ASYMPTOTICALLY OPTIMAL STATISTICS IN SOME MODELS WITH INCREASING FAILURE RATE AVERAGES[1]

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1. Introduction and summary. Birnbaum, Esary and Marshall (1966) have shown that the class $\mathfrak{F}$ of distributions with increasing failure rate averages ($IFRA$) characterizes the concept of wear-out in the sense that $\mathfrak{F}$ is the smallest class that contains the exponential distributions and is closed under the formation of coherent systems.

In this note, statistical inference for models in which the distributions are unknown and $IFRA$ will be considered. Let F and G be defined by

$$F(t) = H(t/\theta) \quad \text{and} \quad G(t) = H(t/\gamma) \tag{1.1}$$

where H is an unknown $IFRA$ distribution with $H(0) = 0$. Then, for the two-sample problem where one tests the equality of the means of F and G, it is shown that the Savage (1956) statistic maximizes the minimum power over $IFRA$ distributions asymptotically. This asymptotic minimax solution is extended to censored samples and it turns out that the Gastwirth (1965) modified version of the Savage statistic is asymptotically minimax for this case. Asymptotic uniqueness of these minimax solutions holds only in a class of rank tests. The results are extended to obtain an estimate of the ratio of the means that minimizes the maximum asymptotic variance over $IFRA$ distributions.

Moreover, the results are shown to hold also for distributions with increasing failure rates (IFR), extensions to the k-sample problem are given, and asymptotic efficiencies of the best test for exponential models are given.

2. The two-sample life-testing problem. $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ are independent random samples from populations with distribution functions F and G. $N = m + n$, $F(t) = H(t/\theta_N)$, $G(t) = H(t/\gamma_N)$, H has the density h and is $IFRA$, i.e., $H(0) = 0$ and for each $t > 0$,

$$d/dt\{-\log [1 - H(t)]/t\} = \ln [1 - H(t)]/t^2 + h(t)/t[1 - H(t)] \geqq 0. \tag{2.1}$$

$r_1, \cdots, r_m$ denote the ranks of the x's in the combined sample. The level α Savage (1956) test ψ_N of $H_0 : \Delta_N = (\theta_N/\gamma_N) = 1$ against $\Delta_N > 1$ rejects for large values of the statistic (see Remark (iii))

$$S_N = m^{-1} \textstyle\sum_{i=1}^{m} -\ln (1 - r_i/(N + 1)). \tag{2.2}$$

Received 25 January 1967.

[1] This research has been partically supported by the Office of Naval Research under Contract Nonr-3656 (18) with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

It is assumed throughout that

$$0 < \lim_{N\to\infty} (m/N) = \lambda < 1. \tag{2.3}$$

Let $0 \leqq c \leqq \infty$ and consider sequences of alternatives $\{\Delta_N\}$ satisfying

$$\lim_{N\to\infty} N^{\frac{1}{2}}(\Delta_N - 1) = c \tag{2.4}$$

Then *the asymptotic power function* $\beta(c; \varphi, H)$ of a test φ_N is defined as the limit of the power for such alternatives, i.e.

$$\beta(c; \varphi, H) = \liminf_{N\to\infty} \beta_N(\varphi_N \mid H), \tag{2.5}$$

where $\beta_N(\varphi_N \mid H) = E(\varphi_N \mid F_N, G_N)$ denotes the power of φ_N when $F_N(t) = H(t/\theta_N)$, $G_N(t) = H(t/\gamma_N)$ and $\Delta_N = \theta_N/\gamma_N$ satisfies (2.4).

Let Φ be the standard normal distribution function. Then the results of Chernoff and Savage (1958), Fatou's Lemma, and a few computations yield.

