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Research Article

The Best Constant of Sobolev Inequality Corresponding to Clamped Boundary Value Problem

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Green's function $G(x, y)$ of the clamped boundary value problem for the differential operator $(-1)^M(d/dx)^{2M}$ on the interval $(-s, s)$ is obtained. The best constant of corresponding Sobolev inequality is given by $\max_{|y|\le s} G(y, y)$. In addition, it is shown that a reverse of the Sobolev best constant is the one which appears in the generalized Lyapunov inequality by Das and Vatsala (1975).

1. Introduction

For $M = 1, 2, 3, \ldots, s > 0$, let $H(= H_0^M(-s, s))$ be a Sobolev (Hilbert) space associated with the inner product $(\cdot, \cdot)_M$:

$$H = H(M) = \left\{u \mid u^{(M)} \in L^2(-s, s),\ u^{(i)}(\pm s) = 0\ (0 \le i \le M-1)\right\},$$
$$(u, v)_M = \int_{-s}^{s} u^{(M)}(x)\overline{v}^{(M)}(x)\,dx, \qquad \|u\|_M^2 = (u, u)_M. \tag{1.1}$$

The fact that $(\cdot,\cdot)_M$ induces the equivalent norm to the standard norm of the Sobolev (Hilbert) space of Mth order follows from Poincaré inequality. Let us introduce the functional $S(u)$ as follows:

$$S(u) = \frac{\left(\sup_{|y|\le s}|u(y)|\right)^2}{\|u\|_M^2}. \tag{1.2}$$

To obtain the supremum of S (i.e., the best constant of Sobolev inequality), we consider the following clamped boundary value problem:

$$\begin{aligned} &(-1)^M u^{(2M)} = f(x) \quad (-s < x < s), \\ &u^{(i)}(\pm s) = 0 \quad (0 \le i \le M-1). \end{aligned} \tag{BVP(M)}$$

Concerning the uniqueness and existence of the solution to (BVP(M)), we have the following proposition. The result is expressed by the monomial $K_j(x)$:

$$K_j(x) = K_j(M;x) = \begin{cases} \dfrac{x^{2M-1-j}}{(2M-1-j)!} & (0 \le j \le 2M-1), \\ 0 & (2M \le j). \end{cases} \tag{1.3}$$

Proposition 1.1. *For any bounded continuous function $f(x)$ on an interval $-s < x < s$,* (BVP(M)) *has a unique classical solution $u(x)$ expressed by*

$$u(x) = \int_{-s}^{s} G(x,y) f(y)\,dy \quad (-s < x < s), \tag{1.4}$$

where Green's function $G(x,y) = G(M;x,y)$ $(-s < x, y < s)$ is given by

$$G(x,y)$$

$$= \frac{(-1)^M}{2}\left[K_0(|x-y|) + D^{-1}\left\{\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s-y) \\ \hline K_j(s+x) & 0 \end{array}\right| + \left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline K_j(s-x) & 0 \end{array}\right|\right\}\right] \tag{1.5}$$

$$= (-1)^M D^{-1}\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s + x\wedge y) \\ \hline K_j(s - x\vee y) & 0 \end{array}\right| \quad (-s < x, y < s). \tag{1.6}$$

D is the determinant of $M \times M$ matrix $(K_{i+j})(2s)$ $(0 \le i, j \le M-1)$, $x \wedge y = \min(x,y)$, and $x \vee y = \max(x,y)$.

With the aid of Proposition 1.1, we obtain the following theorem. The proof of Proposition 1.1 is shown in Appendices A and B.

Theorem 1.2. (i) *The supremum $C(M;-s,s)$ (abbreviated as $C(M)$ if there is no confusion) of the Sobolev functional S is given by*

$$C(M;-s,s) = \sup_{u\in H,\ u\not\equiv 0} S(u) = \max_{|y|\le s} G(y,y) = G(0,0) = \frac{s^{2M-1}}{2^{2M-1}(2M-1)\{(M-1)!\}^2}. \tag{1.7}$$

Concretely,

$$C(1,-s,s) = \frac{s}{2}, C(2,-s,s) = \frac{s^3}{24}, C(3,-s,s) = \frac{s^5}{640}, C(4,-s,s) = \frac{s^7}{32256}, \ldots. \tag{1.8}$$

(ii) *$C(M;-s,s)$ is attained by $u = G(x,0)$, that is, $S(G(x,0)) = C(M;-s,s)$.*

Clearly, Theorem 1.2(i), (ii) is rewritten equivalently as follows.

Corollary 1.3. *Let $u \in H$, then the best constant of Sobolev inequality (corresponding to the embedding of H into $L^\infty(-s,s)$)*

$$\left(\sup_{|y|\le s}|u(y)|\right)^2 \le C\int_{-s}^{s}\left|u^{(M)}(x)\right|^2 dx, \tag{1.9}$$

is $C(M;-s,s)$. Moreover the best constant $C(M;-s,s)$ is attained by $u(x) = cG(x,0)$, where c is an arbitrary complex number.

