SPARSITY ENFORCING EDGE DETECTION METHOD FOR BLURRED AND NOISY FOURIER DATA: IS THE REGULARIZATION PARAMETER IMPORTANT FOR IMAGE SEGMENTATION?

Rosemary Renaut
This is joint work with
Wolfgang Stefan, Rice University,
Aditya Viswanathan, Cal Tech,
and Anne Gelb, Arizona State University

AIP 2011

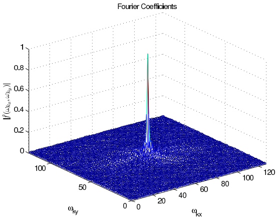
May 2011

Outline

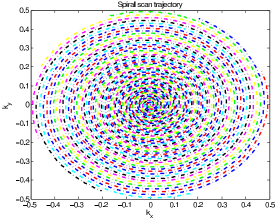
Magnetic Resonance Imaging

- We acquire spectral data (Fourier domain).
- Data may be acquired along **non-Cartesian** sampling trajectories.
  - resistance to motion artifacts
  - ease in generating field gradients
- Data can be degraded by **blur**.
- Data can be **noisy**

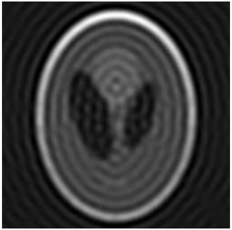
Image Reconstruction



(a) Acquired Fourier Samples



(b) Spiral Sampling Trajectory



(c) Reconstructed Shepp Logan

Figure: Simulated MR no blur or noise, but non Cartesian Data[1]

Image exhibits aliasing and ringing: segmentation would be challenging

[1] Sampling pattern courtesy Dr. Jim Pipe, Barrow Neurological Institute, Phoenix, Arizona

Goal: given spectral data obtain an image segmentation

Estimate edges from blurred , noisy Fourier data on non-equispaced grids.

Assume a finite number of Fourier Coefficients available for a piecewise function.

- These may be **noisy**
- The function may be **blurred**
- Data may be collected at **non Cartesian grid points**

Desire accurate and robust detection of **jump discontinuities** to segment the data.

Validation examine **true classifications** of edges in data.

What do we mean by a jump?

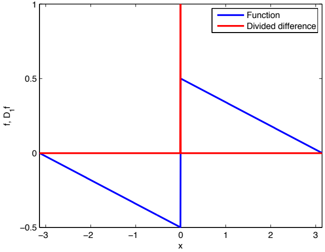
Assume f is piecewise smooth

- Its *jump function* is defined by

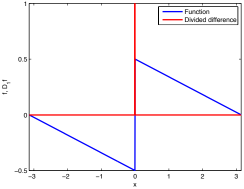
$$[f](x) := f(x^+) - f(x^-)$$

- A jump discontinuity is a local feature; i.e., the jump function at any point x only depends on the values of f at x^+ and x^-.

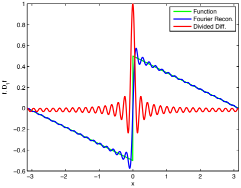
Function in blue and Jump Function in Red



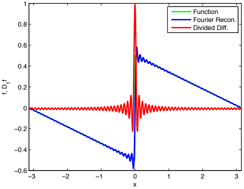
Jump Detection from Reconstructed Fourier data



(a) Physical space jump detection



(b) Fourier space using $N = 32$ modes



(c) Fourier space using $N = 64$ modes

Figure: Jump Detection of the unit ramp function with original data on 1024 grid and using first order divided difference

Gibbs at jump makes edge detection infeasible: alternative required

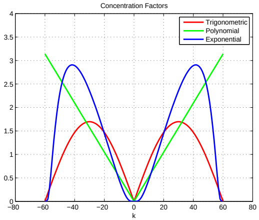
Concentration Factor Method (Gelb, Tadmor)

Concentrate edges using convolution with $C_N^\sigma(x)$

Approximate $[f](x)$ using generalized conjugate partial Fourier sum

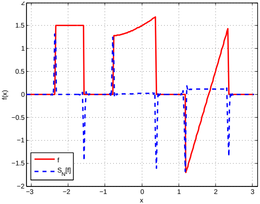
$$S_N^\sigma[f](x) = i \sum_{k=-N}^{N} \hat{f}_k \,\mathrm{sgn}(k)\, \sigma\left(\frac{|k|}{N}\right) e^{ikx} = (f * C_N^\sigma)(x) \tag{1}$$

