX16, Conjecture 6.5].

As an easy consequence of Theorem 1.5, we get the following result. The proof is given in Section 4.1. We remark that Dervan also treated similar problem. See [Der16].

Corollary 1.7 (see also Example 4.2). Let (X, Δ) and (X', Δ') be log Fano pairs. Assume that there exists a finite and surjective morphism $\phi: X' \to X$ such that $\phi^*(K_X + \Delta) = K_{X'} + \Delta'$. If (X', Δ') is uniformly K-stable (resp., K-semistable), then so is (X, Δ) .

We can also show as an application of Theorem 1.5 that the projective plane is K-semistable. The result is well-known (see [Don02]). Moreover, the result has been already proved purely algebraically (see [Kem78, RT07] and [Li16, Blu16, PW16]). However, it is worth writing the proof since our proof is purely birational geometric. The proof is given in Section 4.2.

Corollary 1.8 (see also [Kem78, Don02, Li16, Blu16, PW16]). The projective plane \mathbb{P}^2 is (that is, the pair (\mathbb{P}^2 , 0) is) K-semistable.

This paper is organized as follows. In Section 2, we see the equivalence between the conditions in Theorem 1.5 (2i), Theorem 1.5 (2ii), and Theorem 1.5 (2iii). For the proof, we use the log-convexity of volume functions and restricted volume functions. In Section 3, we see how to replace a primitive divisor by a plt-type prime divisor with smaller $\hat{\beta}$ -invariant. For the proof, we use techniques of minimal model program. Theorem 1.5 follows from those observations. In Section 4, we prove Corollaries 1.7 and 1.8.

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2. UNIFORM K-STABILITY

In this section, we simplify the conditions in Theorem 1.4 (2). In this section, we always assume that (X, Δ) is a log Fano pair of dimension $n, L := -(K_X + \Delta)$, and F is a prime divisor over X.

The proof of the following proposition is essentially same as the proofs of [FO16, Theorem 4.2] and [Fjt16c, Proposition 3.2].

Proposition 2.1. We have the inequality

$$\frac{n}{n+1}\tau(F) \ge \frac{1}{(L^{\cdot n})} \int_0^\infty \operatorname{vol}_X(L-xF) dx.$$

Proof. Take any log resolution $\sigma: Y \to X$ of (X, Δ) such that $F \subset Y$ and there exists a σ -ample Q-divisor A_Y on Y with $\gamma := -\operatorname{ord}_F A_Y > 0$ and $-A_Y$ effective. Then, for any $0 < \varepsilon \ll 1$, $\sigma^*L + (\varepsilon/\gamma)A_Y$ is ample. Hence $\mathbf{B}_+(\sigma^*L + (\varepsilon/\gamma)A_Y) = \emptyset$, where \mathbf{B}_+ is the augmented base locus (see [ELMNP09]). Note that

$$\mathbf{B}_{+}(\sigma^{*}L - \varepsilon F) \subset \mathbf{B}_{+}(\sigma^{*}L + (\varepsilon/\gamma)A_{Y}) \cup \operatorname{Supp}(-(\varepsilon/\gamma)A_{Y} - \varepsilon F).$$

This implies that $F \not\subset \mathbf{B}_+(\sigma^*L - \varepsilon F)$. Thus, by [ELMNP09, Theorem A] (and by [BFJ09, Theorem A and Corollary C]), the restricted volume $\operatorname{vol}_{Y|F}(\sigma^*L - xF)$ on $x \in [0, \tau(F))$ satisfies the log-concavity (in the sense of [ELMNP09, Theorem A]). In particular, for an arbitrary $x_0 \in (0, \tau(F))$, we have

$$\begin{cases} \operatorname{vol}_{Y|F}(\sigma^*L - xF) \ge (x/x_0)^{n-1} \cdot \operatorname{vol}_{Y|F}(\sigma^* - x_0F) & \text{if } x \in [0, x_0], \\ \operatorname{vol}_{Y|F}(\sigma^*L - xF) \le (x/x_0)^{n-1} \cdot \operatorname{vol}_{Y|F}(\sigma^* - x_0F) & \text{if } x \in [x_0, \tau(F)). \end{cases}$$

On the other hand, by [LM09, Corollary 4.27], for an arbitrary $x \in [0, \tau(F)]$, we have the equality

$$\operatorname{vol}_Y(\sigma^*L - xF) = n \int_x^{\tau(F)} \operatorname{vol}_{Y|F}(\sigma^*L - yF) dy.$$

Let us set

$$b := \frac{\int_0^{\tau(F)} y \cdot \operatorname{vol}_{Y|F}(\sigma^*L - yF) dy}{\int_0^{\tau(F)} \operatorname{vol}_{Y|F}(\sigma^*L - yF) dy}.$$

Obviously, $b \in (0, \tau(F))$ holds. Moreover, we get

$$0 = \int_{-b}^{\tau(F)-b} y \cdot \operatorname{vol}_{Y|F}(\sigma^*L - (y+b)F) dy$$

$$\leq \int_{-b}^{\tau(F)-b} y \cdot \left(\frac{y+b}{b}\right)^{n-1} \cdot \operatorname{vol}_{Y|F}(\sigma^*L - bF) dy$$

$$= \frac{\operatorname{vol}_{Y|F}(\sigma^*L - bF) \cdot \tau(F)^n}{n \cdot b^{n-1}} \left(\frac{n}{n+1}\tau(F) - b\right).$$

Thus $b \leq (n/(n+1))\tau(F)$. On the other hand, we have

$$b = \frac{n \int_0^{\tau(F)} \int_x^{\tau(F)} \operatorname{vol}_{Y|F}(\sigma^*L - yF) dy dx}{(L^{\cdot n})}$$
$$= \frac{\int_0^{\tau(F)} \operatorname{vol}_Y(\sigma^*L - xF) dx}{(L^{\cdot n})}.$$

Thus we have proved Proposition 2.1.

