Finite Field Theory on Noncommutative Geometries

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June 20, 2021

Abstract

The propagator is calculated on a noncommutative version of the flat plane and the Lobachevsky plane with and without an extra (euclidean) time parameter. In agreement with the general idea of noncommutative geometry it is found that the limit when the two 'points' coincide is finite and diverges only when the geometry becomes commutative. The flat 4-dimensional case is also considered. This is at the moment less interesting since there has been no curved case developed with which it can be compared.

LMU-TPW 99-06 MPI-PhT/99-12

1 Introduction and motivation

It was postulated some time ago [38, 39] that a noncommutative structure at small length scales could introduce an effective cut-off in field theory similar to a lattice but at the same time maintain Lorentz invariance. Recently there has been a revival of this idea and several new examples [27, 12, 20, 2, 23, 24] have been studied. Models [15, 22, 21, 5] in '1-dimension' have also added to our understanding of the 'lattice' structure. The basic idea is simple and can be illustrated by a classical particle moving in a plane, described by two position coordinates (q^1, q^2) and two momentum coordinates (p_1, p_2) . In the language of quantum mechanics these four classical coordinates are commuting operators. In the presence of a magnetic field *B* normal to the plane the momentum operators are modified and they cease to commute:

$$[p_1, p_2] = i\hbar eB. \tag{1.1}$$

This introduces a cellular structure in the momentum plane. It becomes divided into Landau cells of area proportional to $\hbar eB$. Consider in this case the divergent integral

$$I = \int \frac{dp_1 dp_2}{p^2}.$$

The commutation relation (1.1) does not permit p_1 and p_2 simultaneously to take the eigenvalue zero and the operator $p^2 = p_1^2 + p_2^2$ is bounded below by $\hbar eB$. The magnetic field acts as an infrared cut-off. If the position space were curved, with constant Gaussian curvature K one would obtain again an infrared regularization for I. In an exactly analogous fashion, to obtain an ultraviolet regularization one must replace the coordinates of position space by two operators which do not commute:

$$[q^1, q^2] = i\hbar q^{12}. (1.2)$$

By the new uncertainty relation there is no longer a notion of a point in position space since one cannot measure both coordinates simultaneously but as before, position space can be thought of as divided into Planck cells. It has become fuzzy. This cellular structure serves as an ultraviolet cut-off similar to a lattice structure. If we consider for example the divergent integral I and introduce also a Gaussian curvature we find

$$I \sim \log(kK). \tag{1.3}$$

The integral has become completely regularized. There is however now a new complication; the right-hand side of (1.3) seems not to depend on the operator q^{12} . We have argued elsewhere [28] that, endowed with an appropriate differential structure, each fuzzy spacetime supports a uniquely determined gravitational field and that the latter is a classical manifestation of the commutation relations plus a differential structure. From this point of view what we put on the right-hand side of (1.2) will depend on which gravitational field we wish to regularize the integral with. That is, in fact K does depend on q^{12} .

In Section 2 we shall give a description of how the integral I of (1.3) is to be calculated in the case of a general algebra \mathcal{A} . The propagator is an element of the tensor product $\mathcal{H} \otimes \mathcal{H}$ of two copies of a Hilbert space $\mathcal{H} \subset \mathcal{A}$. We represent $\mathcal{A} \otimes \mathcal{A}$ as an algebra of operators on the tensor product $L^2(V, d\mu) \otimes L^2(V, d\mu)$ of two copies of another Hilbert space $L^2(V, d\mu)$ of functions on a manifold V, square integrable with respect to some measure $d\mu$. We then express $L^2(V, d\mu) \otimes L^2(V, d\mu)$ as the tensor product of a Hilbert space $\mathcal{D} \simeq L^2(V, d\mu)$, which represents the diagonal elements of $\mathcal{A} \otimes \mathcal{A}$, and an extra Hilbert space \mathcal{F} , which describes the off-diagonal expansion. This must be done in a way consistent with the commutation relations. Those of \mathcal{F} effectively force the distance from the diagonal in the tensor product to be 'quantized' and exclude the value zero. In the examples we shall see that if one were to interpret a given set of matrix elements of the propagator of the tensor product as a propagator on an ordinary space then it would appear to be associated to a nonlocal differential operator [45, 36]. In Section 3 we apply the formalism to the case of a noncommutative version [29] of \mathbb{R}^2 with a flat metric obtained by setting $q^{12} = 1$. In Section 4 we shall be interested in a noncommutative version [26, 7] of the Lobachevsky half-plane, the surface of constant negative Gaussian curvature. Finally in Section 5 we examine briefly the extension to dimension 4 and the problem of Lorentz invariance. In this paper we consider infinite-dimensional algebras. There are also models which are described by finite-dimensional algebras [28, 40] where the fact that the *n*-point elements are welldefined is automatic.

2 The general theory

In general consider any *-algebra \mathcal{A} with a trivial center, in some representation with a partial trace and let Δ be a linear operator on \mathcal{A} with a set of eigenvectors $\phi_r \in \mathcal{A}$ and corresponding real eigenvalues λ_r :

$$\Delta \phi_r = \lambda_r \phi_r.$$

The parameter r here designates a point in some parameter space and we write the integral on this space as a sum over r. The corresponding classical action is

$$S = \operatorname{Tr}(\phi^* \Delta \phi), \qquad \phi \in \mathcal{A}.$$
(2.1)

The trace here must be defined in some representation of \mathcal{A} . We shall assume that with respect to this trace

$$\operatorname{Tr}(\phi_r^*\phi_s) = \delta_{rs} \tag{2.2}$$

and we define the Hilbert space $\mathcal{H} \subset \mathcal{A}$ of 1-particle states to be

$$\mathcal{H} = \{ \phi = \sum_r a_r \phi_r : \sum_r |a_r|^2 < \infty \}.$$

As usual the a_r become operators when the field is quantized. For $f \in \mathcal{H}$ the completeness condition can be written as

$$\phi = \sum_{r} \phi_r \operatorname{Tr}(\phi_r^* \phi).$$

If we introduce the element

$$W = \sum_{r} \phi_r \otimes \phi_r^*$$

then the completeness condition can also be written

$$\operatorname{Tr}_2(W \cdot 1 \otimes \phi) = \phi \otimes 1.$$

The tensor product is here over the complex numbers and the subscript on the trace indicates that it is taken over the second factor. The element W is therefore the noncommutative

generalization of the Dirac distribution in the commutative case; it is not an element of $\mathcal{H} \otimes \mathcal{H}$. We introduce also the element G defined by the formal sum

$$G = \sum \lambda_r^{-1} \phi_r \otimes \phi_r^*.$$
(2.3)

Since obviously $\Delta G = W$ this element generalizes the propagator corresponding to Δ . We wish to discuss the conditions under which the sum converges and G can be considered as a well-defined element of a weak closure of $\mathcal{H} \otimes \mathcal{H}$.

It is possible to give a second formal definition of G using the noncommutative version of the euclidean path integral. Let $S[\phi, J] = S[\phi] + \operatorname{Tr}(J\phi)$ be the classical action of an interacting scalar field in the presence of an external source $J \in \mathcal{A}$. The term $S[\phi]$ would be a sum of the kinematical term (2.1) and an interaction term $S_J[\phi] = \operatorname{Tr}(V(\phi))$ with $V(\phi) \in \mathcal{A}$. Define the partition function Z[J] and generating functional W[J] by

$$Z[J] = \int d\phi e^{-S[\phi,J]} = e^{-W[J]}$$

If the algebra is for example a finite matrix algebra then this integral can be considered as well defined. Otherwise we consider it as a mnemonic trick. The theory is to be defined by the Gell-Mann-Low expansion of the *n*-point elements in terms of the propagator, with or without normal ordering. The *n*-point element $G_{(n)}$ is defined to be the functional derivative of W[J] with respect to J:

$$G_{(n)} = -\frac{\delta^n W[J]}{\delta J_1 \cdots \delta J_n}.$$

Here the J_i are different occurrences of J. They are all canonically equal to J but carry an extra index to distinguish them: $J_i = 1 \otimes \cdots \otimes J \otimes \cdots \otimes 1$ is an element of the *n*-fold tensor product of \mathcal{H} . By construction $G_{(n)}$ also is an element of the *n*-fold tensor product of \mathcal{H} . In particular we have

$$\langle \phi \rangle_J = Z[J]^{-1} \int d\phi \, \phi \, e^{-S[\phi,J]} = -Z[J]^{-1} \frac{\delta Z[J]}{\delta J} = \frac{\delta W[J]}{\delta J}$$

and

$$\langle \phi \otimes \phi \rangle_J = Z[J]^{-1} \int d\phi \, \phi \otimes \phi \, e^{-S[\phi,J]} = -\frac{\delta W[J]}{\delta J_1 \delta J_2}.$$

If $S[\phi, J]$ is the free action then $\langle \phi \otimes \phi \rangle_0$ is equal to the (bare) propagator G. The bracket is here the quantum bracket, which we distinguish with the index J. The context will indicate whether ϕ designates a quantum operator or a classical element of \mathcal{A} .