Factor	Expression
Trigonometric	$\sigma_T(\eta) = \dfrac{\pi \sin(\alpha\,\eta)}{Si(\alpha)}$ $Si(\alpha) = \displaystyle\int_0^\alpha \frac{\sin(x)}{x}\,dx$
Polynomial	$\sigma_P(\eta) = -p\,\pi\,\eta^p$ p is the order of the factor
Exponential	$\sigma_{\exp}(\eta) = C\,\eta\,\exp\left(\dfrac{1}{\alpha\,\eta\,(\eta-1)}\right)$ C - normalization; $\alpha > 0$ - order



Envelopes in Fourier Space

Example: No Noise, No Blur, Cartesian Grid



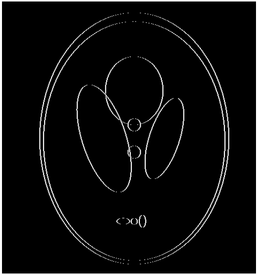
(a) Trigonometric Factor: Jump Response

(b) Shepp Logan phantom

- Apply concentration to each dimension

$$S_N^\sigma[f](x(\bar{y})) = i \sum_{l=-N}^{N} \operatorname{sgn}(l)\, \sigma\left(\frac{|l|}{N}\right) \cdot \sum_{k=-N}^{N} \hat{f}_{k,l}\, e^{i(kx+l\bar{y})}$$

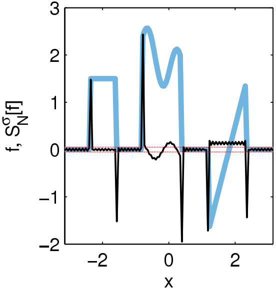
overbar represents constant dimension.



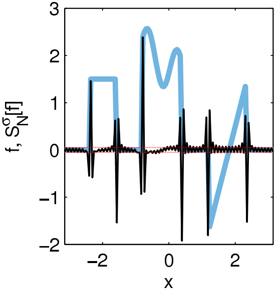
(c) Edge Map

Illustration $N = 64$: Perfect Case. Black line is the jump function

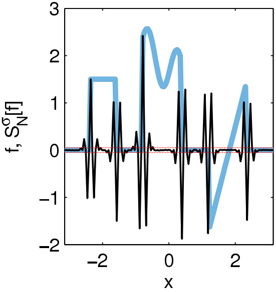
$$f(x) = \begin{cases} 3/2 & \text{for} \quad -\frac{3\pi}{4} \leq x < -\frac{\pi}{2} \\ 7/4 - x/2 + \sin(7x - 1/4) & \text{for} \quad -\frac{\pi}{4} \leq x < \frac{\pi}{8} \\ x11/4 - 5 & \text{for} \quad \frac{3\pi}{8} \leq x < \frac{3\pi}{4} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$



(d) Polynomial p=1 (σ_1)



(e) Polynomial p=2 (σ_2)



(f) Exponential ($\sigma_{\exp}$)

Not sufficient: false positives and false negatives depend on red threshold

Improving jump detection

The minmod to improve the approximation (Gelb and Tadmor (2006))

Use the **minmod** function over different concentration functions

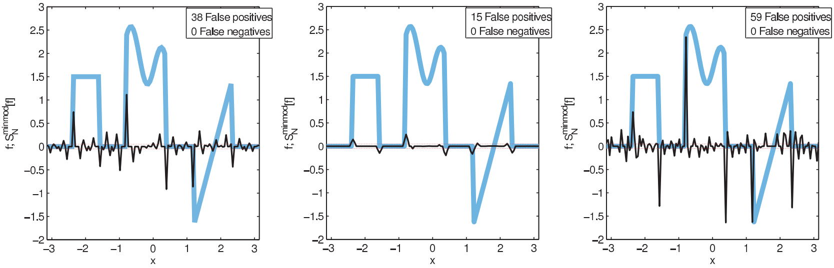
$$\mathbf{minmod}\{a_1, \ldots, a_n\} := \begin{cases} s\min(|a_1|, |a_2|, \ldots, |a_n|) & s := \mathrm{sgn}(a_i), \forall i \\ 0 & \text{otherwise} \end{cases}, \tag{3}$$

yielding the approximation obtained by finding the jump approximation with multiple σ

$$S_N^{MM}[f](x) = \mathbf{minmod}\{S_N^{\sigma_1}[f](x), S_N^{\sigma_2}[f](x), \ldots, S_N^{\sigma_n}[f](x)\}. \tag{4}$$

Apply multiple concentration factors: pick edges detected for all CFs.

Minmod CF edge detection for noisy and blurred functions: 2% threshold



(g) Under sampling (h) Blurring by a Gaussian (i) Noise contamination

Figure: False positives & negatives. (g) 10% missing Fourier Coefficients. (h) Gaussian blur of variance $\tau = 0.05$, for point spread function coefficients $\hat{h}_k = e^{-\frac{k^2\tau^2}{2}}$. (i) Noise of variance .015 applied to Fourier Coefficients.