The following lemma is nothing but a logarithmic version of [FO16, Lemma 2.2]. We give a proof just for the readers' convenience.

Lemma 2.2 ([FO16, Lemma 2.2]). We have the inequality

$$\frac{\tau(F)}{n+1} \le \frac{1}{(L^{\cdot n})} \int_0^\infty \operatorname{vol}_X(L-xF) dx.$$

Proof. By [LM09, Corollary 4.12], we have

$$\operatorname{vol}_X(L - xF) \ge \left(1 - \frac{x}{\tau(F)}\right)^n \cdot (L^{\cdot n}).$$

Lemma 2.2 follows immediately from the above.

Now we are ready to prove the following theorem.

Theorem 2.3. Let (X, Δ) be a log Fano pair. Then the following are equivalent:

- (i) (X, Δ) is uniformly K-stable.
- (ii) There exists ε ∈ (0,1) such that for any prime divisor F over X, the inequality β(F) ≥ ε holds.
- (iii) There exists $\varepsilon \in (0,1)$ such that for any dreamy prime divisor F over (X, Δ) , the inequality $\hat{\beta}(F) \geq \varepsilon$ holds.

Proof. Let F be an arbitrary prime divisor over X. Firstly, we observe that the condition $\beta(F) \geq \delta \cdot j(F)$ for some $\delta \in (0, 1)$ is equivalent to the condition

(1)
$$(1+\delta')A(F) - \delta'\tau(F) \ge \frac{1}{(L^{\cdot n})} \int_0^\infty \operatorname{vol}_X(L-xF) dx,$$

where $\delta' := \delta/(1-\delta) \in (0,\infty)$.

We also observe that the condition $\hat{\beta}(F) \geq \varepsilon$ for some $\varepsilon \in (0, 1)$ is equivalent to the condition

(2)
$$A(F) \ge \frac{1+\varepsilon'}{(L^{\cdot n})} \int_0^\infty \operatorname{vol}_X(L-xF) dx,$$

where $\varepsilon' := \varepsilon/(1-\varepsilon) \in (0,\infty)$.

Claim 2.4 (see [FO16, Theorem 2.3]). If the inequality (2) holds for some $\varepsilon' \in (0, \infty)$, then the inequality (1) holds for $\delta' = \varepsilon'/(n+1)$.

Proof of Claim 2.4. By Lemma 2.2, the inequality (2) implies that

$$A(F) \ge \frac{1}{(L^{\cdot n})} \int_0^\infty \operatorname{vol}_X(\sigma^*L - xF) dx + \frac{\varepsilon'}{n+1} \tau(F).$$

Thus the inequality (1) holds for $\delta' = \varepsilon'/(n+1)$ since A(F) > 0. \Box

Claim 2.5. If the inequality (1) holds for some $\delta' \in (0, \infty)$, then the inequality (2) holds for

$$\varepsilon' = \min\left\{\frac{\delta'\frac{1-\theta}{\theta}}{1-\delta'\frac{1-\theta}{\theta}}, \frac{1}{2n+1}\right\} \in (0,1),$$

where

$$\theta := \max\left\{\frac{2n}{2n+1}, \frac{2\delta'}{2\delta'+1}\right\} \in (0,1).$$

Proof of Claim 2.5. We firstly assume that case $A(F) < \theta \cdot \tau(F)$. Then the inequality (1) implies that

$$\frac{1}{(L^{\cdot n})} \int_0^\infty \operatorname{vol}_X(L - xF) dx < \left(1 - \delta' \frac{1 - \theta}{\theta}\right) A(F).$$

Note that

$$\delta' \frac{1-\theta}{\theta} \in (0, 1/2] \subset (0, 1).$$

Thus the inequality (2) holds for such ε' .

We secondly consider the remaining case $A(F) \ge \theta \cdot \tau(F)$. In this case, by Proposition 2.1, we have

$$\frac{1}{(L^{\cdot n})} \int_0^\infty \operatorname{vol}_X(L - xF) dx \le \frac{n}{n+1} \tau(F) \le \frac{n}{n+1} \frac{1}{\theta} A(F).$$

Note that

$$\frac{n+1}{n}\theta - 1 \ge \frac{1}{2n+1}.$$

Thus the inequality (2) holds for such ε' .

Theorem 2.3 immediately follows from Theorem 1.4 (2), Claims 2.4 and 2.5. $\hfill \Box$

Remark 2.6. Let (X, Δ) be a log Fano pair and F be a prime divisor over X. If $\tau(F) \leq A(F)$, then $\hat{\beta}(F) \geq 1/(n+1)$ by Proposition 2.1. Thus it is enough to consider prime divisors F over X with $\tau(F) > A(F)$ in order to check the conditions in Theorem 1.5.

3. Plt blowups

The following theorem is inspired by [Xu14, Lemma 1].