With our definitions a composite field like $\phi^n \in \mathcal{A}$ can appear in the interaction term $S_I[\phi]$ of the action but $\langle \phi^n \rangle_J$ is not defined. To define such objects we would, as in the commutative case [46], introduce an extra source $J_{(n)}$ in the path integral and a corresponding extra term $\operatorname{Tr}(\phi^n J_{(n)})$ in the action. One might be tempted to define for example $\langle \phi^2 \rangle_J$ as the image of $\langle \phi \otimes \phi \rangle_J$ under the multiplication map $\pi : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ but this will not be consistent in the classical limit. If one tries to define the expectation values of composite fields in terms of J one will come upon the same divergences [17, 12, 4] as in ordinary field theory. In the situations of interest the sum

$$\pi G = \sum_r \lambda_r^{-1} |\phi_r|^2.$$

diverges and it is not to be expected [42] that the noncommutativity of the algebra will alter this fact. We shall find finite results because the noncommutativity 'smears' the vertices, as it does points in general. By definition we have subtracted disconnected 'vacuum bubbles'. These could be singular; in the commutative limit they would be proportional to the volume of space-time. If the center of the algebra is not trivial one could still obtain a divergent result [20].

We shall restrict our attention to algebras which are generated by a set q^{μ} , $1 \leq \mu \leq n$, of *n* hermitian elements. Define $q^{\mu\nu} \in \mathcal{A}$ by

$$[q^{\mu},q^{\nu}]=i\hbar q^{\mu\nu}$$

where k is a parameter which one can suppose to be of the order of the square of the Planck length. This however is not necessary; the experimental bounds are much weaker. We shall suppose that \mathcal{A} is represented as an algebra of operators on a Hilbert space $L^2(V, d\mu)$ and we fix an orthonormal basis $|i\rangle$. We can write then

$$q^{\mu}\left|i\right\rangle = \sum_{j} Q_{ji}^{\mu}\left|j\right\rangle$$

for some set of n matrices Q_{ij}^{μ} . If the algebra is commutative then $Q_{ij}^{\mu} = q_i^{\mu} \delta_{ij}$. As above, the symbol Σ here can represent a sum or an integral depending on the basis $|i\rangle$ it is convenient to choose. The index i belongs again to some parameter space which of course is not to be confused with the space to which the parameters r and s of (2.2) belong. The symbol δ_{ij} can represent therefore the Kronecker or Dirac delta.

Consider the differential d_u of the universal calculus. It is a map of \mathcal{A} into $\mathcal{A} \otimes \mathcal{A}$ given by $d_u f = 1 \otimes f - f \otimes 1$. We define the 'variation' δq^{μ} of the generator q^{μ} as

$$\delta q^{\mu} = \frac{1}{2} d_{u} q^{\mu} = \frac{1}{2} (1 \otimes q^{\mu} - q^{\mu} \otimes 1).$$
(2.4)

We identify $q^{\mu} = q^{\mu} \otimes 1$ in the tensor product and we set $q^{\mu'} = 1 \otimes q^{\mu}$. Thus we can write

$$\delta q^{\mu} = \frac{1}{2} (q^{\mu\prime} - q^{\mu})$$

It follows from the commutation rules of the algebra that

$$[\delta q^{\mu}, \delta q^{\nu}] = \frac{1}{4} i \hbar (q^{\mu\nu} \otimes 1 + 1 \otimes q^{\mu\nu}).$$

Suppose that a set of elements \bar{q}^{μ} of $\mathcal{A} \otimes \mathcal{A}$ can be found such that $\mathcal{A} \otimes \mathcal{A}$ is generated by the set $\{\bar{q}^{\mu}, \delta q^{\mu}\}$ and such that

$$[\bar{q}^{\mu}, \delta q^{\nu}] = 0. \tag{2.5}$$

Then we can write the tensor product $L^2(V, d\mu) \otimes L^2(V, d\mu)$ in the form

$$L^2(V, d\mu) \otimes L^2(V, d\mu) \simeq \mathcal{D} \otimes \mathcal{F}$$
 (2.6)

where \bar{q}^{μ} acts on \mathcal{D} and δq^{μ} on \mathcal{F} . We shall choose accordingly a basis

$$|\bar{i},k\rangle = |\bar{i}\rangle_D \otimes |k\rangle_F$$

of $L^2(V, d\mu) \otimes L^2(V, d\mu)$. If $q^{\mu\nu}$ lies in the center of the algebra then the elements

$$\bar{q}^{\mu} = \frac{1}{2}(q^{\mu} + q^{\mu\prime})$$

are such that Equation (2.5) is satisfied. Further one has

$$q^{\mu} = \bar{q}^{\mu} - \delta q^{\mu}, \qquad q^{\mu\prime} = \bar{q}^{\mu} + \delta q^{\mu}$$

and with the obvious identifications

$$[\bar{q}^{\mu}, \bar{q}^{\nu}] = \frac{1}{2}i\bar{k}q^{\mu\nu}, \qquad [\delta q^{\mu}, \delta q^{\nu}] = \frac{1}{2}i\bar{k}q^{\mu\nu}.$$
(2.7)

The tensor product in the definition of G is now to be considered as a tensor product of a 'diagonal' algebra $\overline{\mathcal{A}}$, acting on \mathcal{D} and a 'variation' $\delta \mathcal{A}$, acting on \mathcal{F} . That is, we rewrite

$$\mathcal{A} \otimes \mathcal{A} = \bar{\mathcal{A}} \otimes \delta \mathcal{A} \tag{2.8}$$

in accordance with (2.6). If (2.5) is not satisfied the factorization (2.6) can still be of interest if δq^{μ} acts only on \mathcal{F} . In general then \bar{q}^{μ} will act non-trivially on the complete tensor product $\mathcal{D} \otimes \mathcal{F}$. We shall suppose that the definition (2.4) of δq^{μ} in terms of the tensor product coincides with the intuitive notion of the 'variation of a coordinate'. One can introduce a new differential calculus $(\bar{\Omega}^*(\mathcal{A}), \bar{d})$ defined by

$$\bar{d}\bar{q}^{\mu} = \delta q^{\mu}. \tag{2.9}$$

We shall see an example of this in Section 4. One would like this new calculus to be isomorphic to the original one if δq^{μ} and dq^{μ} are to be thought of as 'infinitesimal variations'.

Let $\mathcal{C}(M)$ be an algebra of functions on a space M. Let f be a map of M into itself and let f^* be the induced map of $\mathcal{C}(M)$ into itself. We set $\phi' = f^*(\phi)$ and define $\delta \phi = \phi' - \phi$. The ordinary propagator is a function of two points, an element of $\mathcal{C}(M) \otimes \mathcal{C}(M)$ and we are interested in the limit when the two points coincide. This limit must be taken with care since the partial derivative of a function after the limit and the limit of the derived function with respect to one of the variables are not in general equal. We are interested in the latter since the Laplace operator which defines the propagator acts only on one of the variables. If we set $\delta x = x' - x$ where x' = f(x) then we can express the limit $\delta x \to 0$ as $\delta \phi \to 0$. We wish to study the element $G(q^{\mu}; q^{\nu'})$ of the tensor product $\mathcal{H} \otimes \mathcal{H}$ most particularly in the limit $q^{\mu'} \to q^{\mu}$. The q^{μ} are however fixed generators of the algebra and this limit must be defined otherwise. As a possible added complication, which will however not appear explicitly in the examples we shall consider, the generators q^{μ} are in general unbounded operators. We shall give a formal definition of the limit as a weak limit within the tensor product in terms of variations of the basis vectors $|i\rangle$. We shall use a tensor product which is not braided. We shall return to his assumption later.

