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Research Article

A Feedback Retrial Queueing System with Two Types of Batch Arrivals

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A retrial queueing system with two types of batch arrivals, called type I and type II customers, is considered. Type I customers and type II customers arrive in batches of variable sizes according to two different Poisson processes. Service time distributions are identical and independent and are different for both types of customers. If the arriving customers are blocked due to the server being busy, type I customers are queued in a priority queue of infinite capacity, whereas type II customers enter into a retrial group in order to seek service again after a random amount of time. A type I customer who has received service departs the system with a preassigned probability or returns to the priority queue for reservice with the complement probability. A type II call who has received service leaves the system with a preassigned probability or rejoins the retrial group with complement probability. For this model, the joint distribution of the number of customers in the priority queue and in the retrial group is obtained in a closed form. Some particular models and operating characteristics are obtained. A numerical study is also carried out.

1. Introduction

In the last three decades there has been significant contribution in the area of retrial queueing theory. For detailed survey one can see Yang and Templeton [1], Falin [2] and Choi and Chang [3]. Choi and Park [4] investigated an $M/G/1$ retrial queue with two type of customers in which the service time distribution for both types of customers are the same. Khalil et al. [5] investigated the above model at Markovian level in detail. Falin et al. [6] investigated a similar model, in which they assumed different service time distributions for both types of customers. In 1995, Choi et al. [7], studied an $M/G/1$ retrial queue with two types of customers and finite capacity. Atenica and Moreno [8] has analyzed a single server retrial queuing system with infinite buffer, Poisson arrivals, general distribution of service time, and linear retrial policy. If an arriving customer finds the server occupied, he joins a retrial group (called orbit) with probability p and with complementary probability q a priority queue in order to be served. After the customer is served completely, he will decide either to return to the priority queue for another service with probability θ or to leave the system forever with probability $\overline{\theta} = 1 - \theta$, where $0 \leq \theta < 1$. They proved the ergodicity of the embedded Markov chain and obtained its stationary distribution function and the joint generating function of the number of customers in both groups in the steady-state regime. Kalyanaraman and Srinivasan [9] studied an $M/G/1$ retrial queue with geometric loss and with type I batch arrivals and type II single arrivals. Artalejo and Atenica [10] analyzed a single server retrial queue with batch arrivals. Atenica and Moreno [11] considered an $M/G/1$ retrial queue with general retrial time where the blocked customers either join the infinite waiting room (called priority queue) with probability q or with complementary probability p leave the service area and enter the retrial group (called orbit) in accordance with an FCFS discipline. They assume that only the customers at the head of the orbit are allowed to retry for service. They studied the ergodicity of the embedded Markov chain, its stationary distribution function, and the joint generating function of the number of customers in both groups in the steady-state regime. In 2005, Lee [12] considered an $M/G/1$ retrial queueing system with two types of customers and feedback and derived the joint generating function of the number of customers in two groups by using the supplementary variable method. Falin [13] considered a single server batch arrival queue with returning customers. In 2011, Thillaigovindan and Kalyanaraman [14] analyzed a feedback retrial queueing system with two types of arrivals and the type I arrival being in batches of fixed size K.

In this paper, we deal with a feedback retrial queue with two types of customers, in which both types of customers arrive in batches of variable sizes. In Section 2, we describe the system with stability condition. In Section 3, we obtain the joint probability generating function for the number of customers in the priority queue and in the retrial group when server is busy as well as idle. The expressions for some particular models are deduced in Section 4. Some operating characteristics are derived in Section 5 and a numerical study is carried out in Section 6.

2. The Model

A retrial queueing system with two types of customers is considered in this paper. Type I customers arrive in batches of size k with probability c_k and type II customers arrive in batches of size k with probability d_k according to two independent Poisson processes with rates $\lambda_1\overline{c} = \lambda_1 \sum_{k=1}^{\infty} kc_k$ and $\lambda_2\overline{d} = \lambda_2 \sum_{k=1}^{\infty} kd_k$, respectively. If type II customers, upon arrival, find the server busy, they enter in to an orbit of infinite capacity in order to seek service again after random amount of time. All the customers in the retrial group behave independently of each other. The retrial time is exponentially distributed with mean $1/\alpha$. Type I customers are queued in a priority queue of infinite capacity after blocking, if the server is busy. As soon as the server is free, the customers in the priority queue are served using FCFS rule and the customers in the retrial group are served only if there are no customers in the priority queue. A type I call who has received service departs the system with probability $1 - q_1$ or returns to the priority queue for reservice with probability q_1. A type II call who has received service leaves the system with probability $1 - q_2$ or rejoins the retrial group with probability q_2.

The service time distributions for both type of customers are identically and independently distributed random variables and have different distributions. A supplementary variable technique is used for the analysis and the variable considered being the residual service time of a customer in service. The service time density function is $b_i(x)$; $i = 1, 2$ and $B_k^*(s) = \int_0^\infty e^{-sx} b_k(x)dx$, $k = 1, 2$ is the Laplace transformation of the distribution function $b_k(x)$.

The Stochastic process related to the model is $X(t) = \{(\xi(t), N_p(t), N_r(t), S_k(t)) : t \geq 0\}$ where $N_p(t)$ = number of customers in the priority queue at time t, $N_r(t)$ = number of customers in the retrial group at time t, and $\xi(t)$ = the server state at time t where

$$\xi(t) = \begin{cases} 0, \text{ when the server is idle} \\ 1, \text{ when the server is busy with type I customer} \\ 2, \text{ when the server is busy with type II customer,} \end{cases} \tag{2.1}$$

$S_k(t)$ = the residual service time of a type k customer in service at time t. $X(t)$ is a Markov process with state space $\{0, 1, 2\} \times \{0, 1, 2, 3, \ldots\} \times (0, \infty)$ and the corresponding stationary process is $\{(\xi, N_p, N_r, S_k)\}$.

The related probabilities are $q_j(t) = \Pr\{\xi(t) = 0, N_r(t) = j\}$

$$p(k, i, j; x, t)dx = \Pr\{\xi(t) = k, N_p(t) = i, N_r(t) = j, S_k(t)\epsilon(x, x + dx)\}, \quad k = 1, 2. \tag{2.2}$$

In steady state, the corresponding probabilities are

$$\begin{aligned} q_j &= \lim_{t \to \infty} q_j(t), \\ p(k, i, j; x) &= \lim_{t \to \infty} p(k, i, j; x, t), \end{aligned} \tag{2.3}$$

and the Laplace transformation of $p(k, i, j; x)$ is

$$p^*(k, i, j; s) = \int_0^\infty e^{-sx} p(k, i, j; x)dx, \quad i = 1, 2, \ j \geq 0. \tag{2.4}$$

It is clear that

$$p(k, i, j; \ 0) = \int_0^\infty p(k, i, j; x)dx = \Pr\{\xi = k, \ N_p = i, \ N_r = j\} \tag{2.5}$$

is the steady-state probability that there are i customers in the priority queue, j customers in the retrial group, and the server is busy with a kth-type customer.