For blurred functions the edges may be missed, for noisy functions or with missing data too many edges are determined.

The Jump Response: What do we expect to see at a jump

Let $r(x)$ denote the unit ramp function.

$$r(x) = \begin{cases} \frac{x-\pi}{2\pi} & x < 0 \\ \frac{\pi - x}{2\pi} & x > 0 \end{cases}, \qquad [r](x) = \begin{cases} 1 & x = 0 \\ 0 & \text{else} \end{cases}$$

Definition : Jump Response

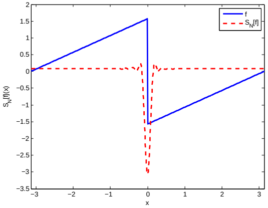
The **jump response**, denoted by $W_N^\sigma(x)$, is defined as the jump function approximation of the unit ramp as generated by the concentration sum, i.e.,

$$W_N^\sigma(x) := S_N^\sigma[r](x) = i \sum_{|k| \le N} \hat{r}(k)\operatorname{sgn}(k)\sigma\left(\frac{|k|}{N}\right) e^{ikx}$$

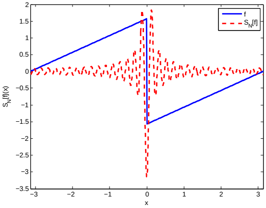
$$= \frac{1}{2\pi} \sum_{0<|k|\le N} \frac{\sigma\left(\frac{|k|}{N}\right)}{|k|} e^{ikx}$$

The JR describes the unique oscillatory pattern of the jump function approximation in the immediate vicinity and away from jumps.

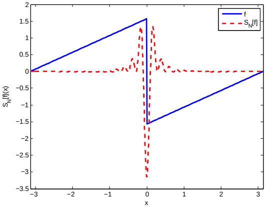
Sample Jump Responses of the Concentration Factors



(a) Trigonometric Factor



(b) Polynomial Factor



(c) Exponential Factor

JR depends on CF

Suggests define the JR as a Matching Waveform (MW)

Correlate Edge Function with the Matching Waveform

Leads to a **Matching Waveform Concentration Factor**

Mathematical Description MWCF (A. Gelb and D. Cates, 2008)

- Jump approximation at $x = \xi$ depends on size $[f]$ and location ξ, not f:

$$S_N^\sigma[f](x) = \frac{[f](\xi)}{\pi} \sum_{k=1}^{N} \sigma_{k,N} \frac{\cos k(x-\xi)}{k} + \mathcal{O}\left(\frac{\log N}{N}\right).$$

- Use JR $W_N^\sigma(x) = \sum_{k=1}^{N} \sigma_{k,N} \frac{\cos kx}{k}$.
- Apply CF and correlate with JR

$$S_N^{\sigma_{mw}}[f](x) = \frac{1}{\gamma_{mw}}(S_N^\sigma[f] * W_N^\sigma)(x), \qquad \gamma_{mw} = \frac{1}{\pi} \sum_{k=1}^{N} \left(\frac{\sigma_{k,N}}{k}\right)^2 \quad (5)$$

- Gives admissible *matching waveform concentration factor* (MWCF)

$$\sigma_{mw}\left(\frac{|k|}{N}\right) := \frac{1}{\gamma_{mw}} \sigma\left(\frac{|k|}{N}\right) \int_{-\pi}^{\pi} W_N^\sigma(\rho) \exp\left(-ik\rho\right) d\rho. \quad (6)$$

MWCF performs better in the presence of noise, does not remove oscillations. Performance deteriorates for nearby jumps.

Mathematical Description: Estimate Jump $[f]$ given $\hat{g}_k$ of Noisy Blurred f

Appealing to sparsity

Given $\hat{g}_k$ for blur function h and noise n, $\hat{g}_k = \hat{h}_k \hat{f}_k + \hat{n}_k$

Approximate $[f]$ from $[g]$, given $\hat{g}_k$: $(\widehat{S_N^\sigma[g]})_k = \left(i\sigma_{|k|,N}\,\mathrm{sgn}(k)\right)\hat{g}_k$

Observe $\hat{g}_k \approx \hat{h}_k \hat{f}_k$ yields
$\left(i\sigma_{|k|,N}\,\mathrm{sgn}(k)\right)\hat{g}_k = (\widehat{S_N^\sigma[g]})_k \approx \hat{h}_k(\widehat{S_N^\sigma[f]})_k$