Using the representation of \mathcal{A} the propagator $G = G(q^{\mu}; q^{\nu})$ can be expressed as a map

$$G: L^2(V, d\mu) \otimes L^2(V, d\mu) \to L^2(V, d\mu) \otimes L^2(V, d\mu)$$

It can be defined in terms of its (classical) matrix elements $\langle j, j' | G(q^{\mu}; q^{\nu'}) | i, i' \rangle$. In the commutative limit $\hbar \to 0$ one would find

$$\langle j, j' | G(q^{\mu}; q^{\nu'}) | i, i' \rangle \to G(q^{\mu}; q^{\nu'}) \, \delta_{ij} \delta_{i'j'}$$

with

$$q^{\mu} \left| i \right\rangle = q_{i}^{\mu} \left| i \right\rangle, \qquad q^{\nu \prime} \left| i^{\prime} \right\rangle = q_{i^{\prime}}^{\nu \prime} \left| i^{\prime} \right\rangle$$

and so, at least in a quasicommutative approximation, we can identify q^{μ} with a point $i \in V = \mathbb{R}^n$ and $q^{\mu'}$ with $i' \in V = \mathbb{R}^n$. We shall therefore represent graphically $G(q^{\mu}; q^{\mu'})$ as a line between i and i':

$$\begin{array}{cccc}
j & j' \\
\circ & & \circ \\
i & i'
\end{array}$$
(2.10)

The extra pair of indices (j, j') is present because in general G acts as an operator on each end of the line. An ordinary propagator on a manifold diverges in the limit $q^{\mu'} \to q^{\mu}$. This limit can be redefined as the limit

$$|i'\rangle \rightarrow |i\rangle$$

This limit makes sense in the noncommutative case but it cannot be attained as we shall see below. We shall use therefore the identification (2.6) to express the limit as

$$|\bar{i},k\rangle \to |\bar{i},0\rangle \equiv |\bar{i}\rangle.$$
 (2.11)

In the graph (2.10) this means that the two ends of the line almost close to form a circle. It does not really follow that $|j\rangle$ and $|j'\rangle$ are related, except in the commutative limit. We shall however suppose that

$$|\bar{j},k\rangle \to |\bar{j},0\rangle \equiv |\bar{j}\rangle$$
 (2.12)

with (2.11).

It is here that the representation, especially the representation of the tensor product, becomes of importance. We shall describe the second copy \mathcal{F} of the Hilbert space using creation and annihilation operators. We choose then the basis $|k\rangle_F$ with $k \in \mathbb{Z}_+$. The states $|\bar{i}, 0\rangle$ are those in which collectively the operators δq^{μ} take their minimum value. If we introduce a distance s by

$$s^2 = g_{\mu\nu} \delta q^\mu \delta q^\nu$$

then we can define the coincidence limit as a state in \mathcal{F} on which s takes its minimum value. In the language of quantum mechanics such a state is an example of a coherent state.

We introduce a set of n annihilation operators a_l with their adjoints a_m^\ast such that, as in quantum mechanics

$$[a_l, a_m^*] = k \delta_{lm}. \tag{2.13}$$

We shall see that each a_l annihilates and each a_l^* creates a unit of separation. The quantum mechanical analogue of this separation would be the energy of the harmonic oscillator. By analogy then we define a diagonal state to be a state annihilated by all the a_l . We define as usual the action of a_l on the diagonal basis element $|\bar{i}, 0\rangle \in \mathcal{D} \otimes \mathcal{F}$ by the condition $a_l |\bar{i}, 0\rangle = 0$ and we set recursively

$$a_l^* |\bar{i}, k_1, \dots, k_l, \dots, k_n\rangle_F = \sqrt{k}\sqrt{k_l+1} |\bar{i}, k_1, \dots, k_l+1, \dots, k_n\rangle_F.$$

The coincidence limit is attained on elements of $L^2(V, d\mu) \otimes L^2(V, d\mu)$ of the form $|\bar{i}, 0\rangle$.

The analogue of the integral I defined in the Introduction is defined then by the equation

$$\langle \bar{j} | G(q^{\mu}; q^{\nu'}) | \bar{i} \rangle = \langle \bar{j} | I(\hbar \mu^2) | \bar{i} \rangle.$$

Here μ is a parameter in the operator Δ with the dimension of mass. In general $I(\hbar\mu^2)$ is an operator acting on \mathcal{D} . In all the examples we shall consider however the space is homogeneous and it reduces to a constant. We can write then

$$\langle \bar{j} | G(q^{\mu}; q^{\nu'}) | \bar{i} \rangle = I(\hbar \mu^2) \langle \bar{j} | \bar{i} \rangle.$$

We represent this by the graph obtained by joining the ends of (2.10) and placing a \bar{j} above and a \bar{i} below the circle which marks the join, as in the center of (2.20) below.

To calculate $\langle \bar{j} | G(q^{\mu}; q^{\mu}) | \bar{i} \rangle$ we must express G in terms of the a_l and their adjoints. For this we write

$$\delta q^{\mu} = \sum_{l=1}^{n} (J_l^{\mu} a_l + J_l^{\mu*} a_l^*)$$
(2.14)

and from (2.7) we conclude that

$$\sum_{l=1}^{n} J_l^{[\mu} J_l^{\nu]*} = \frac{1}{2} i q^{\mu\nu}.$$
(2.15)

The J_l^{μ} appear here as the components of a symplectomorphism. They are fixed only to within a redefinition of the a_l and contain therefore $2n^2 + n$ free parameters. This is the number of elements of $GL(2n, \mathbb{R})$ which leave invariant the right-hand side of (2.15). If we interpret δq^{μ} as a 'string' joining two 'points' q^{μ} and $q^{\mu'}$ then each a_j creates a longitudinal displacement. They would correspond to the rigid longitudinal vibrational modes of the string. Since it requires no energy to separate two points the string tension would be zero.

If the differential calculus $(\bar{\Omega}^*(\mathcal{A}), \bar{d})$ defined in (2.9) has a frame $\bar{\theta}^{\alpha} = \bar{\theta}^{\alpha}_{\lambda}(\bar{q}^{\mu}) \bar{d}\bar{q}^{\lambda}$ then it would seem more appropriate to expand the variation in the form

$$\bar{\theta}^{\alpha}_{\lambda}(\bar{q}^{\mu})\,\delta q^{\lambda} = \sum_{l=1}^{n} (j^{\alpha}_{l}a_{l} + j^{\alpha*}_{l}a^{*}_{l}).$$

$$(2.16)$$

We shall return to this Ansatz in Section 4. We are motivated here by the desire to make δq^{μ} as similar as possible to the element dq^{μ} of the differential calculus. This would suggest, in particular, that the condition (2.5) is fulfilled only if the geometry is flat.

The 'non-local' modification we shall find in the propagator is to be associated not with the propagator but rather with the vertices at its end points. To see this we consider now the matrix elements

$$\langle j, j' | G(q^{\mu}; q^{\rho'}) | i, i' \rangle \langle l', l | G(q^{\sigma'}; q^{\nu}) | k', k \rangle = \langle j | \otimes \langle j' | \otimes \langle l' | \otimes \langle l | G \otimes G | i \rangle \otimes | i' \rangle \otimes | k' \rangle \otimes | k \rangle$$

$$(2.17)$$

of the tensor product of two copies of the propagator, which we represent by the graph

To form a vertex we must 'join' the 'point' k' to the 'point' i'. Following the prescription (2.11) this means that we replace the basis element

$$|i'\rangle \otimes |k'\rangle \in L^2(V,d\mu) \otimes L^2(V,d\mu)$$

by the basis element

$$|\bar{i}'\rangle = |\bar{i}', 0\rangle \in \mathcal{D} \otimes \mathcal{F}.$$

We are prompted then to introduce the projection

$$L^{2}(V, d\mu) \otimes L^{2}(V, d\mu) \otimes L^{2}(V, d\mu) \otimes L^{2}(V, d\mu) \xrightarrow{P} L^{2}(V, d\mu) \otimes \mathcal{D} \otimes L^{2}(V, d\mu)$$

defined by

$$P = \sum_{r,\bar{r}',s} \, |r,\bar{r}',s\rangle \langle r,\bar{r}',s|$$

and to define the propagator $G_2(q^{\mu}, q^{\rho'}, q^{\nu})$ in terms of the matrix elements

$$\langle j, \bar{j}', l | G_2 | i, \bar{i}', k \rangle =$$

$$\sum_{r, \bar{r}', s} \langle j, \bar{j}', l | G \otimes (1 \otimes 1) | r, \bar{r}', s \rangle \langle r, \bar{r}', s | (1 \otimes 1) \otimes G | i, \bar{i}', k \rangle =$$

$$\sum_{r, \bar{r}', s} \langle j, \bar{j}' | G \otimes 1 | r, \bar{r}' \rangle \, \delta_{ls} \delta_{ri} \, \langle \bar{r}', s | 1 \otimes G | \bar{i}', k \rangle =$$

$$\sum_{\bar{r}'} \langle j, \bar{j}' | G \otimes 1 | i, \bar{r}' \rangle \langle \bar{r}', l | 1 \otimes G | \bar{i}', k \rangle \qquad (2.19)$$

which we represent by the graph

We could have also included the dummy multiplication index and written

We have used the identifications

$$G \otimes G = G \otimes (1 \otimes 1) \cdot (1 \otimes 1) \otimes G$$

and the fact that $G \otimes G$ acts on

$$\begin{pmatrix} L^2(V,d\mu) \otimes L^2(V,d\mu) \end{pmatrix} \otimes \left((L^2(V,d\mu) \otimes L^2(V,d\mu)) \right) = L^2(V,d\mu) \otimes \left(L^2(V,d\mu) \otimes L^2(V,d\mu) \right) \otimes L^2(V,d\mu) = L^2(V,d\mu) \otimes (\mathcal{D} \otimes \mathcal{F}) \otimes L^2(V,d\mu).$$

