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Certain integral inequalities involving tensor products, positive linear maps, and operator means

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Abstract

We present a number of integral inequalities involving tensor products of continuous fields of bounded linear operators, positive linear maps, and operator means. In particular, the Kantorovich, Grüss, and reverse Hölder-McCarthy integral inequalities are obtained as special cases.

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1 Introduction

Integral analogs of certain analytic inequalities in terms of continuous fields of operators and positive linear maps were first established in [1]. In this work, we continue developing integral inequalities involving continuous fields of operators related to Kantorovich and Grüss type inequalities.

Recall that the scalar Kantorovich inequality [2] is a reverse weighted arithmetic-harmonic mean inequality. It says that, for positive real numbers a_i and w_i such that $0 < m \le a_i \le M$ and $w_i \ge 0$ for all $1 \le i \le n$, we have

$$\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} \frac{w_i}{a_i}\right) \le \frac{(m+M)^2}{4mM}\left(\sum_{i=1}^{n} w_i\right)^2. \tag{1.1}$$

This inequality is useful in numerical analysis and statistics, especially in the method of steepest descent. Over the years, various variations and extensions of this inequality have been investigated by many authors in several contexts. In fact, this inequality is equivalent to many inequalities, *e.g.* the Cauchy-Schwarz-Bunyakovsky inequality and Wielant's inequality; see also [3, 4]. An integral version of the Kantorovich inequality states that, for any integrable function $f : [\alpha,\beta] \to \mathbb{R}$ with $m \le f(x) \le M$ for all $x \in [\alpha,\beta]$, we have (see *e.g.* [5])

$$\int_{\alpha}^{\beta} f(x)^2\,dx \le \frac{(m+M)^2}{4mM}\left(\int_{\alpha}^{\beta} f(x)\,dx\right)^2. \tag{1.2}$$

The inequality (1.2) is also called an additive version of the Grüss inequality.

Many matrix versions of Kantorovich inequality were obtained in the literature, *e.g.* [6–8]. Denote by $\mathbb{M}_k$ the algebra of k-by-k complex matrices. The Kantorovich inequality can be regarded as a reverse of the Fiedler inequality (see [9]):

$$A \circ A^{-1} \geq I$$

for any positive definite matrix A, here the symbol $\circ$ stands for the Hadamard product (*i.e.* the entrywise product). A matrix analog of inequality (1.1) involving Hadamard products was established as follows.

Theorem 1.1 ([10], Theorem 2.2) *For each $i = 1, 2, \dots, n$, let $A_i \in \mathbb{M}_k$ be a positive definite matrix such that $0 < mI \leq A_i \leq MI$. Let $W_i \in \mathbb{M}_k$ be a positive semidefinite matrix. Then*

$$\sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i W_i^{\frac{1}{2}} \circ \sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i^{-1} W_i^{\frac{1}{2}} \leq \frac{m^2 + M^2}{2mM} \left(\sum_{i=1}^{n} W_i \circ \sum_{i=1}^{n} W_i \right). \tag{1.3}$$

Several operator extensions of Kantorovich and Grüss inequalities were also investigated, for instance, in [11–17] and references therein. Kantorovich type inequalities where the product is replaced by an operator mean were discussed in [18, 19].

In this paper, we establish various integral inequalities involving tensor products of continuous field of Hilbert space operators and positive linear maps. Our results can be viewed as generalizations of Kantorovich and Grüss inequalities. In particular, we obtain operator versions of Theorem 1.1 in which the Hadamard product is replaced by the tensor product. Our results also include reverse Hölder-McCarthy integral inequalities. Moreover, Kantorovich type inequalities involving Kubo-Ando operator means are also investigated. Such integral inequalities include discrete inequalities as special cases.

This paper is organized as follows. We set up basic notations and preliminaries on continuous fields of operators and positive linear maps in Section 2. In Section 3, we establish certain integral inequalities involving tensor product of continuous fields of operators. These inequalities include inequalities of Kantorovich and Grüss types as special cases, which are presented separately in Section 4. In Section 5, after setting up some prerequisites about operator means, we derive certain integral inequalities involving positive linear maps and operator means.

2 Continuous fields of operators and positive linear maps

In this section, we set up basic notations and provide fundamental facts about continuous fields of operators and positive linear maps. Moreover, we establish the Bochner integrability of certain operator-valued maps which is used in later discussions.

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be complex separable Hilbert spaces. Let $\mathbb{A}$ and $\mathbb{B}$ be two unital C*-algebras of bounded linear operators acting on $\mathcal{H}$ and $\mathcal{K}$, respectively. The C*-algebra of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. The cone of positive operators on $\mathcal{H}$ is expressed as $\mathcal{B}(\mathcal{H})^+$. The identity operator is denoted by I, where the underlying space should be clear from the context. The spectrum of an operator A is written as $\mathrm{Sp}(A)$.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $\mathbb{A}$ is called a *continuous field of operators* if the parametrization

$t \mapsto A_t$ is norm continuous on Ω. If, in addition, the norm function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t \, d\mu(t)$, which is the unique operator in $\mathbb{A}$ such that

$$\phi\left(\int_\Omega A_t \, d\mu(t)\right) = \int_\Omega \phi(A_t) \, d\mu(t)$$

for every bounded linear functional ϕ on $\mathbb{A}$ (see *e.g.* [20], pp.75-78).

