ADMISSIBILITY OF QUANTILE ESTIMATES OF A SINGLE LOCATION PARAMETER[1]

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1. Introduction and summary. Let B be the σ-algebra of all Borel subsets of the real line $\mathfrak{X}$ and $\mathfrak{A}$ be a σ-algebra of subsets of a set $\mathfrak{Y}$. Let ν be a probability measure on $\mathfrak{A}$. Let P be a $B \times \mathfrak{A}$ measurable function on $\mathfrak{X} \times \mathfrak{Y}$ such that $P(\cdot, y)$ is a distribution function for each $y \in \mathfrak{Y}$. As usual, $B \times \mathfrak{A}$ is the σ-algebra generated by sets $C \times A$ for $C \in B$ and $A \in \mathfrak{A}$. We observe (X, Y) where ν is the marginal distribution of Y and, for some unknown θ, the conditional distribution function of $X - \theta$ given Y is $P(\cdot, Y)$. It is desired to estimate θ with loss function

$$
\begin{aligned}
L(\theta, d) &= a(\theta - d) && \text{if } d \leqq \theta \\
&= b(d - \theta) && \text{if } d \geqq \theta.
\end{aligned} \tag{1.1}
$$

For any statistical problem, let $\rho(\theta, \delta)$ be the risk when δ is the decision procedure used and θ is the value of the parameter. In all that follows, μ will be Lebesgue measure on the real line. In a problem involving a real parameter, we say that δ is *almost admissible* provided that, given any other procedure δ', if $\rho(\theta, \delta') \leqq \rho(\theta, \delta)$ for all θ, then $\rho(\theta, \delta') = \rho(\theta, \delta)$ a.e. (μ).

The purpose of the present paper is to prove the

Theorem. *Suppose that, in addition to the above assumptions,*

(i) *for each* $y \in \mathfrak{Y}$, *the unique* $(1 - \alpha)$th *quantile of* $P(\cdot, y)$ *is* 0 *where* $\alpha = a/(a + b)$ *and*

$$
\int d\nu(y) \int x^2 \, d_x P(x, y) < \infty. \tag{ii}
$$

Then, under the loss function given in (1.1), X *is an almost admissible estimate of* θ.

The proof of the theorem will be given in Section 2. Section 3 contains the proof of the

Corollary. *If, in addition to the assumptions of the theorem,* $P(\cdot, y)$ *is either absolutely continuous for all* $y \in \mathfrak{Y}$ *or has its points of increase in a fixed lattice for all* $y \in \mathfrak{Y}$, *then* X *is an admissible estimate of* θ.

If θ has "uniform" a priori distribution on $\mathfrak{X}$, then X is the αth quantile of the a posteriori distribution of θ given X. Hence, X is the best invariant estimate of θ. Farrell [3] has shown that when the unicity assumption in the theorem is violated X cannot be almost admissible.

The Theorem and Corollary are analogous to Theorems 1 and 2 of Stein [5]

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for the case of loss function $(\theta - d)^2$. In Stein's theorems (i) is replaced by $E(X \mid Y) = 0$ and (ii) is replaced by $\int d\nu(y)[\int x^2\, d_xP(x, y)]^{\frac{3}{2}} < \infty$. Thus, Stein's result, as well as ours, requires one moment more than is intuitively necessary. In Stein's case, contrary to ours, the extra moment is not required if $\mathcal{Y}$ contains only one element. Similar results for the case of hypothesis testing were obtained by Lehmann and Stein [4] under the assumption of a first moment, again one more moment than is intuitively needed.

The proofs in Sections 2 and 3 are similar to those used by Stein. The method is originally due to Blyth [2]. Since the passage from our corollary to the case of a sample of size n is similar to Stein's, it will be omitted here.

Section 4 contains examples which show that the conclusion of the theorem cannot be strict admissibility. These examples are analogous to Blackwell's [1] example for the case of the loss function $(\theta - d)^2$.

2. Proof of the theorem. Without loss of generality, assume $a + b = 1$. Suppose the theorem is false. That is, suppose there is a real valued function ϕ on $\mathcal{X} \times \mathcal{Y}$ for which

$$\int d\nu(y) \int L(\theta, \phi(x, y))\, d_xP(x - \theta, y) \leqq \int d\nu(y) \int L(\theta, x)\, d_xP(x - \theta, y) \tag{2.1}$$

with strict inequality on a set S of θ's for which $\mu(S) > 0$ where μ is Lebesgue measure.