Theorem 3.1. Let (X, Δ) be a quasi-projective klt pair with Δ effective \mathbb{R} -divisor. Let F be a primitive prime divisor over X and $\sigma: (Y, \Delta_Y + F) \rightarrow (X, \Delta)$ be the associated prime blowup. Assume that F is not plt-type (resp., not lc-type) over (X, Δ) . Then there exists a plt-type prime divisor G over (X, Δ) such that the inequality $A_{(Y,\Delta_Y+F)}(G) \leq 0$ (resp., $A_{(Y,\Delta_Y+F)}(G) < 0$) holds.

Proof. Let $\pi: V \to Y$ be a log resolution of $(Y, \Delta_Y + F)$ and let E_1, \ldots, E_k be the set of π -exceptional divisors on V. We set $F_V := \pi_*^{-1}F$ and $\Delta_V := \pi_*^{-1}\Delta_Y$. We may assume that there exists a $(\sigma \circ \pi)$ -ample \mathbb{Q} -divisor

$$A_V = -\sum_{i=1}^k h_i E_i - h_F F_V$$

on V with $h_1, \ldots, h_k \in \mathbb{Q}_{>0}$ and

 $\begin{cases} h_F \in \mathbb{Q}_{>0} & \text{if } F \text{ is exceptional over } X, \\ h_F = 0 & \text{otherwise.} \end{cases}$

Take a sufficiently ample Cartier divisor L on X such that $\sigma^*L - F$ is ample. Let L_Y be a general effective \mathbb{Q} -divisor with small coefficients such that $L_Y \sim_{\mathbb{Q}} \sigma^*L - F$. Set $L_V := \pi_*^{-1}L_Y$. Then $L_V = \pi^*L_Y$ and the pair $(V, \Delta_V + F_V + \sum_{i=1}^k E_i + L_V)$ is log smooth by generality. Moreover, the \mathbb{Q} -divisor $\sigma_*(L_Y + F)$ is \mathbb{Q} -Cartier with $\sigma_*(L_Y + F) \sim_{\mathbb{Q}} L$ and $\sigma^*\sigma_*(L_Y + F) = L_Y + F$.

Let us set

$$\pi^* F =: F_V + \sum_{i=1}^k c_i E_i \quad (c_i \in \mathbb{Q}_{\geq 0}),$$

$$a_F := A_{(X,\Delta)}(F) \in \mathbb{R}_{>0},$$

$$a_i := A_{(X,\Delta)}(E_i) \in \mathbb{R}_{>0},$$

$$b_i := A_{(Y,\Delta_Y+F)}(E_i) \in \mathbb{R}.$$

Of course, we have the inequality $h_i > h_F c_i$ (from the negativity lemma) and the equality $a_i = b_i + a_F c_i$ for any $1 \le i \le k$. By assumption, the inequality $b_i \le 0$ (resp., $b_i < 0$) holds for some $1 \le i \le k$. In particular, the inequality $c_i > 0$ holds for some $1 \le i \le k$. By changing E_1, \ldots, E_k and by perturbing the coefficients of A_V if necessary, we can assume that the following conditions are satisfied:

- There exists $1 \leq l \leq k$ such that $c_i > 0$ holds if and only if $1 \leq i \leq l$.
- $b_1/c_1 = \min_{1 \le i \le l} \{b_i/c_i\}.$
- The inequality $h_j/c_j < h_1/c_1$ holds for any $2 \le j \le l$ with $b_1/c_1 = b_j/c_j$.

Take a rational number $0 < \varepsilon \ll 1$ and set $t := (a_1 - \varepsilon h_1)/c_1 \in \mathbb{R}_{>0}$. Since ε is very small, we get the following properties:

- $a_1 (tc_1 + \varepsilon h_1) = 0$,
- $a_i (tc_i + \varepsilon h_i) > 0$ holds for any $2 \le i \le k$,
- $1 a_F + t \in (1 a_F, 1)$ and $1 a_F + t + \varepsilon h_F < 1$, and
- $\pi^*(L_Y + F) + \varepsilon A_V$ is ample on V.

Take a general effective \mathbb{R} -divisor L' with small coefficients such that $L' \sim_{\mathbb{R}} \pi^*(L_Y + F) + \varepsilon A_V$. Moreover, we set

$$b_F := \begin{cases} 1 & \text{if } F \text{ is exceptional over } X, \\ 1 - a_F + t & \text{otherwise.} \end{cases}$$

Then

$$K_V + \Delta_V + b_F F_V + tL_V + \sum_{i=1}^k E_i + L'$$

is dlt. (We remark that $1 - a_F + t \ge t > 0$ holds if F is a divisor on X. We also remark that the coefficients of tL_V can be less than 1 by the definition of L_V .) Moreover, we have

$$K_{V} + \Delta_{V} + b_{F}F_{V} + tL_{V} + \sum_{i=1}^{k} E_{i} + L'$$

$$\sim_{\mathbb{R}} (\sigma \circ \pi)^{*}(K_{X} + \Delta) + t\pi^{*}(L_{Y} + F) + (a_{F} + b_{F} - 1)F_{V} - t\pi^{*}F$$

$$+ \sum_{i=1}^{k} a_{i}E_{i} + \pi^{*}(L_{Y} + F) + \varepsilon A_{V}$$

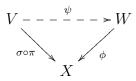
$$\sim_{\mathbb{R},X} (b_{F} - (1 - a_{F} + t) - \varepsilon h_{F})F_{V} + \sum_{i=2}^{k} (a_{i} - (tc_{i} + \varepsilon h_{i}))E_{i}.$$