For $|Z_1|,|Z_2| \leq 1$, the following probability generating functions are defined for the analysis:

$$
\begin{gathered}
Q(Z_2) = \sum_{j=0}^{\infty} q_j Z_2^j, \\
C(Z_1) = \sum_{j=1}^{\infty} c_j Z_1^j, \\
D(Z_2) = \sum_{j=1}^{\infty} d_j Z_2^j, \\
P^*(k,i,s,Z_2) = \sum_{j=0}^{\infty} p^*(k,i,j,s) Z_2^j; \quad i = 0,1,2,\ldots,\ k = 1,2, \\
P^*(k,s,Z_1,Z_2) = \sum_{i=0}^{\infty} P^*(k,i,s,Z_2) Z_1^i; \quad k = 1,2, \\
P(k,i,0,Z_2) = \sum_{j=0}^{\infty} p^*(k,i,j,0) Z_2^j; \quad i = 0,1,2,\ldots,\ k = 1,2, \\
P(k,0,Z_1,Z_2) = \sum_{i=0}^{\infty} P(k,i,0,Z_2) Z_1^i.
\end{gathered} \tag{2.6}
$$

3. The Analysis

Using the mean drift argument of Falin [15], it can be shown that the system is stable if $\rho_1 + \rho_2 < 1$ where $\rho_1 = -\lambda_1 \overline{c} B_1^{*'}(0)$,$\rho_2 = -\lambda_2 \overline{d} B_2^{*'}(0)$.

Now the mathematical equations that govern the system are obtained by employing the remaining service time (vacation time) as the supplementary variable. Relating the state of the system at time t and $t + dt$, the following partial differential-difference equations are obtained.

For $j \geq 0,\ x \geq 0,\ i \geq 0$

$$
\begin{aligned}
(\lambda + j\alpha) \frac{d}{dt} q_j(t) &= (1 - q_1) p(1,0,j;0,t) + (1 - q_2) p(2,0,j;0,t) + q_2 p(2,0,j-1;0,t), \\
\frac{-\partial p(1,0,j;x,t)}{\partial x} + \frac{-\partial p(1,0,j;x,t)}{\partial t} &= -\lambda p(1,0,j;x,t) + q_1 b_1(x) p(1,0,j;0,t) \\
&\quad + \lambda_1 b_1(x) q_j(t) + (1 - q_1) b_1(x) p(1,1,j;0,t) \\
&\quad + \lambda_2 \sum_{k=1}^{j} d_k p(1,0,j-k;x,t),
\end{aligned}
$$

$$
\begin{aligned}
\frac{-\partial p(1,i,j;x,t)}{\partial x}+\frac{-\partial p(1,i,j;x,t)}{\partial t} &= -\lambda p(1,i,j;x,t)+q_1 b_1(x)p(1,i,j;0,t) \\
&\quad +(1-q_1)b_1(x)p(1,i+1,j;0,t)+\lambda_1\sum_{k=1}^{i}c_k p(1,i-k,j;x,t) \\
&\quad +\lambda_2\sum_{k=1}^{j}d_k p(1,i,j-k;x,t), \\
\frac{-\partial p(2,0,j;x,t)}{\partial x}+\frac{-\partial p(2,0,j;x,t)}{\partial t} &= -\lambda p(2,0,j;x,t)+\lambda_2 b_2(x)\sum_{k=0}^{j}d_{k+1}q_{j-k}(t) \\
&\quad +(j+1)\alpha b_2(x)q_{j+1}(t)+\lambda_2\sum_{k=1}^{j}d_k p(2,0,j-k;x,t), \\
\frac{-\partial p(2,i,j;x,t)}{\partial x}+\frac{-\partial p(2,i,j;x,t)}{\partial t} &= -\lambda p(2,i,j;x,t)+\lambda_1\sum_{k=1}^{i}c_k p(2,i-k,j;x,t) \\
&\quad +\lambda_2\sum_{k=1}^{j}d_k p(2,i,j-k;x,t).
\end{aligned}
\tag{3.1}
$$

In steady state, (3.1) becomes

$$
(\lambda+j\alpha)q_j=(1-q_1)p(1,0,j;0)+(1-q_2)p(2,0,j,0)+q_2 p(2,0,j-1,0), \tag{3.2}
$$

$$
\begin{aligned}
-p'(1,0,j;x) &= -\lambda p(1,0,j;x)+q_1 b_1(x)p(1,0,j;0)+\lambda_1 b_1(x)q_j \\
&\quad +(1-q_1)b_1(x)p(1,1,j;0)+\lambda_2\sum_{k=1}^{j}d_k p(1,0,j-k;x),
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
-p'(1,i,j;x) &= -\lambda p(1,i,j;x)+q_1 b_1(x)p(1,i,j;0)+(1-q_1)b_1(x)p(1,i+1,j;0) \\
&\quad +\lambda_1\sum_{k=1}^{i}c_k p(1,i-k,j;x)+\lambda_2\sum_{k=1}^{j}d_k p(1,i,j-k;x),
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
-p'(2,0,j;x) &= -\lambda p(2,0,j;x)+\lambda_2 b_2(x)\sum_{k=0}^{j}d_{k+1}q_{j-k}+(j+1)\alpha b_2(x)q_{j+1} \\
&\quad +\lambda_2\sum_{k=1}^{j}d_k p(2,0,j-k;x),
\end{aligned}
\tag{3.5}
$$

$$
-p'(2,i,j;x)=-\lambda p(2,i,j;x)+\lambda_1\sum_{k=1}^{i}c_k p(2,i-k,j;x)+\lambda_2\sum_{k=1}^{j}d_k p(2,i,j-k;x), \tag{3.6}
$$

and the normalization condition is

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\int_0^{\infty} p(1,i,j;x)dx + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^{\infty} p(2,i,j;x)dx + \sum_{j=0}^{\infty} q_j = 1, \tag{3.7}$$

where $\lambda = \lambda_1 + \lambda_2$.

By taking the Laplace Stieltjes transformation of (3.3) to (3.6), the following equations are obtained:

$$\begin{aligned}
(s-\lambda)P^*(1,0,j;s) &= P(1,0,j;0) - q_1 p(1,0,j;0)B_1^*(s) - \lambda_1 q_j B_1^*(s) \\
&\quad - (1-q_1)p(1,1,j;0)B_1^*(s) - \lambda_2\sum_{k=1}^{j} d_k P^*(1,0,j-k;s), \\
(s-\lambda)P^*(1,i,j;s) &= P(1,i,j;0) - q_1 p(1,i,j;0)B_1^*(s) - (1-q_1)p(1,i+1,j;0)B_1^*(s) \\
&\quad - \lambda_1\sum_{k=1}^{i} c_k P^*(1,i-k,j;s) - \lambda_2\sum_{k=1}^{j} d_k P^*(1,i,j-k;s), \\
(s-\lambda)P^*(2,0,j;s) &= P(2,0,j;0) - \lambda_2 B_2^*(s)\sum_{k=0}^{j} d_{k+1}q_{j-k} - (j+1)\alpha q_{j+1}B_2^*(s) \\
&\quad - \lambda_2\sum_{k=1}^{j} d_k P^*(2,0,j-k;s), \\
(s-\lambda)P^*(2,i,j;s) &= -\lambda_1\sum_{k=1}^{i} c_k P^*(2,i-k,j;x) + \lambda_2\sum_{k=1}^{j} d_k P^*(2,i,j-k;s), \\
P(2,i,j;0) &= 0.
\end{aligned} \tag{3.8}$$