A linear map $\Phi : \mathbb{A} \to \mathbb{B}$ is said to be positive if $\Phi(A)$ is positive whenever $A \in \mathbb{A}$ is positive. It is well known that every positive linear map Φ between unital C*-algebras is a bounded linear operator with

$$\|\Phi\| = \left\|\Phi(I)\right\|.$$

A field $(\Phi_t)_{t\in\Omega}$ of positive linear maps from $\mathbb{A}$ to $\mathbb{B}$ is said to be a *continuous field of positive linear maps* if the function $t \mapsto \Phi_t(A)$ is continuous on Ω for every $A \in \mathbb{A}$.

From now on, assume that μ is a finite Radon measure on Ω.

Proposition 2.1 *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of positive operators in $\mathbb{A}$. Let $(\Phi_t)_{t\in\Omega}$ be a continuous field of positive linear maps from $\mathbb{A}$ into $\mathbb{B}$ such that the function $t \mapsto \|\Phi_t(I)\|$ is Lebesgue integrable. Then $\int_\Omega \Phi_t(A_t) \, d\mu(t)$ is a well-defined positive operator in $\mathbb{B}$.*

Proof Recall that a vector-valued function defined on a finite measure space is Bochner integrable if and only if its norm function is Lebesgue integrable (see *e.g.* [21], p.426). To show that the map $t \mapsto \Phi_t(A_t)$ is Bochner integrable on Ω with respect to the finite measure μ, it suffices to show its continuity and boundedness. To show that this map is continuous, let $x \in \Omega$. Since $t \mapsto \Phi_t(I)$ is continuous at x, there is a neighborhood U of x such that

$$\left\|\Phi_t(I) - \Phi_x(I)\right\| < 1 \quad \forall t \in U.$$

Since the maps $t \mapsto A_t$ and $t \mapsto \Phi_t(A_x)$ are continuous at x, there is a neighborhood V of x such that $V \subseteq U$ and

$$\|A_t - A_x\| < \frac{\epsilon}{2(1 + \|\Phi_x(I)\|)}, \qquad \left\|\Phi_t(A_x) - \Phi_x(A_x)\right\| < \frac{\epsilon}{2} \quad \forall t \in V.$$

It follows that, for each $t \in V$,

$$\begin{aligned}
\left\|\Phi_t(A_t) - \Phi_x(A_x)\right\| &= \left\|\Phi_t(A_t - A_x) + \Phi_t(A_x) - \Phi_x(A_x)\right\| \\
&\leq \left\|\Phi_t(A_t - A_x)\right\| + \left\|\Phi_t(A_x) - \Phi_x(A_x)\right\| \\
&\leq \|\Phi_t\| \|A_t - A_x\| + \left\|\Phi_t(A_x) - \Phi_x(A_x)\right\| \\
&= \left\|\Phi_t(I)\right\| \|A_t - A_x\| + \left\|\Phi_t(A_x) - \Phi_x(A_x)\right\| \\
&\leq \left(1 + \left\|\Phi_x(I)\right\|\right) \|A_t - A_x\| + \left\|\Phi_t(A_x) - \Phi_x(A_x)\right\| \\
&< \epsilon.
\end{aligned}$$

Hence $t \mapsto \Phi_t(A_t)$ is continuous. To see that $t \mapsto \Phi_t(A_t)$ is bounded, note that, for each $t \in \Omega$,

$$\left\| \Phi_t(A_t) \right\| \leq \|\Phi_t\| \|A_t\| = \left\| \Phi_t(I) \right\| \|A_t\|.$$

Since $t \mapsto \|\Phi_t(I)\|$ is Lebesgue integrable and $t \mapsto A_t$ is bounded on Ω, we obtain the boundedness of the family $(\Phi_t(A_t))_{t\in\Omega}$ as desired. The resulting integral is a positive operator since each A_t is positive and each Φ_t preserves positivity. □

For each fixed $X \in \mathbb{B}$, the map $A \mapsto A \otimes X$ is a bounded linear operator from $\mathbb{A}$ to $\mathbb{A} \otimes \mathbb{B}$. It follows that

$$\int_\Omega A_t \, d\mu(t) \otimes X = \int_\Omega (A_t \otimes X) \, d\mu(t). \tag{2.1}$$

Moreover, this map preserves positivity when the multiplier is a positive operator.

