Roman Bondage Number of a Graph^{*}

Fu-Tao Hu, Jun-Ming Xu[†]

Department of Mathematics University of Science and Technology of China Hefei, Anhui, 230026, China

Abstract: The Roman dominating function on a graph G = (V, E) is a function $f: V \to \{0, 1, 2\}$ such that each vertex x with f(x) = 0 is adjacent to at least one vertex y with f(y) = 2. The value $f(G) = \sum_{u \in V(G)} f(u)$ is called the weight of f. The Roman domination number $\gamma_{\mathrm{R}}(G)$ is defined as the minimum weight of all Roman dominating functions. This paper defines the Roman bondage number $b_{\mathrm{R}}(G)$ of a nonempty graph G = (V, E) to be the cardinality among all sets of edges $B \subseteq E$ for which $\gamma_{\mathrm{R}}(G-B) > \gamma_{\mathrm{R}}(G)$. Some bounds are obtained for $b_{\mathrm{R}}(G)$, and the exact values are determined for several classes of graphs. Moreover, the decision problem for $b_{\mathrm{R}}(G)$ is proved to be NP-hard even for bipartite graphs.

Keywords: Roman domination number, Roman bondage number, NP-hardness. **AMS Subject Classification:** 05C69

1 Introduction

In this paper, a graph G = (V, E) is considered as an undirected graph without loops and multi-edges, where V = V(G) is the vertex set and E = E(G) is the edge set. For each vertex $x \in V(G)$, let $N_G(x) = \{y \in V(G) : (x, y) \in E(G)\}, N_G[x] = N_G(x) \cup \{x\}.$

^{*}The work was supported by NNSF of China (No. 11071233).

[†]Corresponding author: xujm@ustc.edu.cn

A subset $S \subseteq V$ is a dominating set of G if $N_G[x] \cap S \neq \emptyset$ for every vertex x in G. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets of G. The Roman dominating function on a graph G = (V, E), proposed by Cockayne et al. [2], is a function $f: V \to \{0, 1, 2\}$ such that each vertex x with f(x) = 0 is adjacent to at least one vertex y with f(y) = 2. Let (V_0, V_1, V_2) be the ordered partition of Vinduced by f, where $V_i = \{v \in V | f(v) = i\}$ for i = 0, 1, 2. It is clear that $V_1 \cup V_2$ is a dominating set of G, called the Roman dominating set, denoted by $D_{\mathbf{R}} = (V_1, V_2)$. For $S \subseteq V$, let $f(S) = \sum_{u \in S} f(u)$. The value f(V(G)) is called the weight of f, denoted by f(G). The Roman domination number, denoted by $\gamma_{\mathbf{R}}(G)$, is defined as the minimum weight of all Roman dominating functions, that is,

 $\gamma_{\rm R}(G) = \min\{f(G) : f \text{ is a Roman dominating function on } G\}.$

It is clear that for a Roman dominating function f on G and a Roman dominating set $D_{\rm R}$ of G, $f(D_{\rm R}) = 2|V_2| + |V_1|$. If $D_{\rm R}$ is a minimum Roman dominating set of graph G, then $f(D_{\rm R}) = \gamma_{\rm R}(G)$. A Roman dominating function f is called a $\gamma_{\rm R}$ -function if $f(G) = \gamma_{\rm R}(G)$. It has been showed by Cockayne *et al.* [2] that for any graph G, $\gamma(G) \leq \gamma_{\rm R}(G) \leq 2\gamma(G)$. A graph G is called to be *Roman* if $\gamma_{\rm R}(G) = 2\gamma(G)$. Roman domination numbers have been studied, for example, in [2, 3, 5, 6, 11, 12, 13, 14, 15, 16, 20].

To measure the vulnerability or the stability of the domination in an interconnection network under edge failure, Fink et at. [4] proposed the concept of the bondage number in 1990. The *bondage number*, denoted by b(G), of G is the minimum number of edges whose removal from G results in a graph with larger domination number of G.

Analogously, we can define the Roman bondage number. The Roman bondage number, denoted by $b_{\rm R}(G)$, of a nonempty graph G is the minimum number of edges whose removal from G results in a graph with larger Roman domination number. Precisely speaking, the Roman bondage number

$$b_{\mathcal{R}}(G) = \min\{|B| : B \subseteq E(G), \gamma_{\mathcal{R}}(G-B) > \gamma_{\mathcal{R}}(G)\}.$$

An edge set B that $\gamma_{\rm R}(G-B) > \gamma_{\rm R}(G)$ is called the Roman bondage set and the minimum one the minimum Roman bondage set. In fact, if B is a minimum Roman

bondage set, then $\gamma_{\rm R}(G-B) = \gamma_{\rm R}(G) + 1$, because the removal of one single edge can not increase the Roman domination number by more than one. If $b_{\rm R}(G)$ does not exist we define $b_{\rm R}(G) = \infty$.

In this paper, we give an original investigation. Some bounds are obtained for $b_{\rm R}(G)$, and the exact values are determined for several classes of graphs. Moreover, the decision problem for $b_{\rm R}(G)$ is proved to be NP-hard even for bipartite graphs.

In the proofs of our results, when a Roman dominating function of a graph is constructed, we only give its nonzero value of some vertices.

2 Some basic results on $\gamma_{\rm R}$

For terminology and notation on graph theory not given here, the reader is referred to [8, 9, 19].

Let G = (V, E) be a graph and $E_G(x) = \{xy \in E(G) : y \in N_G(x)\}$. For two disjoint nonempty sets $S, T \subset V(G), E_G(S,T) = E(S,T)$ denotes the set of edges between Sand T. The degree of x is denoted by $d_G(x)$, which is equal to $|N_G(x)|$, and n_i denotes the number of vertices of degree i in G for $i = 1, 2, \dots, \Delta(G)$. Denote the maximum and the minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively.

The symbols P_n and C_n denote a path and a cycle, respectively, where $V(P_n) = V(C_n) = \{x_1, x_2, \dots, x_n\}, E(P_n) = \{x_i x_{i+1} : i = 1, 2, \dots, n\}$ and $E(C_n) = E(P_n) \cup \{x_1 x_n\}.$

The Cartesian product graph $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph with vertex-set $V = V_1 \times V_2 = \{(x, y) : x \in V_1, y \in V_2\}$, and two vertices (x_1, y_1) and (x_2, y_2) being adjacent if and only if either $x_1 = x_2$, y_1 and y_2 are adjacent in G_2 , or $y_1 = y_2$, x_1 and x_2 are adjacent in G_1 .

In this section, we recall some basic results on $\gamma_{\rm R}$, which will be used in our discussion.

Lemma 2.1 (Cockayne et al. [2]) For a path P_n and a cycle C_n ,

$$\gamma_{\mathrm{R}}(P_n) = \gamma_{\mathrm{R}}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$$

For a grid graph $P_2 \times P_n$,

$$\gamma_{\mathrm{R}}(P_2 \times P_n) = n + 1.$$

For a complete t-partite graph K_{m_1,m_2,\cdots,m_t} with $1 \le m_1 \le m_2 \le \cdots \le m_t$ and $t \ge 2$,

$$\gamma_{\mathrm{R}}(K_{m_1,m_2,\cdots,m_t}) = \begin{cases} 2, & \text{if } m_1 = 1; \\ 3, & \text{if } m_1 = 2; \\ 4, & \text{if } m_1 \ge 3. \end{cases}$$

Lemma 2.2 (Cockayne et al. [2]) If G is a graph of order n and contains vertices of degree n - 1, then $\gamma_{\rm R}(G) = 2$.