Assign a priori distribution $(1/\sigma)q(\theta/\sigma)$ to θ where $q(u) = 1/[\pi(1 + u^2)]$. Let $\mathcal{E}$ stand for expectation with respect to ν, P, and the a priori distribution. By the same proof as that used by Stein, (2.1) and the strict inequality on S imply, for $\epsilon > 0$ sufficiently small and some $\kappa > 0$, that

$$\mathcal{E}L(\theta, \phi(X, Y)) \leqq \mathcal{E}L(\theta, X) - \epsilon\kappa/2\pi\sigma. \tag{2.2}$$

It will be shown that

$$\inf_\psi \mathcal{E}L(\theta, \psi(X, Y)) \geqq \mathcal{E}L(\theta, X) - f(\sigma)/\sigma \tag{2.3}$$

where $f(\sigma) \to 0$ as $\sigma \to \infty$. This contradicts (2.2) and will complete the proof.

Let $\theta^*(x, y; \sigma)$ denote any αth quantile of the a posteriori distribution of θ given $X = x$ and $Y = y$. Since $\theta^*(x, y; \sigma)$ is the Bayes estimate of θ, it follows that

$$\inf_\psi \mathcal{E}L(\theta, \psi\,(X, Y)) = \mathcal{E}L(\theta, \theta^*(X, Y; \sigma)). \tag{2.4}$$

We now wish to take a jointly measurable determination of θ^*. Let

$$\varphi(u, x, y; \sigma) = \int_{-\infty}^{u+} q\left(\frac{\theta}{\sigma}\right) d_\theta P(x - \theta, y) \Big/ \int_{-\infty}^{\infty} q\left(\frac{\theta}{\sigma}\right) d_\theta P(x - \theta, y)$$

and let $\theta^*(x, y; \sigma) = \min\{u : \varphi(u, x, y; \sigma) \geqq \alpha\}$, that is, $\theta^*(x, y; \sigma)$ is the smallest

αth quantile of the a posteriori distribution. In all that follows this determination of $\theta^*(x, y; \sigma)$ is used.

Let

$$
\begin{aligned}
\frac{f(\sigma)}{\sigma} &= \mathcal{E}L(\theta, x) - \inf_\psi L(\theta, \psi(X, Y)) \\
&= \frac{1}{\sigma}\int q\left(\frac{\theta}{\sigma}\right) d\theta \int d\nu(y) \int [L(\theta, x) - L(\theta, \theta^*(x, y; \sigma))]\, d_x P(x-\theta, y) \qquad (2.5)\\
&= -\frac{1}{\sigma}\int d\nu(y) \int dx \int q\left(\frac{\theta}{\sigma}\right) [L(\theta, x) - L(\theta, \theta^*(x, y; \sigma))]\, d_\theta P(x-\theta, y).
\end{aligned}
$$

By Assumption (ii) of the Theorem, the integrand in (2.5) is absolutely integrable so that the change in order of integration is valid.

We wish to show $f(\sigma) \to 0$ as $\sigma \to \infty$.

For $x \geqq \theta^*(x, y; \sigma)$ we have

$$
\begin{aligned}
L(\theta, x) - L(\theta, \theta^*(x, y; \sigma)) &= b[x - \theta^*(x, y; \sigma)] && \text{if } \theta \leqq \theta^*(x, y; \sigma) \\
&= -a[x - \theta^*(x, y; \sigma)] + (a+b)(x-\theta) && \qquad (2.6)\\
& && \text{if } \theta^*(x, y; \sigma) \leqq \theta \leqq x \\
&= -a[x - \theta^*(x, y; \sigma)] && \text{if } \theta \geqq x
\end{aligned}
$$

so that

$$
\begin{aligned}
&\int q\left(\frac{\theta}{\sigma}\right) [L(\theta, x) - L(\theta, \theta^*(x, y; \sigma))]\, d_\theta P(x-\theta, y) \\
&\quad = [x - \theta^*(x, y; \sigma)]\left[b \int_{-\infty}^{\theta^*(x,y;\sigma)} q\left(\frac{\theta}{\sigma}\right) d_\theta P(x-\theta, y)\right. \qquad (2.7)\\
&\quad \left. - a \int_{\theta^*(x,y;\sigma)}^{\infty} q\left(\frac{\theta}{\sigma}\right) d_\theta P(x-\theta, y)\right] + \int_{\theta^*(x,y;\sigma)}^{x} (x-\theta)\, q\left(\frac{\theta}{\sigma}\right) d_\theta P(x-\theta, y)
\end{aligned}
$$

since we have assumed $a + b = 1$.

In (2.7) the point $\theta^*(x, y; \sigma)$ may be included in the range of integration of the first integral or of the second and third integrals or partially in both. In all that follows, integrals with the $(1-\alpha)$th or αth quantile as a limit (which will be obvious in the context) are taken with that limit included to the extent that makes that quantile exact. Thus, if c is the αth quantile of the distribution function F, then

$$\int_d^c f(x)\, dF(x) = \int_d^{c-} f(x)\, dF(x) + f(c)\,[\alpha - F(c-)]$$

for $d < c$ and

$$\int_c^d f(x)\, dF(x) = \int_{c+}^d f(x)\, dF(x) + f(c)\,[F(c) - \alpha]$$

for $d > c$. With this convention, the first term of (2.7) is 0 since $\theta^*(x, y; \sigma)$ is the αth quantile of $q(\theta/\sigma)P(x - \theta, y)$. Thus,

$$\begin{aligned} &\int q\left(\frac{\theta}{\sigma}\right)[L(\theta, x) - L(\theta, \theta^*(x, y; \sigma))]\, d_\theta P(x - \theta, y) \\ (2.8) \qquad &= \int_{\theta^*(x,y;\sigma)}^{x} (x - \theta) q\left(\frac{\theta}{\sigma}\right) d_\theta P(x - \theta, y). \end{aligned}$$