In (3.2) and (3.8), multiplying by Z_2^j and then summing over j, the following equations are obtained:

$$\lambda Q(Z_2) + \alpha Z_2 Q'(Z_2) = (1-q_1)P(1,0;0,Z_2) + (1-q_2)P(2,0;0,Z_2) + q_2 Z_2 P(2,0;0,Z_2), \tag{3.9}$$

$$\begin{aligned}
(s-\lambda+\lambda_2 D(Z_2))P^*(1,0;s,Z_2) &= P(1,0;0,Z_2) - \lambda_1 B_1^*(s)Q(Z_2) - q_1 B_1^*(s)P(1,0;0,Z_2) \\
&\quad - (1-q_1)B_1^*(s)P(1,1;0,Z_2),
\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
(s-\lambda+\lambda_2 D(Z_2))P^*(1,i;s,Z_2) = {} & P(1,i;0,Z_2) - \left(1-q_1\right)B_1^*(s)P(1,i+1;0,Z_2) \\
& - q_1 B_1^*(s)P(1,i;0,Z_2) - \lambda_1 \sum_{k=1}^{i} c_k P^*(1,i-k;s,Z_2),
\end{aligned} \tag{3.11}
$$

$$
(s-\lambda+\lambda_2 D(Z_2))P^*(2,0;s,Z_2) = P(2,0;0,Z_2) - \lambda_2 B_2^*(s)\frac{D(Z_2)}{Z_2}Q(Z_2) - \alpha B_2^*(s)Q'(Z_2), \tag{3.12}
$$

$$
(s-\lambda+\lambda_2 D(Z_2))P^*(2,i;s,Z_2) = -\lambda_1 \sum_{k=1}^{i} c_k P^*(2,i-k;s,Z_2). \tag{3.13}
$$

Multiplying (3.11) and (3.13) by Z_1^i and summing over $i = 1,2,\ldots$ and using (3.10) and (3.12) leads to

$$
\begin{aligned}
(s-\lambda+\lambda_1 C(Z_1)+\lambda_2 D(Z_2))P^*(1;s,Z_1,Z_2) = {} & P(1;0,Z_1,Z_2) - \lambda_1 B_1^*(s)Q(Z_2) \\
& - q_1 B_1^*(s)P(1;0,Z_1,Z_2) \\
& - \frac{(1-q_1)B_1^*(s)}{Z_1}P(1;0,Z_1,Z_2) \\
& + \frac{(1-q_1)B_1^*(s)}{Z_1}P(1;0,0,Z_2),
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
(s-\lambda+\lambda_1 C(Z_1)+\lambda_2 D(Z_2))P^*(2;s,Z_1,Z_2) = {} & P(2,0;0,Z_2) - \lambda_2 B_2^*(s)\frac{D(Z_2)}{Z_2}Q(Z_2) \\
& - \alpha B_2^*(s)Q'(Z_2).
\end{aligned} \tag{3.15}
$$

By substituting $s = \lambda - \lambda_1 C(Z_1) - \lambda_2 D(Z_2)$ in (3.14) and (3.15), we get

$$
P(1,0;0,Z_2) = \frac{\lambda_1 z_1}{1-q_1}Q(Z_2) - \frac{\left(Z_1 - B_1^*(l)\right) - q_1(Z_1-1)B_1^*(l)}{\left(1-q_1\right)B_1^*(l)}P(1;0,Z_1,Z_2), \tag{3.16}
$$

$$
P(2,0;0,Z_2) = B_2^*(l)\left[\lambda_2 \frac{D(Z_2)}{Z_2}Q(Z_2) + \alpha Q'(Z_2)\right], \tag{3.17}
$$

where $l = \lambda - \lambda_1 C(Z_1) - \lambda_2 D(Z_2)$.

Using (3.16) and (3.17) in (3.9) and on simplification, one can get the following equation:

$$
\begin{aligned}
& \alpha\left[Z_2 - (1-q_2+q_2 Z_2)B_2^*(l)\right]Q'(Z_2) \\
& \quad + \left[\lambda - (1-q_1)\frac{\lambda_1 Z_1}{1-q_1} - (1-q_2+q_2 Z_2)\lambda_2 B_2^*(l)\frac{D(Z_2)}{Z_2}\right]Q(Z_2) \\
& = \frac{q_1(Z_1-1)B_1^*(l) - \left(Z_1 - B_1^*(l)\right)}{B_1^*(l)}P(1;0,Z_1,Z_2).
\end{aligned} \tag{3.18}
$$

Define $f(Z_1, Z_2) = (q_1(Z_1 - 1)B_1^*(l) - (Z_1 - B_1^*(l)))/B_1^*(l)$ for each fixed Z_2, $|Z_2| \leq 1$. By Rouche's theorem, there is a unique solution $Z_1 = h(Z_2)$ of the equation $f(Z_1, Z_2) = 0$. Now (3.18) becomes

$$Q'(Z_2) = \frac{1}{\alpha}\frac{\lambda - \lambda_1 h(Z_2) - \lambda_2 U(Z_2)(D(Z_2)/Z_2)}{U(Z_2) - Z_2}Q(Z_2), \tag{3.19}$$

where $h(Z_2)$ is the root of the equation $Z_1 = B_1^*(\lambda - \lambda_1 C(Z_1) - \lambda_2 D(Z_2))$ and $U(Z_2) = (1 - q_2 + q_2 Z_2)B_2^*(l)$.

Using (3.19) in (3.18), it can be seen that

$$P(1;0,Z_1,Z_2) = \frac{\{L[Z_2 - (1 - q_2 + q_2 Z_2)B_2^*(l)] + R[U(Z_2) - Z_2]\}B_1^*(l)}{[q_1(Z_1 - 1)B_1^*(l) - (Z_1 - B_1^*(l))][U(Z_2) - Z_2]}Q(Z_2), \tag{3.20}$$

where $L = \lambda - \lambda_1 h(Z_2) - \lambda_2 U(Z_2)(D(Z_2)/Z_2)$, $R = \lambda - \lambda_1 Z_1 - (1 - q_2 + q_2 Z_2)\lambda_2 B_2^*(l)(D(Z_2)/Z_2)$.

Using (3.20) in (3.16) leads to

$$P(1;0,0,Z_2) = \frac{[\lambda_1 Z_1 + R][U(Z_2) - Z_2] + L[Z_2 - (1 - q_2 + q_2 Z_2)B_2^*(l)]}{(1 - q_1)[U(Z_2) - Z_2]}Q(Z_2). \tag{3.21}$$

Using (3.19) in (3.17) leads to

$$P(2;0,0,Z_2) = \frac{\{\lambda_2(D(Z_2)/Z_2)[U(Z_2) - Z_2] + L\}B_2^*(l)}{[U(Z_2) - Z_2]}Q(Z_2). \tag{3.22}$$

The general solution of the differential equation (3.19) is

$$Q(Z_2) = Q(1)\exp\left\{\frac{-1}{\alpha}\int_{Z_2}^{1}\frac{\lambda - \lambda_1 h(x) - \lambda_2 U(x)(D(x)/x)}{U(x) - x}dx\right\}, \tag{3.23}$$

where $Q(1)$ is a constant, which is the probability that the server is idle.