The right-hand side is effective and its support is equal to the union of $(\sigma \circ \pi)$ -exceptional prime divisors other than E_1 . Furthermore,

$$K_V + \Delta_V + b_F F_V + tL_V + \sum_{i=1}^k E_i + L'$$

$$\sim_{\mathbb{R},X} \quad K_V + \Delta_V + b_F F_V + tL_V + \sum_{i=1}^k E_i + (1-\delta)L' + \delta\varepsilon A_V$$

is klt for $0 < \delta \ll 1$. Thus, by [BCHM10, Corollary 1.4.2], we can run and terminate a $(K_V + \Delta_V + b_F F_V + tL_V + \sum_{i=1}^k E_i + L')$ -MMP with scaling L' over X. Let



be the output of this MMP. The MMP does not contract E_1 . Let $G_W \subset W$ be the image of E_1 . Moreover, by the negativity lemma, any $(\sigma \circ \pi)$ -exceptional prime divisor other than E_1 is contracted by ψ . In particular, we get

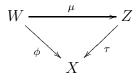
$$\psi_*(K_V + \Delta_V + b_F F_V + tL_V + \sum_{i=1}^k E_i + L') \sim_{\mathbb{R},X} 0.$$

Furthermore, by the definition of MMP with scaling, the \mathbb{R} -divisor

$$\psi_*(K_V + \Delta_V + b_F F_V + tL_V + \sum_{i=1}^k E_i + (1+\lambda)L')$$

$$\sim_{\mathbb{R},X} \lambda \psi_* L' \sim_{\mathbb{R},X} -\lambda \varepsilon h_1 G_W$$

is nef over X for any $0 < \lambda \ll 1$. Moreover, by the base point free theorem, the above \mathbb{R} -divisor admits the ample model over X. Let



be the model and we set $G := \mu_* G_W$. Since -G is τ -ample, the morphism μ is a small morphism. We remark that

$$\psi_*(K_V + \Delta_V + b_F F_V + tL_V + \sum_{i=1}^k E_i + L')$$

= $K_W + \phi_*^{-1} \Delta + t\psi_*(F_V + L_V) + G_W + \psi_*L'$

is dlt, \mathbb{R} -linearly equivalent to zero over X, and G_W is the unique prime divisor whose coefficient is equal to one. Thus this is plt and

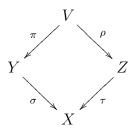
$$K_{Z} + \tau_{*}^{-1}\Delta + t(\mu \circ \psi)_{*}(F_{V} + L_{V}) + G + (\mu \circ \psi)_{*}L'$$

is also plt. Note that $(\mu \circ \psi)_* L'$ is effective \mathbb{R} -Cartier. Moreover, since $\tau_*(\mu \circ \psi)_*(F_V + L_V)$ is \mathbb{R} -Cartier and $\tau^* \tau_*(\mu \circ \psi)_*(F_V + L_V) - (\mu \circ \psi)_*(F_V + L_V)$ is equal to some multiple of G, the \mathbb{R} -divisor $(\mu \circ \psi)_*(F_V + L_V)$ is also effective \mathbb{R} -Cartier. This implies that the pair $(Z, \tau_*^{-1}\Delta + G)$

is also plt. By construction, -G is τ -ample, G is exceptional over X (since E_1 is exceptional over Y), and $A_{(Y,\Delta_Y+F)}(G) = b_1 \leq 0$ (resp., < 0).

Corollary 3.2. Let (X, Δ) be a log Fano pair and F be a primitive prime divisor over X with the associated prime blowup $\sigma: (Y, \Delta_Y + F) \rightarrow (X, \Delta)$. If F is not plt-type over (X, Δ) , then there exists a plt-type prime divisor G over (X, Δ) with $A_{(Y, \Delta_Y + F)}(G) \leq 0$ such that the inequality $\hat{\beta}(F) > \hat{\beta}(G)$ holds.

Proof. We set $n := \dim X$ and $L := -(K_X + \Delta)$. By Theorem 3.1, there exists a plt-type prime divisor G over (X, Δ) with $A_{(Y, \Delta_Y + F)}(G) \leq 0$. Let $\tau : (Z, \Delta_Z + G) \to (X, \Delta)$ be the associated plt blowup. Let



be a common log resolution of $(Y, \Delta_Y + F)$ and $(Z, \Delta_Z + G)$. Since (X, Δ) is klt and $A_{(Y,\Delta_Y+F)}(G) \leq 0$, we have $c_Y(G) \subset F$. Set $f := \operatorname{ord}_{\pi_*^{-1}F}(\rho^*G)$ and $g := \operatorname{ord}_{\rho_*^{-1}G}(\pi^*F)$. Since $c_Y(G) \subset F$, we have the inequality g > 0. Moreover, we have the following equalities:

$$A_{(X,\Delta)}(F) = A_{(Z,\Delta_Z+G)}(F) + f \cdot A_{(X,\Delta)}(G),$$

$$A_{(X,\Delta)}(G) = A_{(Y,\Delta_Y+F)}(G) + g \cdot A_{(X,\Delta)}(F).$$

Claim 3.3. (1) For any $x \in \mathbb{R}_{\geq 0}$, we have the inequality $\operatorname{vol}_X(L - xF) \leq \operatorname{vol}_X(L - gxG)$.

(2) If $A_{(Y,\Delta_Y+F)}(G) = 0$, then, for any $0 < x \ll 1$, we have the inequality $\operatorname{vol}_X(L - xF) < \operatorname{vol}_X(L - gxG)$.

