ON MULTIVARIATE DISTRIBUTION THEORY[1]

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1. Summary. This paper is concerned with a matrix method of deriving the sampling distributions of a large class of statistics directly from the probability law for random samples from a multivariate normal population, that is without assuming the Wishart distribution or the distribution of rectangular coordinates. Two techniques are proposed for evaluating the Jacobians of certain transformations, one based on a theorem on Jacobians [1], and the second based on the introduction of pseudo or extra variables. This matrix approach has a geometrical analog developed in part by one of the authors [2]. Section 3 is concerned with a discussion of these two techniques; in Section 4, the former is applied to obtain the joint distribution of the rectangular coordinates [3], and in Section 5, the second method is applied to obtain the joint distribution of the roots of a determinantal equation [4], [5], [6], and [7].

2. Introduction. Much work on the sampling distributions connected with multivariate normal populations is based on the Wishart distribution as the starting point, from which analytical or geometrical arguments are applied to obtain the desired results. This presupposes that the Wishart distribution is available and that it was somehow derived from the probability law for the raw observations. When the Wishart distribution is unavailable, as in the case of a sample of N observations from a p-variate normal population with $p > N - 1$, other techniques must be applied.

When considering the roots of the determinantal equation $| XX' - \theta YY' | = 0$, where X and Y are p by n and p by m matrices, respectively, consisting of the observations from p-variate normal populations with $n < p < m$, a lemma may be employed [6] to the effect that the nonvanishing roots of $| UU' - \theta I_r | = 0$ are the roots of $| U'U - \theta I_s | = 0$, where U is an r by s matrix, $r > s$. However, by starting with the raw observations, the availability of the Wishart distribution need not be considered. A discussion of the above examples is given in [8] and [9]. The present approach is based on matrix algebra and is proposed as an alternative and unified procedure to obtain most multivariate distributions.

As the shape and properties of the matrices are of importance, the following notation is adopted.

(i) A matrix is denoted by a capital letter, and a column vector by an underlined lower case letter, for example $\underline{x}' = (x_1, \cdots, x_p)$.

(ii) $X:p \times n$ means that the matrix X has p rows and n columns.

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(iii) A triangular matrix with zeros above the main diagonal is denoted by a superior tilde, for example $\tilde{X}$.

(iv) A skew-symmetric matrix is denoted by a superior karat, for example $\hat{X}$.

(v) An orthogonal matrix is denoted by a capital Greek letter.

(vi) The absolute value of the Jacobian, $\partial(y_{11} \cdots y_{pn})/\partial(x_{11} \cdots x_{pn})$ is denoted by $J(Y; X)$.

3. General procedure. Let $X{:}p \times n$ be pn random variables following the multivariate normal probability law as defined by the density function

$$p(X) = (2\pi)^{-pn/2} A^{-n/2} \exp[-\tfrac{1}{2} \operatorname{tr}(A^{-1}XX')]$$

where $A{:}p \times p$ is the population covariance matrix, and the means have already been integrated out. If $X = g(Y)$ is a one-one transformation in the pn variates, the density function of Y is given by

$$p(Y) = p(g(Y))\, J(X; Y).$$

In most cases, we are only interested in the distribution of a subset Z of the pn variates, and we may achieve this by integrating out those variates which are in Y but not in Z. Thus the problem consists in (a) finding an appropriate transformation, (b) evaluating the Jacobian of the transformation, and (c) integrating out any extraneous variates. This paper is concerned with (b) and (c); the requisite transformations (a) are assumed to be available [8].

In the process of evaluating the Jacobian, a difficulty arises whenever the transformation involves an orthogonal matrix, and in particular when the orthogonality is with respect to rows alone, for example $\Gamma{:}p \times n$, $\Gamma\Gamma' = I_p$, $p \leqq n$. Such a matrix contains $pn - p(p+1)/2$ independent variates, and the usual procedure for evaluating the Jacobian requires the solution of $p(p-1)/2$ equations so that the dependent elements may be expressed in terms of the independent elements. This is troublesome even if $p = 3$. The two techniques proposed avoid this arduous task. The first makes use of the following theorem on Jacobians.

