Coherent States in Gravitational Quantum Mechanics

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Abstract

We present the coherent states of the harmonic oscillator in the framework of the generalized (gravitational) uncertainty principle (GUP). This form of GUP is consistent with various theories of quantum gravity such as string theory, loop quantum gravity, and black-hole physics and implies a minimal measurable length. Using a recently proposed formally self-adjoint representation, we find the GUP-corrected Hamiltonian as a generator of the generalized Heisenberg algebra. Then following Klauder's approach, we construct exact coherent states and obtain the corresponding normalization coefficients, weight functions, and probability distributions. We find the entropy of the system and show that it decreases in the presence of the minimal length. These results could shed light on

possible detectable Planck-scale effects within recent experimental tests.

Keywords: Quantum gravity; Generalized uncertainty principle; Coherent States.

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Introduction 1

The canonical quantization and the path integral quantization of gravity are two well-known but old pro-

posals which attempted to quantize gravity and to unify the general relativity with the laws of quantum

mechanics. However, from field theoretical approach, the theory of relativity is not renormalizable and

results in the ultraviolet divergences. Indeed, beyond the Planck energy scale, the effects of gravity are

so important which could lead to discreteness of the very spacetime. This is due to the fact that when

we probe small distances with high energies, the spacetime structure will be significantly disturbed by

the gravitational effects. However, we can solve the normalizability problem of gravity by introducing a

minimal measurable length as an effective cutoff in the ultraviolet domain.

Various candidates of quantum gravity such as string theory, loop quantum gravity and quantum

geometry all agree on the existence of a minimum observable length. In the language of the string

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theory, a string cannot probe distances smaller than its length. Moreover, some Gedanken experiments in black-hole physics and noncommutative geometry imply a minimal length of the order of the Planck length  $\ell_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35} m$  where G is Newton's gravitational constant. For instance, since in string theory the mass of a string is proportional to its length, they expand in size when probed at sufficiently high energies. This suggests an additional momentum dependent uncertainty in the position of a string in the form of [1,2] ( $\hbar=1$ )

$$\Delta X \ge \frac{1}{2\Delta P} + k\ell_{Pl}\Delta P,\tag{1}$$

where k is a dimensionless constant. In order to incorporate the idea of the minimal length into quantum mechanics, we need to change the Heisenberg uncertainty principle to the so-called Generalized Uncertainty Principle (GUP). The introduction of this idea has attracted much attention in recent years and many papers have appeared in the literature to address the effects of GUP on various quantum mechanical systems [3–26].

In this paper, we are interested to find exact coherent states of the harmonic oscillator in the framework of the generalized commutation relation in the form  $[X,P]=i\hbar(1+\beta P^2)$  where  $\beta$  is the GUP parameter. Note that the problem of the GUP-corrected harmonic oscillator is exactly solvable in the momentum space and its exact energy eigenvalues and the eigenfunctions are obtained in Refs. [27,28]. Moreover, the perturbative construction of the corresponding coherent states is discussed in Refs. [29,30] to first-order of the GUP parameter. Here, following Klauder's approach and using the formally self-adjoint representation of the deformed commutation relation, we take the Hamiltonian as a generator of the generalized Heisenberg algebra and find the exact form of the coherent states, weight functions, normalization coefficients, and probability distributions. We show that the entropy of the system reduces in the GUP scenario as a consequence of the minimal observable length and we explain the physical reasoning behind this phenomenon. The connection between our results and the recent progresses in probing Planck-scale physics with quantum optics is discussed finally.

## 2 The Generalized Uncertainty Principle

The Heisenberg uncertainty relation asserts that we can measure the position and momentum of a particle separately with arbitrary precision. So if there is a absolute minimal value for the results of the measurements, the Heisenberg uncertainty relation should be modified. Here we consider a generalized uncertainty principle which implies a minimum observable length

$$\Delta X \Delta P \ge \frac{\hbar}{2} \left( 1 + \beta \left[ (\Delta P)^2 + \langle P \rangle^2 \right] \right), \tag{2}$$