Lemma 2.3 Let G be a nonempty graph with order $n \ge 3$, then $\gamma_{\rm R}(G) = 3$ if and only if $\Delta(G) = n - 2$.

Proof. Assume that u is a vertex of degree n-2 and v is the unique vertex not adjacent to u in G. It is easy to verify that $\gamma_{\mathrm{R}}(G) \geq 3$. Let f be a function from V(G) to $\{0, 1, 2\}$ subject to

$$f(x) = \begin{cases} 2, & \text{if } x = u; \\ 1, & \text{if } x = v; \\ 0, & \text{otherwise} \end{cases}$$

Then f is a Roman dominating function of G with f(G) = 3. Thus, $\gamma_{\rm R}(G) = 3$.

Conversely, assume $\gamma_{\rm R}(G) = 3$, then $\Delta(G) \leq n-2$ by Lemma 2.2. Let f be a $\gamma_{\rm R}$ -function of G.

If there is no vertex u with f(u) = 2, then f(v) = 1 for each vertex $v \in V(G)$, and so n = 3 since $f(G) = \gamma_{\mathbb{R}}(G) = 3$. Sine G is nonempty and not K_3 , G consists of K_2 and an isolated vertex. Thus, $\Delta(G) = 1 = n - 2$.

If there is a vertex u with f(u) = 2, then there is only one vertex $v \in V(G)$ with f(v) = 1 since $f(G) = \gamma_{\rm R}(G) = 3$. The other n-2 vertices assigned 0 are all adjacent to u. Thus, $\Delta(G) \ge d_G(u) \ge n-2$ and hence $\Delta(G) = n-2$.

Lemma 2.4 Let G be an (n-3)-regular graph with order $n \ge 4$. Then $\gamma_{\rm R}(G) = 4$.

Proof. Since G is (n-3)-regular and $n \ge 4$, G is nonempty. It is clear that $\gamma_{\rm R}(G) > 2$. By Lemma 2.3, $\gamma_{\rm R}(G) \ne 3$ since $\Delta(G) = n - 3$. Then $\gamma_{\rm R}(G) \ge 4$. For any vertex

 $x \in V(G)$, let y, z be the only two vertices not adjacent to x in G, let f(x) = 2 and f(y) = f(z) = 1. Then, f is a Roman dominating function of G with f(G) = 4, hence $\gamma_{\mathrm{R}}(G) \leq 4$. Thus, $\gamma_{\mathrm{R}}(G) = 4$.

Lemma 2.5 (Cockayne et al. [2]) For any graph G, $\gamma(G) \leq \gamma_{R}(G) \leq 2\gamma(G)$.

Lemma 2.6 (Cockayne et al. [2]) For any graph G of order n, $\gamma(G) = \gamma_{\rm R}(G)$ if and only if $G = \overline{K}_n$.

Lemma 2.7 (Cockayne et al. [2]) If G is a connected graph of order n, then $\gamma_{\rm R}(G) =$ $\gamma(G) + 1$ if and only if there is a vertex $v \in V(G)$ of degree $n - \gamma(G)$.

A graph G is called to be *vertex domination-critical* (*vc-graph* for short) if $\gamma(G-x) < 1$ $\gamma(G)$ for any vertex x in G.

Lemma 2.8 (Brigham, Chinn and Dutton [1], 1988) A graph G with $\gamma(G) = 2$ is a vcgraph if and only if G is a complete graph K_{2t} $(t \ge 2)$ with a perfect matching removed.

A graph G of order n is vertex Roman domination-critical (vrc-graph for short) if $\gamma_{\rm R}(G) \neq n$ and $\gamma_{\rm R}(G-x) < \gamma_{\rm R}(G)$ for any vertex x in G. For example, for a positive integer k, both C_{3k+1} and C_{3k+2} are vrc-graphs by Lemma 2.1. From the definition, it is clear that $\gamma_{\mathrm{R}}(G) \geq 3$ if G is a vrc-graph with order at least 3.

Lemma 2.9 If G is a vrc-graph with $\gamma_{\rm R}(G) = 3$, then G is a vc-graph with $\gamma(G) = 2$.

Proof. Let G be a vrc-graph with $\gamma_{\rm R}(G) = 3$. From the definition of vrc-graph, $|V(G)| > \gamma_{\mathbb{R}}(G) = 3$. By Lemma 2.3, $\Delta(G) = |V(G)| - 2$ and hence $\gamma(G) = 2$. For any vertex x, if $\gamma_{\rm R}(G-x) < \gamma_{\rm R}(G) = 3$, then $\gamma_{\rm R}(G-x) = 2$ since G-x is nonempty. By Lemma 2.2, G - x contains vertices of degree |V(G - x)| - 1 and, hence, $\gamma(G - x) = 1$, which implies that G is a vc-graph.

3 The exact values of $b_{\rm R}$ for some graphs

Lemma 3.1 Let G be a graph with order $n \ge 3$ and t be the number of vertices of degree n-1 in G. If $t \ge 1$ then $b_{\mathrm{R}}(G) = \lceil \frac{t}{2} \rceil$.

Proof. Let H be a spanning subgraph of G obtained by removing fewer than $\lceil \frac{t}{2} \rceil$ edges from G. Then H contains vertices of degree n-1 and, hence, $\gamma_{\mathrm{R}}(H) = 2 = \gamma_{\mathrm{R}}(G)$ by Lemma 2.2, which implies $b_{\mathrm{R}}(G) \ge \lceil \frac{t}{2} \rceil$.

Since G contains t vertices of degree n-1, it contains a complete subgraph K_t induced by these t vertices. We can remove $\lceil \frac{t}{2} \rceil$ edges such that no vertices have degree n-1 and, hence, $\gamma_{\rm R}(H) \ge 3 > 2 = \gamma_{\rm R}(G)$ since $n \ge 3$. Thus $b_{\rm R}(G) \le \lceil \frac{t}{2} \rceil$, whence $b_{\rm R}(G) = \lceil \frac{t}{2} \rceil$.

Corollary 3.1 For a complete graph K_n $(n \ge 3)$, $b_{\mathrm{R}}(K_n) = \lceil \frac{n}{2} \rceil$.

Theorem 3.1 For a path P_n with $n \ge 3$,

$$b_{\mathrm{R}}(P_n) = \begin{cases} 1, & \text{if } n \equiv 0, 1 \pmod{3}; \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let $P_n = (x_1, x_2, \ldots, x_n)$ be a path. By Lemma 2.1, $\gamma_{\rm R}(P_n) = \lceil \frac{2n}{3} \rceil$.

If $n \equiv 0, 1 \pmod{3}$, then

$$\gamma_{\rm R}(P_n - x_2 x_3) = 2 + \left\lceil \frac{2(n-2)}{3} \right\rceil = 1 + \left\lceil \frac{2n-1}{3} \right\rceil = 1 + \gamma_{\rm R}(P_n),$$

and hence $b_{\mathbf{R}}(P_n) \leq 1$, whence $b_{\mathbf{R}}(P_n) = 1$.