Since P projects $\mathcal{D} \otimes \mathcal{F}$ onto \mathcal{D} we see that

$$G_2: L^2(V, d\mu) \otimes \mathcal{D} \otimes L^2(V, d\mu) \to L^2(V, d\mu) \otimes \mathcal{D} \otimes L^2(V, d\mu).$$

In the commutative limit $k \to 0$ one would find

$$\langle j, \bar{j}', l | G_2 | i, \bar{i}', k \rangle \to G_2 \,\delta_{ij} \delta_{i'j'} \delta_{kl}$$

on the left-hand side of (2.19) and

$$\langle j, \bar{j}' | G \otimes 1 | i, \bar{r}' \rangle \to G \, \delta_{ij} \delta_{r'j'}$$

on the right-hand side. One would normally choose as basis the eigenvectors of the position operator so that $q^{\mu} |i\rangle = q_i^{\mu} |i\rangle$ and one would normally drop the extra index on q^{μ} . The preceeding two limits would be written then respectively

$$\langle j, \overline{j}', l | G_2 | i, \overline{i}', k \rangle \to G_2(q^\mu, q^{\rho'}, q^{\nu})$$

and

$$\langle j, \bar{j}' | G \otimes 1 | i, \bar{r}' \rangle \to G(q^{\mu}, q^{\rho'})$$

The graph (2.20) in turn can be cut into the two graphs

which represent respectively the factors

 $\langle j, \bar{j}' | G \otimes 1 | i, \bar{r}' \rangle, \qquad \langle \bar{r}', l | 1 \otimes G | \bar{i}', k \rangle.$

We are prompted by this to introduce also the graph

$$\begin{array}{cccc}
\bar{j} & \bar{l} \\
\bigcirc & & \\
\bar{i} & & \\
\bar{k}
\end{array}$$
(2.22)

to represent the matrix elements

$$\langle \bar{j}, \bar{l} | 1 \otimes G \otimes 1 | \bar{i}, \bar{k} \rangle.$$

This is the propagator with 'fuzzy' vertices. It is obtained by joining (i, j) to (k, l) in the graph (2.20) and cutting it as in (2.21). We designate it by \overline{G} :

$$\bar{G}: \mathcal{D} \otimes \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$$

If we replace the ends of (2.20) by fuzzy vertices we obtain the graph

$$\begin{array}{c} \bar{j} & \bar{j}' & \bar{l} \\ \supset & \bigcirc \\ \bar{i} & \bar{i}' & \frown \\ \bar{k} \end{array}$$

This is a 2-line vertex. We designate it by \bar{G}_2 :

$$\bar{G}_2: \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \to \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D}.$$

If we join the two ends we obtain a 2-line loop which we write also \bar{G}_2 but now

$$\bar{G}_2: \mathcal{D} \otimes \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}.$$

Normally one vertex will be considered as fixed. If we trace over the remaining one we shall use still the notation \bar{G}_2 . We give a simple example in the following section.

The theory can be readily extended to incorporate a tree-level n-line vertex. Consider as example a triple vertex. To pass from the equivalent of (2.18) to (2.20) we must replace (2.6) by the identification

$$L^{2}(V, d\mu) \otimes L^{2}(V, d\mu) \otimes L^{2}(V, d\mu) \simeq \mathcal{D} \otimes \mathcal{F} \otimes \mathcal{F}$$

which is obtained by introducing

$$\bar{q}^{\mu} = \frac{1}{3}(q^{\mu} \otimes 1 \otimes 1 + 1 \otimes q^{\mu} \otimes 1 + 1 \otimes 1 \otimes q^{\mu})$$

as well as two Fock spaces to describe the variations. Three lines are joined to a vertex by considering the tensor product of three propagators:

$$\langle j, j' | G | i, i' \rangle \langle l, l' | G | k, k' \rangle \langle n, n' | G | m, m' \rangle$$

and projecting the element

$$\begin{aligned} |i\rangle \otimes |i'\rangle \otimes |k\rangle \otimes |k'\rangle \otimes |m\rangle \otimes |m'\rangle \in \\ L^2(V,d\mu) \otimes L^2(V,d\mu) \otimes L^2(V,d\mu) \otimes L^2(V,d\mu) \otimes L^2(V,d\mu) \otimes L^2(V,d\mu) \end{aligned}$$

onto an element

$$|i\rangle \otimes |k\rangle \otimes |m\rangle \otimes |\bar{i}'\rangle \in L^2(V, d\mu) \otimes L^2(V, d\mu) \otimes L^2(V, d\mu) \otimes \mathcal{D}$$

The way this projection is defined will depend on which lines are to be considereed as incoming and which are outgoing. The above construction of joining and cutting would lead to the vextex defined by the matrix elements

$$\langle \bar{j}, \bar{l}, \bar{n} | G_3(q_1^{\mu}; q_2^{\mu}; q_3^{\mu}) | \bar{i}, \bar{j}, \bar{m} \rangle.$$

3 The noncommutative flat plane

The noncommutative flat plane is the algebra \mathcal{A}_{k} generated by two hermitian elements $q^{1} = x$ and $q^{2} = y$ which satisfy the commutation relation [x, y] = ik and which has over it the differential calculus $\Omega^{*}(\mathcal{A}_{k})$ given by $[q^{\mu}, dq^{\nu}] = 0$. If we introduce the two derivations

$$e_1 = -\frac{1}{ik} \operatorname{ad} y, \qquad e_2 = \frac{1}{ik} \operatorname{ad} x$$

dual to dq^{μ} then an appropriate generalization [29] of the Laplace operator Δ with mass μ is given by

$$\Delta = \Delta_{\pounds} + \mu^2, \qquad \Delta_{\pounds} = -(e_1^2 + e_2^2).$$

For each couple $(k_1, k_2) \in \mathbb{R}^2$ we introduce the unitary elements $u(k_1), v(k_2) \in \mathcal{A}_k$ defined by

$$u(k_1) = e^{ik_1x}, \qquad v(k_2) = e^{ik_2y}$$

They satisfy the commutation relations

$$u(k_1)v(k_2) = q^{k_1k_2k}v(k_2)u(k_1), \qquad q = e^{-i}$$

A basis for the Hilbert space \mathcal{H} is given by the eigenvectors

$$\phi_k = u(k_1)v(k_2), \qquad k = (k_1, k_2)$$

of Δ . The corresponding eigenvalues are

$$\lambda_k = k^2 + \mu^2, \qquad k^2 = k_1^2 + k_2^2.$$

The propagator can be written then

$$G(x,y;x',y') = \frac{1}{(2\pi)^2} \int (k^2 + \mu^2)^{-1} \phi'_k \otimes \phi^*_k dk, \qquad dk = dk_1 dk_2$$

We must introduce a partial trace on \mathcal{A}_{k} . This can be done only through a representation. The only properties which we shall need however are the identities

$$\operatorname{Tr}(u^*(k_1')u(k_1)) = 2\pi\delta(k_1'-k_1), \qquad \operatorname{Tr}(v^*(k_2')v(k_2)) = 2\pi\delta(k_2'-k_2).$$

That is:

$$Tr(\phi_{k'}^*\phi_k) = (2\pi)^2 \delta^{(2)}(k'-k).$$

The commutation relations (2.7) become in this case

$$[\bar{x},\bar{y}] = \frac{1}{2}i\hbar, \qquad [\delta x,\delta y] = \frac{1}{2}i\hbar.$$
(3.1)

As in (2.14) we write

$$\delta x = J^1 a + J^{1*} a^*, \qquad \delta y = J^2 a + J^{2*} a^*.$$
 (3.2)

With (2.13) satisfied we have $J^{[1}J^{2]*} = \frac{1}{2}iq^{12}$. By a redefinition of a we can choose

$$J^{1} = \frac{1}{2}, \qquad J^{2} = \frac{1}{2i}, \qquad a = \delta x + i\delta y$$

The freedom here is $SL(2, \mathbb{R})$, the symplectomorphism group in dimension 2. By a renormalization of k we can also choose $q^{12} = 1$.