Seek sparse y which also approximates the jump function of f

Convolve y with $W_N^\sigma(x)$ to approximate jump $S_N^\sigma[f](x)$

$$(\widehat{S_N^\sigma[f]})_k \approx (\widehat{W_N^\sigma * y})_k = (\hat{W}_N^\sigma)_k \hat{y}_k, \tag{7}$$

Obtain for $(\hat{W}_N^\sigma)_k = \frac{\pi}{|k|}\sigma_{|k|,N}$, $|k| \leq N, k \neq 0$

$$\hat{h}_k(\hat{W}_N^\sigma)_k \hat{y}_k \approx i\sigma_{|k|,N}\,\mathrm{sgn}(k)\hat{g}_k$$

A Discrete Variational Formulation: for blur, noise and Cartesian

The Formulation

- Introduce necessary matrices

$$\begin{aligned}
\Sigma &= \operatorname{diag}\left(\sigma\left(\frac{|-N|}{N}\right), \cdots, 0, \cdots, \sigma\left(\frac{|N-1|}{N}\right)\right) \quad \text{Concentration} \\
H &= \operatorname{diag}(\frac{\pi}{|-N|}\hat{h}_{-N}, \cdots, 0, \cdots, \frac{\pi}{|N-1|}\hat{h}_{N-1}) \quad \text{Blur} \\
F_{kj} &= \frac{1}{2N}(-1)^k \exp(\frac{-i\pi jk}{N}) \quad \text{where} \quad \hat{\mathbf{y}} = F\mathbf{y}((x)) \text{ Fourier Transform.}
\end{aligned}$$

- Find discrete approximation $\mathbf{y}$ to $y(x)$ from second order cone problem.

$$\mathbf{y} = \arg\min_{\mathbf{u}} \|\mathbf{u}\|_1 \qquad \text{subject to} \qquad \|\Sigma(HF\mathbf{u} - \mathbf{b})\|_2^2 \le \delta, \tag{8}$$

$\mathbf{b} = (-i\hat{g}_{-N}, \cdots, 0, \cdots, i\hat{g}_{N-1})$. Σ weights the data fit term.

Reconstruct edges with MWCF and Impose Sparsity as a Regularization

Summary of The Approach

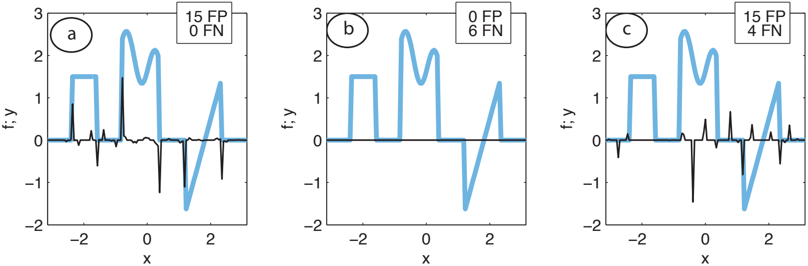
1. Concentration function applied in Fourier domain enhances scales at edges
2. Find Fourier expansion approximating jumps using concentration at the edges
3. Correlate obtained jump approximation to JR. (use MWCF)
4. Impose sparseness in space on the jump function: **Regularization term**
5. Find sparse jump function which matches the jump function of the data: **Data Fit Term**
6. Introduce regularization parameter λ and solve

$$\mathbf{y} = \arg\min_{\mathbf{u}} \{\lambda \|\mathbf{u}\|_1 + \frac{1}{2} \|\Sigma(HF\mathbf{u} - \mathbf{b})\|_2^2\}$$

Recall Goal: **segment data**

How important is the regularization parameter λ?

Experiments with $N = 64$ and under sampling but no noise and no blur.

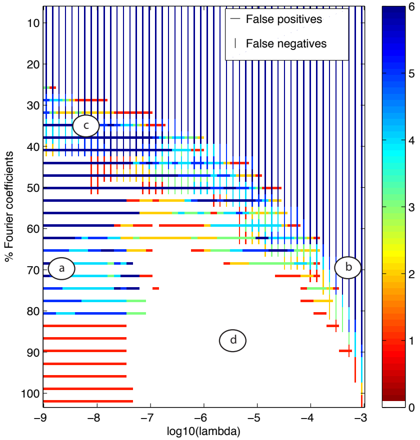


(d) $(\mathrm{FC}\ , \lambda) = 70\%, 10^{-9}$ (e) $(\mathrm{FC}\ , \lambda) = 70\%, 10^{-3}$ (f) $(\mathrm{FC}, \lambda) = 35\%, 10^{-8}$

Figure: Edge detection using σ_{exp}. FC is percentage of Fourier Coefficients used, y is the thin line, unseen in (b). FP and FN are count of misidentified edges, either false positive or false negative, using 2% threshold on $\mathbf{y}$.