THEOREM 1. *If* $y_i = f_i(\dot{\underline{x}}, \ddot{\underline{x}})$, $(i = 1 \cdots m)$, $\dot{\underline{x}}' = (x_1 \cdots x_m)$, $\ddot{\underline{x}}' = (x_{m+1} \cdots x_{m+n})$, *where* $\dot{\underline{x}}$ *and* $\ddot{\underline{x}}$ *are subject to* n *constraints* $f_i(\dot{\underline{x}}, \ddot{\underline{x}}) = 0$, $(i = m+1, \cdots, m+n)$, then

$$J(y_1 \cdots y_m ; \dot{\underline{x}}) = J(f_1 \cdots f_{m+n} ; \dot{\underline{x}}, \ddot{\underline{x}})/J(f_{m+1} \cdots f_{m+n} ; \ddot{x}),$$

provided that the numerator and denominator exist and do not vanish [1].

The conditions $\Gamma\Gamma' - I_p = 0$ constitute the constraints of the theorem.

The second procedure involves the introduction of pseudo or extra variates as follows. Let $\Gamma_1 : p \times n$, $p \leqq n$, $\Gamma_1\Gamma_1' = I_p$, and write

$$\Gamma_1 = (I_p 0)\begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix} = (I_p 0)\Gamma,$$

where $0{:}p \times n - p$, and $\Gamma_2{:}n - p \times n$ is so chosen so that $\Gamma{:}n \times n$ is orthogonal. If $| I + \Gamma | \neq 0$, there is a one-one correspondence between Γ and a skew-symmetric matrix $\hat{S}$ given by $\hat{S} = (I + \Gamma)^{-1}(I - \Gamma)$ [10]. This $\hat{S}$ has the desirable property that all its elements are independent. The transformation may then directly involve $\hat{S}$ rather than Γ.

4. Distribution of rectangular coordinates. The existence of the desired transformation is given by the following theorem.

THEOREM 2. *If $X:p \times n$, $p \leqq n$, there exists an orthogonal matrix $\Gamma:p \times n$, and a triangular matrix $\tilde{T}:p \times p$ with $t_{ii} \geqq 0$, $(i = 1 \cdots p)$ such that*

$$X = \tilde{T}\Gamma. \tag{1}$$

If X is of rank p, then $t_{ii} > 0$ $(i = 1 \cdots p)$, and the representation is unique [8].

We note that X is of rank p with probability 1, and hence the representation is unique with probability 1. The elements of $\tilde{T}$ are the rectangular coordinates [3]. Before proceeding to the evaluation of the Jacobian, we will have occasion to use the following lemma.

LEMMA 3. *Let $\Gamma:p \times n$, $p \leqq n$, $\Gamma\Gamma' = I_p$, and denote the set of $pn - p(p+1)/2$ independent elements by Γ_I. If no γ_{ij} of Γ is zero, then for each Γ_I there are 2^p matrices Γ which can be formed.*

PROOF. Without any loss in generality we can let Γ_I consist of

$$\gamma_{ij}(i = 1 \cdots p; j = 1 \cdots n - i),$$

and the dependent set Γ_D consist of $\gamma_{ij}(i = 1 \cdots p; j = n - i + 1, \cdots, n)$. The matrix Γ has the form

$$\Gamma = \left\{\begin{array}{cccccccc} \gamma_{11} & \gamma_{12} & \cdots & \cdots & \cdots & \gamma_{1,n-1} & \gamma_{1,n-1} & | \; \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots & \cdots & \cdots & \gamma_{2,n-2} \; | & \gamma_{2,n-1} & \gamma_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{p1} & \gamma_{p2} & \cdots & \gamma_{p,n-p} \; | & \cdots & \gamma_{p,n-2} & \gamma_{p,n-1} & \gamma_{pn} \end{array}\right\},$$

where Γ_I consists of all elements to the left of the vertical lines. By the orthogonality conditions on the rows, γ_{in} can take two values $\pm(1 - \sum_{j=1}^{n-1}\gamma_{ij}^2)^{\frac{1}{2}}$ $(i = 1 \cdots p)$. Because the inner product of any pair of row vectors is zero, all the other elements in Γ_D are determined.