Next, we introduce a connection between the best constant of Sobolev- and Lyapunov-type inequalities. Let us consider the second-order differential equation

$$u'' + p(x)u = 0 \quad (-s \le x \le s), \tag{1.10}$$

where $p(x) \in L^1(-s,s) \cap C[-s,s]$. If the above equation has two points s_1 and s_2 in $[-s,s]$ satisfying $u(s_1) = 0 = u(s_2)$, then these points are said to be *conjugate*. It is wellknown that if there exists a pair of conjugate points in $[-s,s]$, then the classical Lyapunov inequality

$$\int_{-s}^{s} p_+(x)dx > \frac{2}{s}, \tag{1.11}$$

holds, where $p_+(x) := \max(p(x),0)$. Various extensions and improvements for the above result have been attempted; see, for example, Ha [1], Yang [2], and references there in. Among these extensions, Levin [3] and Das and Vatsala [4] extended the result for higher order equation

$$(-1)^M u^{(2M)} - p(x)u = 0 \quad (-s \le x \le s). \tag{1.12}$$

For this case, we again call two distinct points s_1 and s_2 *conjugate* if there exists a nontrivial $C^{2M}(-s,s) \cap C^{M-1}[-s,s]$ solution of (1.12) satisfying

$$u^{(i)}(s_1) = 0 = u^{(i)}(s_2) \quad (i = 0, \ldots, M-1). \tag{1.13}$$

We point out that the constant which appears in the generalized Lyapunov inequality by Levin [3] and Das and Vatsala [4] is the reverse of the Sobolev best embedding constant.

Corollary 1.4. *If there exists a pair of conjugate points on* $[-s,s]$ *with respect to* (1.12), *then*

$$\int_{-s}^{s} p_+(x)dx > \frac{1}{C(M;-s,s)}, \tag{1.14}$$

where $C(M;-s,s)$ *is the best constant of the Sobolev inequality* (1.9).

Without introducing auxiliary equation $u^{(2M)} + (-1)^{M-1}p_+u = 0$ and the existence result of conjugate points as [2, 4], we can prove this corollary directly through the Sobolev inequality (the idea of the proof origins to Brown and Hinton [5, page 5]).

Proof of Corollary 1.4. Consider

$$\begin{aligned} \int_{s_1}^{s_2} \left(u^{(M)}(x)\right)^2 dx &= \int_{s_1}^{s_2} p(x)(u(x))^2 dx \le \left(\sup_{s_1 \le x \le s_2} |u(x)|\right)^2 \int_{s_1}^{s_2} p_+(x)dx \\ &\le C(M;s_1,s_2) \int_{s_1}^{s_2} \left(u^{(M)}(x)\right)^2 dx \int_{s_1}^{s_2} p_+(x)dx. \end{aligned} \tag{1.15}$$

In the second inequality, the equality holds for the function which attains the Sobolev best constant, so especially it is not a constant function. Thus, for this function, the first inequality is strict, and hence we obtain

$$\frac{1}{C(M;s_1,s_2)} < \int_{s_1}^{s_2} p_+(x)dx. \tag{1.16}$$

Since

$$\frac{1}{C(M;-s,s)} \le \frac{1}{C(M;s_1,s_2)} < \int_{-s_1}^{s_2} p_+(x)dx \le \int_{-s}^{s} p_+(x)dx, \tag{1.17}$$

we obtain the result. □

Here, at the end of this section, we would like to mention some remarks about (1.12). The generalized Lyapunov inequality of the form (1.14) was firstly obtained by Levin [3] without proof; see Section 4 of Reid [6]. Later, Das and Vatsala [4] obtained the same inequality (1.14) by constructing Green's function for (BVP(M)). The expression of the Green's function of Proposition 1.1 is different from that of [4]. The expression of

[4, Theorem 2.1] is given by some finite series of x and y on the other hand, the expression of Proposition 1.1 is by the determinant. This complements the results of [7–9], where the concrete expressions of Green's functions for the equation $(-1)^M u^{(2M)} = f$ but different boundary conditions are given, and all of them are expressed by determinants of certain matrices as Proposition 1.1.

2. Reproducing Kernel

First we enumerate the properties of Green's function $G(x,y)$ of (BVP(M)). $G(x,y)$ has the following properties.