If $n \equiv 2 \pmod{3}$, then for any edge $e = x_i x_{i+1} \in E(P_n)$,

$$\gamma_{\mathrm{R}}(P_n - e) = \left\lceil \frac{2i}{3} \right\rceil + \left\lceil \frac{2(n-i)}{3} \right\rceil \le \left\lceil \frac{2(n-i)+2i+2}{3} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil = \gamma_{\mathrm{R}}(P_n),$$

and hence $b_{\mathrm{R}}(P_n) \geq 2$. Since

$$\gamma_{\mathrm{R}}(P_n - x_2 x_3 - x_4 x_5) = 2 + 2 + \left\lceil \frac{2(n-4)}{3} \right\rceil = 1 + \left\lceil \frac{2n+1}{3} \right\rceil \ge 1 + \gamma_{\mathrm{R}}(P_n),$$

we have $b_{\mathbf{R}}(P_n) \leq 2$, whence $b_{\mathbf{R}}(P_n) = 2$.

Corollary 3.2 For a cycle C_n with $n \ge 3$,

$$b_{\mathrm{R}}(C_n) = \begin{cases} 2, & \text{if } n = 0, 1 \pmod{3}; \\ 3, & \text{otherwise.} \end{cases}$$

Lemma 3.2 Let $P_n = (x_1, x_2, \ldots, x_n)$ be a path, and use $u_{i,j}$ to denote the vertex (x_i, x_j) in $P_2 \times P_n$, where $1 \le i \le 2$ and $1 \le j \le n$. Then there exists a γ_R -function f on $P_2 \times P_n$ such that $f(u_{1,1}) = 2$ or $f(u_{2,1}) = 2$ or $f(u_{1,n}) = 2$ or $f(u_{2,n}) = 2$.

Proof. Without loss of generality, we only need to find a $\gamma_{\rm R}$ -function f on $P_2 \times P_n$ with $f(u_{1,1}) = 2$. Define a Roman dominating function f as follows. For each non-negative integer i with $1 + 4i \leq n$, let $f(u_{1,1+4i}) = 2$, and for each non-negative integer j with $3 + 4j \le n$, let $f(u_{2,3+4j}) = 2$. If $n \equiv 0 \pmod{4}$, let $f(u_{1,n}) = 1$, and if $n \equiv 2 \pmod{4}$, let $f(u_{2,n}) = 1$. Then $f(P_2 \times P_n) = n + 1$ and, hence by Lemma 2.1, f is a $\gamma_{\rm R}$ -function with $f(u_{1,1}) = 2$.

Theorem 3.2 $b_{\rm R}(P_2 \times P_n) = 2$ for $n \ge 2$.

Proof. By Lemma 2.1, we have $\gamma_{\rm R}(P_2 \times P_n) = n + 1$. Since $\gamma_{\rm R}(P_2 \times P_n - u_{1,1}u_{1,2} - u_{1,2}u_{1,2}) = n + 1$. $u_{2,1}u_{2,2} = 2 + \gamma_{\rm R}(P_2 \times P_{n-1}) = n+2$, we have $b_{\rm R}(P_2 \times P_n) \leq 2$. Next we prove that $\gamma_{\mathrm{R}}(P_2 \times P_n - e) \le \gamma_{\mathrm{R}}(P_2 \times P_n)$ for any edge $e \in E(P_2 \times P_n)$.

Suppose that e is incident with some vertex in $\{u_{1,1}, u_{2,1}, u_{1,n}, u_{2,n}\}$. Without loss of generality let e be incident with $u_{1,1}$. By Lemma 3.2, there exists a $\gamma_{\rm R}$ -function f on $P_2 \times (P_n - P_1)$ such that $f(u_{2,2}) = 2$. Denote $f(u_{1,1}) = 1$ and then f is a Roman dominating function of $P_2 \times P_n - e$ with $f(P_2 \times P_n - e) = n + 1$, thus $\gamma_{\rm R}(P_2 \times P_n - e) \leq 1$ $\gamma_{\rm R}(P_2 \times P_n).$

Suppose that e is incident with some vertex in $\{u_{i,j}: 1 \leq i \leq 2, 2 \leq j \leq n - 1\}$ 1} \ $\{u_{1,1}, u_{2,1}, u_{1,n}, u_{2,n}\}$. Without loss of generality let e be incident with $u_{1,j}$ and not incident with $u_{1,j-1}$. By Lemma 3.2, there exists a $\gamma_{\rm R}$ -function f_1 on $P_2 \times P_{j-1}$ with $f_1(u_{1,j-1}) = 2$ and a $\gamma_{\mathbf{R}}$ -function f_2 on $P_2 \times (P_n - P_j)$ with $f_2(u_{2,j+1}) = 2$. Then $f = f_1 \cup f_2$ is a Roman dominating function on $P_2 \times P_n - e$ with $f(P_2 \times P_n - e) = n + 1$, thus $\gamma_{\mathrm{R}}(P_2 \times P_n - e) \leq \gamma_{\mathrm{R}}(P_2 \times P_n).$

The above two cases yield that $b_{\rm R}(P_2 \times P_n) \ge 2$ and, hence, $b_{\rm R}(P_2 \times P_n) = 2$. The lemma follows.

4 Complexity of Roman bondage number

In this section, we will show that the Roman bondage number problem is NP-hard and the Roman domination number problem is NP-complete even for bipartite graphs. We first state the problem as the following decision problem.

Roman bondage number problem (RBN):

Instance: A nonempty bipartite graph G and a positive integer k.

Question: Is $b_{\mathrm{R}}(G) \leq k$?

Roman domination number problem (RDN):

Instance: A nonempty bipartite graph G and a positive integer k.

Question: Is $\gamma_{\rm R}(G) \leq k$?

Following Garey and Johnson's techniques for proving NP-completeness given in [7], we prove our results by describing a polynomial transformation from the known-well NP-complete problem: 3SAT. To state 3SAT, we recall some terms.

Let U be a set of Boolean variables. A truth assignment for U is a mapping $t: U \rightarrow \{T, F\}$. If t(u) = T, then u is said to be "true" under t; If t(u) = F, then u is said to be "false" under t. If u is a variable in U, then u and \bar{u} are literals over U. The literal u is true under t if and only if the variable u is true under t; the literal \bar{u} is true if and only if the variable u is true under t; the literal \bar{u} is true if and only if the variable u is true under t; the literal \bar{u} is true if and only if the variable u is true under t; the literal \bar{u} is true if and only if the variable u is true under t; the literal \bar{u} is true if and only if the variable u is false.

A clause over U is a set of literals over U. It represents the disjunction of these literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection \mathscr{C} of clauses over U is *satisfiable* if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in \mathscr{C} . Such a truth assignment is called a *satisfying truth assignment* for \mathscr{C} . The 3SAT is specified as follows.

3-satisfiability problem (3SAT):

Instance: A collection $\mathscr{C} = \{C_1, C_2, \dots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \dots, m$.

Question: Is there a truth assignment for U that satisfies all the clauses in C?

Theorem 4.1 (Theorem 3.1 in [7]) 3SAT is NP-complete.