We now obtain the Jacobian of (1) using Theorem 1, where $X = \tilde{T}\Gamma$ and $\Gamma\Gamma' - I_p = 0$ take the place of $y_i = f_i$ and $f_i = 0$, respectively. Thus

$$J(X; \tilde{T}, \Gamma_I) = J(X, \Gamma\Gamma'; \tilde{T}, \Gamma)/J(\Gamma\Gamma'; \Gamma_D), \tag{2}$$

where the right-hand side is expressed in terms of $\tilde{T}$ and Γ_I. Let $\Gamma\Gamma' = K:p \times p$, where K is symmetric. We note that K and $\tilde{T}$ are unrelated, and hence the scheme of partial derivatives for the numerator of (2) is

$$\begin{array}{r|cc} & \tilde{T}(p(p+1)/2) & \Gamma(pn) \\ \hline K(p(p+1)/2) & 0 & M_{12} \\ X(pn) & M_{21} & M_{22} \end{array}, \tag{3}$$

the determinant of which is $|M_{22}|\,|M_{12}M_{22}^{-1}M_{21}|$. Write $X = (\underline{x}_1 \cdots \underline{x}_p)'$, where $\underline{x}_i' = (x_{i1} \cdots x_{in})$ and $\Gamma = (\underline{\gamma}_1 \cdots \underline{\gamma}_p)'$, where $\underline{\gamma}_i' = (\gamma_{i1} \cdots \gamma_{in})$. Then

$$\underline{x}_i' = (t_{i1}, \cdots, t_{ii}, 0, \cdots, 0)\Gamma, \qquad i = 1 \cdots p. \tag{4}$$

M_{21} and M_{22} arise from (4), that is $\partial x_{ij}/\partial t_{ij}$ and $\partial x_{ij}/\partial \gamma_{ij}$, respectively, and have the following forms:

$$M_{21} = \begin{array}{c|ccccccccccc} & t_{11} & t_{21} & \cdots & t_{p1} & t_{22} & \cdots & t_{p2} & \cdots & t_{p-1,p-1} & t_{p,p-1} & t_{pp} \\ \hline \underline{x}_1 & \underline{\gamma}_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \underline{x}_2 & 0 & \underline{\gamma}_1 & \cdots & 0 & \underline{\gamma}_2 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{x}_{p-1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \underline{\gamma}_{p-1} & 0 & 0 \\ \underline{x}_p & 0 & 0 & \cdots & \underline{\gamma}_1 & 0 & \cdots & \underline{\gamma}_2 & \cdots & 0 & \underline{\gamma}_{p-1} & \underline{\gamma}_p \end{array},$$

$$M_{22} = \begin{array}{c|ccccc} & \underline{\gamma}_1 & \underline{\gamma}_2 & \cdots & \underline{\gamma}_{p-1} & \underline{\gamma}_p \\ \hline \underline{x}_1 & D_{t_{11}} & 0 & \cdots & 0 & 0 \\ \underline{x}_2 & D_{t_{21}} & D_{t_{22}} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \underline{x}_{p-1} & D_{t_{p-1,1}} & D_{t_{p-1,2}} & \cdots & D_{t_{p-1,p-1}} & 0 \\ \underline{x}_p & D_{t_{p1}} & D_{t_{p2}} & \cdots & D_{t_{p,p-1}} & D_{t_{p,p}} \end{array},$$

where $D_{t_{ij}}: n \times n (i = 1 \cdots p; j = 1 \cdots i)$ is a diagonal matrix with diagonal elements t_{ij}. From $K_{ij} = \underline{\gamma}_i'\underline{\gamma}_j (i, j = 1 \cdots p)$ we have

$$M_{12} = \begin{array}{c|ccccc} & \underline{\gamma}_1 & \underline{\gamma}_2 & \cdots & \underline{\gamma}_{p-1} & \underline{\gamma}_p \\ \hline k_{11} & 2\underline{\gamma}_1 & 0 & \cdots & 0 & 0 \\ k_{22} & 0 & 2\underline{\gamma}_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{pp} & 0 & 0 & \cdots & 0 & 2\underline{\gamma}_p \\ \hdashline k_{12} & \underline{\gamma}_2 & \underline{\gamma}_1 & \cdots & 0 & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{1p} & \underline{\gamma}_p & 0 & \cdots & 0 & \underline{\gamma}_1 \\ k_{p-1,p} & 0 & 0 & \cdots & \underline{\gamma}_p & \underline{\gamma}_{p-1} \end{array}.$$