LEMMA 2.1. *Suppose H has a density h and that $H(0) = 0$, then the asymptotic power function of the level α Savage test ψ_N is given by*

$$\beta(c; \psi, H) = \Phi(\Phi^{-1}(\alpha) + c[\lambda(1 - \lambda)]^{\frac{1}{2}} \textstyle\int_0^\infty th(t)/)(1 - H(t))\, dH(t)). \tag{2.6}$$

The next result shows that ψ and the exponential distribution $K_\sigma(x) = 1 - \exp(-x/\sigma)$ is a saddle point for the asymptotic power function $\beta(c; \varphi, H)$. In other words, ψ is worst for the exponential distribution, but is better than all other tests for this distribution.

THEOREM 2.1. *For all $0 \leqq c \leqq \infty$ and all $\sigma > 0$,*

$$\sup_\varphi \beta(c; \varphi, K_\sigma) = \beta(c; \psi, K_\sigma) = \inf_H \beta(c; \psi, H), \tag{2.7}$$

where H ranges over the class of IFRA distributions with a density, and φ_N ranges over the class of all level α tests.

PROOF. The left hand equality was proved by Capon (1961) by essentially comparing ψ_N with the Neyman-Pearson test for K_σ. To prove the right hand equality, note that (2.1) yields

$$th(t)/(1 - H(t)) \geqq -\ln [1 - H(t)], \tag{2.8}$$

thus

$$\textstyle\int_0^\infty th(t)/(1 - H(t))\, dH(t) \geqq \int_0^\infty -\ln [1 - H(t)]\, dH(t) = 1.$$

The equality signs hold if and only if H has a constant failure rate average, i.e., if and only if H is exponential, thus

COROLLARY 2.1. *If H is IFRA, has a density, and is not exponential, then*

$$\beta(c; \psi, K_\sigma) < \beta(c; \psi, H). \tag{2.9}$$

The minimax property of the Savage statistic now follows at once from Theorem 2.1.

THEOREM 2.2. *The level α Savage test ψ_N is asymptotically minimax over the class Ω of all IFRA distributions with a density*, i.e. *if H ranges over Ω, then*

$$\inf_H \beta(c; \psi, H) \geqq \inf_H \beta(c; \varphi, H) \tag{2.10}$$

for all level α tests φ_N.

REMARKS:

(i) H is said to have increasing failure rate (IFR) [1] if $H(0) = 0$ and $h(t)/[1 - H(t)]$ is nondecreasing in $t > 0$. The class of IFR distributions contains the class of exponential distributions and is contained in the class of $IFRA$ distributions. It follows that Theorem 2.1, Corollary 2.1 and Theorem 2.2 holds also for this class.

(ii) The "lim inf" in the definition of the asymptotic power (2.5) can be replaced by a limit if one assumes conditions as in Lemma 3 of Hodges and Lehmann (1961). The results hold if "lim inf" is replaced by "lim sup" or partially replaced by "lim sup" as in [6].

(iii) An asymptotically equivalent form of the Savage statistic is (see [9, p. 1127]),

(2.11) $$\sum_{i=1}^{m} J_0(r_i), \quad \text{where}$$

$$J_0(k) = \sum_{j=N-k+1}^{N} j^{-1}.$$

(iv) The results in this section hold if one, instead of considering level α tests, considers tests φ_N with asymptotic level α, i.e. tests for which $E(\varphi_N \mid \theta = \gamma) \to \alpha$ as $N \to \infty$.

(v) The one-sided alternative $\Delta > 1$ can be replaced by the two-sided alternative $\Delta \neq 1$.

(vi) The asymptotic minimax result holds if one, instead of considering all F and G with $F(t) = H(t/\theta)$ and $G(t) = H(t/\gamma)$ for some $IFRA$ distribution H, one considers all F and G with $F(t) \leqq H(t/\theta)$ and $G(t) \geqq H(t/\gamma)$.

(vii) For the k-sample problem with model $F_i(x) = H(x/[1 + \theta c_i])$; $i = 1, \cdots, k$; the Puri (1964) extension of the Savage statistic is asymptotically minimax for testing $H_0^k: \theta = 0$ against $\theta > 0$ (or $\theta \neq 0$).