Putting $s = 0$ in (3.10) and in (3.11), we get

$$\begin{aligned}(-\lambda + \lambda_2 D(Z_2))P^*(1,0;0,Z_2) &= P(1,0;0,Z_2) - \lambda_1 Q(Z_2) - q_1 P(1,0;0,Z_2)\\ &\quad - (1 - q_1)P(1,1;0,Z_2),\end{aligned} \tag{3.24}$$

$$\begin{aligned}(-\lambda + \lambda_2 D(Z_2))P^*(1,i;0,Z_2) &= P(1,i;0,Z_2) - (1 - q_1)P(1,i+1;0,Z_2)\\ &\quad - q_1 P(1,i;0,Z_2) - \lambda_1\sum_{k=1}^{i}c_k P^*(1,i-k;0,Z_2).\end{aligned} \tag{3.25}$$

Summing (3.25) over $i = 1$ to ∞ and then adding (3.24), we get

$$\lambda_2(D(Z_2)-1)\sum_{i=0}^{\infty}P^*(1,i;0,Z_2) = (1-q_1)P(1,0;0,Z_2)-\lambda_1 Q(Z_2). \tag{3.26}$$

Putting $s = 0$ in (3.12) and in (3.13) results in

$$(-\lambda+\lambda_2 D(Z_2))P^*(2,0;0,Z_2) = P(2,0;0,Z_2)-\lambda_2\frac{D(Z_2)}{Z_2}Q(Z_2)-\alpha Q'(Z_2), \tag{3.27}$$

$$(-\lambda+\lambda_2 D(Z_2))P^*(2,i;0,Z_2) = -\lambda_1\sum_{k=1}^{i}c_k P^*(2,i-k;0,Z_2). \tag{3.28}$$

Summing (3.28) over $i = 1$ to ∞ and then adding (3.27), we get

$$\lambda_2(D(Z_2)-1)\sum_{i=0}^{\infty}P^*(2,i;0,Z_2) = P(2,0;0,Z_2)-\lambda_2 Q(Z_2)\frac{D(Z_2)}{Z_2}-\alpha Q'(Z_2). \tag{3.29}$$

Adding (3.28) and (3.29) and using (3.9) leads to

$$\begin{aligned}\lambda_2(D(Z_2)-1)\sum_{i=0}^{\infty}\sum_{k=1}^{2}P^*(k,i;0,Z_2) &= \lambda_2\left(1-\frac{D(Z_2)}{Z_2}-q_2(1-Z_2)B_2^*(l)\frac{D(Z_2)}{Z_2}\right)Q(Z_2)\\ &\quad+\alpha(1-Z_2)\left[q_2B_2^*(l)-1\right]Q'(Z_2).\end{aligned} \tag{3.30}$$

Evaluating at $Z_2 = 1$ and using normalization condition we get

$$Q'(1) = \frac{\lambda_2\overline{d}-\lambda_2(1-q_2)Q(1)}{\alpha(1-q_2)}. \tag{3.31}$$

Using (3.31) in (3.23) gives

$$P_I = Q(1) = \frac{\overline{c}\left[(1-\rho_1)(1-q_2)-\rho_2\right]}{(1-q_2)(\overline{c}+\rho_1-\rho_1\overline{c})}. \tag{3.32}$$

In steady state, the probability generating function of number of customers in the orbit when the server is idle is obtained from (3.32) and (3.23).

Substituting $s = 0$ in (3.14)

$$P^*(1;0,Z_1,Z_2) = \frac{(1-Z_1)(1-q_1)P(1;0,Z_1,Z_2) + \lambda_1 Z_1 Q(Z_2) - (1-q_1)P(1,0,0;Z_2)}{Z_1[\lambda - \lambda_1 C(Z_1) - \lambda_2 D(Z_2)]}. \tag{3.33}$$

Equation (3.33) together with (3.20) and (3.16) yields the joint probability generating function of the number of customers in the priority queue and in the orbit when the server is busy with type I customer as

$$P^*(1;0,Z_1,Z_2) = \frac{[1-B_1^*(l)]\{L[Z_2-(1-q_2+q_2Z_2)B_2^*(l)] + R[U(Z_2)-Z_2]\}}{l[q_1(Z_1-1)B_1^*(l) - (Z_1 - B_1^*(l))][U(Z_2)-Z_2]} Q(Z_2). \tag{3.34}$$

Putting $s = 0$ in (3.15),

$$(-\lambda + \lambda_1 C(Z_1) + \lambda_2 D(Z_2))P^*(2;0,Z_1,Z_2) = P(2,0;0,Z_2) - \lambda_2 \frac{D(Z_2)}{Z_2} Q(Z_2) - \alpha Q'(Z_2). \tag{3.35}$$

Substituting (3.19) in (3.35), we get

$$P^*(2;0,Z_1,Z_2) = \frac{1}{l}\left\{\left[\frac{L}{[U(Z_2)-Z_2]} + \lambda_2 \frac{D(Z_2)}{Z_2}\right] Q(Z_2) - P(2;0,0,Z_2)\right\}. \tag{3.36}$$

Equation (3.36) together with (3.19) and (3.17) yields the joint probability generating function of the number of customers in the priority queue and in the orbit when the server is busy with type II customer as

$$P^*(2;0,Z_1,Z_2) = \frac{[1-B_2^*(l)]}{l}\left\{\frac{\lambda_2 D(Z_2)}{Z_2} + \frac{L}{U(Z_2)-Z_2}\right\} Q(Z_2). \tag{3.37}$$

Thus, we have the following theorem.

Theorem 3.1. *The stationary distribution of* $\{(\xi, N_p, N_r, S_k)\}$ *has the following generating functions:*

$$Q(Z_2) = \frac{\bar{c}[(1-\rho_1)(1-q_2)-\rho_2]}{(1-q_2)(\bar{c}+\rho_1-\rho_1\bar{c})}\exp\left\{\frac{1}{\alpha}\int_1^{Z_2}\frac{\lambda-\lambda_1 h(x)-\lambda_2 U(x)(D(x)/x)}{U(x)-x}dx\right\},$$

$$P^*(1;0,Z_1,Z_2) = \frac{[1-B_1^*(l)]\{L[Z_2-(1-q_2+q_2Z_2)B_2^*(l)]+R[U(Z_2)-Z_2]\}}{l[q_1(Z_1-1)B_1^*(l)-(Z_1-B_1^*(l))][U(Z_2)-Z_2]}Q(Z_2),$$

$$P^*(2;0,Z_1,Z_2) = \frac{[1-B_2^*(l)]}{l}\left\{\frac{\lambda_2 D(Z_2)}{Z_2}+\frac{L}{U(Z_2)-Z_2}\right\}Q(Z_2). \tag{3.38}$$

Corollary 3.2. *The probability that the server busy is*

$$P_B = P^*(1;0,1,1)+P^*(2;0,1,1) = \frac{[\rho_1(1-q_2)+\rho_2\bar{c}]}{(1-q_2)(\bar{c}+\rho_1-\rho_1\bar{c})}. \tag{3.39}$$

4. Particular Models

By taking particular values to some parameters of the above model, the following models can be obtained.

