# Roman Bondage Number of a Graph* 

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#### Abstract

The Roman dominating function on a graph $G=(V, E)$ is a function $f: V \rightarrow\{0,1,2\}$ such that each vertex $x$ with $f(x)=0$ is adjacent to at least one vertex $y$ with $f(y)=2$. The value $f(G)=\sum_{u \in V(G)} f(u)$ is called the weight of $f$. The Roman domination number $\gamma_{\mathrm{R}}(G)$ is defined as the minimum weight of all Roman dominating functions. This paper defines the Roman bondage number $b_{\mathrm{R}}(G)$ of a nonempty graph $G=(V, E)$ to be the cardinality among all sets of edges $B \subseteq E$ for which $\gamma_{\mathrm{R}}(G-B)>\gamma_{\mathrm{R}}(G)$. Some bounds are obtained for $b_{\mathrm{R}}(G)$, and the exact values are determined for several classes of graphs. Moreover, the decision problem for $b_{\mathrm{R}}(G)$ is proved to be NP-hard even for bipartite graphs.


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## 1 Introduction

In this paper, a graph $G=(V, E)$ is considered as an undirected graph without loops and multi-edges, where $V=V(G)$ is the vertex set and $E=E(G)$ is the edge set. For each vertex $x \in V(G)$, let $N_{G}(x)=\{y \in V(G):(x, y) \in E(G)\}, N_{G}[x]=N_{G}(x) \cup\{x\}$.

[^0]A subset $S \subseteq V$ is a dominating set of $G$ if $N_{G}[x] \cap S \neq \emptyset$ for every vertex $x$ in $G$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets of $G$. The Roman dominating function on a graph $G=(V, E)$, proposed by Cockayne et al. [2], is a function $f: V \rightarrow\{0,1,2\}$ such that each vertex $x$ with $f(x)=0$ is adjacent to at least one vertex $y$ with $f(y)=2$. Let $\left(V_{0}, V_{1}, V_{2}\right)$ be the ordered partition of $V$ induced by $f$, where $V_{i}=\{v \in V \mid f(v)=i\}$ for $i=0,1,2$. It is clear that $V_{1} \cup V_{2}$ is a dominating set of $G$, called the Roman dominating set, denoted by $D_{\mathrm{R}}=\left(V_{1}, V_{2}\right)$. For $S \subseteq V$, let $f(S)=\sum_{u \in S} f(u)$. The value $f(V(G))$ is called the weight of $f$, denoted by $f(G)$. The Roman domination number, denoted by $\gamma_{\mathrm{R}}(G)$, is defined as the minimum weight of all Roman dominating functions, that is,

$$
\gamma_{\mathrm{R}}(G)=\min \{f(G): f \text { is a Roman dominating function on } G\}
$$

It is clear that for a Roman dominating function $f$ on $G$ and a Roman dominating set $D_{\mathrm{R}}$ of $G, f\left(D_{\mathrm{R}}\right)=2\left|V_{2}\right|+\left|V_{1}\right|$. If $D_{\mathrm{R}}$ is a minimum Roman dominating set of graph $G$, then $f\left(D_{\mathrm{R}}\right)=\gamma_{\mathrm{R}}(G)$. A Roman dominating function $f$ is called a $\gamma_{\mathrm{R}}$-function if $f(G)=\gamma_{\mathrm{R}}(G)$. It has been showed by Cockayne et al. [2] that for any graph $G, \gamma(G) \leqslant$ $\gamma_{R}(G) \leqslant 2 \gamma(G)$. A graph $G$ is called to be Roman if $\gamma_{\mathrm{R}}(G)=2 \gamma(G)$. Roman domination numbers have been studied, for example, in [2, 3, 5, 6, 6, 11, 12, 13, 14, 15, 16, 20].

To measure the vulnerability or the stability of the domination in an interconnection network under edge failure, Fink et at. [4] proposed the concept of the bondage number in 1990. The bondage number, denoted by $b(G)$, of $G$ is the minimum number of edges whose removal from $G$ results in a graph with larger domination number of $G$.

Analogously, we can define the Roman bondage number. The Roman bondage number, denoted by $b_{\mathrm{R}}(G)$, of a nonempty graph $G$ is the minimum number of edges whose removal from $G$ results in a graph with larger Roman domination number. Precisely speaking, the Roman bondage number

$$
b_{\mathrm{R}}(G)=\min \left\{|B|: B \subseteq E(G), \gamma_{\mathrm{R}}(G-B)>\gamma_{\mathrm{R}}(G)\right\}
$$

An edge set $B$ that $\gamma_{\mathrm{R}}(G-B)>\gamma_{\mathrm{R}}(G)$ is called the Roman bondage set and the minimum one the minimum Roman bondage set. In fact, if $B$ is a minimum Roman
bondage set, then $\gamma_{\mathrm{R}}(G-B)=\gamma_{\mathrm{R}}(G)+1$, because the removal of one single edge can not increase the Roman domination number by more than one. If $b_{\mathrm{R}}(G)$ does not exist we define $b_{\mathrm{R}}(G)=\infty$.

In this paper, we give an original investigation. Some bounds are obtained for $b_{\mathrm{R}}(G)$, and the exact values are determined for several classes of graphs. Moreover, the decision problem for $b_{\mathrm{R}}(G)$ is proved to be NP-hard even for bipartite graphs.

In the proofs of our results, when a Roman dominating function of a graph is constructed, we only give its nonzero value of some vertices.

## 2 Some basic results on $\gamma_{R}$

For terminology and notation on graph theory not given here, the reader is referred to [8, 9, 19].

Let $G=(V, E)$ be a graph and $E_{G}(x)=\left\{x y \in E(G): y \in N_{G}(x)\right\}$. For two disjoint nonempty sets $S, T \subset V(G), E_{G}(S, T)=E(S, T)$ denotes the set of edges between $S$ and $T$. The degree of $x$ is denoted by $d_{G}(x)$, which is equal to $\left|N_{G}(x)\right|$, and $n_{i}$ denotes the number of vertices of degree $i$ in $G$ for $i=1,2, \cdots, \Delta(G)$. Denote the maximum and the minimum degree of $G$ by $\Delta(G)$ and $\delta(G)$, respectively.

The symbols $P_{n}$ and $C_{n}$ denote a path and a cycle, respectively, where $V\left(P_{n}\right)=$ $V\left(C_{n}\right)=\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}, E\left(P_{n}\right)=\left\{x_{i} x_{i+1}: \quad i=1,2, \cdots, n\right\}$ and $E\left(C_{n}\right)=E\left(P_{n}\right) \cup$ $\left\{x_{1} x_{n}\right\}$.

The Cartesian product graph $G_{1} \times G_{2}$ of two graphs $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ is a graph with vertex-set $V=V_{1} \times V_{2}=\left\{(x, y): x \in V_{1}, y \in V_{2}\right\}$, and two vertices $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ being adjacent if and only if either $x_{1}=x_{2}, y_{1}$ and $y_{2}$ are adjacent in $G_{2}$, or $y_{1}=y_{2}, x_{1}$ and $x_{2}$ are adjacent in $G_{1}$.

In this section, we recall some basic results on $\gamma_{\mathrm{R}}$, which will be used in our discussion.

Lemma 2.1 (Cockayne et al. [2]) For a path $P_{n}$ and a cycle $C_{n}$,

$$
\gamma_{\mathrm{R}}\left(P_{n}\right)=\gamma_{\mathrm{R}}\left(C_{n}\right)=\left\lceil\frac{2 n}{3}\right\rceil .
$$

For a grid graph $P_{2} \times P_{n}$,

$$
\gamma_{\mathrm{R}}\left(P_{2} \times P_{n}\right)=n+1
$$

For a complete $t$-partite graph $K_{m_{1}, m_{2}, \cdots, m_{t}}$ with $1 \leq m_{1} \leq m_{2} \leq \cdots \leq m_{t}$ and $t \geq 2$,

$$
\gamma_{\mathrm{R}}\left(K_{m_{1}, m_{2}, \cdots, m_{t}}\right)= \begin{cases}2, & \text { if } m_{1}=1 \\ 3, & \text { if } m_{1}=2 \\ 4, & \text { if } m_{1} \geq 3\end{cases}
$$

Lemma 2.2 (Cockayne et al. [2]) If $G$ is a graph of order $n$ and contains vertices of degree $n-1$, then $\gamma_{\mathrm{R}}(G)=2$.