Lemma 2.1. *Consider the following:*

(1)

$$\partial_x^{2M} G(x,y) = 0 \ (-s < x, y < s,\ x \neq y), \tag{2.1}$$

(2)

$$\partial_x^i G(x,y)\Big|_{x=\pm s} = 0 \ (0 \le i \le M-1,\ -s < y < s), \tag{2.2}$$

(3)

$$\partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} = \begin{cases} 0 & (0 \le i \le 2M-2), \\ (-1)^M & (i = 2M-1)\ (-s < x < s), \end{cases} \tag{2.3}$$

(4)

$$\partial_x^i G(x,y)\Big|_{x=y+0} - \partial_x^i G(x,y)\Big|_{x=y-0} = \begin{cases} 0 & (0 \le i \le 2M-2), \\ (-1)^M & (i = 2M-1)\ (-s < y < s). \end{cases} \tag{2.4}$$

Proof. For k $(1 \le k \le 2M)$ and $-s < x,\ y < s$, $x \neq y$, we have from (1.5)

$$\partial_x^k G(x,y) = \frac{(-1)^M}{2}\Bigg[(\operatorname{sgn}(x-y))^k K_k(|x-y|)$$
$$+D^{-1}\left\{\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s-y) \\ \hline K_{k+j}(s+x) & 0 \end{array}\right| + \left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline (-1)^k K_{k+j}(s-x) & 0 \end{array}\right|\right\}\Bigg]. \tag{2.5}$$

For $k = 2M$, noting the fact $K_j(x) = 0 \ \ (2M \le j)$, we have (1). Next, for $0 \le k \le M-1$ and $-s < y < s$, we have from (2.5)

$$\begin{aligned}
&\partial_x^k G(x,y)\Big|_{x=-s} \\
&\quad = \frac{(-1)^M}{2}\left[(-1)^k K_k(s+y) + D^{-1}\left\{\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s-y) \\ \hline K_{k+j}(0) & 0 \end{array}\right| + \left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline (-1)^k K_{k+j}(2s) & 0 \end{array}\right|\right\}\right].
\end{aligned} \tag{2.6}$$

Since $(K_k(0), \ldots, K_{k+M-1}(0)) = (0, \ldots, 0)$, we have

$$\begin{aligned}
(-1)^{M+k} 2 \ \partial_x^k G(x,y)\Big|_{x=-s} &= K_k(s+y) + D^{-1}\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline K_{k+j}(2s) & 0 \end{array}\right| \\
&= K_k(s+y) + D^{-1}\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline 0 \ \cdots \ 0 & -K_k(s+y) \end{array}\right| = 0.
\end{aligned} \tag{2.7}$$

Note that subtracting the kth row from Mth row, the second equality holds. Equation $\partial_x^k G(x,y)|_{x=s} = 0$ is shown by the same way. Hence, we have (2). For $0 \le k \le 2M-1$, we have

$$\begin{aligned}
&\partial_x^k G(x,y)\Big|_{y=x-0} - \partial_x^k G(x,y)\Big|_{y=x+0} \\
&\quad = \frac{(-1)^M}{2}\left(1 - (-1)^k\right) K_k(0) = \begin{cases} 0 & (0 \le k \le 2M-2), \\ (-1)^M & (k = 2M-1) \ (-s < x < s), \end{cases}
\end{aligned} \tag{2.8}$$

where we used the fact $K_k(0) = 0 \ (k \ne 2M-1), 1 \ (k = 2M-1)$. So we have (3), and (4) follows from (3). □

Using Lemma 2.1, we prove that the functional space H associated with inner norm $(\cdot,\cdot)_M$ is a reproducing kernel Hilbert space.

Lemma 2.2. *For any $u \in H$, one has the reproducing property*

$$u(y) = (u(\cdot), G(\cdot, y))_M = \int_{-s}^{s} u^{(M)}(x) \partial_x^M G(x,y)\, dx \quad (-s \le y \le s). \tag{2.9}$$

Proof. For functions $u = u(x)$ and $v = v(x) = G(x,y)$ with y arbitrarily fixed in $-s \le y \le s$, we have

$$u^{(M)} v^{(M)} - u(-1)^M v^{(2M)} = \left(\sum_{j=0}^{M-1} (-1)^{M-1-j} \ u^{(j)} v^{(2M-1-j)}\right)'. \tag{2.10}$$

Integrating this with respect to x on intervals $-s < x < y$ and $y < x < s$, we have

$$\begin{aligned}
&\int_{-s}^{s} u^{(M)}(x) v^{(M)}(x)dx - \int_{-s}^{s} u(x)(-1)^{M} v^{(2M)}(x)dx \\
&= \left[\sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}(x) v^{(2M-1-j)}(x) \right] \left\{ \Big|_{x=-s}^{x=y-0} + \Big|_{x=y+0}^{x=s} \right\} \\
&= \sum_{j=0}^{M-1} (-1)^{M-1-j} \left[u^{(j)}(s) v^{(2M-1-j)}(s) - u^{(j)}(-s)\ v^{(2M-1-j)}(-s) \right] \\
&\quad + \sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}(y) \left[v^{(2M-1-j)}(y-0) - v^{(2M-1-j)}(y+0) \right].
\end{aligned} \tag{2.11}$$

□

Using (1), (2), and (4) in Lemma 2.1, we have (2.9).