3 Operator integral inequalities involving tensor products and positive linear maps

The main result in this section is an integral inequality concerning positive linear maps and tensor products of a continuous field of operators. Then, putting a positive linear map in suitable forms, we obtain many interesting inequalities including reverse Hölder-McCarthy integral inequalities. These results includes discrete inequalities as special cases.

We start with the following lemma.

Lemma 3.1 *For any positive operators $A, B \in \mathbb{A}$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [m, M] \subseteq (0, \infty)$ and for any positive linear maps $\Phi_1, \Phi_2 : \mathbb{A} \to \mathbb{B}$, we have*

$$\Phi_1(A) \otimes \Phi_2\bigl(B^{-1}\bigr) + \Phi_2(B) \otimes \Phi_1\bigl(A^{-1}\bigr) \leq \frac{m^2 + M^2}{mM} \|\Phi_1\| \|\Phi_2\| I. \tag{3.1}$$

Moreover, the constant bound $(m^2 + M^2)/(mM)$ is best possible.

Proof Note first that, for all real numbers x, y such that $x, y \in [m, M]$, we have

$$\frac{x}{y} + \frac{y}{x} \leq \frac{m}{M} + \frac{M}{m}.$$

Moreover, the constant bound $(m/M) + (M/m)$ is the minimal possibility.

Since $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [m, M]$, we have $\|A\|, \|B\| \in [m, M]$ and $\|A^{-1}\|, \|B^{-1}\| \in [M^{-1}, m^{-1}]$. The previous claim implies that

$$\begin{aligned}
&\left\| \Phi_1(A) \otimes \Phi_2\bigl(B^{-1}\bigr) + \Phi_2(B) \otimes \Phi_1\bigl(A^{-1}\bigr) \right\| \\
&\quad \leq \left\| \Phi_1(A) \otimes \Phi_2\bigl(B^{-1}\bigr) \right\| + \left\| \Phi_2(B) \otimes \Phi_1\bigl(A^{-1}\bigr) \right\| \\
&\quad = \left\| \Phi_1(A) \right\| \left\| \Phi_2\bigl(B^{-1}\bigr) \right\| + \left\| \Phi_2(B) \right\| \left\| \Phi_1\bigl(A^{-1}\bigr) \right\| \\
&\quad \leq \|\Phi_1\| \|A\| \|\Phi_2\| \left\| B^{-1} \right\| + \|\Phi_2\| \|B\| \|\Phi_1\| \left\| A^{-1} \right\|
\end{aligned}$$

$$\begin{aligned}&\leq \|\Phi_1\|\|\Phi_2\|\bigl(Mm^{-1}+mM^{-1}\bigr)\\&=\frac{m^2+M^2}{mM}\|\Phi_1\|\|\Phi_2\|.\end{aligned}$$

Thus, we arrive at inequality (3.1). The best possibility for the constant $(m/M) + (M/m)$ comes from the scalar case $A = xI_{\mathcal{H}}$, $B = yI_{\mathcal{H}}$ and Φ_1, Φ_2 preserve the identity $I_{\mathcal{H}}$. □

Theorem 3.2 *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathbb{A}$ such that $\operatorname{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ for each $t \in \Omega$. Let $(\Phi_t)_{t\in\Omega}$ be a continuous field of positive linear maps from $\mathbb{A}$ into $\mathbb{B}$ such that the function $t \mapsto \|\Phi_t(I)\|$ is Lebesgue integrable. Then*

$$\int_\Omega \Phi_t(A_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\bigl(A_t^{-1}\bigr)\,d\mu(t) \leq K(m,M)\biggl(\int_\Omega \|\Phi_t\|\,d\mu(t)\biggr)^2 I. \tag{3.2}$$

Here, $K(m,M) := \frac{m^2+M^2}{2mM}$ is the best possible constant.

Proof Since $\|A_t\| \leq M$ for all $t \in \Omega$, the field $(A_t)_{t\in\Omega}$ is bounded. It follows from Proposition 2.1 that $\int_\Omega \Phi_t(A_t)\,d\mu(t)$ is a well-defined positive operator. By using property (2.1) and Fubini's theorem for Bochner integrals (see *e.g.* [22]), we have

$$\begin{aligned}&\int_\Omega \Phi_t(A_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\bigl(A_t^{-1}\bigr)\,d\mu(t)\\&\quad=\int_\Omega \Phi_t(A_t)\otimes\biggl(\int_\Omega \Phi_s\bigl(A_s^{-1}\bigr)\,d\mu(s)\biggr)\,d\mu(t)\\&\quad=\iint_{\Omega^2}\Phi_t(A_t)\otimes\Phi_s\bigl(A_s^{-1}\bigr)\,d\mu(s)\,d\mu(t)\\&\quad=\iint_{\Omega^2}\Phi_s(A_s)\otimes\Phi_t\bigl(A_t^{-1}\bigr)\,d\mu(s)\,d\mu(t).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_\Omega \Phi_t(A_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\bigl(A_t^{-1}\bigr)\,d\mu(t)\\&\quad=\frac{1}{2}\iint_{\Omega^2}\bigl[\Phi_t(A_t)\otimes\Phi_s\bigl(A_s^{-1}\bigr)+\Phi_s(A_s)\otimes\Phi_t\bigl(A_t^{-1}\bigr)\bigr]\,d\mu(s)\,d\mu(t).\end{aligned}$$