3. Efficiency of the best test for exponential models. When H equals an exponential distribution $K_\sigma(t) = 1 - \exp(-t/\sigma)$, then the uniformly most powerful level α test [7] φ_N^* of $\theta = \gamma$ against $\theta > \gamma$ rejects when

(3.1) $$T = m^{-1} \sum_{i=1}^{m} X_i / n^{-1} \sum_{i=1}^{n} Y_i > F_{2m,2n}(\alpha),$$

where $F_{2m,2n}(\alpha)$ is obtained from the tables of the F distribution with $2m$ and $2n$ degrees of freedom. In this section the performance of T is investigated when the assumption of exponentiality is violated and H is an $IFRA$ distribution. Upon writing

(3.2) $$N^{\frac{1}{2}}(T - \Delta) = N^{\frac{1}{2}}(\bar{X} - \Delta\bar{Y})/\bar{Y},$$

it is clear that $N^{\frac{1}{2}}(T - \Delta)$ has an asymptotic normal distribution with mean zero and variance

(3.3) $$\sigma^2(T) = \Delta^2\sigma^2(H)/\lambda(1 - \lambda)\mu^2(H) \quad \text{where}$$

$$\mu(H) = \int_0^\infty t\,dH(t) \quad \text{and} \quad \sigma^2(H) = \int_0^\infty t^2\,dH(t) - \mu^2(H).$$

When H is exponential, then $\sigma^2(H) = \mu^2(H)$. It follows that when H is such that $\sigma^2(H) \neq \mu^2(H)$, then φ_N^* does not have level α asymptotically, in fact

$$E(\varphi_N^* \mid \theta = \gamma) \to \Phi(\Phi^{-1}(\alpha)\mu(H)/\sigma(H)) \quad \text{as} \quad N \to \infty. \tag{3.4}$$

Thus when $\alpha < \frac{1}{2}$ and $\mu(H) > \sigma(H)$, then the asymptotic level of φ_N^* is less than α. Barlow, Marshall and Proschan (1963) have essentially shown that for *IFRA* distributions, $\mu(H) \geqq \sigma(H)$. The asymptotic power function of φ_N^* is

$$\beta(c; \varphi^*, H) = \Phi(\{\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)]^{\frac{1}{2}}\}\mu(H)/\sigma(H)) \tag{3.5}$$

φ_N^* can easily be modified to have asymptotic level α by dividing

$$\begin{aligned} &N^{\frac{1}{2}}(T-1) \quad \text{by a consistent (when} \quad \theta = \gamma) \text{ estimate of} \\ &r(H) = \sigma(H)/\mu(H); \quad \text{e.g.} \\ &\hat{r}(H) = \hat{\sigma}(H)/\hat{\mu}(H) \quad \text{with} \\ &\hat{\mu}(H) = N^{-1}(\textstyle\sum x_i + \sum y_i) \quad \text{and} \\ &\hat{\sigma}^2(H) = N^{-1}(\textstyle\sum x_i^2 + \sum y_i^2) - \hat{\mu}^2(H). \end{aligned}$$

For this test, $\hat{\varphi}_N$, one has

$$\beta(c; \hat{\varphi}, H) = \Phi(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda]^{\frac{1}{2}}\mu(H)/\sigma(H)). \tag{3.6}$$

Since $\mu(H) \geqq \sigma(H)$ [1] when H is *IFRA*, since $\mu(K_\sigma) = \sigma(K_\sigma)$ for the exponential distribution K_σ, and since $\beta(c; \psi, K_\sigma) = \beta(c, \hat{\varphi}, K_\sigma)$, then (2.7) yields.

THEOREM 3.1. *For all* $0 \leqq c \leqq \infty$ *and all* $\sigma > 0$,

$$\sup_\varphi \beta(c; \varphi, K_\sigma) = \beta(c; \hat{\varphi}, K_\sigma) = \inf_H \beta(c, \hat{\varphi}, H) \tag{3.7}$$

where H *ranges over the class of IFRA distributions and* φ_N *ranges over the class of all tests with asymptotic level* $\alpha \leqq \frac{1}{2}$.