Similarly, for $x < \theta^*(x, y; \sigma)$, we have

$$\begin{aligned} &\int q\left(\frac{\theta}{\sigma}\right)[L(\theta, x) - L(\theta, \theta^*(x, y; \sigma))]\, d_\theta P(x - \theta, y) \\ (2.9) \qquad &= \int_{x}^{\theta^*(x,y;\sigma)} (\theta - x) q\left(\frac{\theta}{\sigma}\right) d_\theta P(x - \theta, y) \\ &= \int_{\theta^*(x,y;\sigma)}^{x} (x - \theta) q\left(\frac{\theta}{\sigma}\right) d_\theta P(x - \theta, y). \end{aligned}$$

From (2.5), (2.8), and (2.9) it follows that

$$\begin{aligned} f(\sigma) &= \int d\nu(y) \int dx \int_{\theta^*(x,y;\sigma)}^{x} (x - \theta) q\left(\frac{\theta}{\sigma}\right) d_\theta P(x - \theta, y) \\ (2.10) \qquad &= \int d\nu(y) \int dx \int_0^{x-\theta^*(x,y;\sigma)} uq\left(\frac{x - u}{\sigma}\right) d_u P(u, y) \\ &= \int d\nu(y) \int dx \int_0^{x-\theta^*(x,y;\sigma)} \frac{u}{1 + (x - u)^2/\sigma^2}\, d_u P(u, y). \end{aligned}$$

Let

$$K_y = \int |x|\, d_x P(x, y); \qquad K = \int K_y\, d\nu(y);$$

$$K'_y = \int x^2\, d_x P(x, y); \qquad K' = \int K'_y\, d\nu(y).$$

We wish to break up the range of integration in (2.10). The first term is

$$\begin{aligned} J_1(\sigma) &= \int_{K'_y > \epsilon^2\sigma^2} d\nu(y) \int dx \int_0^{x-\theta^*(x,y;\sigma)} \frac{u}{1 + (x - u)^2/\sigma^2}\, d_u P(u, y) \\ (2.11) \qquad &\leqq \sigma^2 \int_{K'_y > \epsilon^2\sigma^2} d\nu(y) \int |u|\, d_u P(u, y) \int_{-\infty}^{\infty} \frac{1}{\sigma^2 + (x - u)^2}\, dx \\ &= \frac{\pi\sigma}{2} \int_{K'_y > \epsilon^2\sigma^2} K_y\, d\nu(y). \end{aligned}$$

But $K_y/K'_y \leqq 1/(K'_y)^{\frac{1}{2}} < 1/(\epsilon\sigma)$ in the range of this integration. Hence,

$$(2.12) \qquad J_1(\sigma) \leqq \frac{\pi}{2\epsilon} \int_{K'_y > \epsilon^2\sigma^2} K'_y\, d\nu(y) \to 0$$

as $\sigma \to \infty$ since $K' < \infty$.

In the remaining part of this section we assume $K_y' \leqq \epsilon^2\sigma^2$. In order to bound the remainder of $f(\sigma)$ we need bounds for the quantity

$$
\begin{aligned}
\tfrac{1}{2}\Delta(x, y; \sigma) &= a \int_0^\infty \frac{1}{1 + (x - u)^2/\sigma^2}\, d_u P(u, y) \\
&\quad - b \int_{-\infty}^0 \frac{1}{1 + (x - u)^2/\sigma^2}\, d_u P(u, y) \qquad (2.13)\\
&= \int_0^{x - \theta^*(x, y; \sigma)} \frac{1}{1 + (x - u)^2/\sigma^2}\, d_u P(u, y).
\end{aligned}
$$