By making use of Lemma 3.1 and property (2.1), we obtain

$$\begin{aligned}&\int_\Omega \Phi_t(A_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\bigl(A_t^{-1}\bigr)\,d\mu(t)\\&\quad\leq\frac{1}{2}\int_\Omega\int_\Omega 2K(m,M)\|\Phi_t\|\|\Phi_s\|I\,d\mu(s)\,d\mu(t)\\&\quad=K(m,M)\biggl(\int_\Omega\|\Phi_t\|\,d\mu(t)\biggr)^2 I.\end{aligned}$$

Therefore, we arrive at the desired inequality (3.2). The best possibility of the constant $K(m,M)$ follows from the discussion in Lemma 3.1. □

Note that $K(m,M)$ is the ratio between the arithmetic mean and the geometric mean of m^2 and M^2. As a special case of Theorem 3.2, we obtain a discrete version of integral inequality (3.2) as follows.

Corollary 3.3 *For each $i = 1,2,\dots,n$, let $A_i \in \mathbb{A}$ be a positive operator such that* $\mathrm{Sp}(A_i) \subseteq [m,M] \subseteq (0,\infty)$ *and let $\Phi_i : \mathbb{A} \to \mathbb{B}$ be a positive linear map. Then we have*

$$\sum_{i=1}^{n} \Phi_i(A_i) \otimes \sum_{i=1}^{n} \Phi_i\bigl(A_i^{-1}\bigr) \le K(m,M)\left(\sum_{i=1}^{n} \|\Phi_i\|\right)^2 I. \tag{3.3}$$

Proof Set μ to be the counting measure on $\Omega = \{1,2,\dots,n\}$ in Theorem 3.2. □

Corollary 3.4 *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathbb{A}$ such that* $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ *for each $t \in \Omega$. Let $(T_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathbb{B}$ such that the function $t \mapsto \|T_t\|$ is Lebesgue integrable on Ω. Then*

$$\int_\Omega T_t \otimes A_t \, d\mu(t) \otimes \int_\Omega T_t \otimes A_t^{-1} \, d\mu(t) \le K(m,M)\left(\int_\Omega \|T_t\| \, d\mu(t)\right)^2 I, \tag{3.4}$$

$$\int_\Omega A_t \otimes T_t \, d\mu(t) \otimes \int_\Omega A_t^{-1} \otimes T_t \, d\mu(t) \le K(m,M)\left(\int_\Omega \|T_t\| \, d\mu(t)\right)^2 I. \tag{3.5}$$

Proof For each $t \in \Omega$, consider the positive linear map

$$\Phi_t : \mathbb{A} \to \mathbb{B} \otimes \mathbb{A}, \qquad X \mapsto T_t \otimes X.$$

Since the map $t \mapsto T_t$ is continuous, so is the map $t \mapsto \Phi_t$. Note that

$$\|\Phi_t\| = \bigl\|\Phi_t(I)\bigr\| = \|T_t \otimes I\| = \|T_t\|.$$

Then $t \mapsto \|\Phi_t(I)\|$ is Lebesgue integrable on Ω. Hence, the family $(\Phi_t)_{t\in\Omega}$ satisfies the hypothesis of Theorem 3.2 and inequality (3.4) follows. To prove another inequality, consider Φ_t: $\mathbb{A} \to \mathbb{A} \otimes \mathbb{B}$, $X \mapsto X \otimes T_t$ for each $t \in \Omega$. □

Our next result concerns the Hadamard product of operators. Recall that the Hadamard product of A and B in $\mathcal{B}(\mathcal{H})$ is defined to be the operator $A \circ B \in \mathcal{B}(\mathcal{H})$ satisfying

$$\bigl\langle (A \circ B)e_j, e_j\bigr\rangle = \langle Ae_j, e_j\rangle \langle Be_j, e_j\rangle \quad \text{for all } j \in \mathbb{N}.$$

Here, $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$. Equivalently, it was shown in [23] that

$$A \circ B = U^*(A \otimes B)U, \tag{3.6}$$

where $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the isometry defined by $Ue_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

Corollary 3.5 *Let $(A_t)_{t\in\Omega}$ and $(T_t)_{t\in\Omega}$ be two continuous fields of positive operators in $\mathbb{A}$ such that* $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ *for each $t \in \Omega$ and the function $t \mapsto \|T_t\|$ is Lebesgue integrable on Ω. Then*

$$\int_\Omega T_t \circ A_t \, d\mu(t) \otimes \int_\Omega T_t \circ A_t^{-1} \, d\mu(t) \le K(m,M) \left(\int_\Omega \|T_t\| \, d\mu(t) \right)^2 I. \tag{3.7}$$