Theorem 4.2 *RBN is NP-hard even for bipartite graphs.*

Proof. The transformation is from 3SAT. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3SAT. We will construct a bipartite graph G and choose an integer k such that \mathscr{C} is satisfiable if and only if $b_{\mathrm{R}}(G) \leq k$. We construct such a graph G as follows.

For each i = 1, 2, ..., n, corresponding to the variable $u_i \in U$, associate a graph H_i with vertex set $V(H_i) = \{u_i, \bar{u}_i, v_i, v'_i, x_i, y_i, z_i, w_i\}$ and edge set $E(H_i) = \{u_i v_i, u_i z_i, \bar{u}_i v'_i, \bar{u}_i z_i, y_i v_i, y_i v'_i, y_i z_i, w_i v_i, w_i v'_i, w_i z_i, x_i v_i, x_i v'_i\}$. For each j = 1, 2, ..., m, corresponding to the clause $C_j = \{p_j, q_j, r_j\} \in \mathscr{C}$, associate a single vertex c_j and add edge set $E_j = \{c_j p_j, c_j q_j, c_j r_j\}$, $1 \leq j \leq m$. Finally, add a path $P = s_1 s_2 s_3$, join s_1 and s_3 to each vertex c_j with $1 \leq j \leq m$ and set k = 1.

Figure 1 shows an example of the graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \bar{u}_3\}, C_2 = \{\bar{u}_1, u_2, u_4\}, C_3 = \{\bar{u}_2, u_3, u_4\}.$

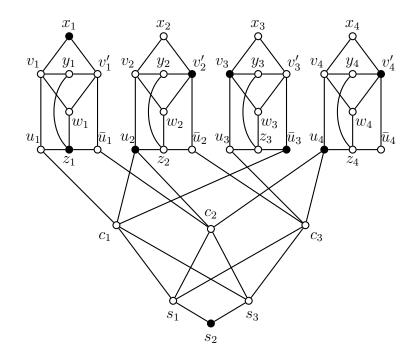


Figure 1: An instance of the Roman bondage number problem resulting from an instance of 3SAT. Here k = 1 and $\gamma_{\rm R}(G) = 18$, where the bold vertex w means a Roman dominating function with f(w) = 2.

To prove that this is indeed a transformation, we only need to show that $b_{\rm R}(G) = 1$ if and only if there is a truth assignment for U that satisfies all clauses in \mathscr{C} . This aim can be obtained by proving the following four claims.

Claim 4.1 $\gamma_{\rm R}(G) \ge 4n + 2$. Moreover, if $\gamma_{\rm R}(G) = 4n + 2$, then for any $\gamma_{\rm R}$ -function fon G, $f(H_i) = 4$ and at most one of $f(u_i)$ and $f(\bar{u}_i)$ is 2 for each i, $f(c_j) = 0$ for each j and $f(s_2) = 2$.

Proof. Let f be a $\gamma_{\rm R}$ -function of G, and let $H'_i = H_i - u_i - \bar{u}_i$.

If $f(u_i) = 2$ and $f(\bar{u}_i) = 2$, then $f(H_i) \ge 4$. Assume either $f(u_i) = 2$ or $f(\bar{u}_i) = 2$, if $f(x_i) = 0$ or $f(y_i) = 0$, then there is at least one vertex t in $\{v_i, \bar{v}_i, z_i\}$ such that f(t) = 2. And hence $f(H'_i) \ge 2$. Thus, $f(H_i) \ge 4$.

If $f(u_i) \neq 2$ and $f(\bar{u}_i) \neq 2$, let f' be a restriction of f on H'_i , then f' is a Roman dominating function of H'_i , and $f'(H'_i) \geq \gamma_{\mathrm{R}}(H'_i)$. Since the maximum degree of H'_i is $V(H'_i) - 3$, by Lemma 2.3, $\gamma_{\mathrm{R}}(H'_i) > 3$ and hence $f'(H'_i) \geq 4$ and $f(H_i) \geq 4$. If $f(s_1) = 0$ or $f(s_3) = 0$, then there is at least one vertex t in $\{c_1, \dots, c_m, s_2\}$ such that f(t) = 2. Then $f(N_G[V(P)]) \geq 2$, and hence $\gamma_{\mathrm{R}}(G) \geq 4n + 2$.

Suppose that $\gamma_{\mathbf{R}}(G) = 4n + 2$, then $f(H_i) = 4$ and since $f(N_G[x_i]) \ge 1$, at most one of $f(u_i)$ and $f(\bar{u}_i)$ is 2 for each i = 1, 2, ..., n, while $f(N_G[V(P)]) = 2$. Then we have $f(s_2) = 2$ since $f(N_G[s_2]) \ge 1$. Consequently, $f(c_j) = 0$ for each j = 1, 2, ..., m.

Claim 4.2 $\gamma_{\rm R}(G) = 4n + 2$ if and only if \mathscr{C} is satisfiable.

Proof. Suppose that $\gamma_{\mathrm{R}}(G) = 4n + 2$ and let f be a γ_{R} -function of G. By Claim 4.1, at most one of $f(u_i)$ and $f(\bar{u}_i)$ is 2 for each $i = 1, 2, \ldots, n$. Define a mapping $t: U \to \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if } f(u_i) = 2 \text{ or } f(u_i) \neq 2 \text{ and } f(\bar{u}_i) \neq 2, \\ F & \text{if } f(\bar{u}_i) = 2. \end{cases} \quad i = 1, 2, \dots, n.$$
(4.1)

We now show that t is a satisfying truth assignment for \mathscr{C} . It is sufficient to show that every clause in \mathscr{C} is satisfied by t. To this end, we arbitrarily choose a clause $C_j \in \mathscr{C}$ with $1 \leq j \leq m$.

By Claim 4.1, $f(c_j) = f(s_1) = f(s_3) = 0$. There exists some *i* with $1 \leq i \leq n$ such that $f(u_i) = 2$ or $f(\bar{u}_i) = 2$ where c_j is adjacent to u_i or \bar{u}_i . Suppose that c_j is adjacent to u_i where $f(u_i) = 2$. Since u_i is adjacent to c_j in *G*, the literal u_i is in the clause C_j by the construction of *G*. Since $f(u_i) = 2$, it follows that $t(u_i) = T$ by (4.1), which implies that the clause C_j is satisfied by *t*. Suppose that c_j is adjacent to \bar{u}_i where $f(\bar{u}_i) = 2$. Since \bar{u}_i is adjacent to c_j in *G*, the literal \bar{u}_i is in the clause C_j . Since $f(\bar{u}_i) = 2$, it follows that $t(u_i) = F$ by (4.1). Thus, *t* assigns \bar{u}_i the truth value *T*, that is, *t* satisfies the clause C_j . By the arbitrariness of *j* with $1 \leq j \leq m$, we show that *t* satisfies all the clauses in \mathscr{C} , that is, \mathscr{C} is satisfiable.