We index the basis of $L^2(V, d\mu) = L^2(\mathbb{R}^2, dp)$ by $p = (p_1, p_2)$ and introduce the basis $|\bar{p}, k\rangle = |\bar{p}\rangle_D \otimes |k\rangle_F$ according to the prescription (2.6) of the previous section. We shall also re-express the tensor product according to (2.8) and drop the tensor-product symbol. We have then

$$u'^{*}(k_{1}) |\bar{p}'\rangle = e^{-ik_{1}x'} |\bar{p}'\rangle = e^{-ik_{1}(\bar{x}+\delta x)} |\bar{p}'\rangle.$$

Since \bar{x} and δx commute we can write this as

$$u^{\prime*}(k_1) |\bar{p}'\rangle = e^{-ik_1\bar{x}} e^{-ik_1(a+a^*)/2} |\bar{p}'\rangle.$$

Using the Baker-Campbell-Hausdorff (BaCH) formula

$$e^{\alpha a + \beta a^*} = e^{\beta a^*} e^{\alpha a} e^{\alpha \beta k/2} = e^{\alpha a} e^{\beta a^*} e^{-\alpha \beta k/2}$$

we find that

$$u^{\prime*}(k_1) \left| \bar{p}^{\prime} \right\rangle = e^{-ik_1 \bar{x}} e^{-k_1^2 k/8} e^{-ik_1 a^*/2} \left| \bar{p}^{\prime} \right\rangle$$

and therefore

$$\begin{split} \phi_k^{\prime*} \left| \bar{p}^{\prime} \right\rangle &= e^{-ik_2 y^{\prime}} e^{-ik_1 \bar{x}} e^{-k_1^2 \hbar/8} e^{-ik_1 a^*/2} \left| \bar{p}^{\prime} \right\rangle \\ &= e^{-ik_2 \bar{y}} e^{-ik_1 \bar{x}} e^{-\hbar k^2/8} e^{k_2 a^*/2} e^{-k_2 a/2} e^{-ik_1 a^*/2} \left| \bar{p}^{\prime} \right\rangle \\ &= e^{-ik_2 \bar{y}} e^{-ik_1 \bar{x}} e^{-\hbar k^2/8} e^{ik_1 k_2 \hbar/4} e^{(k_2 - ik_1) a^*/2} \left| \bar{p}^{\prime} \right\rangle. \end{split}$$

Similarly we find

$$\phi_k^* |\bar{p}\rangle = e^{-ik_2\bar{y}} e^{-ik_1\bar{x}} e^{-\bar{k}k^2/8} e^{ik_1k_2\bar{k}/4} e^{-(k_2-ik_1)a^*/2} |\bar{p}\rangle.$$

From these last two equations we deduce that

$$\langle \bar{p}' | \phi_k \otimes \phi_k^* | \bar{p} \rangle = e^{-kk^2/2} \langle \bar{p}' | \bar{p} \rangle.$$
(3.3)

The product here is the tensor product (2.8). Since the $\bar{\mathcal{A}}_k$ factor reduces in fact to the identity, the product depends only on the second factor $\delta \mathcal{A}_k$. We have dropped the prime on ϕ_k since the information is contained in the position in the tensor product.

The Fourier transform is the map

$$\tilde{\phi}(k) = \frac{1}{(2\pi)^2} \operatorname{Tr}(\phi_k^* \phi) \tag{3.4}$$

from \mathcal{H} to the momentum space $L^2(\mathbb{R}^2, dk)$ and the map

$$\phi = \int \phi_k \,\tilde{\phi}(k) dk = \int e^{ik_2 y} e^{ik_1 x} e^{-ik_1 k_2 k} \,\tilde{\phi}(k) dk \tag{3.5}$$

from $L^2(\mathbb{R}^2, dk)$ to \mathcal{H} . The Plancherel theorem is the completeness relation for the set of ϕ_k . We have the unitary map

$$\tilde{\phi}(l) = \frac{1}{(2\pi)^2} \operatorname{Tr}\left(\phi_l^* \int \tilde{\phi}(k) \,\phi_k \,dk\right)$$

from $L^2(\mathbb{R}^2, dk)$ onto itself and the unitary map

$$\phi \otimes 1 = \frac{1}{(2\pi)^2} \int \operatorname{Tr}_2(\phi_k \otimes \phi_k^* \cdot 1 \otimes \phi)$$

of $1 \otimes \mathcal{H}$ onto $\mathcal{H} \otimes 1$. Introduce

$$\tilde{D}_1 = \tilde{\partial}_1, \qquad \tilde{D}_2 = \tilde{\partial}_2 - ik_1\hbar, \qquad [\tilde{D}_1, \tilde{D}_2] = -i\hbar.$$

The multiplication by x and y are transformed respectively into the operators $i\tilde{D}_1$ and $i\tilde{D}_2$, which are self-adjoint on $L^2(\mathbb{R}^2, dk)$. The Fourier transform respects the commutation relations. The multiplication by $x \pm iy$ are transformed respectively into \tilde{b} and \tilde{b}^* where

$$\tilde{b} = i\tilde{\partial}_1 - \tilde{\partial}_2 + i\hbar k_1.$$

The asymmetry in the Fourier transform of the multiplication operators is due to our convention in the choice of basis ϕ_k . In the analogous calculations in the quantum Hall effect one would speak of a choice of gauge. If one introduces the 'gauge symmetric' operators

$$\tilde{b}' = e^{\hbar k^2/2} \, \tilde{b} \, e^{-\hbar k^2/2} = i \tilde{\partial}_1 - \tilde{\partial}_2 + \frac{1}{2} \hbar (ik_1 + k_2)$$
$$\tilde{b}'^* = e^{-\hbar k^2/2} \, \tilde{b}^* \, e^{\hbar k^2/2} = i \tilde{\partial}_1 + \tilde{\partial}_2 + \frac{1}{2} \hbar (ik_1 + k_2)$$

then \tilde{b}'^* is the adjoint of \tilde{b}' on $L^2(\mathbb{R}^2, e^{-kk^2}dk)$. This symmetric form emphasizes the role of the commutation relations in position space as a cut-off in momentum space.

The Fourier transform defines the map

$$\tilde{\phi}(k,k') = \frac{1}{(2\pi)^4} \operatorname{Tr}(\phi_k^* \otimes \phi_{k'}^* \phi \otimes \phi) = \frac{1}{(2\pi)^4} \operatorname{Tr}(\phi_k^* \phi) \operatorname{Tr}(\phi_{k'}^* \phi)$$

from $\mathcal{H}\otimes\mathcal{H}$ to $L^2(\mathbb{R}^2,dk)\otimes L^2(\mathbb{R}^2,dk)$ and the map

$$\phi \otimes \phi = \int \phi_k \otimes \phi_{k'} \, \tilde{\phi}(k,k') dk dk'$$

from $L^2(\mathbb{R}^2, dk) \otimes L^2(\mathbb{R}^2, dk)$ to $\mathcal{H} \otimes \mathcal{H}$. If we write $\phi \otimes \phi = \overline{\phi} \otimes \delta \phi$ as in (2.8) then (3.3) states that the Fourier transform of the diagonal factor of $\phi_k \otimes \phi_{k'}$ is a constant function and that the projection onto the ground-state in \mathcal{F} produces an exponential damping in momentum space.