Proof For each $t \in \Omega$, consider the positive linear map

$$\Phi_t : \mathbb{A} \to \mathcal{B}(\mathcal{H}), \qquad X \mapsto T_t \circ X.$$

Then the map $t \mapsto \Phi_t$ is continuous. By (3.6), we have

$$\|\Phi_t(I)\| = \|T_t \circ I\| = \|U^*(T_t \otimes I)U\| \le \|U^*\| \|T_t \otimes I\| \|U\| = \|T_t\|.$$

It follows that the function $t \mapsto \|\Phi_t(I)\|$ is Lebesgue integrable on Ω. Now, the desired inequality follows from Theorem 3.2. □

Now, recall the Hölder-McCarthy type inequalities for operators.

Proposition 3.6 ([24] or [15], pp.123-126) *Let A be a positive operator on $\mathcal{H}$ and $x \in \mathcal{H}$ a unit vector. Then*

1. $\langle A^\alpha x, x\rangle \ge \langle Ax, x\rangle^\alpha$ *for any* $\alpha \ge 1$,
2. $\langle A^\alpha x, x\rangle \le \langle Ax, x\rangle^\alpha$ *for any* $\alpha \in [0,1]$.

If A is invertible, then $\langle A^\alpha x, x\rangle \ge \langle Ax, x\rangle^\alpha$ *for any* $\alpha < 0$.

The next result is a reverse Hölder-McCarthy type integral inequality.

Corollary 3.7 *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathbb{A}$ such that* $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ *for each $t \in \Omega$. For each $t \in \Omega$, let u_t be a unit vector in $\mathcal{H}$. Assume that μ is a probability measure on Ω. For any $\lambda > 0$ or $\lambda \le -1$, we have*

$$\int_\Omega \langle A_t^\lambda u_t, u_t\rangle \, d\mu(t) \le K\left(m^\lambda, M^\lambda\right) \left(\int_\Omega \langle A_t^{-\lambda} u_t, u_t\rangle \, d\mu(t) \right)^{-1} \tag{3.8}$$

$$\le K\left(m^\lambda, M^\lambda\right) \left(\int_\Omega \langle A_t u_t, u_t\rangle^{-\lambda} \, d\mu(t) \right)^{-1}. \tag{3.9}$$

Proof For each $t \in \Omega$, consider the positive linear map

$$\Phi_t : \mathbb{A} \to \mathbb{C}, \qquad X \mapsto \langle X u_t, u_t\rangle.$$

We have $\|\Phi_t\| = |\langle u_t, u_t\rangle| = 1$ for each $t \in \Omega$. It is easy to see that the field $(\Phi_t)_{t\in\Omega}$ satisfies the hypothesis of Theorem 3.2 and thus

$$\int_\Omega \langle A_t u_t, u_t\rangle \, d\mu(t) \int_\Omega \langle A_t^{-1} u_t, u_t\rangle \, d\mu(t) \le K(m,M).$$

Since $\int_\Omega \langle A_t u_t, u_t\rangle \, d\mu(t) \ge \int_\Omega m \, d\mu(t) = m > 0$, we get

$$\int_\Omega \langle A_t^{-1} u_t, u_t\rangle \, d\mu(t) \le K(m,M) \left(\int_\Omega \langle A_t u_t, u_t\rangle \, d\mu(t) \right)^{-1}.$$

The inequality (3.8) is now done by replacing A_t with $A_t^{-\lambda}$ in the above inequality. Note that $K(M^{-\lambda}, m^{-\lambda}) = K(m^{-\lambda}, M^{-\lambda}) = K(m^{\lambda}, M^{\lambda})$. The inequality (3.9) comes from Proposition 3.6. □

In finite-dimensional setting, we identify $\mathcal{B}(\mathbb{C}^k)$ with the matrix algebra $\mathbb{M}_k$. In this case, we obtain the following.

Corollary 3.8 *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive definite matrices in $\mathbb{M}_k$ such that $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ for each $t \in \Omega$. Let μ be a probability measure on Ω. Then*

$$\int_\Omega \operatorname{tr}\bigl(A_t^{-1}\bigr)\,d\mu(t) \le k^2 K(m,M)\biggl(\int_\Omega \operatorname{tr}(A_t)\,d\mu(t)\biggr)^{-1}. \tag{3.10}$$

Proof For each $t \in \Omega$, consider the positive linear functional

$$\Phi_t : \mathbb{M}_k \to \mathbb{C}, \qquad A \mapsto \operatorname{tr}(A).$$

Then $\Phi_t(I) = k$ for all $t \in \Omega$. From Theorem 3.2, we have

$$\int_\Omega \operatorname{tr}(A_t)\,d\mu(t) \cdot \int_\Omega \operatorname{tr}\bigl(A_t^{-1}\bigr)\,d\mu(t) \le k^2 K(m,M).$$

Since

$$\int_\Omega \operatorname{tr}(A_t)\,d\mu(t) = \int_\Omega \sum_{\lambda\in\mathrm{Sp}(A_t)} \lambda\,d\mu(t) \ge mk > 0,$$

we obtain inequality (3.10). □

4 Kantorovich and Grüss type integral inequalities

In this section, we extract some interesting consequences of Theorem 3.2, namely, integral inequalities of Kantorovich and Grüss types. The next corollary is an operator extension of Theorem 1.1 in which the Hadamard product is replaced by the tensor product.