Proof of Claim 3.3. The assertion (1) is trivial since we know that

$$H^0(X, kL - jF) \subset H^0(X, kL - gjG)$$

for any sufficiently divisible $k, j \in \mathbb{Z}_{>0}$. We see the assertion (2). We assume that $A_{(Y,\Delta_Y+F)}(G) = 0$. Then we have

$$A_{(X,\Delta)}(G) = gA_{(X,\Delta)}(F)$$

= $gA_{(Z,\Delta_Z+G)}(F) + fgA_{(X,\Delta)}(G) > fgA_{(X,\Delta)}(G).$

This implies the inequality 1 > fg.

Fix any $0 < x \ll 1$ such that both $\sigma^*L - xF$ and $\tau^*L - gxG$ are ample. Note that $\operatorname{vol}_X(L-xF) = ((\sigma^*L-xF)^{\cdot n})$ and $\operatorname{vol}_X(L-gxG) = ((\tau^*L - gxG)^{\cdot n})$. Since

$$-x(\pi^*F - g\rho^*G) = \pi^*(\sigma^*L - xF) - \rho^*(\tau^*L - gxG)$$

is ρ -nef and

$$\rho_*(\pi^*F - g\rho^*G) = (1 - fg)\rho_*\pi_*^{-1}F \ge 0,$$

we have $\pi^*F \ge g\rho^*G$ by the negativity lemma. Thus, for any $0 \le i \le n-1$, we have

$$0 \leq (\pi^{*}(\sigma^{*}L - xF)^{\cdot i} \cdot \rho^{*}(\tau^{*}L - gxG)^{\cdot n - 1 - i} \cdot \pi^{*}(xF) - \rho^{*}(gxG))$$

= $(\pi^{*}(\sigma^{*}L - xF)^{\cdot i} \cdot \rho^{*}(\tau^{*}L - gxG)^{\cdot n - i})$
 $- (\pi^{*}(\sigma^{*}L - xF)^{\cdot i + 1} \cdot \rho^{*}(\tau^{*}L - gxG)^{\cdot n - 1 - i}).$

Moreover, we have

$$\left(\pi^* (\sigma^* L - xF)^{\cdot n-1} \cdot \rho^* (\tau^* L - gxG) \right) - (\pi^* (\sigma^* L - xF)^{\cdot n})$$

= $x(1 - fg) \left((\sigma^* L - xF)^{\cdot n-1} \cdot F \right) > 0.$

Therefore, we have

$$\begin{aligned} & ((\tau^*L - gxG)^{\cdot n}) - ((\sigma^*L - xF)^{\cdot n}) \\ &= \sum_{i=0}^{n-1} \bigg(\left(\pi^* (\sigma^*L - xF)^{\cdot i} \cdot \rho^* (\tau^*L - gxG)^{\cdot n-i} \right) \\ &- \left(\pi^* (\sigma^*L - xF)^{\cdot i+1} \cdot \rho^* (\tau^*L - gxG)^{\cdot n-1-i} \right) \bigg) > 0. \end{aligned}$$

Thus we get Claim 3.3.

From Claim 3.3, we get the inequalities

$$\begin{split} \hat{\beta}(F) &= 1 - \frac{\int_0^\infty \operatorname{vol}_X(L - xF) dx}{A_{(X,\Delta)}(F) \cdot (L^{\cdot n})} \\ &\geq 1 - \frac{\int_0^\infty \operatorname{vol}_X(L - gxG) dx}{A_{(X,\Delta)}(F) \cdot (L^{\cdot n})} = 1 - \frac{\int_0^\infty \operatorname{vol}_X(L - xG) dx}{gA_{(X,\Delta)}(F) \cdot (L^{\cdot n})} \\ &\geq 1 - \frac{\int_0^\infty \operatorname{vol}_X(L - xG) dx}{A_{(X,\Delta)}(G) \cdot (L^{\cdot n})} = \hat{\beta}(G). \end{split}$$

Moreover, at least one of the inequalities is the strict inequality. $\hfill \Box$

Proof of Theorem 1.5. This follows immediately from Remark 1.3 (1), Theorems 1.4, 2.3 and Corollary 3.2. $\hfill \Box$

4. Applications

In this section, we give several applications of Theorem 1.5.

4.1. Finite covers. In this section, we prove Corollary 1.7. To begin with, we show the following easy lemma.

Lemma 4.1. Let $\psi: W \to V$ be a generically finite and surjective morphism between normal projective varieties. For any Cartier divisor D on V, we have the following inequality:

$$\operatorname{vol}_W(\psi^*D) \ge (\deg \psi) \cdot \operatorname{vol}_V(D).$$

Proof. We may assume that D is big. By [Fuj94, Theorem], for any $\varepsilon > 0$, there exists a projective birational morphism $\sigma: V' \to V$ with V' normal, ample \mathbb{Q} -divisor A, and an effective \mathbb{Q} -divisor E such that $\sigma^*D \sim_{\mathbb{Q}} A + E$ and $\operatorname{vol}_V(D) \leq \operatorname{vol}_{V'}(A) + \varepsilon$ hold. Let

$$\begin{array}{ccc} W' & \stackrel{\psi'}{\longrightarrow} & V' \\ \sigma' \downarrow & & \downarrow \sigma \\ W & \stackrel{\psi}{\longrightarrow} & V \end{array}$$

be the normalization of the fiber product. Then we get

$$\operatorname{vol}_{W}(\psi^{*}D) = \operatorname{vol}_{W'}(\psi^{'*}\sigma^{*}D) \ge \operatorname{vol}_{W'}(\psi^{'*}A)$$
$$= (\deg \psi) \cdot \operatorname{vol}_{V'}(A) \ge (\deg \psi) \cdot (\operatorname{vol}_{V}(D) - \varepsilon).$$

The assertion immediately follows from the above inequalities.