3. Sobolev Inequality

In this section, we give a proof of Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2 and Corollary 1.3. Applying Schwarz inequality to (2.9), we have

$$|u(y)|^2 \le \int_{-s}^{s} \left|\partial_x^M\ G(x,y)\right|^2 dx \int_{-s}^{s} \left|u^{(M)}(x)\right|^2 dx = G(y,y) \int_{-s}^{s} \left|u^{(M)}(x)\right|^2 dx. \tag{3.1}$$

Note that the last equality holds from (2.9); that is, substituting (2.9), $u(\cdot) = G(\cdot, y)$. Let us assume that

$$C(M; -s, s) = C(M) = \max_{|y| \le s} G(y,y) = G(0,0), \tag{3.2}$$

holds (this will be proved in the next section). From definition of $C(M)$, we have

$$\left(\sup_{|y|\le s} |u(y)| \right)^2 \le C(M) \int_{-s}^{s} \left|u^{(M)}(x)\right|^2 dx. \tag{3.3}$$

Substituting $u(x) = G(x,0) \in H$ in to the above inequality, we have

$$\left(\sup_{|y|\le s} |G(y,0)| \right)^2 \le C(M) \int_{-s}^{s} \left|\partial_x^M G(x,0)\right|^2 dx = (C(M))^2. \tag{3.4}$$

Combining this and trivial inequality $(C(M))^2 = (G(0,0))^2 \le (\sup_{|y|\le s}|G(y,0)|)^2$, we have

$$(C(M))^2 \le \left(\sup_{|y|\le s}\left|G(y,0)\right|\right)^2 \le C(M)\int_{-s}^{s}\left|\partial_x^M G(x,0)\right|^2 dx \;=\; (C(M))^2. \tag{3.5}$$

Hence, we have

$$\left(\sup_{|y|\le s}|G(y,0)|\right)^2 = C(M)\int_{-s}^{s}\left|\partial_x^M G(x,0)\right|^2 dx, \tag{3.6}$$

which completes the proof of Theorem 1.2 and Corollary 1.3. □

Thus, all we have to do is to prove (3.2).

4. Diagonal Value of Green's Function

In this section, we consider the diagonal value of Green's function, that is, $G(x,x)$. From Proposition 1.1, we have for $M = 1,2,3$

$$G(1;x,x) = \frac{(s^2-x^2)}{2s}, \qquad G(2;x,x) = \frac{(s^2-x^2)^3}{24s^3}, \qquad G(2;x,x) = \frac{(s^2-x^2)^5}{650s^5}. \tag{4.1}$$

Thus, we can expect that $G(x,x)$ takes the form $G(M;x,x) = \text{const.}\, K_0(M;1+x)K_0(M;1-x)$. Precisely, we have the following proposition.

Proposition 4.1. *Consider*

$$\begin{aligned} G(x,x) &= (-1)^M D^{-1}\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s-x) \\ \hline K_j(s+x) & 0 \end{array}\right| = \binom{2(M-1)}{M-1}\frac{1}{K_0(2s)}K_0(s+x)K_0(s-x) \\ &= \binom{2(M-1)}{M-1}\frac{1}{K_0(2s)}\frac{(s^2-x^2)^{2M-1}}{\{(2M-1)!\}^2}. \end{aligned} \tag{4.2}$$

Hence,

$$\begin{aligned} C(M;-s,s) = \sup_{|x|\le s} G(x,x) = G(0,0) &= (-1)^M D^{-1}\left|\begin{array}{c|c} K_{i+j}(2s) & K_i(s) \\ \hline K_j(s) & 0 \end{array}\right| \\ &= \frac{s^{2M-1}}{2^{2M-1}(2M-1)!}\binom{2(M-1)}{M-1} = \frac{s^{2M-1}}{2^{2M-1}(2M-1)((M-1)!)^2}, \end{aligned} \tag{4.3}$$

where i,j satisfy $0 \le i,j \le M-1$.

To prove this proposition, we prepare the following two lemmas.

Lemma 4.2. *Let* $u(x) = c_1 G(x,x)$, *where*

$$c_1^{-1} = (-1)^M \binom{2(2M-1)}{2M-1} D^{-1} \left| \begin{array}{cccc|c} & & & & 1 \\ & K_{i+j}(2s) & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \end{array} \right|, \tag{4.4}$$

(i, j *satisfy* $0 \le i, j \le M-1$), *then it holds that*

$$-u^{(2(2M-1))} = 1 \quad (-s < x < s), \tag{4.5}$$

$$u^{(i)}(\pm s) = 0 \quad (0 \le i \le 2M-2), \tag{4.6}$$

$$u^{(2M-1)}(s) = -\binom{2(M-1)}{M-1} c_1. \tag{4.7}$$

Lemma 4.3. *Let* $\tilde{u}(x) = c_2 K_0(s+x) K_0(s-x)$ $(-s < x < s)$, *where* $c_2^{-1} = \binom{2(2M-1)}{2M-1}$, *then it holds that* (4.6) *and* $\tilde{u}^{(2M-1)}(s) = -K_0(2s) c_2$.