Hence $|M_{22}| = \prod_1^p |D_{t_{ii}}| = |\tilde{T}|^n$. Writing $\tilde{T}^{-1} = (t^{ii})$, $(i = 1 \cdots p;$ $j = 1 \cdots i)$, with $t^{ji} = 0$ for $j > i$, $M_{22}^{-1}M_{21}$ can be shown to be equal to

$$\begin{pmatrix} \gamma_1 t^{11} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \gamma_1 t^{12} & \gamma_2 t^{22} & \cdots & 0 & \gamma_2 t^{22} & \cdots & 0 & \cdots & 0 \\ \gamma_1 t^{1p} & \gamma_2 t^{2p} & \cdots & \gamma_1 t^{pp} & \gamma_2 t^{2p} & \cdots & \gamma_2 t^{pp} & \cdots & \gamma_p t^{pp} \end{pmatrix}$$

$$= M_{21} \begin{pmatrix} \tilde{T}_1 & 0 & \cdots & 0 \\ 0 & \tilde{T}_2 & \cdots & 0 \\ 0 & 0 & \cdots & \tilde{T}_p \end{pmatrix} \equiv M_{21} N,$$

say, where $N: p(p+1)/2 \times p(p+1)/2$, and

$$\tilde{T}_i = \begin{pmatrix} t^{ii} & 0 & \cdots & 0 \\ t^{ip} & t^{i+1,p} & \cdots & t^{pp} \end{pmatrix}, \qquad \tilde{T}_i : (p - i) \times (p - i).$$

Thus $| M_{12} M_{22}^{-1} M_{21} | = | M_{12} M_{21} | \, | N |$. It is easily verified that $| M_{12} M_{21} | = 2^p$ and $| N | = | \tilde{T} |^{-p} \prod_1^p t_{ii}^{p-i} = \prod_1^p t_{ii}^{-i}$. Combining our results we have

$$J(X; \tilde{T}, \Gamma_I) = 2^p \prod_1^p t_{ii}^{n-i} / J(\Gamma\Gamma'; \Gamma_D). \tag{5}$$

Consider the multivariate probability law

$$p(X) = (2\pi)^{-pn/2} \, | A |^{-n/2} \exp\left[-\tfrac{1}{2} \operatorname{tr} (A^{-1} X X')\right] dX. \tag{6}$$

Using transformation (1) and the Jacobian (5), we have the joint distribution of $\tilde{T}$ and Γ_I:

$$\begin{aligned} p(\tilde{T}, \Gamma_I) &= (2\pi)^{-pn/2} \, | A |^{-n/2} \\ &\quad \cdot \exp\left[-\tfrac{1}{2} \operatorname{tr} (A^{-1} \tilde{T} \tilde{T}')\right] 2^p \prod_1^p t_{ii}^{n-i} J^{-1}(\Gamma\Gamma'; \Gamma_D) \{d\tilde{T}\} \{d\Gamma_I\}. \end{aligned} \tag{7}$$

To obtain the distribution of $\tilde{T}$ alone, we must integrate out the variables of Γ_I over the domain $\Gamma\Gamma' = I$, that is we must evaluate

$$L(n, p) = \int_\Omega J^{-1}(\Gamma\Gamma'; \Gamma_D) \, d\Gamma_I, \tag{8}$$

where $\Omega : \Gamma\Gamma' = I$. Since the integral is unity over the space of $\tilde{T}: -\infty < t_{ij} < \infty$ $(i \neq j)$, $0 < t_{ii} < \infty$, we obtain

$$p(\tilde{T}) = c \, | A |^{-n/2} \prod_1^p t_{ii}^{n-i} \exp\left[-\tfrac{1}{2} \operatorname{tr} (A^{-1} \tilde{T} \tilde{T}')\right] d\tilde{T}, \tag{9}$$

where $c = 2^{p - np/2} \pi^{-p(p-1)/4} \prod_1^p \Gamma^{-1}(n - i + 1)/2$. Incidentally, $L(n, p)$ can now easily be evaluated, namely,

$$L(n, p) = \pi^{np/2 - p(p-1)/4} \prod_1^p \Gamma^{-1}(n - i + 1)/2. \tag{10}$$

With respect to the integral (8), it should be noted that if the probability density involves Γ, it will be necessary to add up the probability densities for the 2^p

points in Γ (subject to $\Gamma\Gamma' = I$) which correspond to a particular point of Γ_I, by virtue of Theorem 3. If the probability density is free of Γ a factor of 2^p is sufficient.

5. Distribution of the roots of a certain determinantal equation. The existence of the requisite transformation is given by the following theorem.