Lemma 2.3 Let $G$ be a nonempty graph with order $n \geq 3$, then $\gamma_{\mathrm{R}}(G)=3$ if and only if $\Delta(G)=n-2$.

Proof. Assume that $u$ is a vertex of degree $n-2$ and $v$ is the unique vertex not adjacent to $u$ in $G$. It is easy to verify that $\gamma_{\mathrm{R}}(G) \geq 3$. Let $f$ be a function from $V(G)$ to $\{0,1,2\}$ subject to

$$
f(x)= \begin{cases}2, & \text { if } x=u \\ 1, & \text { if } x=v \\ 0, & \text { otherwise }\end{cases}
$$

Then $f$ is a Roman dominating function of $G$ with $f(G)=3$. Thus, $\gamma_{\mathrm{R}}(G)=3$.
Conversely, assume $\gamma_{\mathrm{R}}(G)=3$, then $\Delta(G) \leq n-2$ by Lemma 2.2, Let $f$ be a $\gamma_{\mathrm{R}}$-function of $G$.

If there is no vertex $u$ with $f(u)=2$, then $f(v)=1$ for each vertex $v \in V(G)$, and so $n=3$ since $f(G)=\gamma_{\mathrm{R}}(G)=3$. Sine $G$ is nonempty and not $K_{3}, G$ consists of $K_{2}$ and an isolated vertex. Thus, $\Delta(G)=1=n-2$.

If there is a vertex $u$ with $f(u)=2$, then there is only one vertex $v \in V(G)$ with $f(v)=1$ since $f(G)=\gamma_{\mathrm{R}}(G)=3$. The other $n-2$ vertices assigned 0 are all adjacent to $u$. Thus, $\Delta(G) \geq d_{G}(u) \geq n-2$ and hence $\Delta(G)=n-2$.

Lemma 2.4 Let $G$ be an $(n-3)$-regular graph with order $n \geq 4$. Then $\gamma_{\mathrm{R}}(G)=4$.

Proof. Since $G$ is $(n-3)$-regular and $n \geq 4, G$ is nonempty. It is clear that $\gamma_{\mathrm{R}}(G)>2$. By Lemma 2.3, $\gamma_{\mathrm{R}}(G) \neq 3$ since $\Delta(G)=n-3$. Then $\gamma_{\mathrm{R}}(G) \geq 4$. For any vertex
$x \in V(G)$, let $y, z$ be the only two vertices not adjacent to $x$ in $G$, let $f(x)=2$ and $f(y)=f(z)=1$. Then, $f$ is a Roman dominating function of $G$ with $f(G)=4$, hence $\gamma_{\mathrm{R}}(G) \leq 4$. Thus, $\gamma_{\mathrm{R}}(G)=4$.

Lemma 2.5 (Cockayne et al. [2]) For any graph $G$, $\gamma(G) \leq \gamma_{\mathrm{R}}(G) \leq 2 \gamma(G)$.

Lemma 2.6 (Cockayne et al. [2]) For any graph $G$ of order $n, \gamma(G)=\gamma_{\mathrm{R}}(G)$ if and only if $G=\bar{K}_{n}$.

Lemma 2.7 (Cockayne et al. [2]) If $G$ is a connected graph of order n, then $\gamma_{\mathrm{R}}(G)=$ $\gamma(G)+1$ if and only if there is a vertex $v \in V(G)$ of degree $n-\gamma(G)$.

A graph $G$ is called to be vertex domination-critical (vc-graph for short) if $\gamma(G-x)<$ $\gamma(G)$ for any vertex $x$ in $G$.

Lemma 2.8 (Brigham, Chinn and Dutton [1], 1988) A graph $G$ with $\gamma(G)=2$ is a vcgraph if and only if $G$ is a complete graph $K_{2 t}(t \geq 2)$ with a perfect matching removed.

A graph $G$ of order $n$ is vertex Roman domination-critical (vrc-graph for short) if $\gamma_{\mathrm{R}}(G) \neq n$ and $\gamma_{\mathrm{R}}(G-x)<\gamma_{\mathrm{R}}(G)$ for any vertex $x$ in $G$. For example, for a positive integer $k$, both $C_{3 k+1}$ and $C_{3 k+2}$ are vrc-graphs by Lemma 2.1. From the definition, it is clear that $\gamma_{\mathrm{R}}(G) \geq 3$ if $G$ is a vrc-graph with order at least 3 .

Lemma 2.9 If $G$ is a vrc-graph with $\gamma_{\mathrm{R}}(G)=3$, then $G$ is a vc-graph with $\gamma(G)=2$.

Proof. Let $G$ be a vrc-graph with $\gamma_{\mathrm{R}}(G)=3$. From the definition of vrc-graph, $|V(G)|>\gamma_{\mathrm{R}}(G)=3$. By Lemma 2.3, $\Delta(G)=|V(G)|-2$ and hence $\gamma(G)=2$. For any vertex $x$, if $\gamma_{\mathrm{R}}(G-x)<\gamma_{\mathrm{R}}(G)=3$, then $\gamma_{\mathrm{R}}(G-x)=2$ since $G-x$ is nonempty. By Lemma 2.2, $G-x$ contains vertices of degree $|V(G-x)|-1$ and, hence, $\gamma(G-x)=1$, which implies that $G$ is a vc-graph.

## 3 The exact values of $b_{R}$ for some graphs

Lemma 3.1 Let $G$ be a graph with order $n \geq 3$ and $t$ be the number of vertices of degree $n-1$ in $G$. If $t \geq 1$ then $b_{\mathrm{R}}(G)=\left\lceil\frac{t}{2}\right\rceil$.

Proof. Let $H$ be a spanning subgraph of $G$ obtained by removing fewer than $\left\lceil\frac{t}{2}\right\rceil$ edges from $G$. Then $H$ contains vertices of degree $n-1$ and, hence, $\gamma_{\mathrm{R}}(H)=2=\gamma_{\mathrm{R}}(G)$ by Lemma 2.2, which implies $b_{\mathrm{R}}(G) \geq\left\lceil\frac{t}{2}\right\rceil$.

Since $G$ contains $t$ vertices of degree $n-1$, it contains a complete subgraph $K_{t}$ induced by these $t$ vertices. We can remove $\left\lceil\frac{t}{2}\right\rceil$ edges such that no vertices have degree $n-1$ and, hence, $\gamma_{\mathrm{R}}(H) \geq 3>2=\gamma_{\mathrm{R}}(G)$ since $n \geq 3$. Thus $b_{\mathrm{R}}(G) \leq\left\lceil\frac{t}{2}\right\rceil$, whence $b_{\mathrm{R}}(G)=\left\lceil\frac{t}{2}\right\rceil$.

Corollary 3.1 For a complete graph $K_{n}(n \geq 3), b_{\mathrm{R}}\left(K_{n}\right)=\left\lceil\frac{n}{2}\right\rceil$.
Theorem 3.1 For a path $P_{n}$ with $n \geq 3$,

$$
b_{\mathrm{R}}\left(P_{n}\right)= \begin{cases}1, & \text { if } n \equiv 0,1(\bmod 3) \\ 2, & \text { otherwise }\end{cases}
$$

Proof. Let $P_{n}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a path. By Lemma 2.1, $\gamma_{\mathrm{R}}\left(P_{n}\right)=\left\lceil\frac{2 n}{3}\right\rceil$.
If $n \equiv 0,1(\bmod 3)$, then

$$
\gamma_{\mathrm{R}}\left(P_{n}-x_{2} x_{3}\right)=2+\left\lceil\frac{2(n-2)}{3}\right\rceil=1+\left\lceil\frac{2 n-1}{3}\right\rceil=1+\gamma_{\mathrm{R}}\left(P_{n}\right)
$$

and hence $b_{\mathrm{R}}\left(P_{n}\right) \leq 1$, whence $b_{\mathrm{R}}\left(P_{n}\right)=1$.
If $n \equiv 2(\bmod 3)$, then for any edge $e=x_{i} x_{i+1} \in E\left(P_{n}\right)$,

$$
\gamma_{\mathrm{R}}\left(P_{n}-e\right)=\left\lceil\frac{2 i}{3}\right\rceil+\left\lceil\frac{2(n-i)}{3}\right\rceil \leq\left\lceil\frac{2(n-i)+2 i+2}{3}\right\rceil=\left\lceil\frac{2 n}{3}\right\rceil=\gamma_{\mathrm{R}}\left(P_{n}\right)
$$

and hence $b_{\mathrm{R}}\left(P_{n}\right) \geq 2$. Since

$$
\gamma_{\mathrm{R}}\left(P_{n}-x_{2} x_{3}-x_{4} x_{5}\right)=2+2+\left\lceil\frac{2(n-4)}{3}\right\rceil=1+\left\lceil\frac{2 n+1}{3}\right\rceil \geq 1+\gamma_{\mathrm{R}}\left(P_{n}\right)
$$

we have $b_{\mathrm{R}}\left(P_{n}\right) \leq 2$, whence $b_{\mathrm{R}}\left(P_{n}\right)=2$.