Proof of Proposition 4.1. From Lemmas 4.2 and 4.3, $u(x) = c_1 G(x,x)$ and $\tilde{u}(x) = c_2 K_0(s+x) K_0(s-x)$ satisfy BVP$(2M-1)$ (in case of $f(x) = 1(-s < x < s)$). So we have

$$c_1 G(x,x) = c_2 K_0(s+x) K_0(s-x) \quad (-s < x < s), \tag{4.8}$$

$$\binom{2(M-1)}{M-1} c_1 = K_0(2s) c_2. \tag{4.9}$$

Inserting (4.9) into (4.8), we have Proposition 4.1. □

Proof of Lemma 4.2. Let

$$u(x) = c_1 G(x,x) = c_1 (-1)^M D^{-1} v(x), \quad v(x) = \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s-x) \\ \hline K_j(s+x) & 0 \end{array} \right|, \tag{4.10}$$

then differentiating $v(x)$ k times we have

$$v^{(k)}(x) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} w_{k,l}(x), \quad w_{k,l}(x) = \left| \begin{array}{c|c} K_{i+j}(2s) & K_{l+i}(s-x) \\ \hline K_{k-l+j}(s+x) & 0 \end{array} \right|. \tag{4.11}$$

At first, for $k = 2(2M-1)$, we have

$$
\begin{aligned}
v^{(2(2M-1))}(x) &= \sum_{l=0}^{2(2M-1)} (-1)^l \binom{2(2M-1)}{l} w_{2(2M-1),l}(x) \\
&= \sum_{l=0}^{2M-2} (-1)^l \binom{2(2M-1)}{l} w_{2(2M-1),l}(x) - \binom{2(2M-1)}{2M-1} w_{2(2M-1),2M-1}(x) \\
&\quad + \sum_{l=2M}^{2(2M-1)} (-1)^l \binom{2(2M-1)}{l} w_{2(2M-1),l}(x).
\end{aligned}
\tag{4.12}
$$

The first term vanishes because

$$
K_{2(2M-1)-l+j}(s+x) = K_{2M+(2M-2-l+j)}(s+x) = 0 \quad (0 \le l \le 2M-2). \tag{4.13}
$$

The third term also vanishes because

$$
K_{l+i}(s-x) = 0 \quad (2M \le l \le 2(2M-1)). \tag{4.14}
$$

Thus, we have

$$
\begin{aligned}
v^{(2(2M-1))}(x) &= -\binom{2(2M-1)}{2M-1} w_{2(2M-1),2M-1}(x), \\
w_{2(2M-1),2M-1}(x) &= \left| \begin{array}{c|c} K_{i+j}(2s) & K_{2M-1+i}(s-x) \\ \hline K_{2M-1+j}(s+x) & 0 \end{array} \right| = \left| \begin{array}{cccc|c} & & & & 1 \\ & K_{i+j}(2s) & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ \hline 1 & 0 & \cdots & 0 & 0 \end{array} \right|.
\end{aligned}
\tag{4.15}
$$

Hence, we have

$$
-\, u^{(2(2M-1))}(x) = -\, c_1(-1)^M D^{-1}\ v^{(2(2M-1))}(x) = 1, \tag{4.16}
$$

by which we obtain (4.5). Next, for $0 \le k \le M-1$, we have

$$
v^{(k)}(s) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} w_{k,l}(s), \quad w_{k,l}(s) = \left| \begin{array}{c|c} K_{i+j}(2s) & K_{l+i}(0) \\ \hline K_{k-l+j}(2s) & 0 \end{array} \right|. \tag{4.17}
$$

Since $0 \le l+i \le 2M-2$, we have $w_{k,l}(s)=0$. Thus, we have $v^{(k)}(s)=0$ $(0\le k\le M-1)$. For $M\le k\le 2M-2$, we have

$$v^{(k)}(s)=\sum_{l=0}^{M-1}(-1)^l\binom{k}{l}w_{k,l}(s)+\sum_{l=M}^{k}(-1)^l\binom{k}{l}w_{k,l}(s). \tag{4.18}$$

The first term vanishes because $K_{l+i}(0)=0$ $(0\le l\le M-1)$. Next, we show that the second term also vanishes. Let

$$w_{k,l}(s)=\left|\begin{array}{c|c} K_j(2s) & 0\\ \vdots & \vdots\\ K_{2M-2-l+j}(2s) & 0\\ K_{2M-1-l+j}(2s) & 1\\ K_{2M-l+j}(2s) & 0\\ \vdots & \vdots\\ K_{M-1+j}(2s) & 0\\ \hline K_{k-l+j}(2s) & 0\end{array}\right| \quad (M\le l\le k\le 2M-2). \tag{4.19}$$