Thus $\hat{\varphi}_N$ is asymptotically minimax in the sense of Theorem 2.2 for the class of *IFRA* distributions and the class of tests with asymptotic level $\alpha \leqq \frac{1}{2}$. To see that this is not true for φ_N^*, let H be an *IFRA* distribution with $\mu(H) > \sigma(H)$, then for each $\alpha < \frac{1}{2}$,

$$\beta(c; \varphi^*, H) < \beta(c; \hat{\varphi}, K_\sigma) \quad \text{for} \quad 0 \leqq c < \infty. \tag{3.8}$$

Let Pitman asymptotic efficiency be as defined in [10]. It follows from (2.6) and (3.6) that the Pitman efficiency of the Savage test ψ_N to the modified classical test $\hat{\varphi}_N$ is

$$e(\psi, \hat{\varphi}) = \sigma^2(H)[\textstyle\int_0^\infty tq(t)\,dH(t)]^2/\mu^2(H) \tag{3.9}$$

where $q(t) = h(t)/[1 - H(t)]$ is the failure rate of H.

The Weibull distribution is defined by

$$\hat{H}(t) = 1 - e^{-at^b}; \qquad a, b > 0; \qquad t \geqq 0. \tag{3.10}$$

If μ_k denotes the kth moment about zero, then

$$(3.11) \qquad \mu_k = a^{-k/b}\Gamma(k/b + 1), \; q(t) = abt^{b-1},$$

and

$$\textstyle\int_0^\infty tq(t)\, d\hat{H}(t) = ab\mu_b = b.$$

Thus for the Weibull distribution

$$(3.12) \quad e(\psi, \hat{\varphi}) = e_b(\psi, \hat{\varphi}) = b^2[\Gamma(2/b + 1) - \Gamma^2(1/b + 1)]/\Gamma^2(1/b + 1).$$

For $b = 1$, the Weibull distribution coincides with the exponential distribution and $e_1(\psi, \hat{\varphi}) = 1$. For $b = 2$, one has the linear failure rate $q(x) = 2ax$ and (3.12) becomes

$$(3.13) \qquad e_2(\psi, \hat{\varphi}) = 16/\pi - 4 \doteq 1.093.$$

Moreover, for $b = 3, 4$ and 10, $e_b(\psi, \hat{\varphi})$ takes on the values 1.20, 1.27 and 15.3. Using L'Hospital's rule, one finds that

$$(3.14) \qquad \lim_{b\to\infty} e_b(\psi, \hat{\varphi}) = \infty.$$

When $b < 1$, the failure rate is decreasing. Stirling's approximation shows that if $r = (1/b)$ is large, then (3.12) is approximately $2^{2r+\frac{1}{2}}r^{r-2}e^{-r} - r^{-2}$, and that

$$(3.15) \qquad \lim_{b\to 0} e_b(\psi, \hat{\varphi}) = \infty.$$

If r is an integer, then

$$(3.16) \qquad e_b(\psi, \hat{\varphi}) = (2r)!/r^2(r!)^2 - 1/r^2.$$

For $r = 2, 3$ and 4 (3.16) becomes 1.25, 2.11 and 4.31 respectively.

It is easy to show ([15] and [9]) that ψ_N is asymptotically most powerful and locally most powerful for the Weibull distribution. Thus

$$(3.17) \qquad e_b(\psi, \varphi) \geqq 1 \quad \text{for all} \quad b > 0$$

and for all test φ_N for which this efficiency is computable. In particular (3.17) holds for $\hat{\varphi}_N$. Thus the Savage test ψ_N is uniformly more efficient than the adjusted classical test $\hat{\varphi}_N$ for the Weibull distribution. Moreover, the Savage test is much better when the failure rate parameter b is large or close to zero. It is conjectured that the Savage statistic is uniformly more efficient than $\hat{\varphi}_N$ for all distributions with monotone failure rate averages; i.e. $e(\psi, \hat{\varphi}) \geqq 1$ for all *IFRA* distributions H with equality iff H is exponential. Here, $e(\psi, \hat{\varphi})$ is given by (3.9) an H is assumed to have a density.