Corollary 3.2 For a cycle $C_{n}$ with $n \geq 3$,

$$
b_{\mathrm{R}}\left(C_{n}\right)= \begin{cases}2, & \text { if } n=0,1(\bmod 3) ; \\ 3, & \text { otherwise }\end{cases}
$$

Lemma 3.2 Let $P_{n}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ be a path, and use $u_{i, j}$ to denote the vertex $\left(x_{i}, x_{j}\right)$ in $P_{2} \times P_{n}$, where $1 \leq i \leq 2$ and $1 \leq j \leq n$. Then there exists a $\gamma_{R}$-function $f$ on $P_{2} \times P_{n}$ such that $f\left(u_{1,1}\right)=2$ or $f\left(u_{2,1}\right)=2$ or $f\left(u_{1, n}\right)=2$ or $f\left(u_{2, n}\right)=2$.

Proof. Without loss of generality, we only need to find a $\gamma_{\mathrm{R}}$-function $f$ on $P_{2} \times P_{n}$ with $f\left(u_{1,1}\right)=2$. Define a Roman dominating function $f$ as follows. For each non-negative integer $i$ with $1+4 i \leq n$, let $f\left(u_{1,1+4 i}\right)=2$, and for each non-negative integer $j$ with $3+4 j \leq n$, let $f\left(u_{2,3+4 j}\right)=2$. If $n \equiv 0(\bmod 4)$, let $f\left(u_{1, n}\right)=1$, and if $n \equiv 2(\bmod 4)$, let $f\left(u_{2, n}\right)=1$. Then $f\left(P_{2} \times P_{n}\right)=n+1$ and, hence by Lemma 2.1, $f$ is a $\gamma_{\mathrm{R}}$-function with $f\left(u_{1,1}\right)=2$.

Theorem 3.2 $b_{\mathrm{R}}\left(P_{2} \times P_{n}\right)=2$ for $n \geq 2$.

Proof. By Lemma [2.1, we have $\gamma_{\mathrm{R}}\left(P_{2} \times P_{n}\right)=n+1$. Since $\gamma_{\mathrm{R}}\left(P_{2} \times P_{n}-u_{1,1} u_{1,2}-\right.$ $\left.u_{2,1} u_{2,2}\right)=2+\gamma_{\mathrm{R}}\left(P_{2} \times P_{n-1}\right)=n+2$, we have $b_{\mathrm{R}}\left(P_{2} \times P_{n}\right) \leq 2$. Next we prove that $\gamma_{\mathrm{R}}\left(P_{2} \times P_{n}-e\right) \leq \gamma_{\mathrm{R}}\left(P_{2} \times P_{n}\right)$ for any edge $e \in E\left(P_{2} \times P_{n}\right)$.

Suppose that $e$ is incident with some vertex in $\left\{u_{1,1}, u_{2,1}, u_{1, n}, u_{2, n}\right\}$. Without loss of generality let $e$ be incident with $u_{1,1}$. By Lemma 3.2, there exists a $\gamma_{\mathrm{R}}$-function $f$ on $P_{2} \times\left(P_{n}-P_{1}\right)$ such that $f\left(u_{2,2}\right)=2$. Denote $f\left(u_{1,1}\right)=1$ and then $f$ is a Roman dominating function of $P_{2} \times P_{n}-e$ with $f\left(P_{2} \times P_{n}-e\right)=n+1$, thus $\gamma_{\mathrm{R}}\left(P_{2} \times P_{n}-e\right) \leq$ $\gamma_{\mathrm{R}}\left(P_{2} \times P_{n}\right)$.

Suppose that $e$ is incident with some vertex in $\left\{u_{i, j}: 1 \leq i \leq 2,2 \leq j \leq n-\right.$ $1\} \backslash\left\{u_{1,1}, u_{2,1}, u_{1, n}, u_{2, n}\right\}$. Without loss of generality let $e$ be incident with $u_{1, j}$ and not incident with $u_{1, j-1}$. By Lemma [3.2, there exists a $\gamma_{\mathrm{R}}$-function $f_{1}$ on $P_{2} \times P_{j-1}$ with $f_{1}\left(u_{1, j-1}\right)=2$ and a $\gamma_{\mathrm{R}}$-function $f_{2}$ on $P_{2} \times\left(P_{n}-P_{j}\right)$ with $f_{2}\left(u_{2, j+1}\right)=2$. Then $f=f_{1} \cup f_{2}$ is a Roman dominating function on $P_{2} \times P_{n}-e$ with $f\left(P_{2} \times P_{n}-e\right)=n+1$, thus $\gamma_{\mathrm{R}}\left(P_{2} \times P_{n}-e\right) \leq \gamma_{\mathrm{R}}\left(P_{2} \times P_{n}\right)$.

The above two cases yield that $b_{\mathrm{R}}\left(P_{2} \times P_{n}\right) \geq 2$ and, hence, $b_{\mathrm{R}}\left(P_{2} \times P_{n}\right)=2$. The lemma follows.

## 4 Complexity of Roman bondage number

In this section, we will show that the Roman bondage number problem is NP-hard and the Roman domination number problem is NP-complete even for bipartite graphs. We first state the problem as the following decision problem.

Roman bondage number problem (RBN):
Instance: A nonempty bipartite graph $G$ and a positive integer $k$.
Question: Is $b_{\mathrm{R}}(G) \leq k$ ?
Roman domination number problem (RDN):
Instance: A nonempty bipartite graph $G$ and a positive integer $k$.
Question: Is $\gamma_{\mathrm{R}}(G) \leq k$ ?
Following Garey and Johnson's techniques for proving NP-completeness given in [7], we prove our results by describing a polynomial transformation from the known-well NP-complete problem: 3SAT. To state 3SAT, we recall some terms.

Let $U$ be a set of Boolean variables. A truth assignment for $U$ is a mapping $t: U \rightarrow$ $\{T, F\}$. If $t(u)=T$, then $u$ is said to be "true" under $t$; If $t(u)=F$, then $u$ is said to be "false" under $t$. If $u$ is a variable in $U$, then $u$ and $\bar{u}$ are literals over $U$. The literal $u$ is true under $t$ if and only if the variable $u$ is true under $t$; the literal $\bar{u}$ is true if and only if the variable $u$ is false.

A clause over $U$ is a set of literals over $U$. It represents the disjunction of these literals and is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathscr{C}$ of clauses over $U$ is satisfiable if and only if there exists some truth assignment for $U$ that simultaneously satisfies all the clauses in $\mathscr{C}$. Such a truth assignment is called a satisfying truth assignment for $\mathscr{C}$. The 3SAT is specified as follows.

3-satisfiability problem (3SAT):
Instance: $A$ collection $\mathscr{C}=\left\{C_{1}, C_{2}, \ldots, C_{m}\right\}$ of clauses over a finite set
$U$ of variables such that $\left|C_{j}\right|=3$ for $j=1,2, \ldots, m$.
Question: Is there a truth assignment for $U$ that satisfies all the clauses
in $\mathscr{C}$ ?

Theorem 4.1 (Theorem 3.1 in [7]) 3SAT is NP-complete.

Theorem 4.2 RBN is NP-hard even for bipartite graphs.

Proof. The transformation is from 3 SAT. Let $U=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $\mathscr{C}=\left\{C_{1}, C_{2}, \ldots\right.$, $\left.C_{m}\right\}$ be an arbitrary instance of 3SAT. We will construct a bipartite graph $G$ and choose an integer $k$ such that $\mathscr{C}$ is satisfiable if and only if $b_{\mathrm{R}}(G) \leq k$. We construct such a graph $G$ as follows.