In the defining formula for $\Delta(x, y; \sigma)$ we include 0 in each term to the extent that makes 0 the exact $(1 - \alpha)$th quantile of $P(\cdot, y)$. In the definitions of G and H given below we do the same. For $E \subset [0, \infty)$, let $G(E, y) = aP(E, y) - bP(-E, y)$ and $H(E, y) = aP(E, y) + bP(-E, y)$. Then, $2P(E, y) = (1/a)[H(E, y) + G(E, y)]$ and $2P(-E, y) = (1/b)[H(E, y) - G(E, y)]$ so that

$$
\begin{aligned}
\Delta(x, y; \sigma) &= \int_0^\infty \left[\frac{1}{1 + (x - u)^2/\sigma^2} - \frac{1}{1 + (x + u)^2/\sigma^2}\right] d_u H(u, y) \\
&\quad + \int_0^\infty \left[\frac{1}{1 + (x - u)^2/\sigma^2} + \frac{1}{1 + (x + u)^2/\sigma^2}\right] d_u G(u, y) \qquad (2.14)\\
&= \int_0^\infty \frac{4\, x\, u/\sigma^2}{1 + 2[(x^2 + u^2)/\sigma^2] + (x^2 - u^2)^2/\sigma^4}\, d_u H(u, y) \\
&\quad + \int_0^\infty \frac{2 + 2[(x^2 + u^2)/\sigma^2]}{1 + 2[(x^2 + u^2)/\sigma^2] + (x^2 - u^2)^2/\sigma^4}\, d_u G(u, y).
\end{aligned}
$$

To verify (2.14) we must see that the inclusion of 0 is correct in the two terms. In the definitions of G and H we have $G(\{0\}, y) = a[P(0, y) - \alpha] - bP[\alpha - P(0-, y)]$ and $H(\{0\}, y) = a[P(0, y) - \alpha] + b[\alpha - P(0-, y)]$. Multiplying both sides of the first equality in (2.13) by 2, in the first integral 0 is included to the extent $2a[P(0, y) - \alpha] = H(\{0\}, y) + G(\{0\}, y)$ which accounts for the first term of each integral in the first expression for $\Delta(x, y; \sigma)$ in (2.14). The second integral includes 0 to the extent $2b[\alpha - P(0-, y)] = H(\{0\}, y) - G(\{0\}, y)$ which accounts for the second term of each of the integrands in (2.14). With this computation of G and H the limit 0 is fully included in all the integrals in (2.14).

Let

$$
I_1(x, y; \sigma) = \int_0^\infty \frac{xu/\sigma^2}{1 + 2[(x^2 + u^2)/\sigma^2] + (x^2 - u^2)^2/\sigma^4}\, d_u H(u, y);
$$

$$
I_2(x, y; \sigma) = \int_0^\infty \frac{x^2u^2/\sigma^4}{(1 + (x^2 + u^2)/\sigma^2)[(1 + (x^2 + u^2)/\sigma^2)^2 - 4x^2u^2/\sigma^4]}\, d_u G(u, y);
$$

and

$$I_3(x, y; \sigma) = \int_0^\infty \frac{u^2/\sigma^2}{(1 + x^2/\sigma^2)(1 + (x^2 + u^2)/\sigma^2)}\, d_u\, G(u, y).$$

Elementary algebraic manipulation yields $\Delta = 4I_1 + 8I_2 - 2I_3$ since the fourth term is

$$\int_0^\infty \frac{1}{1 + x^2/\sigma^2}\, d_u\, G(u, y) = \frac{1}{1 + x^2/\sigma^2}\left[a \int_0^\infty d_u\, P(u, y) - b \int_{-\infty}^0 d_u\, P(u, y)\right] = 0.$$

On the right side of the preceding expression, the limit 0 is included in each term to the extent given in the previous conventions concerning integrals with respect to $P(\cdot, y)$.

Bounds for I_1, I_2 and I_3 are given by

$$\text{(2.15)} \qquad \begin{aligned} |I_1(x, y; \sigma)| &\leqq \frac{|x|}{\sigma^2 + 2x^2} \int_0^\infty u\, d_u\, H(u, y) \\ &= \frac{K_y\, |x|}{\sigma^2 + 2x^2} \leqq \frac{K_y}{2^{3/2}\sigma}; \end{aligned}$$

$$\text{(2.16)} \qquad \begin{aligned} |I_2(x, y; \sigma)| &\leqq \frac{x^2}{(\sigma^2 + x^2)(\sigma^2 + 2x^2)} \int_0^\infty u^2\, |d_u\, G(u, y)| \\ &\leqq \frac{1}{2\sigma^2} \int_0^\infty u^2\, d_u\, H(u, y) = \frac{K_y'}{2\sigma^2}; \end{aligned}$$

$$\text{(2.17)} \qquad \begin{aligned} |I_3(x, y; \sigma)| &\leqq \frac{1}{\sigma^2(1 + x^2/\sigma^2)^2} \int_0^\infty u^2\, |d_u\, G(u, y)| \\ &\leqq \frac{1}{\sigma^2} \int_0^\infty u^2\, d_u\, H(u, y) = \frac{K_y'}{\sigma^2}. \end{aligned}$$