where  $\beta$  is the GUP parameter. We also have  $\beta = \beta_0/(M_{Pl}c)^2$  where  $M_{Pl}$  is the Planck mass and  $\beta_0$  is of the order of unity. GUPs of this type also arise from polymer quantization [31–33]. Note that the deviation from the Heisenberg picture takes place in the high energy domain where the effects of gravity would be significant. Thus, for the energies much lower than the Planck energy  $M_{Pl}c^2 \sim 10^{19}$  GeV, we recover the well-known Heisenberg uncertainty relation. It is easy to check that the above inequality relation (2) implies the existence of an absolute minimum observable length given by  $(\Delta X)_{min} = \hbar \sqrt{\beta}$ . In the context of string theory, we can interpret this length as the length of the string and conclude that the string's length is proportional to the square root of the GUP parameter. In one spatial dimension, the above uncertainty relation can be obtained from the following deformed commutation relation

$$[X, P] = i\hbar(1 + \beta P^2). \tag{3}$$

As it is recently suggested in Ref. [28, 34], we can write X and P in terms of ordinary position and momentum operator as

$$X = x, (4)$$

$$P = \frac{\tan\left(\sqrt{\beta}p\right)}{\sqrt{\beta}},\tag{5}$$

where  $[x,p]=i\hbar$  and X and P are symmetric operators on the dense domain  $S_{\infty}$  with respect to the following scalar product:

$$\langle \psi | \phi \rangle = \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} dp \, \psi^*(p) \phi(p). \tag{6}$$

With this definition, the commutation relation (3) is exactly satisfied. In this representation, the completeness relation and scalar product can be written as

$$\langle p'|p\rangle = \delta(p-p'),\tag{7}$$

$$\int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} dp \, |p\rangle\langle p| = 1. \tag{8}$$

Also the eigenfunctions of the position operator in momentum space is given by the solutions of the eigenvalue equation

$$X u_x(p) = x u_x(p), (9)$$

where  $u_x(p) = \langle p|x\rangle$ . The normalized solution is

$$u_x(p) = \sqrt{\frac{\sqrt{\beta}}{\pi}} \exp\left(-i\frac{p}{\hbar}x\right),$$
 (10)

which can be used to check the scalar product relation (7). Now using (8) we find the wave function in coordinate space as

$$\psi(x) = \sqrt{\frac{\sqrt{\beta}}{\pi}} \int_{-\frac{\pi}{2\sqrt{\beta}}}^{+\frac{\pi}{2\sqrt{\beta}}} e^{\frac{ipx}{\hbar}} \phi(p) dp.$$
 (11)

Note that this representation is equivalent with the seminal proposal by Kempf, Mangano and Mann (KMM) [27]

$$X = (1 + \beta p^2)x, \tag{12}$$

$$P = p, (13)$$

through the following canonical transformation:

$$X \longrightarrow \left[1 + \arctan^2\left(\sqrt{\beta}P\right)\right] X,$$
 (14)

$$P \longrightarrow \arctan\left(\sqrt{\beta}P\right)/\sqrt{\beta},$$
 (15)

which transforms Eqs. (12) and (13) to Eqs. (4) and (5) subjected to Eq. (3). This representation (4-5) preserves the ordinary nature of the position operator and only affects the kinetic part of the Hamiltonian.

Similar to KMM representation the momentum operator P is self-adjoint, while the position operator X is merely symmetric. This is due to the fact that the domain of  $X^{\dagger}$  is much larger than the domain of X. However, this representation is formally self-adjoint, i.e.,  $A = A^{\dagger}$  for  $A \in \{X, P\}$  (see [28] for details). Now P and p can be interpreted as follows: p is the momentum operator at low energies ( $p = -i\hbar\partial/\partial x$ ) and P is the momentum operator at high energies. Obviously, this procedure affects all Hamiltonians in the quantum mechanics, namely

$$H = \frac{P^2}{2m} + V(X),\tag{16}$$

where using Eqs. (4) and (5) can be rewritten as

$$H = \frac{\tan^2\left(\sqrt{\beta}p\right)}{2\beta m} + V(x). \tag{17}$$

Since this Hamiltonian is formally self-adjoint  $(H^{\dagger} = H)$ , we can use the general scheme of the coherent states to find the desirable solutions, whereas this property is absent in other representations [27, 35]. In the quantum domain, this Hamiltonian results in the following generalized Schrödinger equation in coordinate space

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)B_{2n}\hbar^{2(n-1)}\beta^{n-2}}{2m(2n)!} \frac{\partial^{2(n-1)}\psi(x)}{\partial x^{2(n-1)}} + V(x)\psi(x) = E\psi(x), \tag{18}$$

where  $B_n$  is the *n*th Bernoulli number.