(i) When $d_k = 0 = c_k$, $k \neq 1$, $q_1 = q_2 = 0$, and $B_1(x) = B_2(x) = B(x)$, the system coincides with that of Choi and Park [4].

(ii) When $d_k = 0 = c_k$, $k \neq 1$, and $q_1 = q_2 = 0$, the above results coincide with the results of Falin et al. [6].

(iii) When $d_k = 0$, $k \neq 1$, the system coincides with that of Thillaigovindan and Kalyanaraman [14].

5. Operating Characteristics

The operating characteristics like the mean number of customers in the priority queue (N_p) and the mean number of customers in the orbit (N_r) have been calculated using the formulas $N_p = \lim_{Z_1\to 1}P^{*'}(1;0,Z_1,1) + \lim_{Z_1\to 1}P^{*'}(2;0,Z_1,1)$ and $N_r = \lim_{Z_2\to 1}P^{*'}(1;0,1,Z_2) + \lim_{Z_2\to 1}P^{*'}(2;0,1,Z_2) + Q'(1)$. After putting $Z_2 = 1$ in (3.34) and in (3.37), we get

$$\begin{aligned} P^*(1;0,Z_1,1) &= \frac{\{1-B_1^*[\lambda_1(1-C(Z_1))]\}(A_1+A_2)Q(1)}{[\lambda_1(1-C(Z_1))]}, \\ P^*(2;0,Z_1,1) &= \frac{\{1-B_2^*[\lambda_1(1-C(Z_1))]\}\left[\lambda_1 h'(1)+\lambda_2\bar{d}\right]Q(1)}{[\lambda_1(1-C(Z_1))][1-U'(1)]}, \end{aligned} \tag{5.1}$$

where $A_1 = ([1-B_2^*[\lambda_1(1-C(Z_1))]][-\lambda_1 h'(1)-\lambda_2 U'(1)-\lambda_2\bar{d}+\lambda_2])/((1-C(Z_1))[U'(1)-1])$, $A_2 = \lambda-\lambda_1 Z_1-\lambda_2 B_2^*[\lambda_1(1-C(Z_1))]$, $A_3 = q_1(Z_1-1)B_1^*[\lambda_1(1-C(Z_1))]-(Z_1-B_1^*[\lambda_1(1-C(Z_1))])$, $h'(1) = \lambda_2\bar{d}\rho_1/\lambda_1\bar{c}(1-\rho_1)$, $U'(1) = q_2\rho_2/(1-\rho_1)$.

Differentiating (5.1) with respect to Z_1 and then taking the limit as $Z_1 \to 1$, we get

$$\lim_{Z_1 \to 1} P^{*'}(1;0,Z_1,1) = (1-\rho_1)\left[(1-q_2)-\rho_2(1-\bar{c})\right]A_4 + \frac{\lambda_1\lambda_2\beta_2\rho_1\bar{c}\bar{d}}{2(1-\rho_1-q_1)(1-q_2)} - \frac{c_2\rho_1\left[(1-\rho_1)(1-q_2)-\rho_2\right]}{2\bar{c}(1-q_2)(1-\rho_1-q_1)\left[\rho_1+\bar{c}(1-\rho_1)\right]}, \tag{5.2}$$

where $A_4 = ((1-q_1)[\rho_1 c_2 + \lambda_1^2\bar{c}^3\beta_1] + 2\rho_1^2 q_1\bar{c})/(2\bar{c}(1-q_2)(1-\rho_1-q_1)^2[\rho_1+\bar{c}(1-\rho_1)])$

$$\lim_{Z_1 \to 1} P^{*'}(2;0,Z_1,1) = \frac{\lambda_1\lambda_2\bar{c}\bar{d}\beta_2}{2(1-q_2)}. \tag{5.3}$$

After putting $Z_1 = 1$ in (3.34) and in (3.37), we get

$$\begin{aligned} P^*(1;0,1,Z_2) &= \frac{\{L[Z_2-(1-q_2+q_2Z_2)B_2^*[\lambda_2[1-D(Z_2)]]] + R[U(Z_2)-Z_2]\}}{-\lambda_2[1-D(Z_2)][U(Z_2)-Z_2]}Q(1), \\ P^*(2;0,1,Z_2) &= \frac{[1-B_2^*[\lambda_2[1-D(Z_2)]]]\{LZ_2+\lambda_2D(Z_2)[U(Z_2)-Z_2]\}}{\lambda_2[1-D(Z_2)]Z_2[U(Z_2)-Z_2]}Q(1). \end{aligned} \tag{5.4}$$

Differentiating (5.4) with respect to Z_2 and then taking limit as $Z_2 \to 1$, we get

$$\begin{aligned} \lim_{Z_2 \to 1} P^{*'}(1;0,1,Z_2) &= \frac{D_1\left[\lambda_1\bar{c}^2(1-\rho_1)^2D_2-(1-q_2-\rho_2)D_3\right]}{2\lambda_1\bar{c}^2\bar{d}^2(1-\rho_1)^2(q_2-1)\left[\rho_1+\bar{c}(1-\rho_1)\right]D_0} \\ &\quad + \frac{(1-q_2-\rho_2)(D_4+\lambda_2\bar{c}D_5-D_6-D_7)}{2\lambda_1\lambda_2\bar{c}^2\bar{d}^2(1-\rho_1)^2(1-q_2)\left[\rho_1+\bar{c}(1-\rho_1)\right]} \\ &\quad + \frac{D_8\bar{c}\left[(1-\rho_1)(1-q_2)-\rho_2\right]}{2\bar{d}^2(1-q_2)\left[\rho_1+\bar{c}(1-\rho_1)\right]} + \frac{\rho_1\lambda_2D_1(1-q_2-\rho_2-\rho_2\bar{c})}{\alpha D_0(1-q_2)\left[\rho_1+\bar{c}(1-\rho_1)\right]}, \\ D_0 &= \left[(1-q_2)(1-\rho_1)-\rho_2\right], \\ D_1 &= \rho_1\bar{d}+\rho_2\bar{c}+\bar{c}(1-\rho_1)\left(\bar{d}-1+q_2\right), \\ D_2 &= D_0\left[2q_2\rho_2\bar{d}+\rho_2d_2+\lambda_2^2\bar{d}^3\beta_2\right], \\ D_3 &= 2\lambda_1\bar{c}^2\bar{d}q_2\rho_2(1-\rho_1)^2+\lambda_1\lambda_2^2\bar{c}^2\beta_2\bar{d}^3(1-\rho_1)+\lambda_2c_2\bar{d}^2\rho_1^2\rho_2 \\ &\quad + \lambda_1^2\lambda_2\bar{c}^3\bar{d}^2\beta_1\rho_2+\lambda_1\bar{c}^2d_2\rho_2(1-\rho_1)^2, \end{aligned}$$