Conversely, suppose that \mathscr{C} is satisfiable, and let $t: U \to \{T, F\}$ be a satisfying truth assignment for \mathscr{C} . Create a function f on V(G) as follows: if $t(u_i) = T$, then let $f(u_i) = f(v'_i) = 2$, and if $t(u_i) = F$, then let $f(\bar{u}_i) = f(v_i) = 2$. Let $f(s_2) = 2$. Clearly, f(G) = 4n + 2. Since t is a satisfying truth assignment for \mathscr{C} , for each $j = 1, 2, \ldots, m$, at least one of literals in C_j is true under the assignment t. It follows that the corresponding vertex c_j in G is adjacent to at least one vertex wwith f(w) = 2 since c_j is adjacent to each literal in C_j by the construction of G. Thus f is a Roman dominating function of G, and so $\gamma_{\mathrm{R}}(G) \leq f(G) = 4n + 2$. By Claim 4.1, $\gamma_{\mathrm{R}}(G) \geq 4n + 2$, and so $\gamma_{\mathrm{R}}(G) = 4n + 2$.

Claim 4.3 $\gamma_{\mathrm{R}}(G-e) \leq 4n+3$ for any $e \in E(G)$.

Proof. For any edge $e \in E(G)$, it is sufficient to construct a Roman dominating function f with weight 4n + 3 of G. We first assume $e \in E_G(s_1)$ or $e \in E_G(s_3)$ or $e \in E_G(c_j)$ for each j = 1, 2, ..., m, without loss of generality let $e \in E_G(s_1)$ or $e = c_j u_i$ or $e = c_j \bar{u}_i$. Let $f(s_3) = 2, f(s_1) = 1$ and $f(u_i) = f(v'_i) = 2$ for each i = 1, 2, ..., n. For the edge $e \notin E_G(u_i)$ and $e \notin E_G(v'_i)$ or $e = \bar{u}_i z_i$, let $f(s_1) = 2, f(s_3) = 1$ and $f(u_i) = f(v'_i) = 2$. For the edge $e \notin E(\bar{u}_i)$ and $e \notin E(v_i)$ or $e = u_i z_i$, let $f(s_1) = 2, f(s_3) = 1$ and $f(\bar{u}_i) = f(v_i) = 2$. If $e = u_i v_i$ or $e = \bar{u}_i v'_i$, let $f(s_1) = 2, f(s_3) = 1$ and $f(x_i) = f(z_i) = 2$. Then f is a Roman dominating function of G - e with f(G - e) = 4n + 3 and hence $\gamma_R(G - e) \le 4n + 3$.

Claim 4.4 $\gamma_{\mathrm{R}}(G) = 4n + 2$ if and only if $b_{\mathrm{R}}(G) = 1$.

Proof. Assume $\gamma_{\mathrm{R}}(G) = 4n + 2$ and consider the edge $e = s_1 s_2$. Suppose $\gamma_{\mathrm{R}}(G) = \gamma_{\mathrm{R}}(G-e)$. Let f' be a γ_{R} -function of G-e. It is clear that f' is also a γ_{R} -function on G. By Claim 4.1 we have $f'(c_j) = 0$ for each $j = 1, 2, \ldots, m$ and $f'(s_2) = 2$. But then $f'(N_{G-e}[s_1]) = 0$, a contradiction. Hence, $\gamma_{\mathrm{R}}(G) < \gamma_{\mathrm{R}}(G-e)$, and so $b_{\mathrm{R}}(G) = 1$.

Now, assume $b_{\mathrm{R}}(G) = 1$. By Claim 4.1, we have that $\gamma_{\mathrm{R}}(G) \ge 4n+2$. Let e' be an edge such that $\gamma_{\mathrm{R}}(G) < \gamma_{\mathrm{R}}(G-e')$. By Claim 4.3, we have that $\gamma_{\mathrm{R}}(G-e') \le 4n+3$. Thus, $4n+2 \le \gamma_{\mathrm{R}}(G) < \gamma_{\mathrm{R}}(G-e') \le 4n+3$, which yields $\gamma_{\mathrm{R}}(G) = 4n+2$.

By Claim 4.2 and Claim 4.4, we prove that $b_{\rm R}(G) = 1$ if and only if there is a truth assignment for U that satisfies all clauses in \mathscr{C} . Since the construction of the Roman bondage number instance is straightforward from a 3-satisfiability instance, the size of the Roman bondage number instance is bounded above by a polynomial function of the size of 3-satisfiability instance. It follows that this is a polynomial reduction.

The theorem follows.

Corollary 4.1 Roman domination number problem is NP-complete even for bipartite graphs.

Proof. It is easy to see that the Roman bondage problem is in NP since a nondeterministic algorithm need only guess a vertex set pair (V_1, V_2) with $|V_1| + 2|V_2| \le k$ and check in polynomial time whether that for any vertex $u \in V \setminus (V_1 \cup V_2)$ whether there is a vertex in V_2 adjacent to u for a given nonempty graph G.

We use the same method as Theorem 4.2 to prove this conclusion. We construct the same graph G but does not contain the path P. We set k = 4n, then use the same methods as Claim 4.1 and 4.2, we have that $\gamma_{\rm R}(G) = 4n$ if and only if \mathscr{C} is satisfiable.

5 General bounds

Lemma 5.1 Let H be a spanning subgraph obtained by removing k edges from a graph G. Then $b_{\rm R}(G) \leq b_{\rm R}(H) + k$.

Proof. Let $B = E(G) \setminus E(H)$ and B' be a minimum Roman bondage set of H. Then $|B| = k, |B'| = b_{\mathrm{R}}(H)$ and $\gamma_{\mathrm{R}}(H - B') > \gamma_{\mathrm{R}}(H)$. Let $f: V \to \{0, 1, 2\}$ be a Roman dominating function on H with $f(H) = \gamma_{\mathrm{R}}(H)$. Then each vertex x with f(x) = 0 is adjacent to at least one vertex y with f(y) = 2 in H, and so is in G since H = G - B, which implies that f is a Roman dominating function of G, and so $f(G) \ge \gamma_{\mathrm{R}}(G)$. It follows that $\gamma_{\mathrm{R}}(G - B - B') = \gamma_{\mathrm{R}}(H - B') > \gamma_{\mathrm{R}}(H) \ge \gamma_{\mathrm{R}}(G)$ and, hence, $b_{\mathrm{R}}(G) \le |B| + |B'| = b_{\mathrm{R}}(H) + k$.

Theorem 5.1 $b_{\rm R}(G) \leq d_G(x) + d_G(y) + d_G(z) - |N_G(y) \cap N_G(\{x, z\})| - 3$ for any path (x, y, z) of length 2 in a graph G.

Proof. Let $F_y = \{(y, u) \in E(G) : u \in N_G(y) \cap N_G(\{x, z\})\}, B = E_G(x) \cup E_G(z) \cup (E_G(y) \setminus F_y)$. Then

$$|B| = d_G(x) + d_G(y) + d_G(z) - |N_G(y) \cap N_G(\{x, z\})| - 2$$

Let H = G - B + yz. Then x is an isolated vertex and z is a vertex of degree 1 which is only adjacent to y in H. Let f be a minimum Roman dominating function of H, then f(x) = 1 and $1 \le f(y) + f(z) \le 2$.

If f(y) + f(z) = 2, then let f' = f except f'(x) = 0, f'(y) = 2 and f'(z) = 0. Clearly, f' is a Roman dominating function of G with f'(G) < f(H) and, hence, $b_{\rm R}(G) \le |B| - 1$.