From (2.15), (2.16), and (2.17) we obtain

$$\text{(2.18)} \qquad |\Delta(x, y; \sigma)| < 2K_y/\sigma + 13K_y'/\sigma^2.$$

We require another bound on Δ for the case $|x| > \sigma$. It will be necessary to split I_1 into two terms, I_1' and I_1'', the former being the integral from 0 to $|x|/2$. Then, for $|x| > \sigma$, we obtain

$$\text{(2.19)} \qquad \begin{aligned} |I_1'(x, y; \sigma)| &\leqq \frac{|x|\, \sigma^2}{\sigma^4 + 2x^2\sigma^2 + 9x^4/16} \int_0^{|x|/2} u\, d_u\, H(u, y) \\ &\leqq \frac{16\sigma^2 K_y}{9\, |x|^3} < \frac{\sigma^2}{x^2}\left(\frac{16K_y}{9\sigma}\right); \end{aligned}$$

$$\text{(2.20)} \qquad \begin{aligned} |I_1''(x, y; \sigma)| &\leqq \frac{|x|}{\sigma^2 + 2x^2} \int_{|x|/2}^\infty u\, d_u\, H(u, y) \\ &\leqq \frac{2}{\sigma^2 + 2x^2} \int_{|x|/2}^\infty u^2\, d_u\, H(u, y) < \frac{K_y'}{x^2} = \frac{\sigma^2}{x^2}\left(\frac{K_y'}{\sigma^2}\right); \end{aligned}$$

$$
\begin{aligned}
(2.21)\qquad |I_2(x, y; \sigma)| &\leqq \frac{x^2}{(\sigma^2 + x^2)^2} \int_0^\infty u^2 \, |d_u \, G(u, y)| \\
&< \frac{K_y'}{x^2} = \frac{\sigma^2}{x^2}\left(\frac{K_y'}{\sigma^2}\right);
\end{aligned}
$$

and

$$
(2.22)\qquad |I_3(x, y; \sigma)| \leqq \frac{\sigma^2}{(\sigma^2 + x^2)^2} \int_0^\infty u^2 \, |d_u \, G(u, y)| < \frac{K_y'}{2x^2} = \frac{\sigma^2}{x^2}\left(\frac{K_y'}{2\sigma^2}\right).
$$

Thus, for $|x| > \sigma$, from (2.19), (2.20), (2.21), and (2.22) we obtain

$$
\begin{aligned}
(2.23)\qquad |\Delta(x, y; \sigma)| &< (\sigma^2/x^2)(8K_y/\sigma + 13K_y'/\sigma^2) \\
&< [\sigma^2/(\sigma^2 + x^2)](16K_y/\sigma + 26K_y'/\sigma^2).
\end{aligned}
$$

For $|x| \leqq \sigma$, it follows from (2.18) that

$$
(2.24)\qquad |\Delta(x, y; \sigma)| < [\sigma^2/(\sigma^2 + x^2)](4K_y/\sigma + 26K_y'/\sigma^2)
$$

so that, for all x, we obtain

$$
(2.25)\qquad |\Delta(x, y; \sigma)| < [\sigma^2/(\sigma^2 + x^2)](16K_y/\sigma + 26K_y'/\sigma^2).
$$

From (2.25), for ϵ sufficiently small,

$$
(2.26)\qquad |\Delta(x, y; \sigma)| < [\sigma^2/(\sigma^2 + x^2)](16\epsilon + 26\epsilon^2) < 17\epsilon\sigma^2/(\sigma^2 + x^2)
$$

uniformly in y such that $K_y' \leqq \epsilon^2\sigma^2$.

Take $0 \leqq |u| \leqq \rho\sigma$ and $0 < \rho < 1$. Then,

$$
(2.27)\qquad 1 + (x - u)^2/\sigma^2 \leqq 2 + x^2/\sigma^2 + 2\,|x|/\sigma < 3(1 + x^2/\sigma^2).
$$

Thus,

$$
\begin{aligned}
(2.28)\qquad \left| \int_0^{\pm\rho\sigma} \frac{1}{1 + (x - u)^2/\sigma^2} \, d_u \, P(u, y) \right| &\geqq \frac{1}{3}\left(\frac{\sigma^2}{\sigma^2 + x^2}\right) \left| \int_0^{\pm\rho\sigma} d_u \, P(u, y) \right| \\
&\geqq \frac{1}{3}\left(\frac{\sigma^2}{\sigma^2 + x^2}\right)\left[\min(\alpha, 1 - \alpha) - \frac{\epsilon^2}{\rho^2}\right]
\end{aligned}
$$

uniformly in y by the Tchebyschev inequality. But ϵ can be chosen sufficiently small so that $\frac{1}{3}$ [min $(\alpha, 1 - \alpha) - \epsilon^2/\rho^2] > 17\epsilon$. For ϵ sufficiently small, from (2.26) and (2.28) it follows that

$$
(2.29)\qquad \left| \int_0^{\pm\rho\sigma} \frac{1}{1 + (x - u)^2/\sigma^2} \, d_u \, P(u, y) \right| > |\Delta(x, y; \sigma)|
$$

uniformly in y.

The inequality in (2.29) still holds if both limits are entirely included. Thus, it follows from (2.13) that $|x - \theta^*(x, y; \sigma)| \leqq \rho\sigma$ for all y.