THEOREM 4. *If $X:p \times n (p \leqq n)$ and $Y:p \times m (p \leqq m)$ are of rank $r \leqq p$ and p, respectively, then there exist matrices $Z:p \times p$, $D_\theta:r \times r$, $\Gamma_1:p \times n$, $\Delta_1:p \times n$ such that*

$$(1) \qquad X = Z\begin{pmatrix} D_\theta \\ 0 \end{pmatrix}\Gamma_1, \qquad Y = Z\Delta_1,$$

where Z is nonsingular, D_θ is diagonal with elements $\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_r > 0$, the $\theta_i^2 (i = 1 \cdots r)$ are the nonvanishing roots of $| XX' - \theta^2 YY' | = 0$, and Γ_1 and Δ_1 are row orthogonal. If in addition (a) X is of rank p, (b) the roots are distinct, and (c) the elements $(Z_{11} \cdots Z_{1p})$ of Z are positive, the representation is unique [8].

We note that (a) and (b) are satisfied with probability 1, and hence we need only guarantee (c). Assuming (a) and (b) we rewrite (1) and introduce pseudo variates.

$$(2) \qquad \begin{aligned} X &= Z(D_\theta 0)\Gamma, & Y &= Z(I0)\Delta \\ \hat{S}_4 &= \hat{S}_4, & \hat{T}_4 &= \hat{T}_4, \end{aligned}$$

where $\Gamma' = (\Gamma_1'\Gamma_2'):n \times n$ and $\Delta' = (\Delta_1'\Delta_2'):m \times m$, are orthogonal. $\hat{S} = \begin{pmatrix} \hat{S}_1 \hat{S}_2 \\ \hat{S}_3 \hat{S}_4 \end{pmatrix}$ is a skew-symmetric matrix ($\hat{S}_1:p \times p$, $\hat{S}_4:n - p \times n - p$) related to Γ by $\hat{S} = (I + \Gamma)^{-1}(I - \Gamma)$, provided the inverse exists; $\hat{T}$ is similarly related to Δ. We note that the left-hand side contains

$$pn + pm + (n - p)(n - p - 1)/2 + (m - p)(m - p - 1)/2 = p^2 + p + n(n - 1)/2 + m(m - 1)/2$$

variates as does the right-hand side. We now proceed to obtain the Jacobian of (2). Familiarity with the techniques in [11] is assumed. In particular, the Jacobian of a transformation is equal to the Jacobian of the transformation in the differentials. Taking differentials in (2), we have

$$(3a) \qquad (dX) = (dZ)(D_\theta 0)\Gamma + Z(D_{d\theta} 0)\Gamma + Z(D_\theta 0)(d\Gamma),$$

$$(3b) \qquad (dY) = (dZ)(I0)\Delta + Z(I0)(d\Delta),$$

$$(3c) \qquad (d\hat{S}_4) = (d\hat{S}_4),$$

$$(3d) \qquad (d\hat{T}_4) = (d\hat{T}_4).$$

Pre- and post-multiply (3a) by $D_\theta^{-1}Z^{-1}$ and Γ', respectively, and (3b) by Z^{-1} and Δ', respectively.

$$D_\theta^{-1}Z^{-1}(dX)\Gamma' = D_\theta^{-1}Z^{-1}(dZ)(D_\theta 0) + D_\theta^{-1}(D_{d\theta}0) + (I0)(d\Gamma)\Gamma', \tag{4a}$$

$$Z^{-1}(dY)\Delta' = Z^{-1}(dZ)(I0) + (I0)(d\Delta)\Delta', \tag{4b}$$

From $d(\Gamma\Gamma') = (d\Gamma)\Gamma' + \Gamma(d\Gamma') = 0$, it follows that $(d\Gamma)\Gamma'$ is skew-symmetric, and hence we can let

$$\hat{A} = (d\Gamma)\Gamma' = -2(I + \hat{S})^{-1}(d\hat{S})(I - \hat{S})^{-1}, \tag{5a}$$

$$\hat{B} = (d\Delta)\Delta' = -2(I + \hat{T})^{-1}(d\hat{T})(I - \hat{T})^{-1}. \tag{5b}$$

Let

$$\begin{aligned} U &= D_\theta^{-1}Z^{-1}(dX)\Gamma', & V &= Z^{-1}(dY)\Delta', \\ W &= Z^{-1}(dZ), & D_\eta &= D_\theta^{-1}D_{d\theta}. \end{aligned} \tag{5c}$$