4. Censored Samples. Fix $M \leq N$ and wait until the M smallest X's and Y's have been observed. Let $m' \leqq m$ be the number of X's observed, then the ranks $r_1, \cdots, r_{m'}$ of these X's among $X_1, \cdots X_m, Y_1, \cdots, Y_n$ can be computed. The Gastwirth (1965) modified Savage statistic is

$$(4.1) \quad S_M' = -m^{-1}[\textstyle\sum_{i=1}^{m'} \ln(1 - r_i/(N+1)) + m' + (m - m')\ln(1 - M/(N+1))].$$

It is assumed that

$$0 < \lim_{N\to\infty} (M/N) = p < 1. \tag{4.2}$$

The asymptotic power function of the level α test ψ_M that rejects for large values of S_M' can be computed using [9] and [8]. One gets

$$\begin{aligned}(4.3)\quad &\beta(c; \psi_p, H) \\ &= \Phi(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{\frac{1}{2}} \textstyle\int_0^{H^{-1}(p)} th(t)/(1-H(t))\, dH(t)).\end{aligned}$$

From (2.8), it follows that when H ranges over the class of *IFRA* distributions, then

$$\begin{aligned}(4.4)\quad \inf_H \beta(c; \psi_p, H) &= \beta(c; \psi_p, K_\sigma) \\ &= \Phi(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{\frac{1}{2}}[p + (1-p)\ln(1-p)])\end{aligned}$$

Since Hájek (1962) and Gastwirth (1965) have shown that

$$\beta(c; \psi_p, K_\sigma) \geqq \beta(c; \varphi, K_\sigma) \tag{4.5}$$

for all level α tests φ_N, then the results of Section 2 hold for ψ_M.

5. Asymptotic uniqueness. Stein (1956) and Hájek (1962) have shown that one can obtain asymptotically optimal statistics by estimating the underlying distribution. Although these statistics are impractical, they show that one can not hope for asymptotic uniqueness in the class of all tests with asymptotic level α.

Consider the class of one-sided level α rank tests $\mathfrak{I}$ [5] based on statistics of the form

$$T_M = T_M(J_N) = m^{-1} \textstyle\sum_{i=1}^m J_N(r_i/(N+1)), \tag{5.1}$$

where there exists a function J which is continuous except for possibly a finite number of jump discontinuities and which satisfies

$$\textstyle\int_0^1 J^2(u)\, du < \infty \quad \text{and} \quad \lim_{N\to\infty} \int_0^1 [J_N(u) - J(u)]^2\, du = 0 \tag{5.2}$$

and the conditions of Comment 3.8 of Hájek (1962). Let $\mathfrak{F}'$ be the class of *IFRA* distributions H with a density h which has the Radon — Nikodym derivative h' with respect to Lebesgue measure and satisfies

$$\textstyle\int_0^\infty [x^2h'(x)/h(x)]^2\, dH(x) < \infty. \tag{5.3}$$

THEOREM 5.2. *The Savage-Gastwirth test ψ_p is asymptotically uniquely minimax for $\mathfrak{I}$ and $\mathfrak{F}'$*, i.e., *if $\varphi_0 = \varphi_0(J_N) \in \mathfrak{I}$, if H ranges over $\mathfrak{F}'$, and if*

$$\inf_H \beta(c; \varphi_0, H) \geqq \inf_H \beta(c; \varphi, H) \tag{5.4}$$

for all $\varphi \in \mathfrak{I}$, then there exist constants a_N and b_N such that

$$N^{\frac{1}{2}}[S_M' - (a_N T_M(J_N) + b_N)] \to 0 \tag{5.5}$$

in probability as $N \to \infty$ provided (2.3), (4.2) *and* (2.4) *hold with $c < \infty$.*

PROOF. (2.7) and (5.4) show that $\beta(c; \varphi_0, K_\sigma) = \beta(c; \psi_p, K_\sigma)$. Thus φ_0 is asymptotically optimal for the exponential distribution K_σ. From Hájek (1962), it follows that the correlation coefficient ρ_N for S_M' and T_M satisfies

$$\rho_N(S_M', T_M \mid K_\sigma; \Delta = 1) \to 1 \quad \text{as} \quad N \to \infty. \tag{5.6}$$