We are now in a position to calculate the coincidence limit of the propagator. We have

$$\begin{split} \langle \bar{p}' | G(x, y; x', y') | \bar{p} \rangle &= \frac{1}{(2\pi^2)} \int (k^2 + \mu^2)^{-1} \langle \bar{p}' | \phi_k \otimes \phi_k^* | \bar{p} \rangle dk \\ &= \frac{1}{(2\pi^2)} \int \frac{e^{-kk^2/2}}{k^2 + \mu^2} \langle \bar{p}' | \bar{p} \rangle dk. \end{split}$$

The Feynman rules here are the same as the commutative ones except for an extra factor $e^{-kk^2/4}$ at each end of a propagator of momentum k to account for the projection onto the ground state in \mathcal{F} . We find then

$$\langle \bar{p} | G(x, y; x', y') | \bar{p} \rangle = I(\hbar \mu^2) \langle \bar{p} | \bar{p} \rangle$$

where $I(\hbar\mu^2)$ is given by the integral [36, 12, 4]

$$I(\hbar\mu^2) = \frac{1}{(2\pi)^2} \int \frac{e^{-\hbar k^2/2}}{k^2 + \mu^2} dk.$$
 (3.6)

With a change of variable it can be written as

$$I(\hbar\mu^2) = \frac{1}{4\pi} \int_0^\infty \frac{e^{-x}}{x + \hbar\mu^2/2} dx = -\frac{1}{4\pi} e^{\hbar\mu^2/2} \text{Ei}(-\hbar\mu^2/2).$$

where $\operatorname{Ei}(x)$ is the exponential-integral function. When $\hbar\mu^2 \to 0$ one finds

$$I(\hbar\mu^2) = \frac{1}{4\pi} \Big(-\log(\hbar\mu^2) + \log 2 - \gamma - \frac{1}{2}\hbar\mu^2\log(\hbar\mu^2) + o(\hbar\mu^2) \Big)$$

and when $\hbar\mu^2 \to \infty$,

$$I(\hbar\mu^2) = \frac{1}{2\pi\hbar\mu^2} + o((\hbar\mu^2)^{-2}).$$

As a further illustration of the modified Feynman rules, we calculate the 2–point function obtained by integrating over the internal vertex in

$$G_2(q^{\mu}; \bar{q}^{\rho\prime}; q^{\nu}) \in \mathcal{A}_k \otimes \bar{\mathcal{A}}_k \otimes \mathcal{A}_k,$$

represented by (2.20). In terms of the eigenfunctions of the Laplacian, the definition (2.19) of $G_2(q^{\mu}; \bar{q}^{\rho'}; q^{\nu})$ can be written as

$$G_2(q^{\mu};\bar{q}^{\rho\prime};q^{\nu}) = \frac{1}{(2\pi)^4} \int dk dl \lambda_k^{-1} \lambda_l^{-1} \phi_l \otimes {}_F \langle 0 | \phi_l^* \otimes \phi_k | 0 \rangle_F \otimes \phi_k^*.$$

The $_F\langle 0 | \phi_l^* \otimes \phi_k | 0 \rangle_F$ is the projection onto the ground state in \mathcal{F} . Integration over $\bar{q}^{\rho'}$ corresponds to taking the trace over $\bar{\mathcal{A}}_k$. Similarly to (3.3), it is straightforward to calculate

$$\operatorname{Tr}_{\bar{\mathcal{A}}_{\bar{k}}}(_{F}\langle 0| \phi_{l}^{*} \otimes \phi_{k} | 0 \rangle_{F}) = \operatorname{Tr}(e^{-il_{2}\bar{y}}e^{i(k_{1}-l_{1})\bar{x}}e^{ik_{2}\bar{y}}) _{F}\langle 0| e^{ik_{2}\delta y}e^{2ik_{1}\delta x}e^{ik_{2}\delta y} | 0 \rangle_{F}$$

$$= (2\pi)^{2}\delta^{(2)}(k-l)_{F}\langle 0| e^{ik_{2}\delta y}e^{2ik_{1}\delta x}e^{ik_{2}\delta y} | 0 \rangle_{F}$$

$$= (2\pi)^{2}\delta^{(2)}(k-l)e^{-\bar{k}k^{2}/2}.$$
(3.7)

Therefore

$$\operatorname{Tr}_{\bar{\mathcal{A}}_{k}}(G_{2}(q^{\mu};\bar{q}^{\rho};q^{\nu})) = (2\pi)^{-2} \int dk \frac{e^{-kk^{2}/2}}{(k^{2}+\mu^{2})^{2}} \phi_{k} \otimes \phi_{k}^{*}.$$
(3.8)

Again, we see that the Feynman rules are the same as in the commutative case, except for an extra factor $e^{-kk^2/4}$ at the end of a propagator of momentum k. Similarly for higher order vertices, the projection onto the ground state of the relative coordinates will lead to an exponential damping factor with length scale set by k, since the plane-wave factors as in (3.7) act as unitary operators which shift $|0\rangle_F$, reducing the overlap with $_F\langle 0|$.

As a second example consider a 2-line loop with no momentum flowing through it:

$$\begin{split} \langle \bar{p}\bar{p} | \,\bar{G}_2(q^{\mu};q^{\nu\prime}) \, | \bar{p}\bar{p} \rangle = \\ \int \langle \bar{p}\bar{p} | \,\bar{G}(q^{\mu};q^{\nu\prime}) \, | \bar{p}'\bar{p}' \rangle \langle \bar{p}',\bar{p}' | \,\bar{G}(q^{\mu};q^{\nu\prime}) \, | \bar{p},\bar{p} \rangle d\bar{p}'. \end{split}$$

If we set as before

$$\mu^2 \langle \bar{p}, \bar{p} | \bar{G}_2(q^\mu; q^{\nu\prime}) | \bar{p}, \bar{p} \rangle = I_2(\bar{k}\mu^2) \langle \bar{p}, \bar{p} | \bar{p}, \bar{p} \rangle$$

then we find that

$$I_2(\hbar\mu^2) = \frac{\mu^2}{(2\pi)^4} \int \frac{e^{-\hbar k^2}}{(k^2 + \mu^2)^2} dk = \frac{1}{32\pi^3} + \frac{\hbar\mu^2}{8\pi^3} e^{\hbar\mu^2} \operatorname{Ei}(-\hbar\mu^2).$$

When $\hbar \mu^2 \to 0$

$$I_2(\hbar\mu^2) = \frac{1}{16\pi^3} \Big(1 + \hbar\mu^2 \log(\hbar\mu^2) + \cdots \Big)$$

and when $\hbar \mu^2 \to \infty$

$$I_2(\hbar\mu^2) = \frac{1}{32\pi^3\hbar\mu^2} \Big(1 - \frac{2}{\hbar\mu} + \cdots \Big).$$

It is remarkable that this vanishes to the same order in $(\hbar\mu^2)^{-1}$ as $I(\hbar\mu^2)$ when $\hbar\mu^2 \to \infty$.

We have represented only the difference δq^{μ} in terms of annihilation and creation operators. It is possible to represent also \bar{q}^{μ} . We shall argue below that this is necessary on a curved noncommutative geometry. For this we introduce as well as *a* defined in (3.2) the operator $b = \bar{x} + i\bar{y}$. Then it is easy to see that the commutation relations (2.13) hold also for *b* and that *a* and *b* commute with each other and their adjoints. We define as usual the commuting number operators

$$N_a = \frac{1}{\hbar}a^*a, \qquad N_b = \frac{1}{\hbar}b^*b$$

and let $|n_a, n_b\rangle$ be their common eigenvectors. The equations which defined a and b can be inverted to yield

$$x \otimes 1 = \frac{1}{2}(-a - a^* + b + b^*), \quad y \otimes 1 = \frac{1}{2i}(-a + a^* + b - b^*),$$
$$1 \otimes x = \frac{1}{2}(a + a^* + b + b^*), \quad 1 \otimes y = \frac{1}{2i}(a - a^* + b - b^*)$$

and therefore we find

$$\begin{split} & u(k_1) = e^{ik_1(-a-a^*+b+b^*)/2}, \quad u'^*(k_1) = e^{-ik_1(a+a^*+b+b^*)/2}, \\ & v(k_2) = e^{k_2(-a+a^*+b-b^*)/2}, \quad v'^*(k_2) = e^{-k_2(+a-a^*+b-b^*)/2}. \end{split}$$

Now it is straightforward to show that Equation (3.3) can be written more generally as

$$\phi_k \otimes \phi_k^* = e^{-(k_2 + ik_1) a + (k_2 - ik_1) a^*}.$$

The *b*-terms cancel. Using the BaCH formula we find that

$$e^{-(k_2+ik_1)a+(k_2-ik_1)a^*} = e^{-kk^2/2}e^{(k_2-ik_1)a^*}e^{-(k_2+ik_1)a}$$

Thus we obtain

$$\langle m_a, m_b | \phi_k \otimes \phi_k^* | n_a, n_b \rangle$$

= $e^{-\hbar k^2/2} \langle m_a | e^{(k_2 - ik_1)a^*} e^{-(k_2 + ik_1)a} | n_a \rangle \delta_{m_b n_b}.$