If f(y) + f(z) = 1, then f(y) = 0 and f(z) = 1. There is an edge $(u, y) \in F_y$ with f(u) = 2. Let f' = f except f'(x) = 0 if $u \in N_G(x)$ or f'(z) = 0 if $u \in N_G(z) \setminus N_G(x)$. Then f' is a Roman dominating function of G with f'(G) < f(H), and hence $b_{\rm R}(G) \leq |B| - 1$.

Theorem 5.2 $b_{\rm R}(G) \leq d_G(x) + d_G(y) + d_G(z) - |N_G(y) \cap N_G(\{x, z\})| - |N_G(x) \cap N_G(z)| - 1$ for any path (x, y, z) of length 2 in a graph G. **Proof.** Let $F_y = \{(y, u) \in E(G) : u \in N_G(y) \cap N_G(\{x, z\})\}$ and $F_z = \{(z, u) \in E(G) : u \in (N_G(z) \cap N_G(x))\}$, $B = E_G(x) \cup (E_G(z) \setminus F_z) \cup (E_G(y) \setminus F_y)$ and H = G - B. Then x is an isolated vertex in H. Let f be a minimum Roman dominating function of H, then f(x) = 1. We will construct a Roman dominating function f' of G with f'(G) < f(H).

If f(z) = 0, then there is an edge $(z, s) \in F_z$ with f(s) = 2. Thus, if f(y) = 2 or f(z) = 0, let f' = f except f'(x) = 0. In the following, let $f(y) \neq 2$ and $f(z) \neq 0$.

If f(y) = 0. Then there is a vertex $s \in F_y$ such that f(s) = 2. If $s \in N_G(x)$, let f' = f except f'(x) = 0. If $s \in N_G(x) \setminus N_G(x)$, let f' = f except f'(z) = 0.

If f(y) = 1. If f(z) = 1, let f' = f except f'(x) = f'(z) = 0 and f'(y) = 2. If f(z) = 2, let f' = f except f'(y) = 0.

Then f' is a Roman dominating function of G with f'(G) < f(H), and hence $b_{\mathbb{R}}(G) \le |B| \le d_G(x) + d_G(y) + d_G(z) - |N_G(y) \cap N_G(\{x, z\})| - |N_G(x) \cap N_G(z)| - 1.$

Corollary 5.1 $b_{\rm R}(G) \leq \min\{d_G(x) + d_G(y) + d_G(z) - |N_G(y) \cap N_G(\{x, z\})| - 3, d_G(x) + d_G(y) + d_G(z) - |N_G(y) \cap N_G(\{x, z\})| - |N_G(x) \cap N_G(z)| - 1\}$ for any path (x, y, z) of length 2 in a graph G.

Corollary 5.2 $b_{\rm R}(G) \leq 2\Delta(G) + \delta(G) - 3$ for any graph with diameter at least two.

Corollary 5.3 For any tree T of order at least 3, then $b_{\rm R}(T) \leq \Delta(T)$.

Proof. If there is a vertex x adjacent to at least two vertices of degree one in T, say u_1 and u_2 , then (u_1, x, u_2) is a path of length 2 in T. By Lemma 5.1, $b_{\rm R}(T) \leq d_T(u_1) + d_T(x) + d_T(u_2) - 3 \leq \Delta(T) - 1$.

Assume now that each vertex of T is adjacent to at most one vertex of degree one. Then T has a vertex u of degree 2 adjacent to exactly one vertex, say v, of degree one. Let w be the other vertex adjacent to u. Then (v, u, w) is a path of length 2 in T. By Lemma 5.1, $b_{\rm R}(T) \leq d_T(v) + d_T(u) + d_T(w) - 3 \leq \Delta(T)$.

Lemma 5.2 Let G be a connected graph of order $n \geq 3$ and $\gamma_{\rm R}(G) = \gamma(G) + 1$. If there is an set B of edges with $\gamma_{\rm R}(G-B) = \gamma_{\rm R}(G)$, then $\Delta(G) = \Delta(G-B)$. **Proof.** Since G is connected and $n \ge 3$, $\gamma_{\rm R}(G) = \gamma(G) + 1 \le n - 1$. Since $\gamma_{\rm R}(G - B) = \gamma_{\rm R}(G) \le n - 1$, G - B is nonempty. By Lemma 2.5 and Lemma 2.6, $\gamma_{\rm R}(G - B) \ge \gamma(G - B) + 1$. Since $\gamma_{\rm R}(G - B) = \gamma_{\rm R}(G) = \gamma(G) + 1 \le \gamma(G - B) + 1$, $\gamma_{\rm R}(G - B) = \gamma(G - B) + 1$ and $\gamma(G - B) = \gamma(G)$.

If G - B is connected, then by Lemma 2.7, $\Delta(G - B) = n - \gamma(G - B) = n - \gamma(G) = \Delta(G)$.

If G - B is disconnected, then let G_1 be a nonempty connected component of G - B. By Lemma 2.5 and 2.6, $\gamma_{\mathrm{R}}(G_1) \geq \gamma(G_1) + 1$. Then $\gamma(G) + 1 = \gamma_{\mathrm{R}}(G - B) \geq \gamma_{\mathrm{R}}(G_1) + \gamma_{\mathrm{R}}(G - G_1) \geq \gamma(G_1) + 1 + \gamma(G - G_1) \geq \gamma(G) + 1$, thus $\gamma_{\mathrm{R}}(G_1) = \gamma(G_1) + 1$, $\gamma_{\mathrm{R}}(G - G_1) = \gamma(G - G_1)$ and $\gamma(G) = \gamma(G_1) + \gamma(G - G_1)$. By Lemma 2.6, $G - G_1$ is empty and hence $\gamma(G - G_1) = |V(G - G_1)|$. By Lemma 2.7, $\Delta(G_1) = |V(G_1)| - \gamma(G_1) = n - |V(G - G_1)| - \gamma(G_1) = n - \gamma(G - G_1) - \gamma(G_1) = n - \gamma(G) = \Delta(G)$.

Theorem 5.3 Let G be a connected graph of order $n \geq 3$ and $\gamma_{\mathrm{R}}(G) = \gamma(G) + 1$. Then $b_{\mathrm{R}}(G) \leq \min\{b(G), n_{\Delta}\}$, where n_{Δ} is the number of vertices with maximum degree Δ in G.

Proof. Since $n \ge 3$ and G is connected, $\Delta(G) \ge 2$ and hence $\gamma(G) \le n-2$. Let B be a minimum bondage set of G. Then G-B is nonempty and by Lemma 2.5 and Lemma 2.6. Thus, $\gamma_{\rm R}(G-B) \ge \gamma(G-B) + 1 > \gamma(G) + 1 = \gamma_{\rm R}(G)$ and hence $b_{\rm R}(G) \le b(G)$.

We now prove that $b_{\mathrm{R}}(G) \leq n_{\Delta}$. By Lemma 2.7, $\gamma_{\mathrm{R}}(G) = \gamma(G) + 1$ if and only if there is a vertex of degree $n - \gamma(G)$. If there is a vertex s in G such that $d_G(s) > n - \gamma(G)$, let f(s) = 2 and f(w) = 1 for any vertex w not in $N_G[s]$, then f is a Roman dominating function of G with $f(G) = \gamma(G)$, a contradiction. Thus, $\Delta(G) = n - \gamma(G)$. We can remove a smallest edge set B with $|B| \leq n_{\Delta}$ edges from G such that $\Delta(G - B) <$ $\Delta(G) = n - \gamma(G)$ and G - B is nonempty. Since G - B is nonempty, by Lemma 2.5 and Lemma 2.6, $\gamma_{\mathrm{R}}(G - B) \geq \gamma(G - B) + 1$. Assume $\gamma_{\mathrm{R}}(G - B) = \gamma_{\mathrm{R}}(G)$, then by Lemma 5.2, $\Delta(G - B) = \Delta(G) = n - \gamma(G)$, a contradiction. Hence $b_{\mathrm{R}}(G) \leq |B| \leq n_{\Delta}$.