Corollary 4.1 *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathbb{A}$ such that $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ for each $t \in \Omega$. Let $(W_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathbb{A}$ such that the function $t \mapsto \|W_t\|$ is Lebesgue integrable on Ω. Then*

$$\int_\Omega W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}}\,d\mu(t) \otimes \int_\Omega W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}}\,d\mu(t) \le K(m,M)\biggl(\int_\Omega \|W_t\|\,d\mu(t)\biggr)^2 I. \tag{4.1}$$

Proof For each $t \in \Omega$, consider Φ_t: $\mathbb{A} \to \mathbb{A}$, $X \mapsto W_t^{\frac{1}{2}} X W_t^{\frac{1}{2}}$. It is straightforward to verify that $(\Phi_t)_{t\in\Omega}$ is a continuous field of positive linear maps such that the function $t \mapsto \|\Phi_t(I)\| = \|W_t\|$ is Lebesgue integrable on Ω. Now, inequality (4.1) follows from Theorem 3.2. □

Corollary 4.2 *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be continuous fields of positive operators in $\mathbb{A}$ such that*

(i) $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ *for each* $t \in \Omega$,
(ii) *the function* $t \mapsto \|B_t\|$ *is Lebesgue integrable on* Ω, *and*
(iii) $A_t B_t = B_t A_t$ *for each* $t \in \Omega$.

Then

$$\int_\Omega A_t B_t \, d\mu(t) \otimes \int_\Omega A_t^{-1} B_t \, d\mu(t) \le K(m,M) \left(\int_\Omega \|B_t\| \, d\mu(t) \right)^2 I. \tag{4.2}$$

Proof Set $W_t = B_t$ for each $t \in \Omega$ in Corollary 4.1. □

Example 4.3 Let $f, \phi : \Omega \to [0,\infty)$ be continuous functions. Assume that Range(f) $\subseteq [m,M] \subseteq (0,\infty)$ and ϕ is a weight function, that is, ϕ integrable with $\int_\Omega \phi \, d\mu = 1$. Then we have the following bound for the weighted integral of f:

$$\int_\Omega \phi f \, d\mu \le K(m,M) \left(\int_\Omega \frac{\phi}{f} \, d\mu \right)^{-1}.$$

Proof Set $\mathbb{A} = \mathbb{C}$ in Corollary 4.2. Note that $\int_\Omega (\phi/f) \, d\mu > 0$. □

The next result is an operator version of additive Grüss inequality.

Corollary 4.4 *Let* $(A_t)_{t\in\Omega}$ *be a continuous field of positive operators in* $\mathbb{A}$ *such that* $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ *for each* $t \in \Omega$. *Suppose that* μ *is a probability measure on* Ω. *Then*

$$\int_\Omega A_t^2 \, d\mu(t) \le K(m,M) \left(\int_\Omega \|A_t\| \, d\mu(t) \right)^2 I. \tag{4.3}$$

Proof Set $W_t = A_t$ for each $t \in \Omega$ in Corollary 4.1. Note that $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω since it is continuous and bounded. □

5 Integral inequalities involving tensor products and operator means

In this section, we establish certain integral inequalities involving continuous fields of operators and operator means. To begin with, recall some prerequisites from Kubo-Ando theory of operator means [25]; see also [26], Section 3 and [27], Chapter 5.

A *(Kubo-Ando) connection* is a binary operation σ assigned to each pair of positive operators such that, for all $A, B, C, D \ge 0$,

(M1) monotonicity: $A \le C, B \le D \implies A \,\sigma\, B \le C \,\sigma\, D$,
(M2) transformer inequality: $C(A \,\sigma\, B)C \le (CAC) \,\sigma\, (CBC)$,
(M3) upper semi-continuity: for any sequences $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ in $B(\mathcal{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$. Here, $X_n \downarrow X$ indicates that (X_n) is a decreasing sequence converging strongly to X.

From these axioms, every connection attains the following properties:

$$X(A \,\sigma\, B)X = (XAX) \,\sigma\, (XBX), \tag{5.1}$$

$$(A + B) \,\sigma\, (C + D) \ge (A \,\sigma\, C) + (B \,\sigma\, D), \tag{5.2}$$

for any $A, B, C, D \ge 0$ and $X > 0$.