## 3 GUP and the Harmonic oscillator

For the harmonic oscillator, because of the quadratic form of the potential  $V(x) = (1/2) m\omega^2 x^2$ , we obtain a second-order differential equation in the momentum space, namely [28]

$$-\frac{\partial^2 \phi(p)}{\partial p^2} + \frac{\tan^2 \left(\sqrt{\gamma}p\right)}{\gamma}\phi(p) = \epsilon \phi(p), \tag{19}$$

where  $p \to \sqrt{m\hbar\omega} p$ ,  $\gamma = m\hbar\omega\beta$ , and  $\epsilon = \frac{2E}{\hbar\omega}$ . In terms of the new variable  $y = \sqrt{\gamma}p$ , it reads

$$\left[ -\frac{\partial^2}{\partial y^2} + \nu(\nu - 1)\tan^2(y) - \bar{\epsilon}(\nu) \right] \phi(y; \nu) = 0, \tag{20}$$

where by definition

$$\nu = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4}{\gamma^2}} \right), \qquad \bar{\epsilon} = \frac{\epsilon}{\gamma}, \tag{21}$$

and the boundary condition is

$$\phi(y;\nu)\Big|_{y=\pm\pi/2} = 0. \tag{22}$$

The above differential equation is exactly solvable and the eigenfunctions can be obtained in terms of the Gauss hypergeometric functions [28]

$$\phi_{2k}(p;\gamma) = \mathcal{A}_k(\nu) \left[\cos(\sqrt{\gamma}p)\right]^{\left(1+\sqrt{1+\frac{4}{\gamma^2}}\right)/2}$$

$$\times {}_2F_1\left(-k,\nu+k;\nu+\frac{1}{2};\cos^2(\sqrt{\gamma}p)\right), \quad k=0,1,2,\dots,$$
(23)

for even states and

$$\phi_{2k+1}(p;\gamma) = \mathcal{B}_{k}(\nu)\sin(\sqrt{\gamma}p)\left[\cos(\sqrt{\gamma}p)\right]^{\left(1+\sqrt{1+\frac{4}{\gamma^{2}}}\right)/2} \times {}_{2}F_{1}\left(-k,\nu+k+1;\nu+\frac{1}{2};\cos^{2}(\sqrt{\gamma}p)\right), \quad k=0,1,2,\dots,$$
 (24)

for odd states. Moreover, the exact GUP-corrected energy spectrum is given by

$$\epsilon_n = (2n+1)\left(\sqrt{1+\frac{\gamma^2}{4}} + \frac{\gamma}{2}\right) + \gamma n^2, \qquad n = 0, 1, 2, \dots$$
(25)

The generalization of this problem to arbitrary dimension is also discussed in Ref. [35].

### 4 The Generalized Coherent States

Coherent states were originally introduced by Schrödinger in 1926 [36] have applications in many areas of physics [37–42]. To construct coherent states, we follow Klauder's approach [43,44] and use the version of the generalized Heisenberg algebra (GHA) [45–49] given in Refs. [48,49]. This version of GHA consists of the generators  $J_0$ , A, and  $A^{\dagger}$  that satisfy [48]

$$J_0 A^{\dagger} = A^{\dagger} f(J_0), \tag{26}$$

$$AJ_0 = f(J_0)A, (27)$$

$$[A^{\dagger}, A] = J_0 - f(J_0).$$
 (28)