Theorem 5.4 For Roman graph G, $b_{\mathrm{R}}(G) \ge b(G)$.

Proof. Let *B* be a minimum Roman bondage set of *G*, then $\gamma_{\mathrm{R}}(G-B) > \gamma_{\mathrm{R}}(G) = 2\gamma(G)$. By Lemma 2.5, $\gamma_{\mathrm{R}}(G-B) \leq 2\gamma(G-B)$, then $\gamma(G-B) > \gamma(G)$ and hence $b_{\mathrm{R}}(G) \geq b(G)$.

The equality in Theorem 5.4 can hold, for example, $b(C_{3k}) = 2 = b_R(C_{3k})$, and the strict inequality can also hold, for example, $b(C_{3k+2}) = 2 < 3 = b_R(C_{3k+2})$.

Theorem 5.5 Let G be a nonempty graph with $\gamma_{\mathrm{R}}(G) \geq 3$. Then $b_{\mathrm{R}}(G) \leq (\gamma_{\mathrm{R}}(G) - 2)\Delta(G) + 1$.

Proof. The proof proceeds by induction on $\gamma_{\rm R}(G)$.

We first assume that $\gamma_{\mathrm{R}}(G) = 3$. Then by Lemma 2.3, $\Delta(G) = |V(G)| - 2$. Assume that $b_{\mathrm{R}}(G) \geq \Delta(G) + 2$. Let u be a vertex of maximum degree in G. We have $\gamma_{\mathrm{R}}(G-u) =$ $\gamma_{\mathrm{R}}(G) - 1 = 2$. There is a vertex v that is adjacent to every vertex in G - u and hence $vu \notin E(G)$. Since $b_{\mathrm{R}}(G-u) \geq 2$, then for any edge $e \in E_{G-u}(v)$, $\gamma_{\mathrm{R}}(G-u-e) = 2$. Thus there is a vertex w that is adjacent to every vertex of G - u - e. But, since vis the only vertex of G that is not adjacent to u, $wu \in E(G)$, $d_G(w) = |V(G)| - 1$, a contradiction. Thus, $b_{\mathrm{R}}(G) \leq \Delta(G) + 1$ if $\gamma_{\mathrm{R}}(G) = 3$.

Assume the induction hypothesis for any integer k and any graph H with $\gamma_{\rm R}(H) = k \geq 3$. Let G be a nonempty graph with $\gamma_{\rm R}(G) = k + 1$, and assume that $b_{\rm R}(G) \geq (k-1)\Delta(G) + 2$. For any vertex u of G, let H = G - u. Then, $\gamma_{\rm R}(H) = \gamma_{\rm R}(G) - 1 = k$ since $d_G(u) < b_{\rm R}(G)$. By the inductive hypothesis and by Lemma 5.1, we have

$$b_{\mathrm{R}}(G) \leq b_{\mathrm{R}}(H) + d_{G}(u)$$

$$\leq (k-2)\Delta(H) + 1 + d_{G}(u)$$

$$\leq (k-2)\Delta(G) + 1 + \Delta(G)$$

$$= (k-1)\Delta(G) + 1,$$

a contradiction. Thus, $b_{\rm R}(G) \leq (k-1)\Delta(G) + 1$, and by the principle of mathematical induction, $b_{\rm R}(G) \leq (\gamma_{\rm R}(G) - 2)\Delta(G) + 1$.

Use $\kappa(G)$ (resp. $\lambda(G)$) to denote the vertex-connectivity (resp. the edge-connectivity) of a connected graph G which is the minimum number of vertices (resp, edges) whose removal results in G disconnected. The famous Whitney's inequality can be stated as $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for any graph G. A subset $F \subseteq E(G)$ is called a λ -cut if $|F| = \lambda(G)$ and G - F is disconnected. **Theorem 5.6** If G is a connected graph with order at least 3, then $b_{\rm R}(G) \leq 2\Delta(G) + \lambda(G) - 3$, where $\lambda(G)$ is the edge-connectivity of G.

Proof. Let G be a connected graph with edge-connectivity $\lambda(G)$ and F be λ -cut of G. Then H = G - F has exact two connected components. Let $x, y \in V(G), xy \in F$, and H_x and H_y denote the components of G - F containing x and y, respectively. Without loss of generality, let z be adjacent to x in H_x since $|V(G)| \geq 3$. Let $B = F \cup E_{H_x}(x) \cup E_{H_x}(z) - xz$ and f be a $\gamma_{\mathbb{R}}$ -function of G' = G - B. Then x and z is only adjacent to each other in G', and so we can assume f(x) = 2 and f(z) = 0. We construct a Roman dominating function f' of G with f'(G) < f(G').

If $V(H_y) = \{y\}$, then f(y) = 1. Let f' = f except f'(y) = 0. Then f' is a Roman dominating function of G with f'(G) < f(G'). Thus, $b_{\rm R}(G) \le |B| \le 2\Delta(G) + \lambda(G) - 3$. In the following, we assume $|V(H_y)| \ge 2$.

If $\gamma_{\mathrm{R}}(H_y - y) \ge \gamma_{\mathrm{R}}(H_y)$, then

$$\gamma_{\mathbf{R}}(G - (F \cup E_{H_y}(y))) \ge \gamma_{\mathbf{R}}(H_x) + \gamma_{\mathbf{R}}(H_y) + 1 \ge \gamma_{\mathbf{R}}(G) + 1.$$

Thus

$$b_{\mathbf{R}}(G) \leq |F \cup E_{H_y}(y)| \leq \Delta(G) + \lambda(G) - 1$$

$$\leq 2\Delta(G) + \lambda(G) - 3.$$

If $\gamma_{\rm R}(H_y - y) = \gamma_{\rm R}(H_y) - 1$, we can assume that f(y) = 1. Let f' = f except f'(y) = 0. Then f' is a Roman dominating function of G with f'(G) < f(G'). Thus,

$$b_{\mathrm{R}}(G) \le |B| \le 2\Delta(G) + \lambda(G) - 3.$$

The theorem follows.

Considering vertex rather than edge-connectivity, we could conjecture an analogy of Theorem 5.6 by a similar argument.

Conjecture 5.1 If G is a connected graph with order no less than 3, then $b_{\rm R}(G) \leq 2\Delta(G) + \kappa(G) - 3$, where $\kappa(G)$ is the vertex-connectivity of G.

Theorem 5.7 If G is a nonempty graph with a unique minimum Roman dominating function, then $b_{\rm R}(G) = 1$.