$$D_4 = \lambda_1^2\lambda_2^2\bar{c}^3\bar{d}^3\beta_1 + \lambda_2^2\bar{d}^3\rho_1^3c_2 + \lambda_1\lambda_2 d_2\bar{d}\rho_1\bar{c}^2(1-\rho_1)^2 + 2\lambda_1\lambda_2\bar{c}^3\bar{d}q_2\rho_2(1-\rho_1)^2,$$

$$D_5 = \lambda_1\lambda_2^2\bar{c}^2\bar{d}^3\beta_2(1-\rho_1) + \lambda_2 c_2\bar{d}^2\rho_1^2\rho_2 + \lambda_1^2\lambda_2\bar{c}^3\bar{d}^2\beta_1\rho_2 + \lambda_1\bar{c}^2 d_2\rho_2(1-\rho_1)^2,$$

$$D_6 = 2\lambda_1\lambda_2\bar{c}^3\bar{d}\left(\bar{d}-1\right)(1-\rho_1)^2[1-\rho_1-\rho_2-q_2(1-\rho_1)],$$

$$D_7 = \lambda_1\lambda_2\bar{c}^2 d_2(1-\rho_1)^2\left[\rho_1\bar{d} + \bar{c}[q_2(1-\rho_1)+\rho_2] - \bar{c}(1-\rho_1)\right],$$

$$D_8 = 2\bar{d}\left(1-\bar{d}\right)(q_2+\rho_2-1) - \lambda_2^2\bar{d}^3\beta_2 - d_2 + q_2\left[-2\rho_2\bar{d} + d_2\right],$$

$$\lim_{Z_2\to 1} P^{*'}(2;0,1,Z_2) = \frac{\lambda_2^2\bar{d}^2\beta_2}{2(1-q_2)} + \frac{\lambda_2\rho_2 D_1}{\alpha\bar{c}(1-q_2)D_0} + \frac{\rho_2^2 q_2}{(1-q_2)[(1-\rho_1)(1-q_2)-\rho_2]}$$
$$+ \frac{\rho_2[((1-\rho_1)(1-q_2)-\rho_2)D_9 + (\bar{c}+\rho_1-\rho_1\bar{c})D_{10}]}{2\lambda_1\bar{d}\bar{c}^2(1-\rho_1)^2 D_0(\bar{c}+\rho_1-\rho_1\bar{c})(1-q_2)},$$

$$D_9 = \lambda_1^2\lambda_2\bar{c}^3\bar{d}^2\beta_1 + \lambda_2\bar{d}^2\rho_1^3c_2 + \lambda_1\bar{c}^2 d_2(1-\rho_1)(\bar{c}+\rho_1-\rho_1\bar{c}),$$

$$D_{10} = \lambda_1\lambda_2^2\bar{c}^2\bar{d}^3\beta_2(1-\rho_1) + \lambda_2\bar{d}^2\rho_1^2\rho_2 c_2 + \lambda_1^2\lambda_2\bar{c}^3\bar{d}^2\beta_1\rho_2$$
$$+ \lambda_1\bar{c}^2 d_2\rho_2(1-\rho_1)^2. \tag{5.5}$$

From (3.31) and (3.32)

$$Q'(1) = \frac{\lambda_2}{\alpha}\left[\frac{\bar{d}}{1-q_2} - \frac{\bar{c}[(1-\rho_1)(1-q_2)-\rho_2]}{(1-q_2)(\bar{c}+\rho_1-\rho_1\bar{c})}\right]. \tag{5.6}$$

(i) Mean number of customers in the priority queue is

$$N_p = \lim_{Z_1\to 1} P^{*'}(1;0,Z_1,1) + \lim_{Z_1\to 1} P^{*'}(2;0,Z_1,1). \tag{5.7}$$

(ii) Mean number of customers in the orbit is

$$N_r = \lim_{Z_2\to 1} P^{*'}(1;0,1,Z_2) + \lim_{Z_2\to 1} P^{*'}(2;0,1,Z_2) + Q'(1). \tag{5.8}$$

(iii) Mean busy period.

Busy period T_b is the length of the time interval that keeps the server busy continuously and this continues till the instant server becomes free again and let T_0 be the

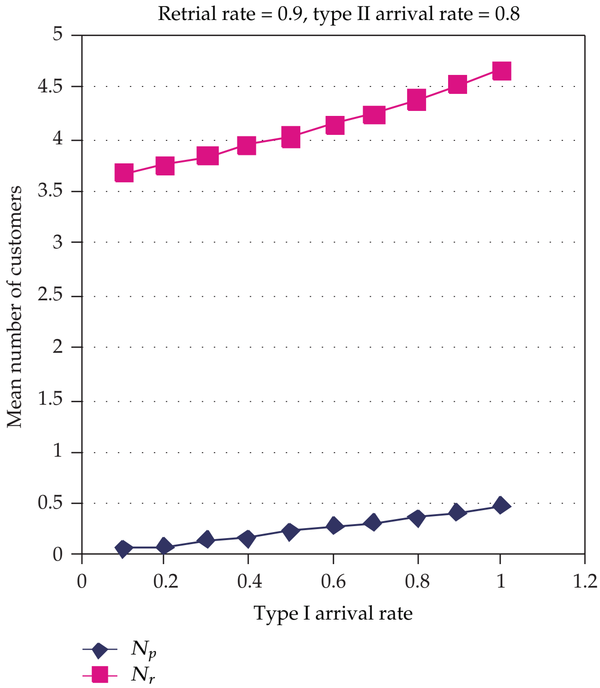

Figure 1: Type I arrival rate versus mean number of customers.

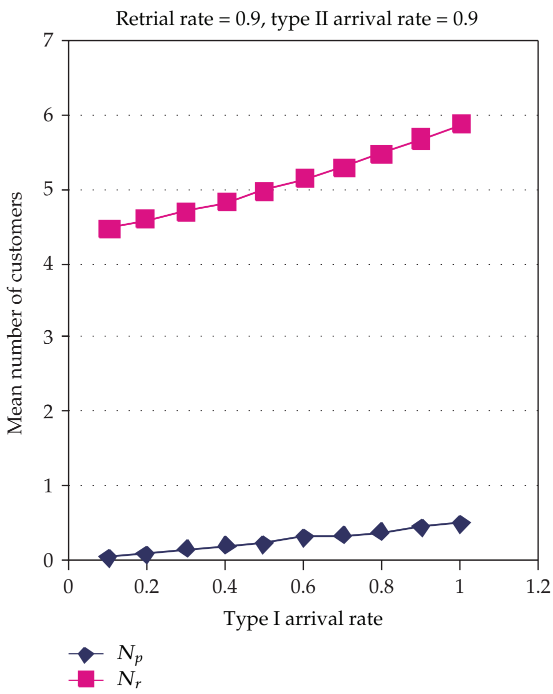

Figure 2: Type I arrival rate versus mean number of customers.