Here  $A = (A^{\dagger})^{\dagger}$  and  $J_0 = J_0^{\dagger}$  is the Hamiltonian of the system. Moreover,  $f(J_0)$  is an analytic function of  $J_0$  and is called the characteristic function of the algebra. These generators also satisfy

$$J_0|m\rangle = \alpha_m|m\rangle, \tag{29}$$

$$A^{\dagger}|m\rangle = N_m|m+1\rangle, \tag{30}$$

$$A|m\rangle = N_{m-1}|m-1\rangle, \tag{31}$$

where  $N_m^2 = \alpha_{m+1} - \alpha_0$ . For instance, for the linear function f(x) = x + 1, we obtain the harmonic oscillator algebra, and for f(x) = qx + 1, the algebra in Eqs. (26)-(28) becomes the deformed Heisenberg algebra [48]. It is shown that the quantum systems having the energy spectrum

$$\epsilon_{n+1} = f(\epsilon_n), \tag{32}$$

where  $\epsilon_n$  and  $\epsilon_{n+1}$  are successive energy levels and f(x) is a distinct function for each physical system, are described by these generalized Heisenberg algebras. Also, This function is exactly the same function that appears in the construction of the algebra, i.e., the characteristic function of the algebra. In the algebraic language,  $J_0$  is the Hamiltonian, A is the annihilation operator, and  $A^{\dagger}$  is the creation operator. These operators are related to the Casimir operator of the GHA through the relation

$$C = A^{\dagger} A - J_0 = A A^{\dagger} - f(J_0). \tag{33}$$

Now the coherent states are given by [49]

$$|z\rangle = N(z) \sum_{n=0}^{\infty} \frac{z^n}{N_{n-1}!} |n\rangle, \tag{34}$$

where  $A|z\rangle = z|z\rangle$ , N(z) is the normalization coefficient, by definition  $N_n! \equiv N_0 N_1 \cdots N_n$ , and by consistency  $N_{-1}! \equiv 1$ . Note that Klauder's coherent states should satisfy the following minimal set of conditions [49]:

#### 1. normalizability condition

$$\langle z|z\rangle = 1. \tag{35}$$

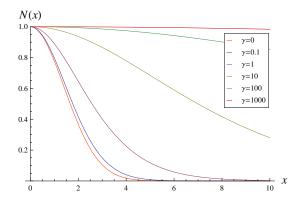


Figure 1: Normalization function for the GUP-corrected harmonic oscillator and for  $\gamma = \{0, 0.1, 1, 10, 100, 1000\}.$ 

2. continuity in the label

$$|z - z'| \to 0,$$
  $|||z\rangle - |z'\rangle|| \to 0.$  (36)

3. completeness relation

$$\int d^2z \, w(z) \, |z\rangle\langle z| = 1. \tag{37}$$

For the harmonic oscillator with the minimal length uncertainty, using Eq. (25) we obtain

$$\epsilon_{n+1} = \epsilon_n + 2\gamma(n+1) + \sqrt{4+\gamma^2},$$

$$= \epsilon_n + 2\sqrt{\gamma\epsilon_n + 1} + \gamma.$$
(38)

Thus, the characteristic function reads

$$f(x) = x + 2\sqrt{\gamma x + 1} + \gamma. \tag{39}$$

Since the above algebraic formalism implies  $\alpha_n = \epsilon_n$ , we find

$$N_{n-1}^{2} = \alpha_{n} - \alpha_{0} = \gamma \left[ n^{2} + \left( \sqrt{1 + \frac{4}{\gamma^{2}}} + 1 \right) n \right],$$

$$= \gamma (n + 2\nu) n, \tag{40}$$

which results in

$$N_{n-1}! = \gamma^{n/2} \sqrt{\frac{n!(n+2\nu)!}{(2\nu)!}}.$$
(41)

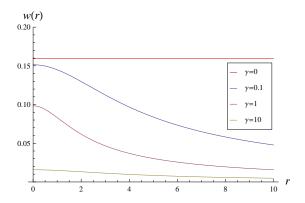


Figure 2: Weight function for the GUP-corrected harmonic oscillator and for  $\gamma = \{0, 0.1, 1, 10\}$ .