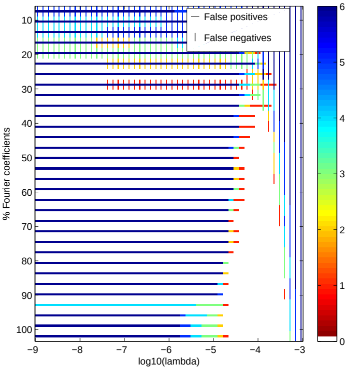
One sees the effect of the regularization parameter comparing (a) and (b), and of reducing the number of Fourier Coefficients comparing (a) and (c).

Under sampling, no noise, no blur. False Positives and False Negatives with Waveform

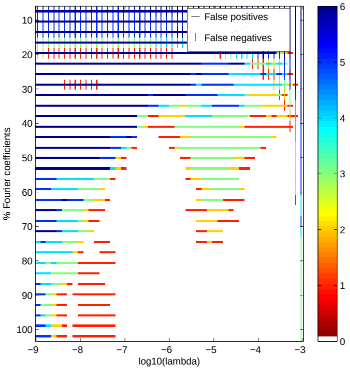


Illustrating impact of choice of regularization parameter in relation to the number of Fourier Coefficients sampled, and impact on the number of False Positives and False Negatives
Region (d) shows that there is a range of regularization parameters for which the method is robust with respect to correct identification of edges provided up to about 70% of coefficients are retained.

Polynomial Concentration $N = 64$, no noise, no blur, missing Fourier data



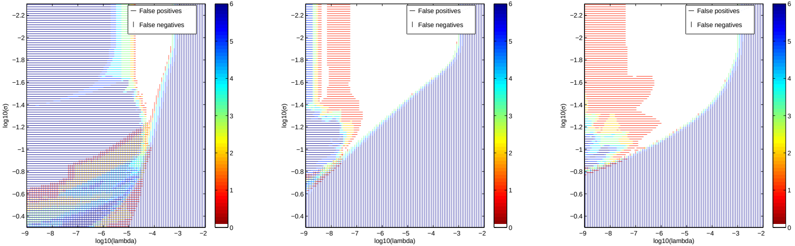
(a) N=64, σ_1



(b) N=64, σ_2

Higher order concentration factors perform better at capturing the edges correctly for a wider range of regularization parameters.

Edge detection in the presence of blur in the coefficients. $N = 64$.

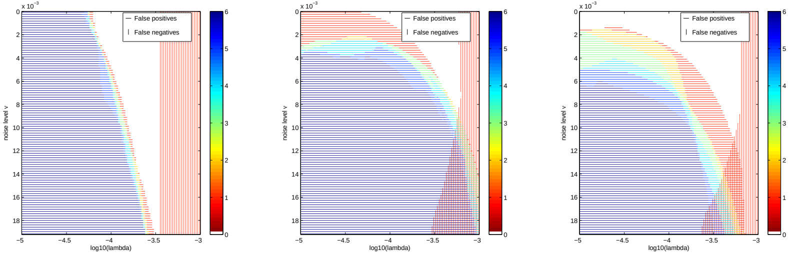


(c) σ_1 (d) σ_2 (e) σ_{exp}

Figure: Edge detection in blurred signals using σ_p, for $p = 1, 2$, and σ_{exp}. All plots show that the method can handle blurring where the traditional CF method fails.

Gaussian blur of variance $\tau = 0.05$, for point spread function coefficients $\hat{h}_k = e^{-\frac{k^2\tau^2}{2}}$. The higher order concentration factors again perform better at capturing the edges correctly for a wider range of regularization parameters.

Edge detection in the presence of additive noise in the coefficients. $N = 64$.

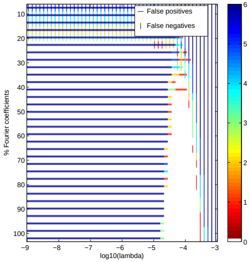


(a) σ_1 (b) σ_2 (c) σ_{exp}

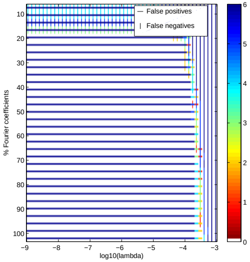
Figure: Edge detection in signals with noise of variance .015 applied to Fourier Coefficients. All plots show that the method can handle noise where the traditional CF method fails.

In this case the higher order exponential concentration factor performs better than the quadratic, perhaps due to its inherent filtering of coefficients contaminated with noise.