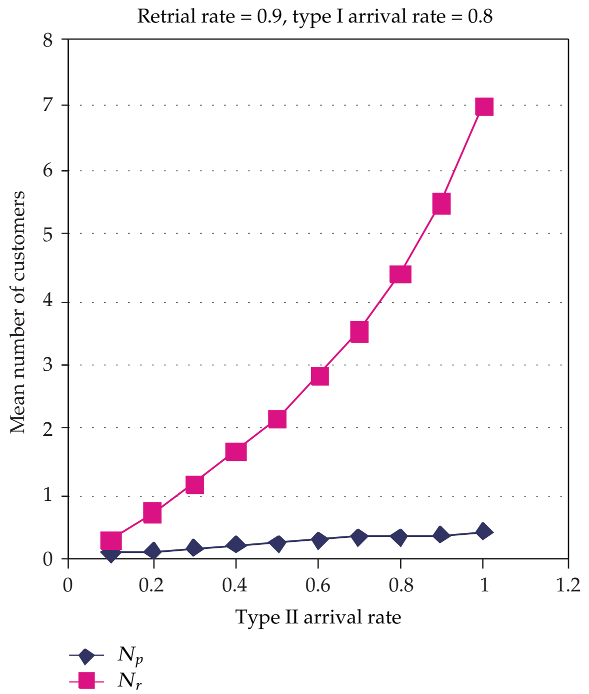

Figure 3: Type II arrival rate versus mean number of customers.

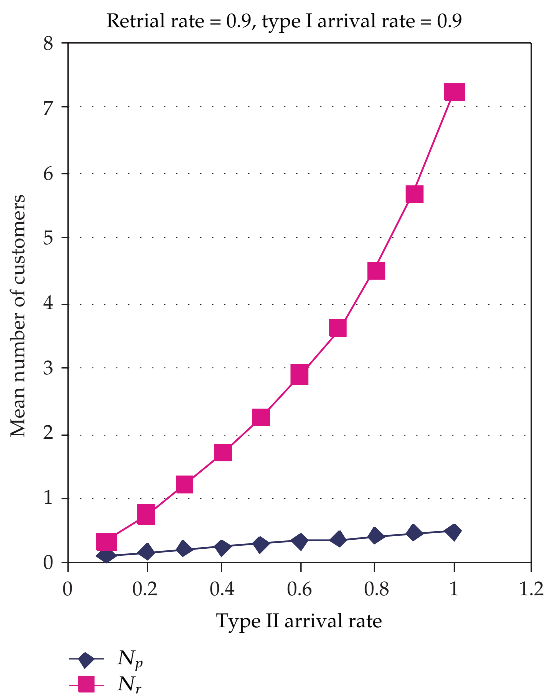

Figure 4: Type II arrival rate versus mean number of customers.

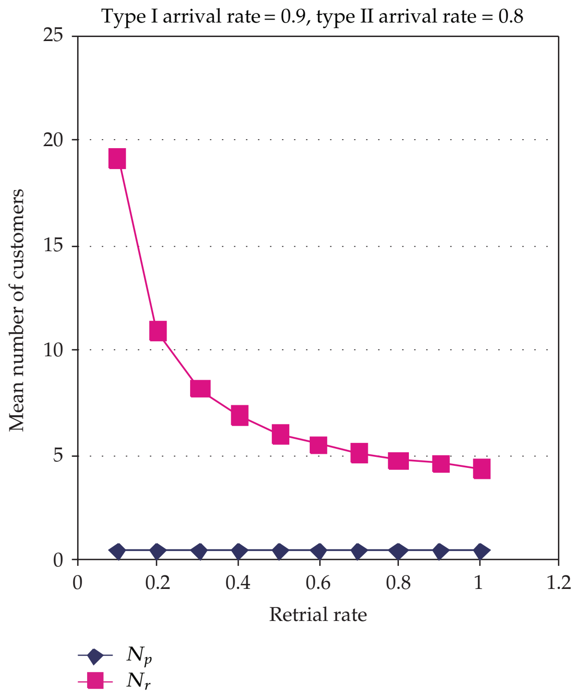

Figure 5: Retrial rate versus mean number of customers.

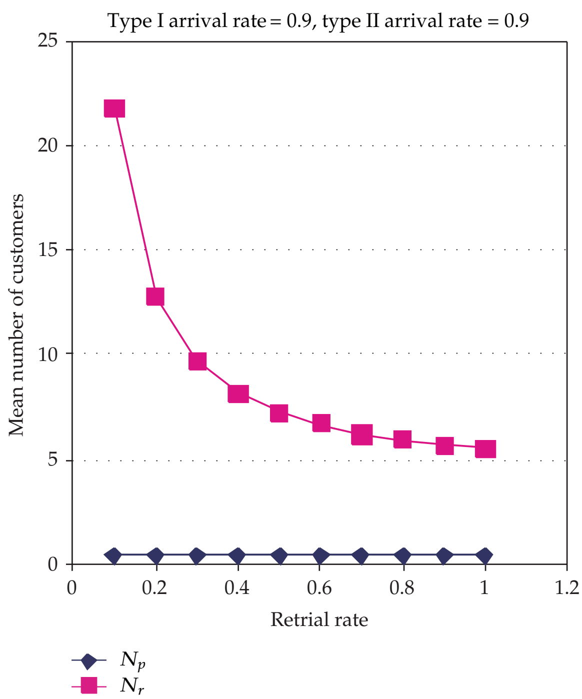

Figure 6: Retrial rate versus mean number of customers.

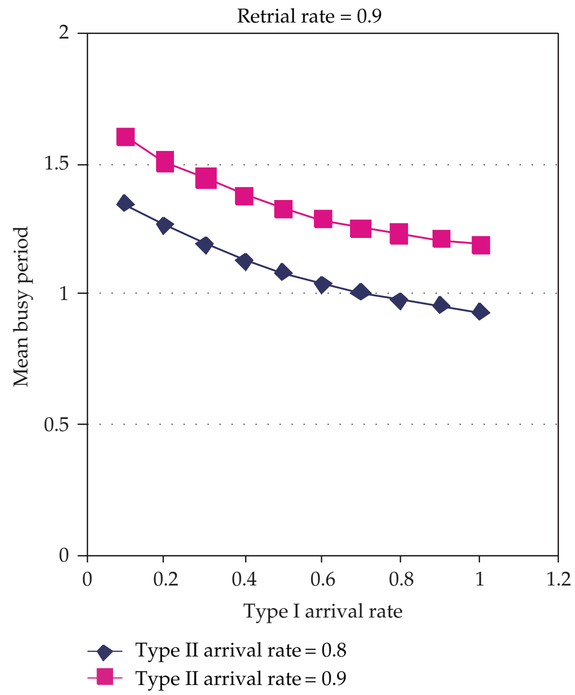

Figure 7: Type I arrival rate versus mean busy period.