Transformation (4) becomes easily

$$U = D_\theta^{-1}WD_\theta(I0) + D_\eta(I0) + (I0)\hat{A}, \tag{6a}$$

$$V = W(I0) + (I0)\hat{B} \tag{6b}$$

$$(d\hat{S}_4) = \tfrac{1}{2}(I + \hat{S}_4)\hat{A}_4(I - \hat{S}_4) + \cdots, \tag{6c}$$

$$(d\hat{T}_4) = \tfrac{1}{2}(I + \hat{T}_4)\hat{B}_4(I - \hat{T}_4) + \cdots. \tag{6d}$$

where the additional terms in (6c) and (6d) are independent of $\hat{A}_4$ and $\hat{B}_4$, respectively. The Jacobian can be written as follows:

$$\begin{aligned} J(X, Y, \hat{S}_4, \hat{T}_4; Z, \theta, \hat{S}, \hat{T}) &= J(dX, dY, d\hat{S}_4, d\hat{T}_4; dZ, d\theta, d\hat{S}, d\hat{T}) \\ &= J(dX, dY, d\hat{S}_4, d\hat{T}_4; U, V, \hat{A}_4, \hat{B}_4)J(U, V, \hat{A}_4, \hat{B}_4; W, \eta, \hat{A}, \hat{B}) \\ &\quad J(W, \eta, \hat{A}, \hat{B}; dZ, d\theta, d\hat{S}, d\hat{T}) \equiv J_1J_2J_3. \end{aligned} \tag{7}$$

We now evaluate the components using [11]. J_1 arises from (5):

$$\begin{aligned} J_1 &= J(dX; U)J(dY; V)J(d\hat{S}_4; \hat{A}_4)J(d\hat{T}_4; \hat{B}_4) \\ &= (|D_\theta|^n |Z|^n |\Gamma|^p)(|Z|^m |\Delta|^p)(2^{-(n-p)(n-p-1)/2} |I + \hat{S}_4|^{n-p-1}) \\ &\quad (2^{-(m-p)(m-p-1)/2} |I + \hat{T}_4|^{m-p-1}). \end{aligned} \tag{8}$$

For simplicity the last two terms will be denoted by $a(\hat{S}_4, n - p)$ and $a(\hat{T}_4, m - p)$, respectively. J_2 arises from (6) yielding the scheme of partial derivatives shown in Figure 1, where $I_1 = I_{(n-p)(n-p-1)/2}$, $I_2 = I_{(m-p)(m-p-1)/2}$, and all other identity matrices I are $p(p - 1)/2$. The determinant of the matrix given in Figure 1 is equal to

$$\left| \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & 0 \end{pmatrix} - \begin{pmatrix} I & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & -I \end{pmatrix} \right| = |D_1 - D_2|.$$

	η_i	a_{ij} $i, j \leqq p$	b_{ij} $i, j \leqq p$	z_{ii}	z_{ij} $i < j$	z_j $i > j$	a_{ij} $j > p > i$	b_{ij} $j > p > i$	$\hat{A}_4$	$\hat{B}_4$
u_{ii}	I_p	0	0	I_p	0	0	0	0	0	0
$u_{ij}(i < j \leqq p)$	0	I	0	0	D_1	0	0	0	0	0
$u_{ij}(i > j)$	0	−I	0	0	0	D_2	0	0	0	0
v_{ii}	0	0	0	I_p	0	0	0	0	0	0
$v_{ij}(i < j \leqq p)$	0	0	I	0	I	0	0	0	0	0
$v_{ij}(i > j)$	0	0	$-I$	0	0	I	0	0	0	0
$u_{ij}(i < j, j > p)$	0	0	0	0	0	0	$I_{n(n-p)}$	0	0	0
$v_{ij}(i < j, j > p)$	0	0	0	0	0	0	0	$I_{m(m-p)}$	0	0
$\hat{A}_4$	0	0	0	0	0	0	0	0	I_1	0
$\hat{B}_4$	0	0	0	0	0	0	0	0	0	I_2

FIG. 1

D_1 arises from that part of (5a) which contains $u_{ij} = (\theta_j/\theta_i)z_{ij}$ + terms independent of z_{ij}. $\partial u_{ij}/\partial z_{ij} = \theta_j\theta_i$ and hence $D_1:p(p-1)/2 \times p(p-1)/2$ is diagonal with elements

$$\frac{\theta_2}{\theta_1}, \frac{\theta_3}{\theta_1}, \cdots, \frac{\theta_p}{\theta_1}, \frac{\theta_3}{\theta_2}, \cdots, \frac{\theta_p}{\theta_2}, \cdots, \frac{\theta_p}{\theta_{p-1}}.$$