¿From the expansion

$$e^{\alpha a} |n_a\rangle = |n_a\rangle + \alpha \sqrt{k} \sqrt{n_a} |n_a - 1\rangle + \dots + \frac{(\alpha \sqrt{k})^{n_a}}{n_a!} \sqrt{n_a!} |0\rangle.$$

it follows then that

$$\langle m_a | e^{(k_2 - ik_1)a^*} e^{-(k_2 + ik_1)a} | n_a \rangle$$

can be calculated for any two given states. We are especially interested in the case when $m_a = n_a$. In this case

$$\langle n_a, m_b | \phi_k \otimes \phi_k^* | n_a, n_b \rangle$$

= $e^{-kk^2/2} \Big(1 - kk^2 n_a + \dots + \frac{(-k)^{n_a} (k^2)^{n_a}}{(n_a!)^2} n_a! \Big) \delta_{m_b n_b}.$

The propagator is given therefore by

$$\langle n_a, m_b | G(x, y; x', y') | n_a, n_b \rangle$$

$$= \frac{1}{(2\pi)^2} \int \frac{e^{-kk^2/2}}{k^2 + \mu^2} \Big(1 - kk^2 n_a + \dots + \frac{(-k)^{n_a} (k^2)^{n_a}}{(n_a!)^2} n_a! \Big) dk \, \delta_{m_b n_b}.$$

This equation generalizes Equation (3.6) for $I(\hbar\mu^2)$ and reduces to it when $n_a = 0$.

A more elegant formulation can be given with a more explicit use [16, 22, 21, 4] of the coherent state formalism. Define

$$a = \frac{1}{\sqrt{2}}(x + iy) \otimes 1, \qquad b = \frac{1}{\sqrt{2}} \ 1 \otimes (x + iy).$$

Then again it is easy to see that the commutation relations (2.13) hold for a and b and that a and b commute with each other and their adjoints. Introduce, for $\tilde{x}, \tilde{y} \in \mathbb{R}$

$$z = \frac{1}{\sqrt{2\hbar}}(\tilde{x} + i\tilde{y}), \qquad T(z) = e^{za^* - \bar{z}a}$$

and similarly for b. Then the coherent states are given by $|z\rangle = T(z)|0\rangle$. It is straightforward to see that $a|z\rangle = \hbar z |z\rangle$ and that (x, y) are related to (\tilde{x}, \tilde{y}) by $\langle z|x|z\rangle = \tilde{x}$, $\langle z|y|z\rangle = \tilde{y}$.

We argued above that we can express the variations δx and δy using the tensor product of two copies of the algebra. Since

$$\begin{aligned} \langle z | \phi_k | z \rangle &= e^{-i\hbar k_1 k_2/2} e^{-\hbar k^2/4} e^{-i(k_1 \tilde{x} + k_2 \tilde{y})}, \\ \langle z' | \phi_k'^* | z' \rangle &= e^{i\hbar k_1 k_2/2} e^{-\hbar k^2/4} e^{i(k_1 \tilde{x}' + k_2 \tilde{y}')}, \end{aligned}$$

we obtain

$$\langle z, z' | \phi_k \otimes \phi_k^* | z, z' \rangle = e^{-kk^2/2} e^{ik_1(\tilde{x}' - \tilde{x}) + ik_2(\tilde{y}' - \tilde{y})},$$

and therefore

$$\langle z, z' | G(x, y; x', y') | z, z' \rangle = \frac{1}{4\pi^2} \int \frac{e^{-kk^2/2}}{k^2 + \mu} e^{ik_1(\tilde{x}' - \tilde{x}) + ik_2(\tilde{y}' - \tilde{y})} dk.$$

When $(\tilde{x}', \tilde{y}') \to (\tilde{x}, \tilde{y})$ it follows that

$$\langle z, z' | G(x, y; x', y') | z, z' \rangle \rightarrow I(\hbar \mu^2).$$

The results we have obtained using only the abstract algebraic structure of the noncommutative flat plane can be of course found also using a specific representation. One such is the standard irreducible representation of \mathcal{A}_k as an I_{∞} factor on $L^2(\mathbb{R}, d\alpha)$ given on $f(\alpha) \in L^2(\mathbb{R}, d\alpha)$ by

$$u(k_1)f(\alpha) = e^{ik_1\alpha}f(\alpha), \qquad v(k_2)f(\alpha) = f(\alpha + k_2\hbar).$$

A convenient basis for $L^2(\mathbb{R}, d\alpha)$ is $|p_1\rangle = e^{ip_1\alpha}$ with $p_1 \in \mathbb{R}$. We have then

$$u(k_1) |p_1\rangle = |p_1 + k_1\rangle, \quad v(k_2) |p_1\rangle = e^{ip_1k_2k} |p_1\rangle.$$

The parameter p_1 can be thought of as the momentum conjugate to x but this fact plays no role here. The eigenvectors $\phi_k = u(k_1)v(k_2)$ have matrix elements defined by

$$\phi_k \left| p_1 \right\rangle = e^{i p_1 k_2 k} \left| p_1 + k_1 \right\rangle.$$

This representation has a bad 'classical' limit. The generator x can be identified then with the parameter α but the generator y tends to zero as $k \to 0$. To obtain a sensible classical limit one needs two copies of $L^2(\mathbb{R}, d\alpha)$. To see this we define $u(k_1)$ and $v(k_2)$ on $L^2(\mathbb{R}^2, d\alpha d\beta)$ as the operators

$$(u(k_1)f)(\alpha,\beta) = e^{i(ak_1\alpha+bk_1\beta)}f(\alpha+ck_1\hbar,\beta+dk_1\hbar),$$

$$(v(k_2)f)(\alpha,\beta) = e^{i(a'k_2\alpha+b'k_2\beta)}f(\alpha+c'k_2\hbar,\beta+d'k_2\hbar).$$
(3.9)

If we choose a + b = 1, a' + b' = 1, we obtain a representation with a non-degenerate limit with

$$u(k_1)v(k_2) = q^{(d-c')k_1k_2k}v(k_2)u(k_1).$$

We can conclude then that if d - c' = 1 one obtains a representation of the algebra. We conclude also that if c' = d then u and v commute. The representation is therefore not irreducible since the commutant is non-trivial. We shall choose

$$a = 1,$$
 $a' = 0,$ $b = 0,$ $b' = 1.$

The propagator can be calculated directly in any one of the representations (3.9). One obtains

$$(\phi_k f)(\alpha,\beta) = e^{idk_1k_2\hbar}e^{i(k_1\alpha+k_2\beta)}f(\alpha+ck_1\hbar+c'k_2\hbar,\,\beta+dk_1\hbar+d'k_2\hbar)$$

and therefore

$$\begin{aligned} (\phi_k \otimes \phi_k^* f f')(\alpha, \beta; \alpha', \beta') &= e^{-ik_1 k_2 \hbar} e^{i(k_1(\alpha - \alpha') + k_2(\beta - \beta'))} \times \\ f(\alpha + ck_1 \hbar + c' k_2 \hbar, \beta + dk_1 \hbar + d' k_1 \hbar) f'(\alpha' - ck_1 \hbar - c' k_2 \hbar, \beta' - dk_1 \hbar - d' k_1 \hbar). \end{aligned}$$

Consider in particular the 'plane-wave' basis $|p\rangle = e^{ip_1\alpha + ip_2\beta}$ of $L^2(\mathbb{R}^2, d\alpha d\beta)$. Then we find

$$\phi_k |p\rangle = e^{idk_1k_2\hbar} e^{ip_1(ck_1+c'k_2)\hbar} e^{ip_2(dk_1+d'k_1)\hbar} |p+k\rangle$$

and therefore

$$\begin{split} \phi_k \otimes \phi_k^* | p; p' \rangle &= \\ e^{-ik_1k_2\hbar} e^{i(p_1 - p'_1)(ck_1 + c'k_2)\hbar} e^{i(p_2 - p'_2)(dk_1 + d'k_1)\hbar} | p + k; p' - k \rangle. \end{split}$$

We are interested in the limit $p' \to p$:

$$\phi_k \otimes \phi_k^* | p; p \rangle = e^{-ik_1k_2k} | p+k; p-k \rangle$$

The decomposition (2.8) of the tensor product $\mathcal{A}_k \otimes \mathcal{A}_k$ is equivalent to a reparametrization of the tensor product $L^2(\mathbb{R}^2, d\alpha \, d\beta) \otimes L^2(\mathbb{R}^2, d\alpha' \, d\beta')$ induced by the linear transformation

$$(\alpha, \beta, \alpha', \beta') \rightarrow (\frac{1}{2}(\alpha' + \alpha), \frac{1}{2}(\beta' + \beta), \alpha' - \alpha, \beta' - \beta)$$

of the parameter space. The first two of the new coordinates yield the representation space \mathcal{D} of \bar{x} and \bar{y} and the second two the representation space \mathcal{F} of δx and δy .