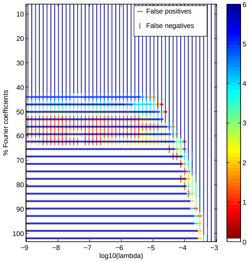
Is waveform correlation required? Examples without waveform $N = 64$



(a) σ_1



(b) σ_3

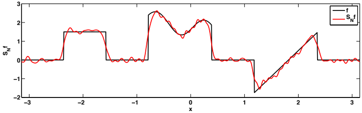


(c) $\sigma_{\exp}$

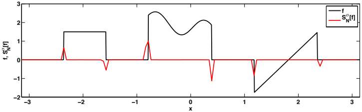
Figure: No blur, no noise missing Fourier data: classification capability robustness with respect to choice of λ.

When using the low order polynomial concentration factor the method is quite robust, but for higher order concentration factors the method is more sensitive to the choice of λ and the ability to correctly detect edges is limited.

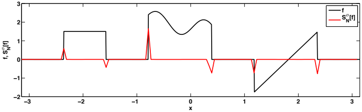
Example for blurred and noisy non-harmonic Fourier data, $N = 64$



(a) Fourier reconstruction of blurred noisy data



(b) Jittered spectral data using σ_1



(c) Log spectral data using σ_{exp}, $\alpha = 2$

- Gaussian blur variance $\tau = .05$.
- Additive white complex Gaussian noise, variance .015.
- Regularization parameters .002 in (e) and .0013 in (f).
- Solution is more sensitive to choice of regularization parameter λ than for the harmonic case.
- Determination of λ is harder for the log than jittered data.

A Two Dimensional Example (Stefan and Yin)

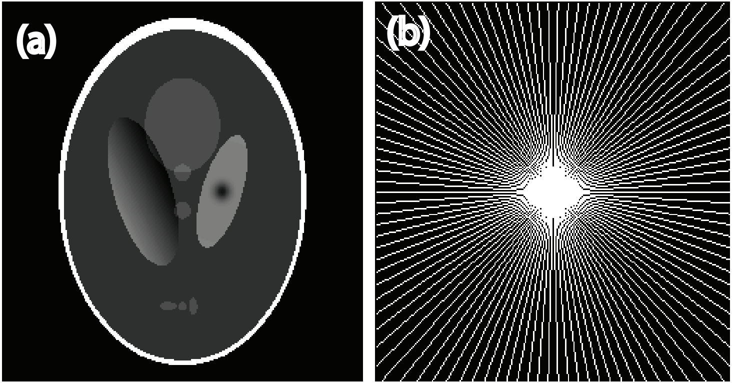


Figure: A modified Shepp logan phantom with gradients and a radial sampling pattern.

Some Two Dimensional Results (Stefan and Yin)

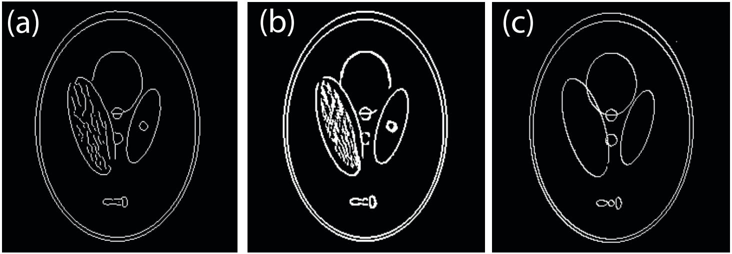


Figure: Edge detection using (a) Canny edge detector (matlab) after reconstruction from the radial samples using TV. (b) Wavelet edge detector on TV reconstruction. (c) A 5th order FD edge detector

Conclusions

A New Formulation for Image Segmentation

- Approach is variational formulation employing sparsity of JF to find edges for noisy/blurred signals.
- Approach is a regularized deconvolution of the approximate jump function.
- Approach requires the MW to improve robustness with respect to choice of the regularization parameter.
- Approach is successful for missing Fourier data.
- Approach suggests new way to think of the Regularized Parameter Estimation Problem:

Is the Parameter Robust for Segmentation

References

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5. E. TADMOR AND J. ZOU, *Novel edge detection methods for incomplete and noisy spectral data*, in J. Four. Analy. App. 14(5) (2008), 744-763.
6. W. STEFAN, A. VISWANATHAN, A. GELB, AND R. A. RENAUT, *Sparsity enforcing edge detection method for blurred and noisy Fourier data*, (2010).

THANK YOU!