This implies that for regression coefficients a_N and b_N,

$$E(N[S_M' - (a_N TN + b_N)]^2 \mid K_\sigma; \Delta = 1) \to 0. \tag{5.7}$$

Since S_M' and T_M are distribution free, (5.7) holds not only for K_σ, but for general H. The result now follows from the contiguity arguments of LeCam and Hájek (e.g., [9]).

REMARK. As in [6], this asymptotic uniqueness result can be extended to the class of all tests that are based on statistics that are appropriately asymptotically normal and distribution-free (see [6], pages 623–624).

6. Estimation. Barlow and Proschan (1966) have shown that the estimates of the mean that are optimal for exponential models are not robust for IFR distributions. Here an asymptotically robust estimate of the ratio μ_1/μ_2 of the means of X and Y is constructed using the methods of Hodges and Lehmann (1963). Write $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_n)$, $ax = (ax_1, \cdots, ax_m)$ etc., and let

$$s(x, y) = S_M' \tag{6.1}$$

be the Savage-Gastwirth statistic (4.1). $\mu_1/\mu_2 = \theta\mu(H)/\gamma\mu(H) = \theta/\gamma = \Delta$, so one estimates Δ.

Note that $N^{\frac{1}{2}}s(X, \Delta Y)$ asymptotically tends to be normally distributed about the point 0 [9]. Let

$$\Delta^* = \sup\{\Delta: s(x, \Delta y) \geqq 0\} \quad \text{and} \tag{6.2}$$

$$\Delta^{**} = \inf\{\Delta: s(x, \Delta y) \leqq 0\}$$

and define the estimate $\hat{\Delta}$ of Δ by

$$\hat{\Delta} = \hat{\Delta}(x, y) = \tfrac{1}{2}(\Delta^* + \Delta^{**}). \tag{6.3}$$

Since $s(ax, ay) = s(x, y)$ for each $a > 0$ by the invariance properties of ranks, then

$$\hat{\Delta}(ax, ay) = \hat{\Delta}(x, y) \quad \text{for all} \quad a > 0. \tag{6.4}$$

Moreover, using this, the Definition (6.2), and noting that $s(x, \Delta y)$ is decreasing in Δ, one concludes

$$\hat{\Delta}(ax, by) = (a/b)\hat{\Delta}(x, y) \quad \text{for all} \quad a, b > 0; \tag{6.5}$$

i.e. $\hat{\Delta}$ is scale invariant,

$$P_\Delta(\hat{\Delta}/\Delta \leqq t) = P_1(\hat{\Delta} \leqq t), \tag{6.6}$$

$$\Delta^* \leqq \Delta^{**}, \tag{6.7}$$

$$(6.8) \qquad P(\Delta^* < t) = P(s(x, ty) < 0),$$

$$(6.9) \qquad P(\Delta^{**} \leqq t) = P(s(x, ty) \leqq 0),$$

$$(6.10) \quad P(s(x, ty) < 0) \leqq P(\hat{\Delta} \leqq t) \leqq P(s(x, ty) \leqq 0).$$

LEMMA 6.1. *If H satisfies (5.3) and $H(0) = 0$, then*

$$\lim_{N\to\infty} P_\Delta(N^{\frac{1}{2}}[(\hat{\Delta}/\Delta) - 1] \leqq t)$$
$$= \Phi(t[\lambda(1-\lambda)/p]^{\frac{1}{2}} \textstyle\int_0^{H^{-1}(p)} xh(x)/(1 - H(x))\, dH(x)).$$

PROOF. (6.6) shows that one can let $\Delta = 1$. From (6.10) it follows that

$$\lim_{N\to\infty} P_1(N^{\frac{1}{2}}(\hat{\Delta} - 1) \leqq t) = \lim_{N\to\infty} P_1(\hat{\Delta} \leqq 1 + tN^{-\frac{1}{2}})$$
$$= \lim_{N\to\infty} P_1(s(X, (1 + tN^{-\frac{1}{2}})Y) \leqq 0)$$
$$= \lim_{N\to\infty} P_{\Delta_N}(s(X, Y) \leqq 0),$$

where $\Delta_N = 1/(1 + tN^{-\frac{1}{2}})$. Since $N^{\frac{1}{2}}(\Delta_N - 1) \to t$ as $N \to \infty$, the result follows from (4.3).