Proof of Corollary 1.7. We set $n := \dim X$, $L := -(K_X + \Delta)$, $L' := -(K_{X'} + \Delta')$ and $d := \deg \phi$. From Theorem 1.5, there exists $\varepsilon > 0$ (resp., ≥ 0) such that $\hat{\beta}(F') \geq \varepsilon$ holds for any prime divisor F' over X'. Take any plt blowup $\sigma : (Y, \Delta_Y + F) \to (X, \Delta)$. From Theorem 1.5, it is enough to show the inequality $\hat{\beta}(F) \geq \varepsilon$. Let

$$\begin{array}{cccc} Y' & \stackrel{\psi}{\longrightarrow} & Y \\ & \sigma' & & & \downarrow \sigma \\ & X' & \stackrel{\phi}{\longrightarrow} & X \end{array}$$

be the normalization of the fiber product. (Note that the morphism ψ is a finite morphism.) Let

$$\psi^*F = \sum_{i=1}^m r_i F_i$$

be the irreducible decomposition of the pullback of F (see [KM98, Proposition 5.20]), where $r_i \in \mathbb{Z}_{>0}$. By [KM98, Proposition 5.20], we have the equality

$$A_{(X',\Delta')}(F'_i) = r_i A_{(X,\Delta)}(F)$$

for any $1 \leq i \leq m$. Moreover, for any $x \in \mathbb{R}_{>0}$, we have

$$\psi^*(\sigma^*L - xF) = \sigma'^*L' - x\sum_{i=1}^m r_i F_i' \le \sigma'^*L' - xr_1F_1'.$$

Therefore, from Lemma 4.1, we have the following inequalities:

$$\begin{split} 1-\varepsilon &\geq 1-\hat{\beta}(F_1') = \frac{\int_0^\infty \operatorname{vol}_{Y'}(\sigma'^*L' - yF_1')dy}{A_{(X',\Delta')}(F_1') \cdot (L'^n)} \\ &= \frac{\int_0^\infty \operatorname{vol}_{Y'}(\sigma'^*L' - xr_1F_1')dx}{A_{(X,\Delta)}(F) \cdot d(L^n)} \\ &\geq \frac{\int_0^\infty \operatorname{vol}_{Y'}(\psi^*(\sigma^*L - xF))dx}{A_{(X,\Delta)}(F) \cdot d(L^n)} \\ &\geq \frac{\int_0^\infty \operatorname{vol}_Y(\sigma^*L - xF)dx}{A_{(X,\Delta)}(F) \cdot (L^n)} = 1 - \hat{\beta}(F). \end{split}$$

As a consequence, we have proved Corollary 1.7.

We remark that the converse of Corollary 1.7 is not true in general. See the following example.

Example 4.2. Let $X := \mathbb{P}^1$, $X' := \mathbb{P}^1$ and let us consider the morphism $\phi \colon X' \to X$ with $t \mapsto t^2$. Take any $d \in (0, 1) \cap \mathbb{Q}$ and set

$$\begin{cases} \Delta := \frac{1}{2}[0] + \frac{1}{2}[\infty] + d[1] & \text{on } X, \\ \Delta' := d[1] + d[-1] & \text{on } X'. \end{cases}$$

Then we know that (X, Δ) and (X', Δ') are log Fano pairs and the equality $\phi^*(K_X + \Delta) = K_{X'} + \Delta'$ holds from the ramification formula. By [Fjt16b, Example 6.6], (X, Δ) is uniformly K-stable. However, again by [Fjt16b, Example 6.6], (X', Δ') is not uniformly K-stable (but K-semistable).

4.2. K-semistability of the projective plane. In this section, we show Corollary 1.8. Take any plt blowup $\sigma: (Y, F) \to (\mathbb{P}^2, 0)$. It is enough to show the inequality $\hat{\beta}(F) \geq 0$ by Theorem 1.5. Assume that F is a divisor on \mathbb{P}^2 . Set $d := \deg_{\mathbb{P}^2} F$. Then

$$\operatorname{vol}_{\mathbb{P}^2}(-K_{\mathbb{P}^2} - xF) = (3 - dx)^2$$

for $x \in [0, 3/d]$ and A(F) = 1. Thus we get $\hat{\beta}(F) = (d-1)/d \ge 0$. (See also [Fjt16a, Corollary 9.3].)

From now on, we assume that F is an exceptional divisor over \mathbb{P}^2 . Set $\{p\} := c_{\mathbb{P}^2}(F)$. Of course, the Picard rank of Y is equal to two. We may assume that the divisor $-(K_Y + F)$ is big by Remark 2.6. By [Pro01, Proposition 6.2.6 and Remark 6.2.7], the morphism σ is a weighted blowup with weights a, b for some local parameters s, t of $\mathcal{O}_{\mathbb{P}^2,p}$, where $a, b \in \mathbb{Z}_{>0}, a \geq b$ and a, b are mutually prime. (Note that the variety Y is not a toric variety in general.) We know that $(F^{\cdot 2})_Y = -1/(ab)$ and A(F) = a + b.