For each $i=1,2, \ldots, n$, corresponding to the variable $u_{i} \in U$, associate a graph $H_{i}$ with vertex set $V\left(H_{i}\right)=\left\{u_{i}, \bar{u}_{i}, v_{i}, v_{i}^{\prime}, x_{i}, y_{i}, z_{i}, w_{i}\right\}$ and edge set $E\left(H_{i}\right)=\left\{u_{i} v_{i}, u_{i} z_{i}, \bar{u}_{i} v_{i}^{\prime}\right.$, $\left.\bar{u}_{i} z_{i}, y_{i} v_{i}, y_{i} v_{i}^{\prime}, y_{i} z_{i}, w_{i} v_{i}, w_{i} v_{i}^{\prime}, w_{i} z_{i}, x_{i} v_{i}, x_{i} v_{i}^{\prime}\right\}$. For each $j=1,2, \ldots, m$, corresponding to the clause $C_{j}=\left\{p_{j}, q_{j}, r_{j}\right\} \in \mathscr{C}$, associate a single vertex $c_{j}$ and add edge set $E_{j}=$ $\left\{c_{j} p_{j}, c_{j} q_{j}, c_{j} r_{j}\right\}, 1 \leq j \leq m$. Finally, add a path $P=s_{1} s_{2} s_{3}$, join $s_{1}$ and $s_{3}$ to each vertex $c_{j}$ with $1 \leq j \leq m$ and set $k=1$.

Figure 1 shows an example of the graph obtained when $U=\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}$ and $\mathscr{C}=\left\{C_{1}, C_{2}, C_{3}\right\}$, where $C_{1}=\left\{u_{1}, u_{2}, \bar{u}_{3}\right\}, C_{2}=\left\{\bar{u}_{1}, u_{2}, u_{4}\right\}, C_{3}=\left\{\bar{u}_{2}, u_{3}, u_{4}\right\}$.


Figure 1: An instance of the Roman bondage number problem resulting from an instance of 3SAT. Here $k=1$ and $\gamma_{\mathrm{R}}(G)=18$, where the bold vertex $w$ means a Roman dominating function with $f(w)=2$.

To prove that this is indeed a transformation, we only need to show that $b_{\mathrm{R}}(G)=1$ if and only if there is a truth assignment for $U$ that satisfies all clauses in $\mathscr{C}$. This aim can be obtained by proving the following four claims.

Claim 4.1 $\gamma_{\mathrm{R}}(G) \geq 4 n+2$. Moreover, if $\gamma_{\mathrm{R}}(G)=4 n+2$, then for any $\gamma_{\mathrm{R}}$-function $f$ on $G, f\left(H_{i}\right)=4$ and at most one of $f\left(u_{i}\right)$ and $f\left(\bar{u}_{i}\right)$ is 2 for each $i, f\left(c_{j}\right)=0$ for each $j$ and $f\left(s_{2}\right)=2$.

Proof. Let $f$ be a $\gamma_{\mathrm{R}}$-function of $G$, and let $H_{i}^{\prime}=H_{i}-u_{i}-\bar{u}_{i}$.
If $f\left(u_{i}\right)=2$ and $f\left(\bar{u}_{i}\right)=2$, then $f\left(H_{i}\right) \geq 4$. Assume either $f\left(u_{i}\right)=2$ or $f\left(\bar{u}_{i}\right)=2$, if $f\left(x_{i}\right)=0$ or $f\left(y_{i}\right)=0$, then there is at least one vertex $t$ in $\left\{v_{i}, \bar{v}_{i}, z_{i}\right\}$ such that $f(t)=2$. And hence $f\left(H_{i}^{\prime}\right) \geq 2$. Thus, $f\left(H_{i}\right) \geq 4$.

If $f\left(u_{i}\right) \neq 2$ and $f\left(\bar{u}_{i}\right) \neq 2$, let $f^{\prime}$ be a restriction of $f$ on $H_{i}^{\prime}$, then $f^{\prime}$ is a Roman dominating function of $H_{i}^{\prime}$, and $f^{\prime}\left(H_{i}^{\prime}\right) \geq \gamma_{\mathrm{R}}\left(H_{i}^{\prime}\right)$. Since the maximum degree of $H_{i}^{\prime}$ is $V\left(H_{i}^{\prime}\right)-3$, by Lemma 2.3, $\gamma_{\mathrm{R}}\left(H_{i}^{\prime}\right)>3$ and hence $f^{\prime}\left(H_{i}^{\prime}\right) \geq 4$ and $f\left(H_{i}\right) \geq 4$. If $f\left(s_{1}\right)=0$ or $f\left(s_{3}\right)=0$, then there is at least one vertex $t$ in $\left\{c_{1}, \cdots, c_{m}, s_{2}\right\}$ such that $f(t)=2$. Then $f\left(N_{G}[V(P)]\right) \geq 2$, and hence $\gamma_{\mathrm{R}}(G) \geq 4 n+2$.

Suppose that $\gamma_{\mathrm{R}}(G)=4 n+2$, then $f\left(H_{i}\right)=4$ and since $f\left(N_{G}\left[x_{i}\right]\right) \geq 1$, at most one of $f\left(u_{i}\right)$ and $f\left(\bar{u}_{i}\right)$ is 2 for each $i=1,2, \ldots, n$, while $f\left(N_{G}[V(P)]\right)=2$. Then we have $f\left(s_{2}\right)=2$ since $f\left(N_{G}\left[s_{2}\right]\right) \geq 1$. Consequently, $f\left(c_{j}\right)=0$ for each $j=1,2, \ldots, m$.

Claim $4.2 \gamma_{\mathrm{R}}(G)=4 n+2$ if and only if $\mathscr{C}$ is satisfiable.
Proof. Suppose that $\gamma_{\mathrm{R}}(G)=4 n+2$ and let $f$ be a $\gamma_{\mathrm{R}}$-function of $G$. By Claim 4.1, at most one of $f\left(u_{i}\right)$ and $f\left(\bar{u}_{i}\right)$ is 2 for each $i=1,2, \ldots, n$. Define a mapping $t: U \rightarrow\{T, F\}$ by

$$
t\left(u_{i}\right)= \begin{cases}T & \text { if } f\left(u_{i}\right)=2 \text { or } f\left(u_{i}\right) \neq 2 \text { and } f\left(\bar{u}_{i}\right) \neq 2, \quad i=1,2, \ldots, n .  \tag{4.1}\\ F & \text { if } f\left(\bar{u}_{i}\right)=2\end{cases}
$$

We now show that $t$ is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by $t$. To this end, we arbitrarily choose a clause
$C_{j} \in \mathscr{C}$ with $1 \leq j \leq m$.
By Claim 4.1, $f\left(c_{j}\right)=f\left(s_{1}\right)=f\left(s_{3}\right)=0$. There exists some $i$ with $1 \leq i \leq n$ such that $f\left(u_{i}\right)=2$ or $f\left(\bar{u}_{i}\right)=2$ where $c_{j}$ is adjacent to $u_{i}$ or $\bar{u}_{i}$. Suppose that $c_{j}$ is adjacent to $u_{i}$ where $f\left(u_{i}\right)=2$. Since $u_{i}$ is adjacent to $c_{j}$ in $G$, the literal $u_{i}$ is in the clause $C_{j}$ by the construction of $G$. Since $f\left(u_{i}\right)=2$, it follows that $t\left(u_{i}\right)=T$ by (4.1), which implies that the clause $C_{j}$ is satisfied by $t$. Suppose that $c_{j}$ is adjacent to $\bar{u}_{i}$ where $f\left(\bar{u}_{i}\right)=2$. Since $\bar{u}_{i}$ is adjacent to $c_{j}$ in $G$, the literal $\bar{u}_{i}$ is in the clause $C_{j}$. Since $f\left(\bar{u}_{i}\right)=2$, it follows that $t\left(u_{i}\right)=F$ by (4.1). Thus, $t$ assigns $\bar{u}_{i}$ the truth value $T$, that is, $t$ satisfies the clause $C_{j}$. By the arbitrariness of $j$ with $1 \leq j \leq m$, we show that $t$ satisfies all the clauses in $\mathscr{C}$, that is, $\mathscr{C}$ is satisfiable.

Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t: U \rightarrow\{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. Create a function $f$ on $V(G)$ as follows: if $t\left(u_{i}\right)=T$, then let $f\left(u_{i}\right)=f\left(v_{i}^{\prime}\right)=2$, and if $t\left(u_{i}\right)=F$, then let $f\left(\bar{u}_{i}\right)=f\left(v_{i}\right)=2$. Let $f\left(s_{2}\right)=2$. Clearly, $f(G)=4 n+2$. Since $t$ is a satisfying truth assignment for $\mathscr{C}$, for each $j=1,2, \ldots, m$, at least one of literals in $C_{j}$ is true under the assignment $t$. It follows that the corresponding vertex $c_{j}$ in $G$ is adjacent to at least one vertex $w$ with $f(w)=2$ since $c_{j}$ is adjacent to each literal in $C_{j}$ by the construction of $G$. Thus $f$ is a Roman dominating function of $G$, and so $\gamma_{\mathrm{R}}(G) \leq f(G)=4 n+2$. By Claim 4.1, $\gamma_{\mathrm{R}}(G) \geq 4 n+2$, and so $\gamma_{\mathrm{R}}(G)=4 n+2$.