So the coherent states given in Eq. (34) can be written as

$$|z\rangle = N(|z|)\sqrt{(2\nu)!} \sum_{n=0}^{\infty} \frac{\gamma^{-n/2} z^n}{\sqrt{n!}\sqrt{(n+2\nu)!}} |n\rangle.$$

$$(42)$$

The normalizability condition is given by

$$1 = N^{2}(|z|)(2\nu)! \sum_{n=0}^{\infty} \frac{\gamma^{-n}|z|^{2n}}{n!(n+2\nu)!}.$$
 (43)

Since we have

$$\sum_{n=0}^{\infty} \frac{\gamma^{-n}|z|^{2n}}{n!(n+2\nu)!} = \frac{I_{2\nu}\left(\frac{2|z|}{\sqrt{\gamma}}\right)}{\gamma^{-\nu}|z|^{2\nu}},\tag{44}$$

where  $I_p(z)$  is the modified Bessel function of the first kind of order p and  $0 \le |z| < \infty$ , the normalizability condition reads

$$N^{2}(|z|) = \frac{\gamma^{-\nu}|z|^{2\nu}}{(2\nu)! I_{2\nu}\left(\frac{2|z|}{\sqrt{\gamma}}\right)}.$$
 (45)

In Fig. 1 we have depicted N(x) for various values of  $\gamma$ . Note that for  $\gamma \to 0$  (harmonic oscillator without GUP), N(x) goes to  $e^{-x^2/4}$  and for  $\gamma \to \infty$  ( $\nu \to 1$ ) (particle in a box), it tends to  $\frac{x^2/\gamma}{2I_2(2x/\sqrt{\gamma})}$ .

To satisfy the completeness relation, we need to find the adequate weight function w(r),  $z = re^{i\theta}$ , implying the equality

$$2\pi \sum_{n=0}^{\infty} |n\rangle \langle n| \frac{(2\nu)! \, \gamma^{-n}}{n!(n+2\nu)!} \int_0^{\infty} \mathrm{d}r \, N^2(r) w(r) r^{2n+1} = 1.$$
 (46)

If we take  $x = r^2$ , we obtain

$$\pi(2\nu)! \sum_{n=0}^{\infty} |n\rangle \langle n| \frac{2\gamma^{-n-\nu-1}}{n!(n+2\nu)!} \int_{0}^{\infty} dx \, \frac{\gamma^{\nu+1} N^{2}(\sqrt{x}) w(\sqrt{x})}{2x^{\nu}} x^{n+\nu} = 1.$$
 (47)

So by taking

$$\frac{\pi(2\nu)! \, \gamma^{\nu+1} N^2(\sqrt{x}) w(\sqrt{x})}{2x^{\nu}} = K_{2\nu} \left( 2\sqrt{\frac{x}{\gamma}} \right),\tag{48}$$

where  $K_p(x)$  is the modified Bessel function of the second kind of order p and using

$$\int_0^\infty dx \, K_{2\nu} \left( 2\sqrt{\frac{x}{\gamma}} \right) x^{n+\nu} = \frac{1}{2} \gamma^{n+\nu+1} n! (n+2\nu)!, \tag{49}$$

Eq. (47) is satisfied which gives the following weight function

$$w(\sqrt{x}) = \frac{2x^{\nu} K_{2\nu}(2\sqrt{x})}{\pi(2\nu)! \, \gamma^{\nu+1} N^2(\sqrt{x})},\tag{50}$$

where can be finally expressed as

$$w(r) = \frac{2}{\pi \gamma} I_{2\nu} \left(\frac{2r}{\sqrt{\gamma}}\right) K_{2\nu} \left(\frac{2r}{\sqrt{\gamma}}\right). \tag{51}$$