Since $0\le k-l\le 2M-2-l$, two rows, including the last row, coincide, and hence we have $w_{k,l}(s)=0$. Thus, we have $v^{(k)}(s)=0$ $(M\le k\le 2M-2)$. So we have obtained $u^{(k)}(s)=0$ $(0\le k\le 2M-2)$. By the same argument, we have $u^{(k)}(-s)=0$ $(0\le k\le 2M-2)$. Hence, we have (4.6). Finally, we will show (4.7). For $k=2M-1$, noting $K_{l+i}(0)=0$ $(0\le l\le M-1)$, we have

$$v^{(2M-1)}(s)=\sum_{l=M}^{2M-1}(-1)^l\binom{2M-1}{l}w_{2M-1,l}(s), \tag{4.20}$$

where

$$\begin{aligned} w_{2M-1,l}(s)&=\left|\begin{array}{c|c} K_{i+j}(2s) & K_{l+i}(0)\\ \hline K_{2M-1-l+j}(2s) & 0\end{array}\right|\\ &=\left|\begin{array}{c|c} K_j(2s) & 0\\ \vdots & \vdots\\ K_{2M-2-l+j}(2s) & 0\\ K_{2M-1-l+j}(2s) & 1\\ K_{2M-l+j}(2s) & 0\\ \vdots & \vdots\\ K_{M-1+j}(2s) & 0\\ \hline K_{2M-1-l+j}(2s) & 0\end{array}\right| = \left|\begin{array}{ccc|c} & K_j(2s) & & 0\\ & \vdots & & \vdots\\ & K_{2M-2-l+j}(2s) & & 0\\ & K_{2M-1-l+j}(2s) & & 1\\ & K_{2M-l+j}(2s) & & 0\\ & \vdots & & \vdots\\ & K_{M-1+j}(2s) & & 0\\ \hline 0 & \cdots & 0 & -1\end{array}\right| = -D. \end{aligned} \tag{4.21}$$

Thus, we obtain $w_{2M-1,l}(s) = -D\ (M \le l \le 2M-1)$. Hence we have

$$\begin{aligned} v^{(2M-1)}(1) &= \sum_{l=M}^{2M-1}(-1)^l\binom{2M-1}{l}w_{2M-1,l}(s) = -\,D\sum_{l=M}^{2M-1}(-1)^l\binom{2M-1}{l} \\ &= -D\sum_{l=M}^{2M-2}(-1)^l\left\{\binom{2M-2}{l-1}+\binom{2M-2}{l}\right\}+D = (-1)^{M+1}D\binom{2(M-1)}{M-1}, \end{aligned} \tag{4.22}$$

that is,

$$u^{(2M-1)}(s) = c_1(-1)^M D^{-1}v^{(2M-1)}(s) = -\binom{2(M-1)}{M-1}c_1. \tag{4.23}$$

This completes the proof of Lemma 4.2. □

Proof of Lemma 4.3. Let

$$u(x) = c_2K_0(s+x)K_0(s-x) = \frac{c_2}{((2M-1)!)^2}\left(s^2-x^2\right)^{2M-1}. \tag{4.24}$$

Differentiating $u(x)$ k times, we have

$$u^{(k)}(x) = c_2\sum_{l=0}^{k}(-1)^l\binom{k}{l}K_{k-l}(s+x)K_l(s-x). \tag{4.25}$$

For $k = 2(2M-1)$, noting $K_{2(2M-1)-l}(s+x) = 0\ (0 \le l \le 2M-2)$, $K_{2M-1}(s+x) = K_{2M-1}(s-x) = 1$, and $K_l(s-x) = 0\ \ (2M \le l \le 2(2M-1))$, we have

$$-\ u^{(2(2M-1))}(x) = c_2\binom{2(2M-1)}{2M-1} = 1. \tag{4.26}$$

Thus, we have (4.5). If $0 \le k \le 2M-2$, then we have

$$u^{(k)}(s) = c_2\sum_{l=0}^{k}(-1)^l\binom{k}{l}K_{k-l}(2s)K_l(0) = 0. \tag{4.27}$$

Since $u^{(k)}(-x) = (-1)^k u^{(k)}(x)$, we have $u^{(k)}(-s) = 0\ (0 \le k \le 2M-2)$. Hence, we have (4.6). If $k = 2M-1$, then we have

$$u^{(2M-1)}(s) = c_2\sum_{l=0}^{2M-1}(-1)^l\binom{2M-1}{l}K_{2M-1-l}(2s)K_l(0) = -\ c_2K_0(2s). \tag{4.28}$$

This proves Lemma 4.3. □

Appendices

A. Deduction of (1.5)

In this section, (1.5) in Proposition 1.1 is deduced. Suppose that (BVP(M)) has a classical solution $u(x)$. Introducing the following notations:

$$\begin{aligned}
\mathbf{u} &= {}^t(u_0, \ldots, u_{2M-1}), \quad u_i = u^{(i)} \quad (0 \le i \le 2M-1), \\
\mathbf{e} &= {}^t(0, \ldots, 0, 1) \quad (2M \times 1 \text{ matrix}), \\
\mathbf{N} &= \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \quad (2M \times 2M \text{ nilpotent matrix}),
\end{aligned} \tag{A.1}$$

(BVP(M)) is rewritten as

$$\begin{aligned}
\mathbf{u}' &= \mathbf{N}\mathbf{u} + \mathbf{e}(-1)^M f(x) \quad (-s < x < s), \\
u_i(\pm s) &= 0 \quad (0 \le i \le M-1).
\end{aligned} \tag{A.2}$$