Claim $4.3 \gamma_{\mathrm{R}}(G-e) \leq 4 n+3$ for any $e \in E(G)$.
Proof. For any edge $e \in E(G)$, it is sufficient to construct a Roman dominating function $f$ with weight $4 n+3$ of $G$. We first assume $e \in E_{G}\left(s_{1}\right)$ or $e \in E_{G}\left(s_{3}\right)$ or $e \in E_{G}\left(c_{j}\right)$ for each $j=1,2, \ldots, m$, without loss of generality let $e \in E_{G}\left(s_{1}\right)$ or $e=c_{j} u_{i}$ or $e=c_{j} \bar{u}_{i}$. Let $f\left(s_{3}\right)=2, f\left(s_{1}\right)=1$ and $f\left(u_{i}\right)=f\left(v_{i}^{\prime}\right)=2$ for each $i=1,2, \ldots, n$. For the edge $e \notin E_{G}\left(u_{i}\right)$ and $e \notin E_{G}\left(v_{i}^{\prime}\right)$ or $e=\bar{u}_{i} z_{i}$, let $f\left(s_{1}\right)=2, f\left(s_{3}\right)=1$ and $f\left(u_{i}\right)=f\left(v_{i}^{\prime}\right)=2$. For the edge $e \notin E\left(\bar{u}_{i}\right)$ and $e \notin E\left(v_{i}\right)$ or $e=u_{i} z_{i}$, let $f\left(s_{1}\right)=2, f\left(s_{3}\right)=1$ and $f\left(\bar{u}_{i}\right)=f\left(v_{i}\right)=2$. If $e=u_{i} v_{i}$ or $e=\bar{u}_{i} v_{i}^{\prime}$,
let $f\left(s_{1}\right)=2, f\left(s_{3}\right)=1$ and $f\left(x_{i}\right)=f\left(z_{i}\right)=2$. Then $f$ is a Roman dominating function of $G-e$ with $f(G-e)=4 n+3$ and hence $\gamma_{\mathrm{R}}(G-e) \leq 4 n+3$.

Claim $4.4 \gamma_{\mathrm{R}}(G)=4 n+2$ if and only if $b_{\mathrm{R}}(G)=1$.
Proof. Assume $\gamma_{\mathrm{R}}(G)=4 n+2$ and consider the edge $e=s_{1} s_{2}$. Suppose $\gamma_{\mathrm{R}}(G)=$ $\gamma_{\mathrm{R}}(G-e)$. Let $f^{\prime}$ be a $\gamma_{\mathrm{R}}$-function of $G-e$. It is clear that $f^{\prime}$ is also a $\gamma_{\mathrm{R}}$-function on $G$. By Claim 4.1 we have $f^{\prime}\left(c_{j}\right)=0$ for each $j=1,2, \ldots, m$ and $f^{\prime}\left(s_{2}\right)=2$. But then $f^{\prime}\left(N_{G-e}\left[s_{1}\right]\right)=0$, a contradiction. Hence, $\gamma_{\mathrm{R}}(G)<\gamma_{\mathrm{R}}(G-e)$, and so $b_{\mathrm{R}}(G)=1$.

Now, assume $b_{\mathrm{R}}(G)=1$. By Claim 4.1, we have that $\gamma_{\mathrm{R}}(G) \geq 4 n+2$. Let $e^{\prime}$ be an edge such that $\gamma_{\mathrm{R}}(G)<\gamma_{\mathrm{R}}\left(G-e^{\prime}\right)$. By Claim 4.3, we have that $\gamma_{\mathrm{R}}\left(G-e^{\prime}\right) \leq 4 n+3$. Thus, $4 n+2 \leq \gamma_{\mathrm{R}}(G)<\gamma_{\mathrm{R}}\left(G-e^{\prime}\right) \leq 4 n+3$, which yields $\gamma_{\mathrm{R}}(G)=4 n+2$.

By Claim 4.2 and Claim 4.4, we prove that $b_{\mathrm{R}}(G)=1$ if and only if there is a truth assignment for $U$ that satisfies all clauses in $\mathscr{C}$. Since the construction of the Roman bondage number instance is straightforward from a 3-satisfiability instance, the size of the Roman bondage number instance is bounded above by a polynomial function of the size of 3 -satisfiability instance. It follows that this is a polynomial reduction.

The theorem follows.

Corollary 4.1 Roman domination number problem is NP-complete even for bipartite graphs.

Proof. It is easy to see that the Roman bondage problem is in NP since a nondeterministic algorithm need only guess a vertex set pair $\left(V_{1}, V_{2}\right)$ with $\left|V_{1}\right|+2\left|V_{2}\right| \leq k$ and check in polynomial time whether that for any vertex $u \in V \backslash\left(V_{1} \cup V_{2}\right)$ whether there is a vertex in $V_{2}$ adjacent to $u$ for a given nonempty graph $G$.

We use the same method as Theorem 4.2 to prove this conclusion. We construct the same graph $G$ but does not contain the path $P$. We set $k=4 n$, then use the same methods as Claim 4.1 and 4.2, we have that $\gamma_{\mathrm{R}}(G)=4 n$ if and only if $\mathscr{C}$ is satisfiable.

## 5 General bounds

Lemma 5.1 Let $H$ be a spanning subgraph obtained by removing $k$ edges from a graph $G$. Then $b_{\mathrm{R}}(G) \leq b_{\mathrm{R}}(H)+k$.

Proof. Let $B=E(G) \backslash E(H)$ and $B^{\prime}$ be a minimum Roman bondage set of $H$. Then $|B|=k,\left|B^{\prime}\right|=b_{\mathrm{R}}(H)$ and $\gamma_{\mathrm{R}}\left(H-B^{\prime}\right)>\gamma_{\mathrm{R}}(H)$. Let $f: V \rightarrow\{0,1,2\}$ be a Roman dominating function on $H$ with $f(H)=\gamma_{\mathrm{R}}(H)$. Then each vertex $x$ with $f(x)=0$ is adjacent to at least one vertex $y$ with $f(y)=2$ in $H$, and so is in $G$ since $H=G-B$, which implies that $f$ is a Roman dominating function of $G$, and so $f(G) \geq \gamma_{\mathrm{R}}(G)$. It follows that $\gamma_{\mathrm{R}}\left(G-B-B^{\prime}\right)=\gamma_{\mathrm{R}}\left(H-B^{\prime}\right)>\gamma_{\mathrm{R}}(H) \geq \gamma_{\mathrm{R}}(G)$ and, hence, $b_{\mathrm{R}}(G) \leq$ $|B|+\left|B^{\prime}\right|=b_{\mathrm{R}}(H)+k$.

Theorem 5.1 $b_{\mathrm{R}}(G) \leq d_{G}(x)+d_{G}(y)+d_{G}(z)-\left|N_{G}(y) \cap N_{G}(\{x, z\})\right|-3$ for any path $(x, y, z)$ of length 2 in a graph $G$.

Proof. Let $F_{y}=\left\{(y, u) \in E(G): u \in N_{G}(y) \cap N_{G}(\{x, z\})\right\}, B=E_{G}(x) \cup E_{G}(z) \cup$ $\left(E_{G}(y) \backslash F_{y}\right)$. Then

$$
|B|=d_{G}(x)+d_{G}(y)+d_{G}(z)-\left|N_{G}(y) \cap N_{G}(\{x, z\})\right|-2 .
$$

Let $H=G-B+y z$. Then $x$ is an isolated vertex and $z$ is a vertex of degree 1 which is only adjacent to $y$ in $H$. Let $f$ be a minimum Roman dominating function of $H$, then $f(x)=1$ and $1 \leq f(y)+f(z) \leq 2$.

If $f(y)+f(z)=2$, then let $f^{\prime}=f$ except $f^{\prime}(x)=0, f^{\prime}(y)=2$ and $f^{\prime}(z)=0$. Clearly, $f^{\prime}$ is a Roman dominating function of $G$ with $f^{\prime}(G)<f(H)$ and, hence, $b_{\mathrm{R}}(G) \leq|B|-1$.