Non-harmonic Fourier data

Motivation: Modern MRI scanners optimize data collection

- Fourier data collected on non-cartesian representations of the k-space.
- Non-harmonic Fourier data, $\hat{f}(\omega_k)$, for piecewise-analytic $f \in L^2(\mathbb{R}(-\pi,\pi))$ are defined by

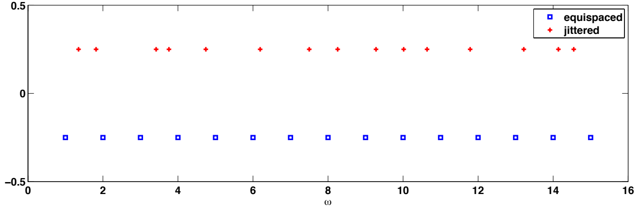
$$\hat{f}(\omega_k) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-i\omega_k x}dx, \;\; \omega_k \notin \mathbb{Z}. \tag{9}$$

- Extension of convolution form of jump approximation (1)

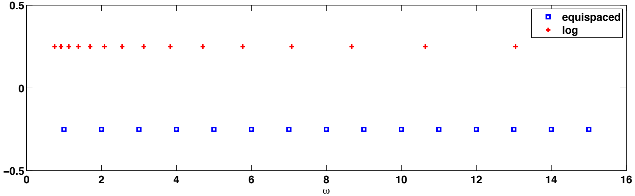
$$\tilde{S}_N^\sigma[f](x) = (f * \tilde{C}_N^\sigma)(x) := i\sum_{k=-N}^{N} \alpha_k \hat{f}(\omega_k)\mathrm{sgn}(\omega_k)\sigma\left(\frac{|\omega_k|}{N}\right)e^{i\omega_k x}.$$

Coefficients α_k are weights for non-uniform trapezoidal rule approximation of inverse Fourier integral. (*convolutional gridding*).

Examples: Non-harmonic sampling (right half plane), $N = 16$

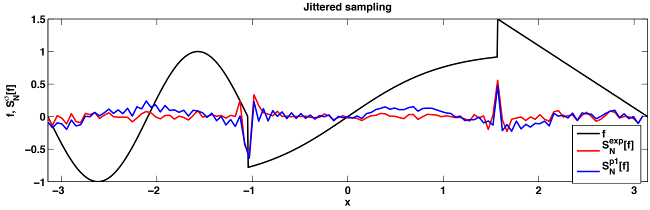


(a) Jittered sampling $\omega_k = k \pm \zeta_k,\ \zeta_k \sim U\left[0, \theta\right),\ k = -N, -(N-1), \ldots, N.$

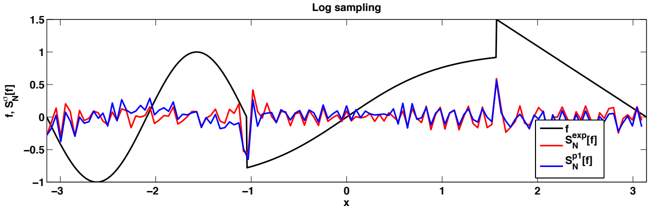


(b) Log sampling

Applying the Edge Detector with the non-harmonic concentration sum



(a) Edges from jittered sampling using σ_1 and $\sigma_{\exp}$ with $\alpha = 2$.



(b) Edges from log sampling using σ_1 and $\sigma_{\exp}$ with $\alpha = 2$.

Extending the Sparsity Approach

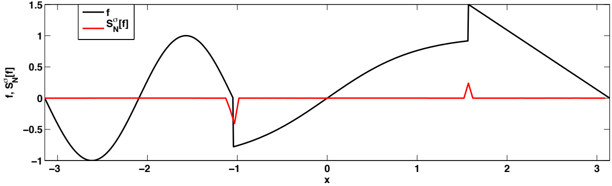
- $\mathbf{g} = (\hat{g}(\omega_{-N}), ..., \hat{g}(\omega_{N-1}))^T$ non-harmonic measurements.
- $\mathbf{y}$, approximates $[f]$ on equispaced grid $x_j = \frac{\pi j}{N} - \pi,\ j = 0, \ldots, 2N-1$.
- Introduce Σ diagonal matrix of concentration factors, H diagonal matrix of blur coefficients, $F \in \mathbb{C}^{2N\times 2N}$ discrete non-harmonic Fourier matrix, and W a Toeplitz matrix whose rows contain shifted replicates of the jump waveform $W_N^\sigma(x)$

$$\begin{aligned}\Sigma &= i\mathrm{diag}\left(\mathrm{sgn}(\omega_k)\,\sigma\left(\frac{|\omega_k|}{N}\right)\right),\ H = \mathrm{diag}\left(\hat{h}(\omega_k)\right)\\ F_{kj} &= \exp\left[i\left(-\pi + \frac{\pi j}{N}\right)\omega_k\right], \quad k = -N, \ldots, N-1,\ j = 0, ..., 2N\end{aligned}$$