Let the fundamental solution $\mathbf{E}(x)$ be expressed as $\mathbf{E}(x) = \exp(\mathbf{N}x) = \mathbf{K}(x)\mathbf{K}(0)^{-1}$, where

$$\mathbf{K}(x) = \left(K_{i+j}\right)(x), \qquad \mathbf{K}(0) = \begin{pmatrix} & & 1 \\ & \cdots & \\ 1 & & \end{pmatrix} = \mathbf{K}(0)^{-1}, \tag{A.3}$$

then i, j satisfy $0 \le i, j \le 2M-1$. $\mathbf{E}(x)$ satisfies the initial value problem $\mathbf{E}' = \mathbf{N}\mathbf{E}$, $\mathbf{E}(0) = \mathbf{I}$. $\mathbf{I}$ is a unit matrix. Solving (A.2), we have

$$\begin{aligned}
\mathbf{u}(x) &= \mathbf{E}(x+s)\mathbf{u}(-s) + \int_{-s}^{x} \mathbf{E}(x-y)\mathbf{e}(-1)^M f(y)\,dy, \\
\mathbf{u}(x) &= \mathbf{E}(x-s)\mathbf{u}(s) - \int_{x}^{s} \mathbf{E}(x-y)\mathbf{e}(-1)^M f(y)\,dy,
\end{aligned} \tag{A.4}$$

or equivalently, for $0 \le i \le 2M-1$, we have

$$\begin{aligned}
u_i(x) &= \sum_{j=0}^{2M-1} K_{i+j}(x+s)u_{2M-1-j}(-s) + \int_{-s}^{x} (-1)^M K_i(x-y) f(y)\,dy, \\
u_i(x) &= \sum_{j=0}^{2M-1} K_{i+j}(x-s)u_{2M-1-j}(s) - \int_{x}^{s} (-1)^M K_i(x-y) f(y)\,dy.
\end{aligned} \tag{A.5}$$

Employing the boundary conditions (A.2), we have

$$u_i(x) = \sum_{j=0}^{M-1} K_{i+j}(x+s)u_{2M-1-j}(-s) + \int_{-s}^{x} (-1)^M K_i(x-y)f(y)dy,$$
$$u_i(x) = \sum_{j=0}^{M-1} K_{i+j}(x-s)u_{2M-1-j}(s) - \int_{x}^{s} (-1)^M K_i(x-y)f(y)dy. \tag{A.6}$$

In particular, if $i=0$, then we have

$$u_0(x) = \sum_{j=0}^{M-1} K_j(x+s)u_{2M-1-j}(-s) + \int_{-s}^{x} (-1)^M K_0(x-y)f(y)dy,$$
$$u_0(x) = \sum_{j=0}^{M-1} K_j(x-s)u_{2M-1-j}(s) - \int_{x}^{s} (-1)^M K_0(x-y)f(y)dy. \tag{A.7}$$

On the other hand, using the boundary conditions (A.2) again, we have

$$0 = u_i(s) = \sum_{j=0}^{M-1} K_{i+j}(2s)u_{2M-1-j}(-s) + \int_{-s}^{s} (-1)^M K_i(s-y)f(y)dy,$$
$$0 = u_i(-s) = \sum_{j=0}^{M-1} K_{i+j}(-2s)u_{2M-1-j}(s) - \int_{-s}^{s} (-1)^M K_i(-s-y)f(y)dy. \tag{A.8}$$

Solving the above linear system of equations with respect to $u_{2M-1-i}(-s)$, $u_{2M-1-i}(s)$ $(0 \le i \le M-1)$, we have

$$(u_{2M-1-i})(-s) = -\int_{-s}^{s} (-1)^M (K_{i+j})^{-1}(2s)(K_i)(s-y)f(y)dy,$$
$$(u_{2M-1-i})(s) = \int_{-s}^{s} (-1)^M (K_{i+j})^{-1}(-2s)(K_i)(-s-y)f(y)dy. \tag{A.9}$$

Substituting (A.9) into (A.7), we have

$$\begin{aligned} u_0(x) = & -\int_{-s}^{s} (-1)^M (K_j)(x+s)(K_{i+j})^{-1}(2s)(K_i)(s-y)f(y)dy \\ & + \int_{-s}^{x} (-1)^M K_0(|x-y|)f(y)dy, \\ u_0(x) = & \int_{-s}^{s} (-1)^M (K_j)(x-s)(K_{i+j})^{-1}(-2s)(K_i)(-s-y)f(y)dy \\ & + \int_{x}^{s} (-1)^M K_0(|x-y|)f(y)dy. \end{aligned} \tag{A.10}$$

Taking an average of the above two expressions and noting $u(x) = u_0(x)$, we obtain (1.4), where Green's function $G(x,y)$ is given by

$$G(x,y) = \frac{(-1)^M}{2}\Big[K_0(|x-y|) - (K_j)(x+s)(K_{i+j})^{-1}(2s)(K_i)(s-y) \\ +(K_j)(x-s)(K_{i+j})^{-1}(-2s)(K_i)(-s-y)\Big]. \tag{A.11}$$