Let $\pi: \tilde{Y} \to Y$ be the minimal resolution of Y and let $E_1 \subset \tilde{Y}$ be the strict transform of the exceptional divisor of the ordinary blowup of $p \in \mathbb{P}^2$. Then we can check that $\operatorname{ord}_{E_1} \pi^* F = 1/a$. Let $\hat{l} \subset \tilde{Y}$ be the strict transform of a general line on \mathbb{P}^2 passing though $p \in \mathbb{P}^2$. Since \hat{l} is movable, we have

$$0 \le \left(\pi^* (\sigma^* (-K_{\mathbb{P}^2} - \tau(F)F)) \cdot \hat{l}\right) = 3 - \tau(F) \cdot \frac{1}{a}$$

This implies that $\tau(F) \leq 3a$. We can write $K_{\tilde{Y}} = \pi^* K_Y - E$ for some effective and π -exceptional \mathbb{Q} -divisor E on \tilde{Y} (see [KM98, Corollary 4.3]). Thus $-K_{\tilde{Y}}$ is big. This implies that \tilde{Y} and Y are Mori dream spaces in the sense of [HK00] by [TVAV11, Theorem 1]. In particular, F is dreamy over ($\mathbb{P}^2, 0$) (see [ELMNP06, Lemma 4.8]).

Let us set

$$\varepsilon(F) := \max\{\varepsilon \in \mathbb{R}_{\geq 0} \,|\, \sigma^*(-K_{\mathbb{P}^2}) - \varepsilon F \text{ nef}\}.$$

Then $\varepsilon(F) \in (0, \tau(F)].$

Claim 4.3. (1) We have the equality $\varepsilon(F)\tau(F) = 9ab$. (2) We get

$$\operatorname{vol}_{\mathbb{P}^2}(-K_{\mathbb{P}^2} - xF) = \begin{cases} 9\left(1 - \frac{x^2}{\varepsilon(F)\tau(F)}\right) & \text{if } x \in [0, \varepsilon(F)], \\ 9\frac{(\tau(F) - x)^2}{\tau(F)(\tau(F) - \varepsilon(F))} & \text{if } x \in (\varepsilon(F), \tau(F)]. \end{cases}$$

(3) We have the equality

$$\hat{\beta}(F) = 1 - \frac{\varepsilon(F) + \tau(F)}{3(a+b)}$$

Proof of Claim 4.3. The assertion (3) follows from (1) and (2). We prove (1) and (2).

We know that

$$\operatorname{vol}_{\mathbb{P}^2}(-K_{\mathbb{P}^2} - xF) = \left((\sigma^*(-K_{\mathbb{P}^2}) - xF)^{\cdot 2} \right) = 9 - \frac{x^2}{ab}$$

for $x \in [0, \varepsilon(F)]$. If $\varepsilon(F) = \tau(F)$, then $\operatorname{vol}_{\mathbb{P}^2}(-K_{\mathbb{P}^2} - \varepsilon(F)F) = 0$. Thus $\varepsilon(F)^2 = 9ab$. Hence we can assume that $\varepsilon(F) < \tau(F)$. In this case, $\varepsilon(F) \in \mathbb{Q}$ and the divisor $\sigma^*(-K_{\mathbb{P}^2}) - \varepsilon(F)F$ gives a nontrivial birational contraction morphism $\mu: Y \to Z$ since F is dreamy over $(\mathbb{P}^2, 0)$ (see [Fjt16c, Lemma 3.1 (4)]). Moreover, since the Picard rank of Z is one, $\mu_*(\sigma^*(-K_{\mathbb{P}^2}))$ and μ_*F are numerically proportional. Thus, for $x \in [\varepsilon(F), \tau(F)]$, we can write

$$\operatorname{vol}_{\mathbb{P}^2}(-K_{\mathbb{P}^2} - xF) = \left(\mu_* \left(\sigma^*(-K_{\mathbb{P}^2}) - xF\right)^{\cdot 2}\right)_Z = c(\tau(F) - x)^2$$

for some $c \in \mathbb{R}_{>0}$. Note that

$$\operatorname{vol}_{\mathbb{P}^2}(-K_{\mathbb{P}^2} - \varepsilon(F)F) = 9 - \frac{\varepsilon(F)^2}{ab} = c(\tau(F) - \varepsilon(F))^2$$

and, by [BFJ09, Theorem A],

$$\frac{d}{dx}\Big|_{x=\varepsilon(F)}\operatorname{vol}_{\mathbb{P}^2}(-K_{\mathbb{P}^2}-xF) = -\frac{2\varepsilon(F)}{ab} = -2c(\tau(F)-\varepsilon(F)).$$

This implies that $ab = \varepsilon(F)\tau(F)/9$ and $c = 9/(\tau(F)(\tau(F)-\varepsilon(F)))$. \Box

Since $\varepsilon(F) \leq \tau(F) \leq 3a$ and $\varepsilon(F)\tau(F) = 9ab$, we have $\tau(F) \in [3\sqrt{ab}, 3a]$. Moreover,

$$\varepsilon(F) + \tau(F) = \tau(F) + \frac{9ab}{\tau(F)}$$

and the function x+9ab/x is monotonically increasing on $x \in [3\sqrt{ab}, 3a]$. Thus $\varepsilon(F) + \tau(F) \leq 9ab/(3a) + 3a = 3(a+b)$. This implies that

$$\hat{\beta}(F) = 1 - \frac{\varepsilon(F) + \tau(F)}{3(a+b)} \ge 0$$

As a consequence, we have completed the proof of Corollary 1.8.