In Fig. 2, the behavior of the weight function is shown for  $\gamma = \{0, 0.1, 1, 10\}$ . As we have expected, for  $\gamma \to 0$  (harmonic oscillator without GUP), w(r) tends to  $\frac{1}{2\pi}$  and for  $\gamma \to \infty$  ( $\nu \to 1$ ) (particle in a box), it goes to  $\frac{2}{\pi\gamma}I_2\left(\frac{2r}{\sqrt{\gamma}}\right)K_2\left(\frac{2r}{\sqrt{\gamma}}\right)$ . It is worth to mention that a potential application of our results is in quantum optics. Indeed the coherent states as the states of the light field can be used to approximately describe the output of a single-frequency laser well above the laser threshold. In the absence of GUP, the probability of detecting n photons is given by Poisson distribution, namely

$$P(n;\lambda) = |\langle n|z\rangle|^2 = \frac{e^{-\lambda/2}}{n!} \left(\frac{\lambda}{2}\right)^n, \tag{52}$$

where  $\lambda = |z|^2 = \langle z|A^{\dagger}A|z\rangle$  and  $\lambda/2$  is the average photon number  $\overline{n}$  in a coherent state for  $\gamma = 0$   $(\epsilon_n = 2n + 1)$ .<sup>1</sup> In the presence of the minimal length, the probability distribution is

$$P(n; \lambda, \gamma) = |\langle n|z\rangle|^2 = N^2(\sqrt{\lambda}) \frac{\lambda^n}{(N_{n-1}!)^2},$$
(53)

$$= \frac{\gamma^{-n-\nu}}{I_{2\nu} \left(2\sqrt{\frac{\lambda}{\gamma}}\right)} \frac{\lambda^{n+\nu}}{n!(n+2\nu)!},\tag{54}$$

<sup>&</sup>lt;sup>1</sup>In terms of  $\overline{n}$  we have  $P(n) = \frac{e^{-\overline{n}} \overline{n}^n}{n!}$ 

which encodes the effects of GUP on the harmonic oscillator statistics and satisfies

$$\sum_{n=0}^{\infty} P(n; \lambda, \gamma) = 1. \tag{55}$$

Also we have

$$\frac{\lambda}{\gamma} = \overline{n^2} + 2\nu\overline{n},\tag{56}$$

which results in

$$P(n; \overline{n}, \nu) = \frac{\left(\overline{n^2} + 2\nu\overline{n}\right)^{n+\nu}}{I_{2\nu} \left(2\sqrt{\overline{n^2} + 2\nu\overline{n}}\right) n!(n+2\nu)!}.$$
 (57)

For  $\gamma \to 0$ , we have  $N^2(\sqrt{\lambda}) = e^{-\lambda/2}$  (see Fig. 1) and using

$$\lim_{\nu \to \infty} \frac{\nu^{-n} (n + 2\nu)!}{(2\nu)!} = 2^n, \tag{58}$$

Eq. (41) reads

$$N_{n-1}! \simeq \sqrt{2^n \, n!} \ . \tag{59}$$

So Eq. (53) gives the Poisson distribution (52) at this limit. For  $\gamma \to \infty$  ( $\nu \to 1$ ), Eq. (54) is expressed as

$$P(n; \lambda, \gamma \to \infty) = \frac{\gamma^{-n-1}}{I_2\left(2\sqrt{\frac{\lambda}{\gamma}}\right)} \frac{\lambda^{n+1}}{n!(n+2)!},\tag{60}$$

where  $\frac{\lambda}{\gamma} \simeq \overline{n^2} + 2\overline{n}$ . Fig. 3 shows the schematic behavior of the probability distribution for the GUP-corrected harmonic oscillator and for various values of the GUP parameter. As the figure shows,  $n_{max}$  which gives the maximum probability, decreases as  $\gamma$  increases and is given by the root of the following equation:

$$H_{n_{max}} + H_{n_{max}}^{(2\nu)} + \ln\frac{\gamma}{\lambda} - 2\xi = 0,$$
 (61)

where  $H_p$  is pth harmonic number,  $H_p^{(r)}$  is pth harmonic number of order r, and  $\xi$  is Euler's constant. Note that Eqs. (54) and (57) define the probability of detecting n photons in a laser beam subject to

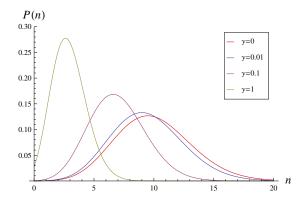


Figure 3: The probability distribution for the GUP-corrected harmonic oscillator. We set  $\lambda = 20$  ( $\overline{n} = 10$  for  $\beta = 0$ ) and  $\gamma = \{0, 0.01, 0.1, 1\}$ .