We seek a bound on $|x - \theta^*(x, y; \sigma)|$. From (2.13), (2.25), and (2.27) we obtain

$$\frac{\sigma^2}{3(\sigma^2 + x^2)} \left| \int_0^{x-\theta^*(x,y;\sigma)} d_u\, P(u, y) \right| < \frac{\sigma^2}{\sigma^2 + x^2} \left(\frac{16K_y}{\sigma} + \frac{26K_y'}{\sigma^2} \right) \tag{2.30}$$

so that

$$\left| \int_0^{x-\theta^*(x,y;\sigma)} d_u\, P(u, y) \right| < 6 \left(\frac{8K_y}{\sigma} + \frac{13K_y'}{\sigma^2} \right). \tag{2.31}$$

Hence,

$$\begin{aligned} \int_{x-\theta^*(x,y;\sigma)}^{\infty} d_u\, P(u, y) &> \min(\alpha, 1 - \alpha) - 6 \left(\frac{8K_y}{\sigma} + \frac{13K_y'}{\sigma^2} \right) \\ &> \min(\alpha, 1 - \alpha) - 6(8\epsilon + 13\epsilon^2). \end{aligned} \tag{2.32}$$

For ϵ sufficiently small, it follows that

$$\int_{x-\theta^*(x,y;\sigma)}^{\infty} d_u\, P(u, y) > \tfrac{1}{2} \min(\alpha, 1 - \alpha) \tag{2.33}$$

so

$$K_y' > \tfrac{1}{2} \min(\alpha, 1 - \alpha)[x - \theta^*(x, y; \sigma)]^2 \tag{2.34}$$

and hence

$$|x - \theta^*(x, y; \sigma)| < |2K_y'/\min(\alpha, 1 - \alpha)|^{\frac{1}{2}}. \tag{2.35}$$

Let $E_\sigma = \{y : K_y' \leqq \epsilon^2\sigma^2$ and $|x - \theta^*(x, y; \sigma)| > \tau$ for some $x\}$ with $\tau > 0$ fixed. Fix $\delta > 0$. By the unicity of the $(1 - \alpha)$th quantile of $P(\cdot, y)$, for λ sufficiently small,

$$\nu\{y : P(\tau-, y) - (1 - \alpha) < \lambda \quad \text{and} \quad (1 - \alpha) - P(-\tau, y) < \lambda\} < \delta. \tag{2.36}$$

In (2.31) the inequality is certainly satisfied if we take the integral so as to completely omit the limit $x - \theta^*(x, y; \sigma)$. In what follows, we will take $P(\tau, y)$ to exclude the probability concentrated at τ if $\tau > 0$ and include that probability otherwise. Then, (2.31) yields

$$|P(x - \theta^*(x, y; \sigma), y) - (1 - \alpha)| < 6(8\epsilon + 13\epsilon^2) < \lambda \tag{2.37}$$

for ϵ sufficiently small. In order for (2.37) to not be a violation of (2.36) we must have $\nu(E_\sigma) < \delta$. But, by increasing σ, for any fixed $\epsilon > 0$ we may decrease λ in (2.37) hence decreasing δ in (2.36). Thus, $\nu(E_\sigma) \to 0$ as $\sigma \to \infty$.

Let

$$J_2(\sigma) = \int_{E_\sigma} d\nu(y) \int dx \int_0^{x-\theta^*(x,y;\sigma)} \frac{u}{1 + (x - u)^2/\sigma^2}\, d_u\, P(u, y).$$

Then,

$$\begin{aligned} |J_2(\sigma)| \leqq \sigma \int_{E_\sigma} \left[\frac{2K_y'}{\min(\alpha, 1 - \alpha)} \right]^{\frac{1}{2}} \left(16\, |K_y| + \frac{26K_y'}{\sigma} \right) d\nu(y) \\ \cdot \int_{-\infty}^{\infty} \frac{dx}{\sigma^2 + x^2} \end{aligned} \tag{2.38}$$

by (2.13), (2.24), and (2.35). Thus,

$$(2.39) \qquad |J_2(\sigma)| \leqq \pi \left[\frac{2}{\min(\alpha, 1-\alpha)}\right]^{\frac{1}{2}} (16 + 26\epsilon) \int_{E_\sigma} K'_y \, d\nu(y) \to 0$$

since $K' < \infty$ and $\nu(E_\sigma) \to 0$ as $\sigma \to \infty$.