Proof. Let f be the unique γ_{R} -function on G, and let x be a vertex in G with f(x) = 0. Then there is a vertex $y \in N_G(x)$ with f(y) = 2. If there are at least two vertices $y, z \in N_G(x)$ such that f(y) = f(z) = 2 for each vertex x with f(x) = 0. Then let f' = f except that f'(x) = 2 and f'(y) = 0 and f' is a γ_{R} -function on G as well, which is a contradiction to the uniqueness of f. Thus, there is a unique $y \in N_G(x)$ with f(y) = 2 for a vertex x with f(x) = 0. Then $\gamma_{\mathrm{R}}(G - xy) > \gamma_{\mathrm{R}}(G)$, which implies that $b_{\mathrm{R}}(G) = 1$.

Theorem 5.8 If G is a vrc-graph with $\gamma_{\rm R}(G) = 3$, then $b_{\rm R}(G) \leq \Delta(G) + 1$.

Proof. By Lemma 2.9, G is a vc-graph with $\gamma(G) = 2$. By Lemma 2.8, G is a complete $K_{2t}(t \ge 2)$ with a perfect matching M removed. Thus, G is $\Delta(G)$ -regular, where $\Delta(G) = 2t - 2$. Let $uv \in M$. Then v is the only vertex not adjacent to u in G. Let H = G - u. Then $\gamma_{\rm R}(H) = 2$ since G is a vrc-graph with $\gamma_{\rm R}(G) = 3$. Note that the vertex v is the only vertex adjacent to all the other vertices in H adjacent to each of other vertices in H. Thus H has a unique minimum Roman dominating function f with $f(v) = 2 = \gamma_{\rm R}(H)$. By Theorem 5.7, $b_{\rm R}(H) = 1$ and hence $b_{\rm R}(G) \le \Delta(G) + 1$.

Theorem 5.9 If there exists at least one vertex u in a graph G with $\gamma_{\mathrm{R}}(G-u) \ge \gamma_{\mathrm{R}}(G)$, then $b_{\mathrm{R}}(G) = d_G(x) \le \Delta(G)$.

Proof. Since $\gamma_{\mathrm{R}}(G - E_G(u)) = \gamma_{\mathrm{R}}(G - u) + 1 > \gamma_{\mathrm{R}}(G), \ b_{\mathrm{R}}(G) = d_G(x) \le \Delta(G).$

Corollary 5.4 Let G be a graph of order n. If $\gamma_{\mathrm{R}}(G) = 3 \neq n$, then $b_{\mathrm{R}}(G) \leq \Delta + 1$.

Problem 5.1 Whether or not there exits a positive integer c such that $b_{\mathrm{R}}(G) \leq \Delta(G) + c$ for any graph G of order n and $\gamma_{\mathrm{R}}(G) \neq n$.

The vertex covering number $\beta(G)$ of G is the minimum number of vertices that are incident with all edges in G. If G has no isolated vertices, then $\gamma_{\mathrm{R}}(G) \leq 2\gamma(G) \leq 2\beta(G)$. If $\gamma_{\mathrm{R}}(G) = 2\beta(G)$, then $\gamma_{\mathrm{R}}(G) = 2\gamma(G)$ and hence G is a Roman graph. In [17], Volkmann gave a lot of graphs with $\gamma(G) = \beta(G)$. **Theorem 5.10** Let G be a graph with $\gamma_{\mathrm{R}}(G) = 2\beta(G)$. Then (1) $b_{\mathrm{R}}(G) \ge \delta(G)$; (2) $b_{\mathrm{R}}(G) \ge \delta(G) + 1$ if G is a vrc-graph.

Proof. Let G be a graph with $\gamma_{\rm R}(G) = 2\beta(G)$.

(1) Without loss of generality, Assume $\delta(G) \ge 2$. Let $B \subseteq E(G)$ with $|B| \le \delta(G) - 1$. Then $\delta(G - B) \ge 1$ and so $\gamma_{\mathrm{R}}(G) \le \gamma_{\mathrm{R}}(G - B) \le 2\beta(G - B) \le 2\beta(G) = \gamma_{\mathrm{R}}(G)$. Thus, B is not a Roman bondage set of G, and so $b_{\mathrm{R}}(G) \ge \delta(G)$.

(2) From the above proof, every Roman bondage set B contains at least all edges incident with some vertex x, so that G - B has an isolated vertex. On the other hand, if G is a vrc-graph, then $\gamma_{\mathrm{R}}(G - x) < \gamma_{\mathrm{R}}(G)$ for any vertex x, which implies that the removal of all edges incident with x can not enlarge the Roman domination number. Hence $b_{\mathrm{R}}(G) \ge \delta(G) + 1$.

References

- R. C. Brigham, P. Z. Chinn and R. D. Dutton, Vertex domination-cirtical graphs. Networks, 18 (1988), 173-179.
- [2] E. J. Cockayne, P. A. Dreyer, Jr., S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, Roman domination in graphs. *Discrete Mathematics*, 278 (1-3) (2004), 11-22.
- [3] E. W. Chambers, B. Kinnersley, N. Prince, D. B. West, Extremal problems for Roman domination. SIAM J. Discrete Math. 23 (2009), 1575-1586.
- [4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph. Discrete Mathematics, 86 (1990), 47-57.
- [5] O. Favaron, H. Karami, R. Khoeilar and S. M. Sheikholeslami, On the Roman domination number of a graph. *Discrete Mathematics*, **309** (2009), 3447-3451.
- [6] X. L. Fu, Y. S. Yang and B. Q. Jiang, Roman domination in regular graphs. Discrete Mathematics, 309 (2009), 1528-1537.

- [7] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.
- [8] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1997.
- [9] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1997.
- [10] J. H. Hattingh, A. R. Plummer, Restrained bondage in graphs. Discrete Mathematics, 308 (2008), 5446-5453.
- [11] M. Liedloff, T. Kloks, J. P. Liu and S. L. Peng, Efficient algorithms for Roman domination on some classes of graphs, *Discrete Applied Mathematics*, **156** (2008), 3400-3415.
- [12] M. Liedloff, T. Kloks, J. P. Liu and S. L. Peng, Roman domination over some graph classes, *Lecture Notes in Computer Science*, 2005, Volume **3787**, Graph-Theoretic Concepts in Computer Science, Pages 103-114.
- [13] A. Pagourtzis, P. Penna, K. Schlude, K. Steinhofel, D. Taylor and P. Widmayer, Server placements, Roman domination and other dominating set variants, in Proc. Second International Conference on Theoretical Computer Science (2002), 280-291.
- [14] R. R. Rubalcaba, P. J. Slater, Roman dominating influence parameters. Discrete Mathematics, 307 (2007), 3194-3200.
- [15] W. P. Shang and X. D. Hu, The roman domination problem in unit disk graphs, Lecture Notes in Computer Science, 2007, Volume 4489/2007, 305-312.
- [16] W. P. Shang and X. D. Hu, Roman domination and its variants in unit disk graphs, Discrete Mathematics, Algorithms and Applications, 2 (2010), 99-105.
- [17] L. Volkmann, On graphs with equal domination and covering numbers. Discrete Applied Mathematics, 51 (1994), 211-217.

- [18] J.-M. Xu, Toplogical Structure and Analysis of Interconnection Networks. Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.
- [19] J.-M. Xu, Theory and Application of Graphs. Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
- [20] H. M. Xing, X. Chen and X. G. Chen, A note on Roman domination in graphs. Discrete Mathematics, 306 (2006), 3338-3340.