A *(Kubo-Ando) mean* is a connection σ satisfying

$$A \,\sigma\, A = A \quad \text{for all } A \geq 0. \tag{5.3}$$

A major core of Kubo-Ando theory is the one-to-one correspondence between connections and operator monotone functions. Recall (*e.g.* [27], Chapter 4) that a continuous function $f : [0,\infty) \to \mathbb{R}$ is said to be *operator monotone* if

$$A \leq B \implies f(A) \leq f(B)$$

holds for any positive operators A and B.

Proposition 5.1 ([25], Theorem 3.4) *Given an operator connection σ, there is a unique operator monotone function $f : [0,\infty) \to [0,\infty)$ such that*

$$f(A) = I \,\sigma\, A, \quad A \geq 0. \tag{5.4}$$

In fact, the map $\sigma \mapsto f$ is a bijection. In addition, σ is a mean if and only if $f(1) = 1$.

Such a function f is called the *representing function* of σ. It follows that there is a one-to-one correspondence between the connections on $\mathcal{B}(\mathcal{H})^+$ and the connections on $\mathcal{B}(\mathcal{K})^+$ where $\mathcal{H}$ and $\mathcal{K}$ are any different Hilbert spaces. A connection σ and its corresponding connection on a different space have the same formula, and thus can be written in the same notation.

In order to prove the main result in this section, recall the following fact.

Lemma 5.2 ([28], Proposition 4) *For any connection σ and positive operators A and B, we have*

$$\|A \,\sigma\, B\| \leq \|A\| \,\sigma\, \|B\|.$$

We say that a linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is strictly positive if $\Phi(A) > 0$ for any $A > 0$. By continuity, every strictly positive linear map is positive.

Lemma 5.3 ([29]) *If $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a positive linear map, then for any connection σ and for each $A, B \geq 0$,*

$$\Phi(A \,\sigma\, B) \leq \Phi(A) \,\sigma\, \Phi(B). \tag{5.5}$$

We say that a function $f : [0,\infty) \to \mathbb{R}$ is *super-multiplicative* if $f(xy) \geq f(x)f(y)$ for all $x, y \geq 0$.

Lemma 5.4 (See *e.g.* [30], Chapter 5) *Let σ be a connection associated with an operator monotone function $f : [0,\infty) \to [0,\infty)$. If f is super-multiplicative, then*

$$(A \,\sigma\, C) \otimes (B \,\sigma\, D) \leq (A \otimes B) \,\sigma\, (C \otimes D)$$

for any $A, B, C, D \geq 0$.

The next theorem is the main result in this section.

Theorem 5.5 *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be two continuous fields of positive operators in $\mathcal{B}(\mathcal{H})$ such that $\operatorname{Sp}(A_t), \operatorname{Sp}(B_t) \subseteq [m,M] \subseteq (0,\infty)$ for each $t \in \Omega$. Let $(\Phi_t)_{t\in\Omega}$ be a continuous field of positive linear maps from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ such that the function $t \mapsto \|\Phi_t(I)\|$ is Lebesgue integrable. Let σ be a mean with a super-multiplicative representing function. Then*

$$\int_\Omega \Phi_t(A_t \,\sigma\, B_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\big(A_t^{-1} \,\sigma\, B_t^{-1}\big)\,d\mu(t) \leq K(m,M)\bigg(\int_\Omega \|\Phi_t\|\,d\mu(t)\bigg)^2 I. \quad (5.6)$$

Proof The assumption and the norm estimate in Lemma 5.2 together imply that

$$\begin{aligned}
\int_\Omega \big\|\Phi_t(A_t \,\sigma\, B_t)\big\|\,d\mu(t) &\leq \int_\Omega \|\Phi_t\| \cdot \|A_t \,\sigma\, B_t\|\,d\mu(t) \\
&\leq \int_\Omega \|\Phi_t\| \cdot \big(\|A_t\| \,\sigma\, \|B_t\|\big)\,d\mu(t) \\
&\leq \int_\Omega \|\Phi_t\| \cdot (M \,\sigma\, M)\,d\mu(t) \\
&= M\int_\Omega \big\|\Phi_t(I)\big\|\,d\mu(t) \\
&< \infty.
\end{aligned}$$

This shows that the function $t \mapsto \Phi_t(A_t \,\sigma\, B_t)$ is Bochner integrable since (Ω,μ) is a finite measure space. Similarly, the function $t \mapsto \Phi_t(A_t^{-1} \,\sigma\, B_t^{-1})$ is Bochner integrable. It follows that