There remains one term of $f(\sigma)$. Let $E^*_\sigma = (\sim E_\sigma) \cap \{y: K'_y \leqq \epsilon^2\sigma^2\}$. The remaining term is

$$J_3(\sigma) = \int_{E^*_\sigma} d\nu(y) \int dx \int_0^{x-\theta^*(x,y;\sigma)} \frac{u}{1 + (x-u)^2/\sigma^2} \, d_u P(u, y).$$

Then, from the definition of E_σ and (2.13) we obtain

$$(2.40) \qquad |J_3(\sigma)| \leqq \tau \int_{E^*_\sigma} d\nu(y) \int_{-\infty}^{\infty} |\Delta(x, y; \sigma)| \, dx.$$

The first bound in each of (2.19), (2.20), (2.21), and (2.22) is independent of the assumption that $|x| > \sigma$. Hence, they can be used to obtain

$$(2.41) \qquad \begin{aligned} \int_{-\infty}^{\infty} |I'_1(x, y; \sigma)| \, dx &\leqq \sigma^2 K_y \int_{-\infty}^{\infty} \frac{|x|}{\sigma^4 + 2x^2\sigma^2 + \frac{9}{16}x^4} \, dx \\ &< 2\sigma^2 K_y \int_0^{\infty} \frac{x}{(\sigma^2 + \frac{3}{4}x^2)^2} \, dx = \frac{4K_y}{3}; \end{aligned}$$

$$(2.42) \qquad \int_{-\infty}^{\infty} |I''_1(x, y; \sigma)| \, dx \leqq 2K'_y \int_{-\infty}^{\infty} \frac{1}{\sigma^2 + 2x^2} \, dx = \frac{2^{\frac{1}{2}}\pi K'_y}{\sigma};$$

$$(2.43) \qquad \int_{-\infty}^{\infty} |I_2(x, y; \sigma)| \, dx \leqq K'_y \int_{-\infty}^{\infty} \frac{x^2}{(\sigma^2 + x^2)^2} \, dx = \frac{\pi K'_y}{2\sigma};$$

and

$$(2.44) \qquad \int_{-\infty}^{\infty} |I_3(x, y; \sigma)| \, dx \leqq \sigma^2 K'_y \int_{-\infty}^{\infty} \frac{1}{(\sigma^2 + x^2)^2} \, dx = \frac{\pi K'_y}{2\sigma}.$$

Hence,

$$(2.45) \qquad \int_{-\infty}^{\infty} |\Delta(x, y; \sigma)| < 3K_y + \frac{21K'_y}{\sigma}.$$

From (2.40) and (2.45) we obtain

$$(2.46) \qquad |J_3(\sigma)| \leqq \tau \int_{E^*_\sigma} \left(3K_y + \frac{21K'_y}{\sigma}\right) d\nu(y) \leqq \tau\left(3K + \frac{21K'}{\sigma}\right).$$

But τ was chosen arbitrarily and can be made as small as we please by taking σ sufficiently large. Hence, $J_3(\sigma) \to 0$ as $\sigma \to \infty$.

This completes the proof of the theorem.

3. Proof of the corollary. The assumption in the corollary is equivalent to the condition that $P(\cdot, y)$ has a density with respect to either Lebesgue measure or counting measure for each $y \varepsilon \mathcal{Y}$. We will assume that for each y the density, denoted by $p(\cdot, y)$, exists with respect to Lebesgue measure. The proof, in general, is along the same lines provided we recall the convention adopted in Section 2 concerning integrals with quantiles as limits of integration.

We know that X is almost admissible. Suppose the risk when we use $\phi(X, Y)$ as an estimate is the same for a.e. θ as that using X and that, on the exceptional set, $\phi(X, Y)$ has smaller risk. Let $\phi_0(x, y) = \frac{1}{2}[x + \phi(x, y)]$. Then,

$$
\begin{aligned}
\tfrac{1}{2}[L(\theta, \phi(x, y)) &+ L(\theta, x)] \\
&= a[\theta - \tfrac{1}{2} - \tfrac{1}{2}\phi(x, y)] && \text{if } x, \phi(x, y) \leqq \theta \\
&= b[\tfrac{1}{2}x + \tfrac{1}{2}\phi(x, y) - \theta] && \text{if } x, \phi(x, y) \geqq \theta \\
&= \tfrac{1}{2}a[\theta - \phi(x, y)] + \tfrac{1}{2}b(x - \theta) && \text{if } \phi(x, y) < \theta < x \\
&= \tfrac{1}{2}b[\phi(x, y) - \theta] + \tfrac{1}{2}a(\theta - x) && \text{if } x < \theta < \phi(x, y) \\
&\geqq L(\theta, \phi_0(x, y))
\end{aligned}
\tag{3.1}
$$

with strict inequality if, and only if, θ is strictly between x and $\phi(x, y)$. From (3.1) we obtain

$$
\begin{aligned}
\int d\nu(y) \int L(\theta, \phi_0(x, y))p(x - \theta, y)\, dx \\
\leqq \tfrac{1}{2} \int d\nu(y) \int [L(\theta, \phi(x, y)) + L(\theta, x)]p(x - \theta, y)\, dx \\
\leqq \int d\nu(y) \int L(\theta, x)p(x - \theta, y)\, dx.
\end{aligned}
\tag{3.2}
$$

To complete the proof it suffices to show the strict inequality in (3.2) holds for θ in a non-null set. To do so we show that

$$
\int d\theta \int d\nu(y) \int_{\theta \varepsilon (x, \phi(x,y))} p(x - \theta, y)\, dx > 0 \tag{3.3}
$$

because of the condition for strict inequality in (3.1).