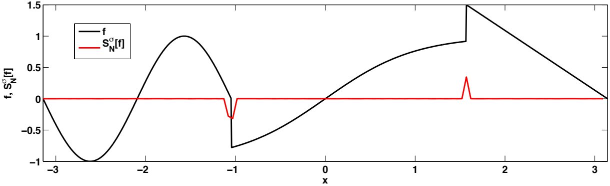
- Compute the jump approximation by solving

$$\mathbf{y} = \arg\min_{\mathbf{u}}\{\lambda\|\mathbf{u}\|_1 + \frac{1}{2}\|HFW\mathbf{u} - \Sigma\mathbf{g}\|_2^2\}. \tag{10}$$

Example for exact data: Detects the location but not the height, $N = 64$



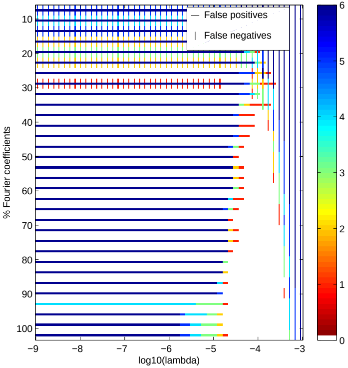
(c) Jittered sampling, σ_1, $\lambda = .0017$



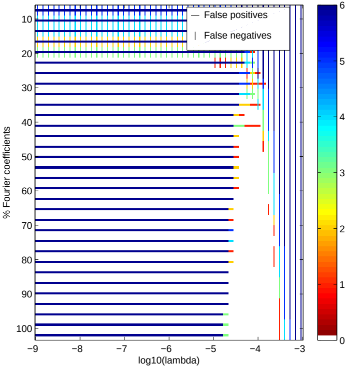
(d) Log sampling, $\sigma_{\exp}$, $\alpha = 2$, $\lambda = .00091$

Approximations: non-harmonic Fourier data using variational formulation

Comparing the performance of the waveform correlation $N = 64$, for σ_1



(e) With W

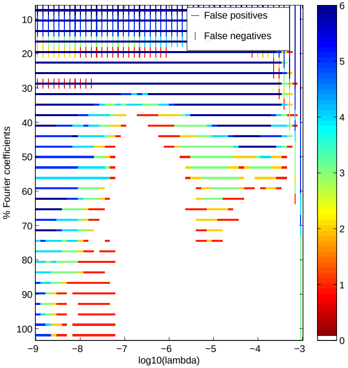


(f) Without W

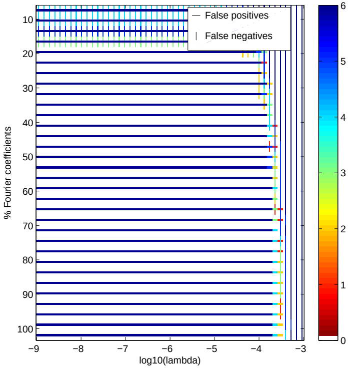
Figure: No blur, no noise, missing data, first order with/without waveform weighting

Two approaches are comparable for low order polynomial concentration

Comparing the performance of the waveform correlation $N = 64$, for σ_3



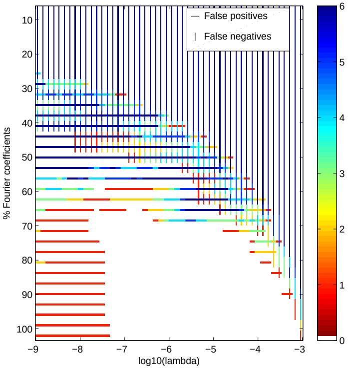
(a) With W



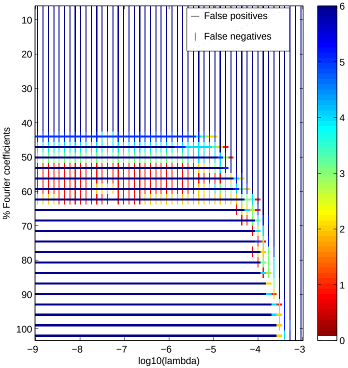
(b) Without W

No blur, no noise, missing data, third order with/without waveform. Clearly waveform is required. Higher order polynomial CF introduces oscillations that need to be suppressed.

Comparing the performance of the waveform correlation $N = 64$, for $\sigma_{\exp}$



(c) With W



(d) Without W

No blur, no noise, missing data. Exponential concentration factor with/without the waveform. Waveform is required. Higher order CF introduces oscillations that need to be suppressed.