References

- [BBJ15] R. Berman, S. Boucksom and M. Jonsson, A variational approach to the Yau-Tian-Donaldson conjecture, arXiv:1509.04561v1.
- [BCHM10] C. Birkar, P. Cascini, C. D. Hacon and J. M^cKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468.
- [BFJ09] S. Boucksom, C. Favre and M. Jonsson, Differentiability of volumes of divisors and a problem of Teissier, J. Algebraic Geom. 18 (2009), no. 2, 279– 308.
- [BHJ15] S. Boucksom, T. Hisamoto and M. Jonsson, Uniform Kstability, Duistermaat-Heckman measures and singularities of pairs, arXiv:1504.06568v3; to appear in Ann. Inst. Fourier.
- [Blu16] H. Blum, Existence of Valuations with Smallest Normalized Volume, arXiv:1606.08894v2.

- [Der15] R. Dervan, Uniform stability of twisted constant scalar curvature Kähler metrics, Int. Math. Res. Notices., DOI: 10.1093/imrn/rnv291.
- [Der16] R. Dervan, On K-stability of finite covers, Bull. Lond. Math. Soc. 48 (2016), no. 4, 717–728.
- [Don02] S. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), no. 2, 289–349.
- [ELMNP06] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 6, 1701–1734.
- [ELMNP09] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, Restricted volumes and base loci of linear series, Amer. J. Math. 131 (2009), no. 3, 607–651.
- [Fjt16a] K. Fujita, On K-stability and the volume functions of Q-Fano varieties, Proc. Lond. Math. Soc. 113 (2016), no. 5, 541–582.
- [Fjt16b] K. Fujita, A valuative criterion for uniform K-stability of Q-Fano varieties, J. Reine Angew. Math., DOI: 10.1515/crelle-2016-0055.
- [Fjt16c] K. Fujita, K-stability of Fano manifolds with not small alpha invariants, arXiv:1606.08261v1.
- [FO16] K. Fujita and Y. Odaka, On the K-stability of Fano varieties and anticanonical divisors, arXiv:1602.01305v2; accepted by Tohoku Math. J.
- [Fuj94] T. Fujita, Approximating Zariski decomposition of big line bundles, Kodai Math. J. 17 (1994), no. 1, 1–3.
- [HK00] Y. Hu and S. Keel, Mori dream spaces and GIT, Michigan Math. J. 48 (2000), 331–348.
- [Ish04] S. Ishii, Extremal functions and prime blow-ups, Comm. Algebra 32 (2004), no. 3, 819–827.
- [Kem78] G. Kempf, Instability in invariant theory, Ann. of Math. 108 (1978), no. 2, 299–316.
- [KM98] J. Kollár and S. Mori, Birational geometry of algebraic varieties, With the collaboration of C. H. Clemens and A. Corti. Cambridge Tracts in Math., 134, Cambridge University Press, Cambridge, 1998.
- [Laz04a] R. Lazarsfeld, Positivity in algebraic geometry, I: Classical setting: line bundles and linear series, Ergebnisse der Mathematik und ihrer Grenzgebiete.
 (3) 48, Springer, Berlin, 2004.
- [Laz04b] R. Lazarsfeld, Positivity in algebraic geometry, II: Positivity for Vector Bundles, and Multiplier Ideals, Ergebnisse der Mathematik und ihrer Grenzgebiete. (3) 49, Springer, Berlin, 2004.
- [Li16] C. Li, K-semistability is equivariant volume minimization, arXiv:1512.07205v4.
- [LM09] R. Lazarsfeld and M. Mustaţă, Convex bodies associated to linear series, Ann. Sci. Éc. Norm. Supér. 42 (2009), no. 5, 783–835.
- [LX16] C. Li and C. Xu, Stability of Valuations and Kollár Components, arXiv:1604.05398v3.
- [Pro00] Y. Prokhorov, Blow-ups of canonical singularities, Algebra (Moscow, 1998), 301–317, de Gruyter, Berlin, 2000.
- [Pro01] Y. Prokhorov, Lectures on complements on log surfaces, MSJ Memoirs, 10. Mathematical Society of Japan, Tokyo, 2001.

- [PW16] J. Park and J. Won, K-stability of smooth del Pezzo surfaces, arXiv:1608.06053v1.
- [RT07] J. Ross and R. Thomas, A study of the Hilbert-Mumford criterion for the stability of projective varieties, J. Algebraic Geom. 16 (2007), no. 2, 201–255.
- [Sho96] V. V. Shokurov, 3-fold log models, Algebraic geometry, 4. J. Math. Sci. 81 (1996), no. 3, 2667–2699.
- [Szé06] G. Székelyhidi, *Extremal metrics and K-stability*, Ph.D Thesis, arXiv:math/0611002.
- [Szé15] G. Székelyhidi, Filtrations and test-configurations, with an appendix by S. Boucksom, Math. Ann. 362 (2015), no. 1-2, 451–484.
- [Tia97] G. Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), no. 1, 1–37.
- [TVAV11] D. Testa, A. Várilly-Alvarado and M. Velasco, Big rational surfaces, Math. Ann. 351 (2011), no. 1, 95–107.
- [Xu14] C. Xu, Finiteness of algebraic fundamental groups, Compos. Math. 150 (2014), no. 3, 409–414.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: fujita@kurims.kyoto-u.ac.jp