Since $D_2 = D_1^{-1}$,

$$J_2 = \prod_{i<j}\left(\frac{\theta_j}{\theta_i} - \frac{\theta_i}{\theta_j}\right) = \prod_{i<j}(\theta_j^2 - \theta_i^2)\prod_1^p \theta_i^{-(p-1)}. \tag{9}$$

Finally,

$$\begin{aligned} J_3 &= J(W; dZ)J(\eta; d\theta)J(\hat{A}; d\hat{S})J(\hat{B}; d\hat{T}) \\ &= (|Z|^{-p})(|D_\theta|^{-1})a^{-1}(\hat{S}, n)a^{-1}(\hat{T}, m). \end{aligned} \tag{10}$$

Combining our results (8) to (10):

$$J = \prod_1^p \theta_i^{n-p}\prod_{i<j}(\theta_i^2 - \theta_j^2)\,|Z|^{n+m-p}\,g(\hat{S})h(\hat{T}), \tag{11}$$

where $g(\hat{S})$ and $h(\hat{T})$ are functions of $\hat{S}$ and $\hat{T}$, respectively.

Consider $p(n+m)$ random variables X^* and Y^* following the multivariate normal probability law as defined by the density function

$$p(X^*, Y^*) = c_1 \exp\left[-\tfrac{1}{2}\operatorname{tr}(A^{-1}X^*X^{*\prime} + B^{-1}Y^*Y^{*\prime})\right], \tag{12}$$

where $c_1 = (2\pi)^{-p(n+m)/2}|A|^{-n/2}|B|^{-m/2}$, $X^*:p \times n(p \leqq n)$, $Y^*:p \times m(p \leqq m)$, $A:p \times p$, and $B:p \times p$ are the population covariance matrices and are positive definite. The roots of $|X^*X^{*\prime} - \theta^2 Y^*Y^{*\prime}| = 0$ are invariant under a nonsingular transformation $X^* = LX$, $Y^* = LY$, where $X:p \times n$, $Y:p \times m$, and $L:p \times p$ is nonsingular and is chosen so that $LAL' = D_\rho$, $LBL' = I$, and $\rho_1, \cdots, \rho_p > 0$ are the population roots of $|A - \rho B| = 0$. The Jacobian $J(X^*, Y^*; X, Y)$ of the transformation is $|L|^{-(n+m)}$, and $|A| = |D_\rho||L|^2$, $|B| = |L|^2$, which yields the density function of X and Y:

$$p(X, Y) = c_2 \exp\left[-\tfrac{1}{2}\operatorname{tr}(D_\rho XX' + YY')\right], \tag{13}$$

where $c_2 = (2\pi)^{-p(n+m)/2}|D_\rho|^{-n/2}$. We now make use of the main transformation (2) and the Jacobian (11) which leads to the density function of Z, θ, $\hat{S}$, $\hat{T}$:

$$\begin{aligned} p(Z, \theta, \hat{S}, \hat{T}) &= c_3\prod_1^p \theta_i^{n-p}\prod_{i<j}(\theta_i^2 - \theta_j^2)\,|Z|^{n+m-p} \\ &\qquad \exp\left[-\tfrac{1}{2}\operatorname{tr}(D_\rho Z D_{\theta^2} Z' + ZZ')\right], \end{aligned} \tag{14}$$

where $c_3 = (2\pi)^{-p(n+m)/2}|D_\rho|^{-n/2}g(\hat{S})h(\hat{T})$. We now integrate out the extraneous variates to obtain the distribution of $\theta_1 \cdots \theta_p$. The integrations will be indicated for the null case, that is when $A = B$ or equivalently $D_\rho = I$. One of the integrations involved is based on the following lemma.

LEMMA.

$$I(r, K, A) = \int_\Omega |WW'|^k \exp[-\tfrac{1}{2}\operatorname{tr}(AWW')]\{dW\}$$
$$= |A|^{-(2k+r)/2} 2^{rk+r^2/2} \pi^{r^2/2} \prod_1^r \left(\Gamma k + \frac{r-i+1}{2}\right)\Gamma^{-1}\left(\frac{r-i+1}{2}\right),$$

where $W: r \times r$; $A: r \times r$ *is positive definite,* and $\Omega: -\infty < w_{ij} < \infty$, $(i, j = 1 \cdots r)$.