If $f(y)+f(z)=1$, then $f(y)=0$ and $f(z)=1$. There is an edge $(u, y) \in F_{y}$ with $f(u)=2$. Let $f^{\prime}=f$ except $f^{\prime}(x)=0$ if $u \in N_{G}(x)$ or $f^{\prime}(z)=0$ if $u \in N_{G}(z) \backslash N_{G}(x)$. Then $f^{\prime}$ is a Roman dominating function of $G$ with $f^{\prime}(G)<f(H)$, and hence $b_{\mathrm{R}}(G) \leq$ $|B|-1$.

Theorem 5.2 $b_{\mathrm{R}}(G) \leq d_{G}(x)+d_{G}(y)+d_{G}(z)-\left|N_{G}(y) \cap N_{G}(\{x, z\})\right|-\left|N_{G}(x) \cap N_{G}(z)\right|-1$ for any path $(x, y, z)$ of length 2 in a graph $G$.

Proof. Let $F_{y}=\left\{(y, u) \in E(G): u \in N_{G}(y) \cap N_{G}(\{x, z\})\right\}$ and $F_{z}=\{(z, u) \in E(G)$ : $\left.u \in\left(N_{G}(z) \cap N_{G}(x)\right)\right\}, B=E_{G}(x) \cup\left(E_{G}(z) \backslash F_{z}\right) \cup\left(E_{G}(y) \backslash F_{y}\right)$ and $H=G-B$. Then $x$ is an isolated vertex in $H$. Let $f$ be a minimum Roman dominating function of $H$, then $f(x)=1$. We will construct a Roman dominating function $f^{\prime}$ of $G$ with $f^{\prime}(G)<f(H)$.

If $f(z)=0$, then there is an edge $(z, s) \in F_{z}$ with $f(s)=2$. Thus, if $f(y)=2$ or $f(z)=0$, let $f^{\prime}=f$ except $f^{\prime}(x)=0$. In the following, let $f(y) \neq 2$ and $f(z) \neq 0$.

If $f(y)=0$. Then there is a vertex $s \in F_{y}$ such that $f(s)=2$. If $s \in N_{G}(x)$, let $f^{\prime}=f$ except $f^{\prime}(x)=0$. If $s \in N_{G}(z) \backslash N_{G}(x)$, let $f^{\prime}=f$ except $f^{\prime}(z)=0$.

If $f(y)=1$. If $f(z)=1$, let $f^{\prime}=f$ except $f^{\prime}(x)=f^{\prime}(z)=0$ and $f^{\prime}(y)=2$. If $f(z)=2$, let $f^{\prime}=f$ except $f^{\prime}(y)=0$.

Then $f^{\prime}$ is a Roman dominating function of $G$ with $f^{\prime}(G)<f(H)$, and hence $b_{\mathrm{R}}(G) \leq$ $|B| \leq d_{G}(x)+d_{G}(y)+d_{G}(z)-\left|N_{G}(y) \cap N_{G}(\{x, z\})\right|-\left|N_{G}(x) \cap N_{G}(z)\right|-1$.

Corollary $5.1 b_{\mathrm{R}}(G) \leq \min \left\{d_{G}(x)+d_{G}(y)+d_{G}(z)-\left|N_{G}(y) \cap N_{G}(\{x, z\})\right|-3, d_{G}(x)+\right.$ $\left.d_{G}(y)+d_{G}(z)-\left|N_{G}(y) \cap N_{G}(\{x, z\})\right|-\left|N_{G}(x) \cap N_{G}(z)\right|-1\right\}$ for any path $(x, y, z)$ of length 2 in a graph $G$.

Corollary $5.2 b_{\mathrm{R}}(G) \leq 2 \Delta(G)+\delta(G)-3$ for any graph with diameter at least two.

Corollary 5.3 For any tree $T$ of order at least 3, then $b_{\mathrm{R}}(T) \leq \Delta(T)$.

Proof. If there is a vertex $x$ adjacent to at least two vertices of degree one in $T$, say $u_{1}$ and $u_{2}$, then $\left(u_{1}, x, u_{2}\right)$ is a path of length 2 in $T$. By Lemma 5.1, $b_{\mathrm{R}}(T) \leq$ $d_{T}\left(u_{1}\right)+d_{T}(x)+d_{T}\left(u_{2}\right)-3 \leq \Delta(T)-1$.

Assume now that each vertex of $T$ is adjacent to at most one vertex of degree one. Then $T$ has a vertex $u$ of degree 2 adjacent to exactly one vertex, say $v$, of degree one. Let $w$ be the other vertex adjacent to $u$. Then $(v, u, w)$ is a path of length 2 in $T$. By Lemma 5.1. $b_{\mathrm{R}}(T) \leq d_{T}(v)+d_{T}(u)+d_{T}(w)-3 \leq \Delta(T)$.

Lemma 5.2 Let $G$ be a connected graph of order $n(\geq 3)$ and $\gamma_{\mathrm{R}}(G)=\gamma(G)+1$. If there is an set $B$ of edges with $\gamma_{\mathrm{R}}(G-B)=\gamma_{\mathrm{R}}(G)$, then $\Delta(G)=\Delta(G-B)$.

Proof. Since $G$ is connected and $n \geq 3, \gamma_{\mathrm{R}}(G)=\gamma(G)+1 \leq n-1$. Since $\gamma_{\mathrm{R}}(G-B)=$ $\gamma_{\mathrm{R}}(G) \leq n-1, G-B$ is nonempty. By Lemma 2.5 and Lemma 2.6, $\gamma_{\mathrm{R}}(G-B) \geq \gamma(G-$ $B)+1$. Since $\gamma_{\mathrm{R}}(G-B)=\gamma_{\mathrm{R}}(G)=\gamma(G)+1 \leq \gamma(G-B)+1, \gamma_{\mathrm{R}}(G-B)=\gamma(G-B)+1$ and $\gamma(G-B)=\gamma(G)$.

If $G-B$ is connected, then by Lemma 2.7, $\Delta(G-B)=n-\gamma(G-B)=n-\gamma(G)=$ $\Delta(G)$.

If $G-B$ is disconnected, then let $G_{1}$ be a nonempty connected component of $G-$ B. By Lemma [2.5 and [2.6, $\gamma_{\mathrm{R}}\left(G_{1}\right) \geq \gamma\left(G_{1}\right)+1$. Then $\gamma(G)+1=\gamma_{\mathrm{R}}(G-B) \geq$ $\gamma_{\mathrm{R}}\left(G_{1}\right)+\gamma_{\mathrm{R}}\left(G-G_{1}\right) \geq \gamma\left(G_{1}\right)+1+\gamma\left(G-G_{1}\right) \geq \gamma(G)+1$, thus $\gamma_{\mathrm{R}}\left(G_{1}\right)=\gamma\left(G_{1}\right)+1$, $\gamma_{\mathrm{R}}\left(G-G_{1}\right)=\gamma\left(G-G_{1}\right)$ and $\gamma(G)=\gamma\left(G_{1}\right)+\gamma\left(G-G_{1}\right)$. By Lemma 2.6, $G-G_{1}$ is empty and hence $\gamma\left(G-G_{1}\right)=\left|V\left(G-G_{1}\right)\right|$. By Lemma 2.7, $\Delta\left(G_{1}\right)=\left|V\left(G_{1}\right)\right|-\gamma\left(G_{1}\right)=$ $n-\left|V\left(G-G_{1}\right)\right|-\gamma\left(G_{1}\right)=n-\gamma\left(G-G_{1}\right)-\gamma\left(G_{1}\right)=n-\gamma(G)=\Delta(G)$.

Theorem 5.3 Let $G$ be a connected graph of order $n(\geq 3)$ and $\gamma_{\mathrm{R}}(G)=\gamma(G)+1$. Then $b_{\mathrm{R}}(G) \leq \min \left\{b(G), n_{\Delta}\right\}$, where $n_{\Delta}$ is the number of vertices with maximum degree $\Delta$ in $G$.

Proof. Since $n \geq 3$ and $G$ is connected, $\Delta(G) \geq 2$ and hence $\gamma(G) \leq n-2$. Let $B$ be a minimum bondage set of $G$. Then $G-B$ is nonempty and by Lemma 2.5 and Lemma 2.6, Thus, $\gamma_{\mathrm{R}}(G-B) \geq \gamma(G-B)+1>\gamma(G)+1=\gamma_{\mathrm{R}}(G)$ and hence $b_{\mathrm{R}}(G) \leq b(G)$.