(56) which is a result of the deformed generalized uncertainty relation (expected to be significant at very high energies where gravity is considerable). Therefore, for a fixed  $\lambda$ , the averaged number of photons decreases as the GUP parameter increases.

Now we can define the entropy of this system as the logarithmic measure of the density of states:

$$S(\lambda; \gamma) = -k_B \sum_{n=0}^{\infty} P(n; \lambda, \gamma) \ln P(n; \lambda, \gamma), \tag{62}$$

where  $k_B$  is the Boltzmann constant. In Fig. 4, we have depicted the entropy of the GUP-corrected harmonic oscillator for  $\gamma = \{0, 0.1, 1\}$ . As the figure shows, for fixed  $\lambda$ , the entropy decreases as  $\gamma$  increases and tends to the Poisson entropy for  $\gamma \to 0$ 

$$S(\lambda;0) = k_B \left[ \frac{\lambda}{2} \left( 1 - \ln \frac{\lambda}{2} \right) + e^{-\lambda/2} \sum_{n=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^n \ln n!}{n!} \right].$$
 (63)

The reason behind this behavior can be understood from the GUP commutation relation (3). Indeed, we can consider the right hand side of this equation effectively as the GUP-corrected Planck's constant which is always greater than  $\hbar$ . Therefore, in the language of the statistical mechanics, the size of the unit cell in the phase space increases and consequently the number of the accessible states and the entropy of the system decrease with respect to the absence of GUP.

As we state before, a potential application of our calculations is in quantum optics. But the question is: could the relation between quantum optics and Plank-scale uncertainty relations have some detectable

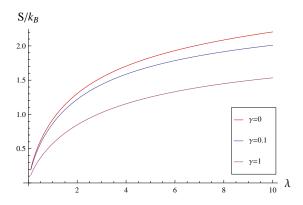


Figure 4: The entropy of the GUP-corrected harmonic oscillator for  $\gamma = \{0, 0.1, 1\}$ .

effects? To answer this question we should mention that in recent years various approaches are developed to test the effects of quantum gravity and to explore possible quantum gravitational phenomena. These attempts range from astronomical observations [50,51] to table-top experiments [52]. Amelino-Camelia and Lammerzahl proposed some laser interferometric setups to explain puzzling observations of ultrahigh energy cosmic rays in the context of quantum gravity modified laws of particle propagation [53]. The implications of high intensity Laser projects for quantum gravity phenomenology are also discussed by Magueijo based on deformed special relativity [54].

Recently, Pikovski et al. have introduced a scheme to experimentally test the existence of a minimal length scale as a modification of the Heisenberg uncertainty relation for various GUP scenarios in the context of quantum optics [52]. They utilized quantum optical control to probe possible deviations from the quantum commutation relation at the Planck scale and showed that their scheme is within reach of current technology. In fact, the idea is direct measurement of the canonical commutator of a massive object using a quantum optical ancillary system that produces nonlinear enhancement without need for Planck-scale accuracy of position measurements. So the possible Planck-scale commutator deformations can be observed with very high accuracy by optical interferometric techniques that are within experimental reach. These experimental progresses support the implication of our calculations which could shed light on possible detectable Planck-scale effects with quantum optics.

# 5 Conclusions

In this paper, we have studied the construction of the coherent states of the harmonic oscillator in the context of the generalized uncertainty principle which implies a minimal length uncertainty proportional to the Planck length. Following Klauder's approach, we constructed the generalized Heisenberg algebra where the Hamiltonian was its formally self-adjoint generator. Then, after finding the characteristic function of the algebra, we obtained the exact expression for the coherent states, weight functions, normalization coefficients, and probability distributions and studied their behavior in terms of the GUP parameter. We showed that because of the gravitational uncertainty relation the ordinary quantum description of laser light should be modified and the Poisson probability distribution will not be exactly preserved. Also, the entropy of the system decreased in the presence of the minimal length uncertainty due to the increase of the size of the unit cell in the phase space. Finally, we indicated that recent progresses to experimentally test the existence of a minimal length scale could reveal some evidence of possible quantum gravitational effects.

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