PROOF. Since A is positive definite we may write $A = \tilde{U}'\tilde{U}$ and make the successive transformations $X = \tilde{T}\Gamma$, $\Gamma = (I + \hat{S})^{-1}(I - \hat{S})$ and $\tilde{V} = \tilde{U}\tilde{T}$. The Jacobians [11] are

$$J(X; \tilde{T}, \hat{S}) = \prod_1^p t_{ii}^{p-i} 2^{p(p-1)/2} |I + \hat{S}|^{-(p-1)}, \quad J(\tilde{T}; \tilde{V}) = \prod_1^p u_{ii}^{-i}.$$

Hence

$$I(r, k, a) = \int_{\Omega_1} \prod_1^r u_{ii}^{-(2k+r)} v_{ii}^{2k+r-i} \exp[-\tfrac{1}{2}\operatorname{tr}\tilde{V}\tilde{V}']\{d\tilde{V}\}$$
$$\times \int_{\Omega_2} 2^p 2^{p(p-1)/2} |I + \hat{S}|^{-(p-1)}\{dS\},$$

where $\Omega_1: -\infty < v_{ij} < \infty$ $(i \neq j)$, $0 < v_{ii} < \infty$, and Ω_2 is the total space of $r(r-1)/2$ dimensions. The first integral is a product of gamma functions and the second integral is evaluated in [12] and [13], giving

$$I(r, k, a) = |A|^{-(2k+r)/2} \prod_1^r \left[2^{(2k+r-i+1)/2}\Gamma\left(\frac{2k+r-i+1}{2}\right)\right]$$
$$\times (2\pi)^{r(r-1)/4} 2^r \pi^{r(r+1)/4} \prod_1^r \Gamma^{-1}\left(\frac{r-i+1}{2}\right).$$

The integral over the Z space is thus $2^{-p}I(p, (n + m - p)/2, D_{\theta^2+1})$, where the 2^{-p} arises from the restriction $(z_{11} \cdots z_{1p}) > 0$. The constants arising from integrating over $\hat{S}$ and $\hat{T}$ can be obtained directly using [12] and [13], taking account of the introduction of the pseudo-variates, or indirectly from the distribution of rectangular coordinates. They turn out to be $2^pL(n, p)$ and $2^pL(m, p)$ of the previous section. Combining all results we obtain:

$$p(\theta) = k \prod_1^p \theta_i^{n-p}(1 + \theta_i^2)^{-(n+m)/2} \prod_{i<j} (\theta_i^2 - \theta_j^2),$$

where

$$k = 2^p \pi^{p/2} \prod_1^p \Gamma\left(\frac{n+m-i+1}{2}\right)$$
$$\Big/ \Gamma\left(\frac{n-i+1}{2}\right)\Gamma\left(\frac{m-i+1}{2}\right)\Gamma\left(\frac{p-i+1}{2}\right),$$

noting that

$$[(2\pi)^{-p(n+m)/2}]\Big[2^{-p}2^{p(n+m-p)/2+p^2/2}\pi^{p^2/2} \times \prod_1^p \Gamma\Big(\frac{n+m-p}{2}+\frac{p-i+1}{2}\Big)\Gamma^{-1}\Big(\frac{p-i+1}{2}\Big)\Big][2^pL(n,p)][2^pL(m,p)] = k.$$

If we let $\theta_i^2 = \alpha_i$ or $\theta_i^2 = \beta_i/(1+\beta_i)$, we may obtain other familiar forms for the density function of the roots of related determinantal equations.

Added in proof. A. T. James, "Normal multivariate analysis and the orthogonal group", Ann. Math. Stat., Vol. 25 (1954), pp. 40–75, obtained a similar result as an application of Grassman and Stiefel manifolds.

6. Conclusion. In the present paper the methods proposed have been illustrated by giving a new derivation of the joint distribution of rectangular coordinates and the joint distribution of the roots of a certain determinantal equation. These methods also apply to other situations, for example the singular case in the above examples, the joint distribution of canonical correlations; multiple and partial correlations; inverse and adjoint of certain matrices. In essence, the problem consists in obtaining the requisite transformation which will lead to the desired variates, and then applying the proposed techniques to evaluate the Jacobian and integrate out any extraneous variates.

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