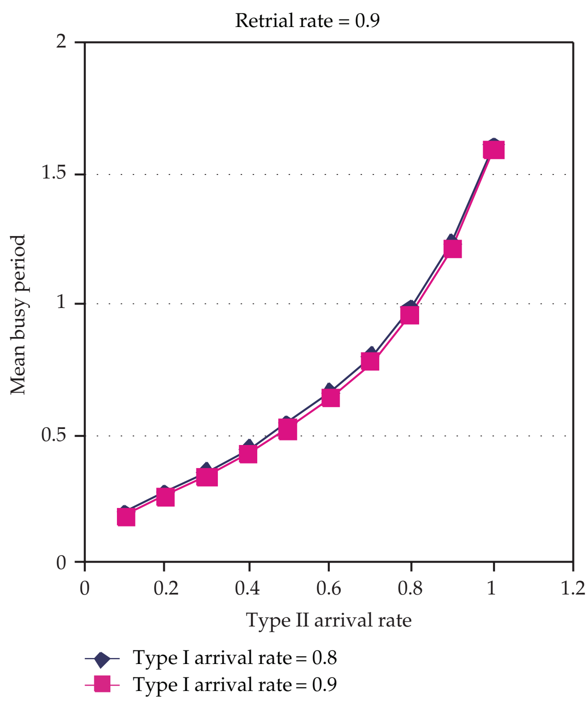

Figure 8: Type I arrival rate versus mean busy period.

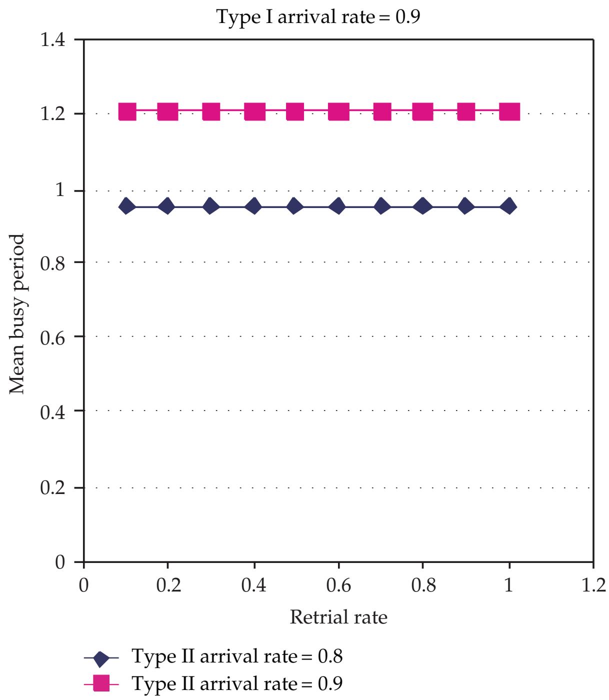

Figure 9: Retrial rate versus mean busy period.

Table 1: The probabilities P_I and P_B.

λ_1	$\alpha = 0.9$, $\lambda_2 = 0.8$		$\alpha = 0.9$, $\lambda_2 = 0.9$	
	P_I	P_B	P_I	P_B
0.1	0.4511	0.0089	0.3837	0.0088
0.2	0.4422	0.0178	0.3750	0.0175
0.3	0.4334	0.0266	0.3663	0.0262
0.4	0.4246	0.0354	0.3576	0.0349
0.5	0.4158	0.0442	0.3490	0.0435
0.6	0.4071	0.0529	0.3404	0.0521
0.7	0.3984	0.0616	0.3319	0.0606
0.8	0.3898	0.0702	0.3233	0.0692
0.9	0.3811	0.0789	0.3148	0.0777
1.0	0.3725	0.0875	0.3064	0.0861

length of the idle period. For this model, T_b and T_0 generates an alternating renewal process and therefore

$$\frac{E(T_b)}{E(T_0)} = \frac{Pr\{T_b\}}{1 - Pr\{T_b\}} = \frac{P_B}{1 - P_B}. \tag{5.9}$$

Table 2: The probabilities P_I and P_B.

λ_2	$\alpha = 0.9$, $\lambda_1 = 0.8$		$\alpha = 0.9$, $\lambda_1 = 0.9$	
	P_I	P_B	P_I	P_B
0.1	0.8548	0.0777	0.8453	0.0872
0.2	0.7884	0.0766	0.7790	0.0860
0.3	0.7219	0.0756	0.7127	0.0848
0.4	0.6555	0.0745	0.6464	0.0836
0.5	0.5891	0.0734	0.5801	0.0824
0.6	0.5226	0.0724	0.5138	0.0812
0.7	0.4562	0.0713	0.4474	0.0801
0.8	0.3898	0.0702	0.3811	0.0789
0.9	0.3233	0.0692	0.3148	0.0777
1.0	0.2569	0.0681	0.2485	0.0765

But $E(T_0) = 1/\lambda$

$$E(T_b) = \frac{P_B}{\lambda(1 - P_B)}. \tag{5.10}$$

Using (3.39) on (5.10), we get

$$E(T_b) = \frac{[\rho_1(1 - q_2) + \rho_2\overline{c}]}{\lambda\overline{c}[(1 - \rho_1)(1 - q_2) - \rho_2]}. \tag{5.11}$$

6. Numerical Study

In this section, some numerical examples related to the model analyzed in this paper are given. By varying type I arrival rate, type II arrival rate, and the retrial rate, the mean number of customers in the priority queue, the mean number of customers in the orbit, the mean busy period, the probability that the server is idle, and the probability that the server is busy are calculated. For the analysis, the parameters q_1, q_2, $\overline{c}$, $c''(1)$, $\overline{d}$, $d''(1)$, β_1, β_2, $B_1^{*(1)}{}_{(0)}$ and $B_2^{*(1)}{}_{(0)}$ are fixed. In Figures 1 and 2, the retrial rate is taken as 0.9 and type II arrival rate is taken as 0.8 and 0.9, respectively; by varying values of type I arrival rate the graphs of the mean number of customers in the priority queue and the mean number of customers in the orbit are drawn. In Figures 3 and 4, the retrial rate has been fixed as 0.9 but type I arrival rates are 0.8 and 0.9, respectively. The graphs of the mean number of customers in the priority queue and the mean number of customers in the orbit are drawn against varying values of type II arrival rate. In Figures 5 and 6, the same graphs with respect to varying retrial rate are drawn for fixed values of type I and type II arrival rates ($\lambda_1 = 0.9$, $\lambda_2 = 0.8$ and 0.9). Figures 7, 8, and 9 show the graphs of mean busy period for varying values of type I arrival rate and type II arrival rate and retrial rate, respectively. From Figures 7 and 8, it is seen that the mean busy period is an decreasing function with respect to type I arrival rate and the mean busy period is an increasing function with respect to type II arrival rate. Whereas from Figure 9, the mean busy period is a constant function with respect to retrial rate. In Tables 1 and 2, the probability that the server is idle (P_I) and the probability that the server is busy (P_B) are presented. From the graphs, it is clear that as type I arrival rate (type II arrival rate) increases the mean number of customers in the priority queue and the mean number of customers in the orbit also increase, whereas as the retrial rate increases the mean number of customers in the orbit decreases and the mean number of customers in the priority queue is a constant function.

7. Conclusion

In the foregoing analysis, an $M/G/1$ queue with retrial queueing system with two types of batch arrivals is considered to obtain queue length distribution and mean queue length. Extensive numerical work has been carried out to observe the trends of the operating characters of the system.

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