We now prove that $b_{\mathrm{R}}(G) \leq n_{\Delta}$. By Lemma 2.7, $\gamma_{\mathrm{R}}(G)=\gamma(G)+1$ if and only if there is a vertex of degree $n-\gamma(G)$. If there is a vertex $s$ in $G$ such that $d_{G}(s)>n-\gamma(G)$, let $f(s)=2$ and $f(w)=1$ for any vertex $w$ not in $N_{G}[s]$, then $f$ is a Roman dominating function of $G$ with $f(G)=\gamma(G)$, a contradiction. Thus, $\Delta(G)=n-\gamma(G)$. We can remove a smallest edge set $B$ with $|B| \leq n_{\Delta}$ edges from $G$ such that $\Delta(G-B)<$ $\Delta(G)=n-\gamma(G)$ and $G-B$ is nonempty. Since $G-B$ is nonempty, by Lemma 2.5 and Lemma 2.6, $\gamma_{\mathrm{R}}(G-B) \geq \gamma(G-B)+1$. Assume $\gamma_{\mathrm{R}}(G-B)=\gamma_{\mathrm{R}}(G)$, then by Lemma 5.2, $\Delta(G-B)=\Delta(G)=n-\gamma(G)$, a contradiction. Hence $b_{\mathrm{R}}(G) \leq|B| \leq n_{\Delta}$. 】

Theorem 5.4 For Roman graph $G, b_{\mathrm{R}}(G) \geq b(G)$.

Proof. Let $B$ be a minimum Roman bondage set of $G$, then $\gamma_{\mathrm{R}}(G-B)>\gamma_{\mathrm{R}}(G)=2 \gamma(G)$. By Lemma 2.5, $\gamma_{\mathrm{R}}(G-B) \leq 2 \gamma(G-B)$, then $\gamma(G-B)>\gamma(G)$ and hence $b_{\mathrm{R}}(G) \geq b(G)$.

The equality in Theorem 5.4 can hold, for example, $b\left(C_{3 k}\right)=2=b_{\mathrm{R}}\left(C_{3 k}\right)$, and the strict inequality can also hold, for example, $b\left(C_{3 k+2}\right)=2<3=b_{\mathrm{R}}\left(C_{3 k+2}\right)$.

Theorem 5.5 Let $G$ be a nonempty graph with $\gamma_{\mathrm{R}}(G) \geq 3$. Then $b_{\mathrm{R}}(G) \leq\left(\gamma_{\mathrm{R}}(G)-\right.$ 2) $\Delta(G)+1$.

Proof. The proof proceeds by induction on $\gamma_{\mathrm{R}}(G)$.
We first assume that $\gamma_{\mathrm{R}}(G)=3$. Then by Lemma [2.3, $\Delta(G)=|V(G)|-2$. Assume that $b_{\mathrm{R}}(G) \geq \Delta(G)+2$. Let $u$ be a vertex of maximum degree in $G$. We have $\gamma_{\mathrm{R}}(G-u)=$ $\gamma_{\mathrm{R}}(G)-1=2$. There is a vertex $v$ that is adjacent to every vertex in $G-u$ and hence $v u \notin E(G)$. Since $b_{\mathrm{R}}(G-u) \geq 2$, then for any edge $e \in E_{G-u}(v), \gamma_{\mathrm{R}}(G-u-e)=2$. Thus there is a vertex $w$ that is adjacent to every vertex of $G-u-e$. But, since $v$ is the only vertex of $G$ that is not adjacent to $u, w u \in E(G), d_{G}(w)=|V(G)|-1$, a contradiction. Thus, $b_{\mathrm{R}}(G) \leq \Delta(G)+1$ if $\gamma_{\mathrm{R}}(G)=3$.

Assume the induction hypothesis for any integer $k$ and any graph $H$ with $\gamma_{\mathrm{R}}(H)=$ $k \geq 3$. Let $G$ be a nonempty graph with $\gamma_{\mathrm{R}}(G)=k+1$, and assume that $b_{\mathrm{R}}(G) \geq$ $(k-1) \Delta(G)+2$. For any vertex $u$ of $G$, let $H=G-u$. Then, $\gamma_{\mathrm{R}}(H)=\gamma_{\mathrm{R}}(G)-1=k$ since $d_{G}(u)<b_{\mathrm{R}}(G)$. By the inductive hypothesis and by Lemma 5.1, we have

$$
\begin{aligned}
b_{\mathrm{R}}(G) & \leq b_{\mathrm{R}}(H)+d_{G}(u) \\
& \leq(k-2) \Delta(H)+1+d_{G}(u) \\
& \leq(k-2) \Delta(G)+1+\Delta(G) \\
& =(k-1) \Delta(G)+1
\end{aligned}
$$

a contradiction. Thus, $b_{\mathrm{R}}(G) \leq(k-1) \Delta(G)+1$, and by the principle of mathematical induction, $b_{\mathrm{R}}(G) \leq\left(\gamma_{\mathrm{R}}(G)-2\right) \Delta(G)+1$.

Use $\kappa(G)$ (resp. $\lambda(G))$ to denote the vertex-connectivity (resp. the edge-connectivity) of a connected graph $G$ which is the minimum number of vertices (resp, edges) whose removal results in $G$ disconnected. The famous Whitney's inequality can be stated as $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$ for any graph $G$. A subset $F \subseteq E(G)$ is called a $\lambda$-cut if $|F|=\lambda(G)$ and $G-F$ is disconnected.

Theorem 5.6 If $G$ is a connected graph with order at least 3, then $b_{\mathrm{R}}(G) \leq 2 \Delta(G)+$ $\lambda(G)-3$, where $\lambda(G)$ is the edge-connectivity of $G$.

Proof. Let $G$ be a connected graph with edge-connectivity $\lambda(G)$ and $F$ be $\lambda$-cut of $G$. Then $H=G-F$ has exact two connected components. Let $x, y \in V(G), x y \in F$, and $H_{x}$ and $H_{y}$ denote the components of $G-F$ containing $x$ and $y$, respectively. Without loss of generality, let $z$ be adjacent to $x$ in $H_{x}$ since $|V(G)| \geq 3$. Let $B=F \cup E_{H_{x}}(x) \cup E_{H_{x}}(z)-x z$ and $f$ be a $\gamma_{\mathrm{R}}$-function of $G^{\prime}=G-B$. Then $x$ and $z$ is only adjacent to each other in $G^{\prime}$, and so we can assume $f(x)=2$ and $f(z)=0$. We construct a Roman dominating function $f^{\prime}$ of $G$ with $f^{\prime}(G)<f\left(G^{\prime}\right)$.

If $V\left(H_{y}\right)=\{y\}$, then $f(y)=1$. Let $f^{\prime}=f$ except $f^{\prime}(y)=0$. Then $f^{\prime}$ is a Roman dominating function of $G$ with $f^{\prime}(G)<f\left(G^{\prime}\right)$. Thus, $b_{\mathrm{R}}(G) \leq|B| \leq 2 \Delta(G)+\lambda(G)-3$. In the following, we assume $\left|V\left(H_{y}\right)\right| \geq 2$.

If $\gamma_{\mathrm{R}}\left(H_{y}-y\right) \geq \gamma_{\mathrm{R}}\left(H_{y}\right)$, then

$$
\gamma_{\mathrm{R}}\left(G-\left(F \cup E_{H_{y}}(y)\right)\right) \geq \gamma_{\mathrm{R}}\left(H_{x}\right)+\gamma_{\mathrm{R}}\left(H_{y}\right)+1 \geq \gamma_{\mathrm{R}}(G)+1
$$

Thus

$$
\begin{aligned}
b_{\mathrm{R}}(G) & \left.\leq \mid F \cup E_{H_{y}}(y)\right) \mid \leq \Delta(G)+\lambda(G)-1 \\
& \leq 2 \Delta(G)+\lambda(G)-3 .
\end{aligned}
$$

If $\gamma_{\mathrm{R}}\left(H_{y}-y\right)=\gamma_{\mathrm{R}}\left(H_{y}\right)-1$, we can assume that $f(y)=1$. Let $f^{\prime}=f$ except $f^{\prime}(y)=0$. Then $f^{\prime}$ is a Roman dominating function of $G$ with $f^{\prime}(G)<f\left(G^{\prime}\right)$. Thus,

$$
b_{\mathrm{R}}(G) \leq|B| \leq 2 \Delta(G)+\lambda(G)-3
$$

The theorem follows.
Considering vertex rather than edge-connectivity, we could conjecture an analogy of Theorem 5.6 by a similar argument.