Using properties $K_i(-x) = (-1)^{i+1}K_i(x)$, we have

$$\begin{aligned}(K_j)(x-s) &= -\ (K_j)(s-x)\Big((-1)^i\delta_{ij}\Big),\\ (K_{i+j})(-2s) &= \Big((-1)^{i+j+1}K_{i+j}\Big)(2s) = -\Big((-1)^i\delta_{ij}\Big)(K_{i+j})(2s)\Big((-1)^i\delta_{ij}\Big),\\ (K_i)(-s-y) &= \Big((-1)^{i+1}K_i\Big)(s+y) = -\Big((-1)^i\delta_{ij}\Big)(K_i)(s+y),\end{aligned} \tag{A.12}$$

where δ_{ij} is Kronecker's delta defined by $\delta_{ij} = 1\ (i = j),\ 0\ (i \neq j)$. Inserting these three relations into (A.11), we have

$$G(x,y) = \frac{(-1)^M}{2}\Big[K_0(|x-y|) - (K_j)(s+x)(K_{i+j})^{-1}(2s)(K_i)(s-y) \\ -(K_j)(s-x)(K_{i+j})^{-1}(2s)(K_i)(s+y)\Big]. \tag{A.13}$$

Applying the relation

$$ {}^t\mathbf{a}\,\mathbf{A}^{-1}\mathbf{b} = -\frac{\left|\begin{array}{c|c}\mathbf{A} & \mathbf{b}\\ \hline {}^t\mathbf{a} & 0\end{array}\right|}{|\mathbf{A}|}, \tag{A.14}$$

where $\mathbf{A}$ is any $N \times N$ regular matrix and $\mathbf{a}$ and $\mathbf{b}$ are any $N \times 1$ matrices, we have (1.5).

B. Deduction of (1.6)

To prove (1.6), we show

$$K_0(x-y) = -D^{-1}\left\{\left|\begin{array}{c|c}K_{i+j}(2s) & K_i(s-y)\\ \hline K_j(s+x) & 0\end{array}\right| - \left|\begin{array}{c|c}K_{i+j}(2s) & K_i(s+y)\\ \hline K_j(s-x) & 0\end{array}\right|\right\} \quad (-s < x, y < s). \tag{B.1}$$

Let $x \geq y$. If (B.1) holds, substituting it to (1.5), replacing x with $x \vee y$, y with $x \wedge y$, then we obtain (1.6). The case $x \leq y$ is shown in a similar way. Let y $(-s \leq y \leq s)$ be fixed, and let $u(x) = K_0(x-y)$. Then u satisfies

$$\begin{gathered} u^{(2M)} = 0 \quad (-s < x < s), \\ u^{(i)}(-s) = (-1)^{i+1} K_i(s+y), \quad u^{(i)}(s) = K_i(s-y) \quad (0 \leq i \leq M-1). \end{gathered} \tag{B.2}$$

On the other hand, let

$$v(x) = -D^{-1}\left\{ \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s-y) \\ \hline K_j(s+x) & 0 \end{array} \right| - \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline K_j(s-x) & 0 \end{array} \right| \right\}. \tag{B.3}$$

Differentiating v k times with respect to x, we have

$$v^{(k)}(x) = -D^{-1}\left\{ \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s-y) \\ \hline K_{k+j}(s+x) & 0 \end{array} \right| - (-1)^k \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline K_{k+j}(s-x) & 0 \end{array} \right| \right\}. \tag{B.4}$$

For $k = 2M$, noticing $K_{k+j}(s+x) = K_{k+j}(s-x) = 0$, we have $v^{(2M)}(x) = 0$. For $0 \leq k \leq M-1$, we have

$$\begin{aligned} v^{(k)}(-s) &= -D^{-1}\left\{ \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s-y) \\ \hline K_{k+j}(0) & 0 \end{array} \right| - (-1)^k \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline K_{k+j}(2s) & 0 \end{array} \right| \right\} \\ &= (-1)^k D^{-1} \left| \begin{array}{c|c} K_{i+j}(2s) & K_i(s+y) \\ \hline 0 \ \cdots \ 0 & -K_k(s+y) \end{array} \right| = (-1)^{k+1} K_k(s+y), \end{aligned} \tag{B.5}$$

where we used $K_{k+j}(0) = 0$. Similarly, for $0 \leq k \leq M-1$, we have $v^{(k)}(s) = K_k(s-y)$. So $v(x)$ satisfies

$$\begin{gathered} v^{(2M)} = 0 \quad (-s < x < s), \\ v^{(i)}(-s) = (-1)^{i+1} K_i(s+y), \quad v^{(i)}(s) = K_i(s-y) \quad (0 \leq i \leq M-1). \end{gathered} \tag{B.6}$$

which is the same equation as (B.2). Hence, we have $v(x) = u(x)$.

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