Let M denote the left side of (3.3). Then,

$$
\begin{aligned}
M &= \int d\nu(y) \int dx \left| \int_x^{\phi(x,y)} p(x - \theta, y)\, d\theta \right| \\
&= \int d\nu(y) \int |P(x - \phi(x, y), y) - (1 - \alpha)|\, dx.
\end{aligned}
\tag{3.4}
$$

Note that the existence of a density has been used in obtaining the last expression in (3.4). But there exists a set A such that $\lambda\nu(A) > 0$ and $\phi(x, y) \neq x$ for $(x, y) \varepsilon A$. By the unicity of the $(1 - \alpha)$th quantile of $P(\cdot, y)$, it follows that $|P(x - \phi(x, y), y) - (1 - \alpha)| > 0$ for all $(x, y) \varepsilon A$. Hence, $M > 0$. This completes the proof of the corollary.

4. Examples. Two examples are presented in this section to show that a condition such as that of the corollary is needed to guarantee strict admissibility. For both examples we take $\mathcal{Y}$ to be a singleton and, hence, drop y as an argument of P. For both examples let $L(\theta, d) = |\theta - d|$.

Example 1. Let P assign probability $\frac{1}{3}$ to each of the points -1 and $+1$ and assign its remaining probability according to any density with unique median at 0. Then, the conditions of the theorem are satisfied. However, the estimate

$$\begin{aligned} \phi(x) &= x - 1 && \text{if } x \text{ if a positive integer} \\ &= x + 1 && \text{if } x \text{ is a negative integer} \\ &= x && \text{otherwise} \end{aligned}$$

has smaller risk when $\theta = 0, \pm 1$ and the same risk as x for all other θ.

Example 2. Let P assign the Cantor distribution with probability $\frac{1}{2}$ to each of $[0, 1]$ and $[-1, 0]$. Then, P satisfies the conditions of the theorem. Let C be the union of the Cantor set on $[0, 1]$ and all its integer translates. Let $C_+ = \{x : x \varepsilon C, x > 0\}$ and $C_- = \{x : x \varepsilon C, x < 0\}$. The estimate

$$\begin{aligned} \phi(x) &= [x] && \text{if } x \varepsilon C_+ \\ &= -[-x] && \text{if } x \varepsilon C_- \\ &= x && \text{otherwise} \end{aligned}$$

has smaller risk than x when $\theta = 0$. We will show that $\phi(x)$ never has a larger risk than does x.

Let $C_+^* = C_+ \cap [0, 1]$ and $C_-^* = C_- \cap [-1, 0]$. Suppose θ is such that $(C_-^* + \theta) \cap C = \phi$. Equivalently, θ is such that $(C_+^* + \theta) \cap C = \phi$. Then, $E|\theta - \phi(X)| = E|\theta - X|$ since $\phi(X) = X$ a.e. (P).

Assume $\theta > 0$. The argument is similar otherwise. Let $x \varepsilon (C_-^* + \theta)$. If $x < 0$, then $|\theta - \phi(x)| = \theta < |\theta - x|$. If $x > 0$, then $(x + 1) \varepsilon (C_+^* + \theta)$. In this case $|\theta - \phi(x)| - |\theta - x| = -(x - [x])$ while $|\theta - \phi(x + 1)| - |\theta - (x + 1)| = x - [x]$ so that the poorer estimate when we observe x is balanced by an improvement when we observe $x + 1$. Finally, for all $x > \theta$, $|\theta - \phi(x)| = \theta < |\theta - x|$. But this covers the case $x \varepsilon (C_+^* + \theta)$. Since with probability 1 we have $x \varepsilon (C_- + \theta) \cup (C_+ + \theta)$, this implies $E|\theta - \phi(X)| \leqq E|\theta - X|$. Note that equality occurs if $\theta > 1$ and if $(C_-^* + \theta) \cap C$ is a $P-$null set.

5. Comment on the proof of the theorem. It is interesting to note why the Cauchy distribution works so well in both Stein's and our proofs. In our first attempts to find a proof of our theorem we used the uniform distribution on $(-R, R)$. This approach failed due to the fact that the Bayes estimate differs too much from X when $|X|$ is very large compared with R. Thus, the difference between the Bayes risk using X and using the Bayes estimate is too large to go to zero faster than $1/\sigma$. However, the tails of the Cauchy distribution go to zero slowly enough so that the Bayes estimate cannot differ too much from X. This is exhibited in the bound obtained for $|x - \theta^*(x, y; \alpha)|$ in (2.35).

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