Lemma 6.1 shows that the asymptotic variance of $N^{\frac{1}{2}}[(\hat{\Delta}/\Delta) - 1]$ is

$$(6.11) \quad V(\hat{\Delta}, H) = 1/[\lambda(1-\lambda)/p]^{\frac{1}{2}} \textstyle\int_0^{H^{-1}(p)} th(t)/(1 - H(t))\, dH(t).$$

Moreover, (4.4) shows that the maximum asymptotic variance over *IFRA* distributions is

$$(6.12) \quad \sup_H V(\hat{\Delta}, H) = V(\hat{\Delta}, K_\sigma) = 1/[\lambda(1-\lambda)/p]^{\frac{1}{2}}[p + (1-p)\ln(1-p)].$$

Let $\mathfrak{F}'$ be as in Section 5, then the results of the previous sections yield:

THEOREM 6.1. *$\hat{\Delta}$ is asymptotically minimax over $\mathfrak{F}'$ and the class $\mathcal{E}$ of scale invariant estimates that are asymptotically normal;* i.e., *if $V(\tilde{\Delta}, H)$ denotes the asymptotic variance of $n^{\frac{1}{2}}[(\tilde{\Delta}/\Delta) - 1]$ for each estimate $\tilde{\Delta} \in \mathcal{E}$, then*

$$(6.13) \quad \sup_H \{V(\hat{\Delta}, H): H \in \mathfrak{F}'\} \leqq \sup_H \{V(\tilde{\Delta}, H): H \in \mathfrak{F}'\} \quad \textit{for all} \quad \tilde{\Delta} \in \mathcal{E}.$$

Moreover, $V(\hat{\Delta}, H)$ has the saddle-point property

$$(6.14) \qquad \sup_H V(\hat{\Delta}, H) = V(\hat{\Delta}, K_\sigma) = \inf_{\tilde{\Delta}} V(\tilde{\Delta}, K_\sigma),$$

where H ranges over $\mathfrak{F}'$ and $\tilde{\Delta}$ over $\mathcal{E}$.

A different approach to the problem of obtaining asymptotic minimax estimates is given by Huber (1963). As in his case, the above minimax result can be extended to the class of all non-superefficient estimates, e.g. the class of all scale invariant estimates (see [12, pages 81–82]).

REMARK. One possible method of computing $\hat{\Delta}$ will now be illustrated for the situation in Section 2. Using Remark (iii), one writes $s'(x, y) = m^{-1} \sum J_0(r_i) - 1$, with $J_0(i) = \sum_{j=N-i+1}^{N} (1/j)$. $\hat{\Delta}$ is computed by trial and error as follows: Compute $\hat{\Delta}_1 = (\bar{x}/\bar{y})$ and $s'(x, \hat{\Delta}_1 y)$. If $s'(x, \hat{\Delta}_1 y) > 0$ (<0), adjust $\hat{\Delta}_1$ to obtain $\hat{\Delta}_2$ by multiplying $\hat{\Delta}_1$ by a number greater (less) than one. In general, if $s'(x, \hat{\Delta}_k y) > 0$ (<0), adjust $\hat{\Delta}_k$ to obtain $\hat{\Delta}_{k+1}$ by multiplying $\hat{\Delta}_k$ by a number greater (less) than one. Repeat until $s'(x, \hat{\Delta}_k y)$ is zero or sufficiently close to zero. $\hat{\Delta}_k$ is then $\hat{\Delta}$ or approximately $\hat{\Delta}$.

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