$$\begin{aligned}
&\int_\Omega \Phi_t(A_t \,\sigma\, B_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\big(A_t^{-1} \,\sigma\, B_t^{-1}\big)\,d\mu(t) \\
&\quad\leq \int_\Omega \Phi_t(A_t) \,\sigma\, \Phi_t(B_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\big(A_t^{-1}\big) \,\sigma\, \Phi_t\big(B_t^{-1}\big)\,d\mu(t) \quad \text{(by Lemma 5.3)} \\
&\quad\leq \bigg[\int_\Omega \Phi_t(A_t)\,d\mu(t)\sigma \int_\Omega \Phi_t(B_t)\,d\mu(t)\bigg] \otimes \bigg[\int_\Omega \Phi_t\big(A_t^{-1}\big)\,d\mu(t)\sigma \int_\Omega \Phi_t\big(B_t^{-1}\big)\,d\mu(t)\bigg] \\
&\qquad \text{(by property (5.2))} \\
&\quad\leq \bigg[\int_\Omega \Phi_t(A_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\big(A_t^{-1}\big)\,d\mu(t)\bigg] \,\sigma\, \bigg[\int_\Omega \Phi_t(B_t)\,d\mu(t) \otimes \int_\Omega \Phi_t\big(B_t^{-1}\big)\,d\mu(t)\bigg] \\
&\qquad \text{(by Lemma 5.4)} \\
&\quad\leq \bigg[K(m,M)\bigg(\int_\Omega \|\Phi_t\|\,d\mu(t)\bigg)^2 I\bigg] \,\sigma\, \bigg[K(m,M)\int_\Omega \big(\|\Phi_t\|\,d\mu(t)\big)^2 I\bigg] \\
&\qquad \text{(by Theorem 3.2)} \\
&\quad= K(m,M)\bigg(\int_\Omega \|\Phi_t\|\,d\mu(t)\bigg)^2 I \quad \text{(by property (5.3))}.
\end{aligned}$$

□

Theorem 5.5 can be reduced to Theorem 3.2 by setting $A_t = B_t$ for all $t \in \Omega$.

Corollary 5.6 *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathcal{B}(\mathcal{H})$ such that $\operatorname{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ for each $t \in \Omega$. Let $(\Phi_t)_{t\in\Omega}$ be a continuous field of positive* *linear maps from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ such that the function $t \mapsto \|\Phi_t(I)\|$ is Lebesgue integrable. Suppose that $1 \in [m,M]$. For any super-multiplicative operator monotone function $f : [0,\infty) \to [0,\infty)$ such that $f(1) = 1$, we have*

$$\int_\Omega \Phi_t\big(f(A_t)\big)\, d\mu(t) \otimes \int_\Omega \Phi_t\big(f\big(A_t^{-1}\big)\big)\, d\mu(t) \le K(m,M)\left(\int_\Omega \|\Phi_t\|\, d\mu(t)\right)^2 I. \tag{5.7}$$

Proof By Proposition 5.1, there is a mean σ such that $f(A) = I\ \sigma\ A$ for any $A \geq 0$. The desired result now follows from Theorem 5.5 by considering $I\ \sigma\ A_t$ instead of $A_t\ \sigma\ B_t$. □

Corollary 5.7 *Let $(A_t)_{t\in\Omega}$ be a continuous field of positive operators in $\mathcal{B}(\mathcal{H})$ such that* $\mathrm{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ *for each $t \in \Omega$. Let $(W_t)_{t\in\Omega}$ be a continuous field of operators in $\mathcal{B}(\mathcal{H})$ such that the function $t \mapsto \|W_t\|$ is square integrable on Ω. Suppose that $1 \in [m,M]$. For any $\alpha \in [-1,1]$, we have*

$$\int_\Omega W_t^* A_t^\alpha W_t\, d\mu(t) \otimes \int_\Omega W_t^* A_t^{-\alpha} W_t\, d\mu(t) \le K(m,M)\left(\int_\Omega \|W_t\|^2\, d\mu(t)\right)^2 I. \tag{5.8}$$

Proof Let $\alpha \in [0,1]$ and consider the operator monotone function $f(x) = x^\alpha$. Note that this function is super-multiplicative and satisfies $f(1) = 1$. The desired inequality (5.8) now follows by setting

$$\Phi_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \qquad \Phi_t(X) = W_t^* X W_t$$

in Corollary 5.6. Note that the function $t \mapsto \|\Phi_t(I)\| = \|W_t\|^2$ is integrable on Ω. For $\alpha \in [-1,0]$, replace A_t by A_t^{-1} in the previous claim and use the fact that $K(M^{-1},m^{-1}) = K(M,m)$. □

Example 5.8 Under the hypothesis of Corollary 5.7, we have an interesting operator inequality. For each $\lambda \in \mathbb{R}$, putting $W_t = A_t^{\frac{\lambda}{2}}$ in (5.8) yields

$$\int_\Omega A_t^{\lambda+\alpha}\, d\mu(t) \otimes \int_\Omega A_t^{\lambda-\alpha}\, d\mu(t) \le K(m,M)\left(\int_\Omega \|A_t\|^\lambda\, d\mu(t)\right)^2 I. \tag{5.9}$$

Discrete versions for every inequality in this paper can be obtained by considering Ω to be a finite space equipped with the counting measure.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The work as a whole is a contribution of the author.

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