Conjecture 5.1 If $G$ is a connected graph with order no less than 3, then $b_{\mathrm{R}}(G) \leq$ $2 \Delta(G)+\kappa(G)-3$, where $\kappa(G)$ is the vertex-connectivity of $G$.

Theorem 5.7 If $G$ is a nonempty graph with a unique minimum Roman dominating function, then $b_{\mathrm{R}}(G)=1$.

Proof. Let $f$ be the unique $\gamma_{\mathrm{R}}$-function on $G$, and let $x$ be a vertex in $G$ with $f(x)=0$. Then there is a vertex $y \in N_{G}(x)$ with $f(y)=2$. If there are at least two vertices $y, z \in N_{G}(x)$ such that $f(y)=f(z)=2$ for each vertex $x$ with $f(x)=0$. Then let $f^{\prime}=f$ except that $f^{\prime}(x)=2$ and $f^{\prime}(y)=0$ and $f^{\prime}$ is a $\gamma_{\mathrm{R}}$-function on $G$ as well, which is a contradiction to the uniqueness of $f$. Thus, there is a unique $y \in N_{G}(x)$ with $f(y)=2$ for a vertex $x$ with $f(x)=0$. Then $\gamma_{\mathrm{R}}(G-x y)>\gamma_{\mathrm{R}}(G)$, which implies that $b_{\mathrm{R}}(G)=1$.

Theorem 5.8 If $G$ is a vrc-graph with $\gamma_{\mathrm{R}}(G)=3$, then $b_{\mathrm{R}}(G) \leq \Delta(G)+1$.

Proof. By Lemma 2.9, $G$ is a vc-graph with $\gamma(G)=2$. By Lemma 2.8, $G$ is a complete $K_{2 t}(t \geq 2)$ with a perfect matching $M$ removed. Thus, $G$ is $\Delta(G)$-regular, where $\Delta(G)=$ $2 t-2$. Let $u v \in M$. Then $v$ is the only vertex not adjacent to $u$ in $G$. Let $H=G-u$. Then $\gamma_{\mathrm{R}}(H)=2$ since $G$ is a vrc-graph with $\gamma_{\mathrm{R}}(G)=3$. Note that the vertex $v$ is the only vertex adjacent to all the other vertices in $H$ adjacent to each of other vertices in $H$. Thus $H$ has a unique minimum Roman dominating function $f$ with $f(v)=2=\gamma_{\mathrm{R}}(H)$. By Theorem 5.7, $b_{\mathrm{R}}(H)=1$ and hence $b_{\mathrm{R}}(G) \leq \Delta(G)+1$.

Theorem 5.9 If there exists at least one vertex $u$ in a graph $G$ with $\gamma_{\mathrm{R}}(G-u) \geq \gamma_{\mathrm{R}}(G)$, then $b_{\mathrm{R}}(G)=d_{G}(x) \leq \Delta(G)$.

Proof. Since $\gamma_{\mathrm{R}}\left(G-E_{G}(u)\right)=\gamma_{\mathrm{R}}(G-u)+1>\gamma_{\mathrm{R}}(G), b_{\mathrm{R}}(G)=d_{G}(x) \leq \Delta(G)$.

Corollary 5.4 Let $G$ be a graph of order $n$. If $\gamma_{\mathrm{R}}(G)=3 \neq n$, then $b_{\mathrm{R}}(G) \leq \Delta+1$.

Problem 5.1 Whether or not there exits a positive integer $c$ such that $b_{\mathrm{R}}(G) \leq \Delta(G)+c$ for any graph $G$ of order $n$ and $\gamma_{\mathrm{R}}(G) \neq n$.

The vertex covering number $\beta(G)$ of $G$ is the minimum number of vertices that are incident with all edges in $G$. If $G$ has no isolated vertices, then $\gamma_{\mathrm{R}}(G) \leq 2 \gamma(G) \leq 2 \beta(G)$. If $\gamma_{\mathrm{R}}(G)=2 \beta(G)$, then $\gamma_{\mathrm{R}}(G)=2 \gamma(G)$ and hence $G$ is a Roman graph. In [17], Volkmann gave a lot of graphs with $\gamma(G)=\beta(G)$.

Theorem 5.10 Let $G$ be a graph with $\gamma_{\mathrm{R}}(G)=2 \beta(G)$. Then
(1) $b_{\mathrm{R}}(G) \geq \delta(G)$;
(2) $b_{\mathrm{R}}(G) \geq \delta(G)+1$ if $G$ is a vrc-graph.

Proof. Let $G$ be a graph with $\gamma_{\mathrm{R}}(G)=2 \beta(G)$.
(1) Without loss of generality, Assume $\delta(G) \geq 2$. Let $B \subseteq E(G)$ with $|B| \leq \delta(G)-1$. Then $\delta(G-B) \geq 1$ and so $\gamma_{\mathrm{R}}(G) \leq \gamma_{\mathrm{R}}(G-B) \leq 2 \beta(G-B) \leq 2 \beta(G)=\gamma_{\mathrm{R}}(G)$. Thus, $B$ is not a Roman bondage set of $G$, and so $b_{\mathrm{R}}(G) \geq \delta(G)$.
(2) From the above proof, every Roman bondage set $B$ contains at least all edges incident with some vertex $x$, so that $G-B$ has an isolated vertex. On the other hand, if $G$ is a vrc-graph, then $\gamma_{\mathrm{R}}(G-x)<\gamma_{\mathrm{R}}(G)$ for any vertex $x$, which implies that the removal of all edges incident with $x$ can not enlarge the Roman domination number. Hence $b_{\mathrm{R}}(G) \geq \delta(G)+1$.

## References

[1] R. C. Brigham, P. Z. Chinn and R. D. Dutton, Vertex domination-cirtical graphs. Networks, 18 (1988), 173-179.
[2] E. J. Cockayne, P. A. Dreyer, Jr., S. M. Hedetniemi, S. T. Hedetniemi, A. A. McRae, Roman domination in graphs. Discrete Mathematics, 278 (1-3) (2004), 11-22.
[3] E. W. Chambers, B. Kinnersley, N. Prince, D. B. West, Extremal problems for Roman domination. SIAM J. Discrete Math. 23 (2009), 1575-1586.
[4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph. Discrete Mathematics, 86 (1990), 47-57.
[5] O. Favaron, H. Karami, R. Khoeilar and S. M. Sheikholeslami, On the Roman domination number of a graph. Discrete Mathematics, 309 (2009), 3447-3451.
[6] X. L. Fu, Y. S. Yang and B. Q. Jiang, Roman domination in regular graphs. Discrete Mathematics, 309 (2009), 1528-1537.
[7] M. R. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.
[8] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1997.
[9] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1997.
[10] J. H. Hattingh, A. R. Plummer, Restrained bondage in graphs. Discrete Mathemat$i c s, 308$ (2008), 5446-5453.
[11] M. Liedloff, T. Kloks, J. P. Liu and S. L. Peng, Efficient algorithms for Roman domination on some classes of graphs, Discrete Applied Mathematics, 156 (2008), 3400-3415.
[12] M. Liedloff, T. Kloks, J. P. Liu and S. L. Peng, Roman domination over some graph classes, Lecture Notes in Computer Science, 2005, Volume 3787, Graph-Theoretic Concepts in Computer Science, Pages 103-114.
[13] A. Pagourtzis, P. Penna, K. Schlude, K. Steinhofel, D. Taylor and P. Widmayer, Server placements, Roman domination and other dominating set variants, in Proc. Second International Conference on Theoretical Computer Science (2002), 280-291.
[14] R. R. Rubalcaba, P. J. Slater, Roman dominating influence parameters. Discrete Mathematics, 307 (2007), 3194-3200.
[15] W. P. Shang and X. D. Hu, The roman domination problem in unit disk graphs, Lecture Notes in Computer Science, 2007, Volume 4489/2007, 305-312.
[16] W. P. Shang and X. D. Hu, Roman domination and its variants in unit disk graphs, Discrete Mathematics, Algorithms and Applications, 2 (2010), 99-105.
[17] L. Volkmann, On graphs with equal domination and covering numbers. Discrete Applied Mathematics, 51 (1994), 211-217.
[18] J.-M. Xu, Toplogical Structure and Analysis of Interconnection Networks. Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.
[19] J.-M. Xu, Theory and Application of Graphs. Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[20] H. M. Xing, X. Chen and X. G. Chen, A note on Roman domination in graphs. Discrete Mathematics, 306 (2006), 3338-3340.


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