The basis given above for the representation space is singular and it is appropriate to change it, at least for the factor \mathcal{F} . This is equivalent to the introduction of a form factor $F(\alpha' - \alpha, \beta' - \beta)$. For each choice of F we introduce $I_F(k\mu^2)$ defined by the equation

$$\langle p; p | G(x, y; x', y') F | p; p \rangle = I_F(\hbar \mu^2) \langle p; p | F | p; p \rangle$$

The 'coherent-state' basis has a fundamental cell of minimal area and the distance between two closest 'points' is minimal. So normally one might expect that every choice of F would yield a value of $I_F(k\mu^2)$ strictly less than $I(k\mu^2)$. However this is not the case. For example a sequence of F which tends to the product of two δ -functions,

$$F \to \delta(\alpha' - \alpha) \,\delta(\beta' - \beta),$$

will yield a value of $I_{\delta}(\hbar\mu^2)$ which is smaller than $I(\hbar\mu^2)$ for sufficiently small values of $\hbar\mu^2$. We obtain in fact

$$I_{\delta}(\hbar\mu^2) = \frac{1}{4\pi^2} \int \frac{e^{ik_1k_2\hbar}}{k^2 + \mu^2} dk.$$

A change of variables yields the expression

$$I_{\delta}(\hbar\mu^2) = \frac{1}{4\pi} \int_0^\infty \frac{J_0(x)}{x + \hbar\mu^2/2} dx = \frac{1}{8} (\mathbf{H}_0(\hbar\mu^2/2) - Y_0(\hbar\mu^2/2)).$$
(3.10)

Here \mathbf{H}_0 is a Struve function and Y_0 is a Neumann function. When $\hbar\mu^2 \to 0$ one finds

$$I_{\delta}(\hbar\mu^2) = \frac{1}{4\pi} \Big(-\log(\hbar\mu^2) + 2\log 2 - \gamma + o(\hbar\mu^2) \Big)$$

and when $\hbar\mu^2 \to \infty$,

$$I_{\delta}(\hbar\mu^2) = \frac{1}{2\pi\hbar\mu^2} + o((\hbar\mu^2)^{-2}.$$

Comparing the two asymptotic expansions we find

$$I(\hbar\mu^2) - I_{\delta}(\hbar\mu^2) = -\frac{1}{8\pi} \Big(\log 2 + \frac{1}{2}\hbar\mu^2 \log(\hbar\mu^2) + o(\hbar\mu^2) \Big),$$

$$I(\hbar\mu^2) - I_{\delta}(\hbar\mu^2) = -\frac{1}{2\pi(\hbar\mu^2)^2} + o((\hbar\mu^2)^{-3}).$$

The two functions agree to the dominant term in $\hbar \mu^2$ for large and small values but at least to the sub-dominant terms it is rather $I(\hbar \mu^2)$ which is the smaller.

The modification of the propagator which we have found is due to the noncommutativity of the algebra. However we saw that we could define the variation of an element of a noncommutative algebra using the tensor product of two copies of it. The effect then was formally encoded in the difference between a product and a tensor product; the generator x does not commute with y but it does commute with y'. In a subsequent article we shall discuss also a braided tensor product, which has all of the properties of an ordinary product. Although it is somewhat formal, one could consider an analog in the present situation by setting also $[x, y'] = i\hbar$. In this case the properties of the variation of an element would not be correctly encoded in the tensor product. One would find that the commutation relations (3.1) were in fact replaced by

$$[\bar{x}, \bar{y}] = i\hbar, \qquad [\delta x, \delta y] = 0.$$

The noncommutative propagator is seen to be exactly the classical propagator. The propagator depends, we have seen, only on the variations δx and δy .

The self-energy of a scalar particle of total charge e and minimal radius [45] is given by

$$E = \frac{1}{2}e^2 I(\hbar\mu^2).$$

If we set this equal to the mass we find the equation

$$e^2 \simeq -\frac{8\pi\mu}{\log(k\mu^2)}$$

for the charge.

The energy density of a uniform, static, free scalar field is given by

$$T_{00} = \frac{1}{2}\mu^2 \phi^2.$$

The extra contribution due to vacuum fluctuations is

$$\langle T_{00} \rangle_0 = \frac{1}{2} \mu^3 I(\hbar \mu^2).$$

We have included an extra factor μ to account for the physical dimensions of the field. Considered just as a constant the vacuum energy is not a very useful quantity unless somehow it can be connected to the gravitational field equations. It is characteristic of all vacuumfluctuation calculations that the result is too large to be a realistic source of a cosmological solution. In some way 'most' of this very large constant must be subtracted. One way of doing this is to consider the variation with respect to the space-time metric just as the Casimir energy is calculated as that part of the vacuum energy which depends on the distance.

Interpreted as a propagator on an ordinary manifold, G would be seen as associated to the non-local differential operator [36]

$$\Delta_{NL} = e^{\bar{k}\bar{\Delta}/2}(\bar{\Delta} + \mu^2).$$

This effective non-locality is due to the 'quantization' of the distance between the two points. We have defended elsewhere [29] the point of view [11, 18, 10, 12, 35] that the regularization can be considered in fact as being due to the gravitational field. To make this point of view consistent with the results of the present section one must consider the vacuum fluctuations as giving rise to a microscopic field which disappears in the mean. In fact we shall argue in Section 5 that flat space is to be considered as an idealized limit.

There is a simple solid-state model for the space we have just considered which has been used in the study of the fractional quantum Hall effect. The x and y correspond to the cartesian components of the guiding centers of the Landau orbits and the factor $e^{-kk^2/2}$ which arises here because of the effective non-locality acts like the Debye-Waller factor. We refer, for example, to Meissner [31] for further details.

It is straightforward to add a time coordinate and consider the euclidean Laplace operator

$$\Delta = -\partial_t^2 + \Delta_k + \mu^2$$

on the algebra $\mathcal{A} = \mathcal{C}(\mathbb{R}) \otimes \mathcal{A}_k$ generated by the three hermitian elements (t, x, y) and their inverses. The differential calculus $\Omega^*(\mathcal{A})$ is constructed by adding to the 1-forms dx and dy the extra 1-form dt. The density of euclidean vacuum action is given by

$$I_E(\hbar\mu^2) = \frac{1}{(2\pi)^3\mu} \int \frac{e^{-\hbar k^2/2}}{\omega^2 + k^2 + \mu^2} d\omega dk = \frac{1}{\sqrt{32\pi\hbar\mu^2}} e^{\hbar\mu^2/2} \left(1 - \operatorname{Erf}(\sqrt{\hbar\mu^2/2})\right)$$

where $\operatorname{Erf}(x)$ is the error function. When $\hbar \mu^2 \to 0$

$$I_E(\hbar\mu^2) = \frac{1}{\sqrt{32\pi\hbar\mu^2}} (1 - \sqrt{2\hbar\mu^2/\pi} + \cdots)$$

and when $\hbar \mu^2 \to \infty$

$$I_E(\hbar\mu^2) = \frac{1}{4\pi\hbar\mu^2} + \cdots.$$

4 The noncommutative Lobachevsky plane

We shall define the noncommutative Lobachevsky plane to be the formal *-algebra \mathcal{A}_h generated by two hermitian elements x and y which satisfy the commutation relation

$$[x,y] = -2ihy \tag{4.1}$$

where $h \in \mathbb{R}$ and the factor -2 is present for historical reasons. We shall suppose that h > 0. Both x and y are without physical dimensions here. We define a differential calculus $(\Omega^*(\mathcal{A}_h), d)$ over \mathcal{A}_h by introducing [7] a frame or Stehbein θ^a defined by

$$\theta^1 = ry^{-1}dx, \qquad \theta^2 = ry^{-1}dy,$$
(4.2)

where r is a real parameter with the units of length. The structure of the calculus is given by the commutation relations

$$f\theta^a = \theta^a f, \qquad f \in \mathcal{A}_h \tag{4.3}$$

as well as the quadratic relations

$$(\theta^1)^2 = 0, \qquad (\theta^2)^2 = 0, \qquad \theta^1 \theta^2 + \theta^2 \theta^1 = 0.$$
 (4.4)

More details of this have been given elsewhere [7].

We shall define [13] a metric g as a bilinear map

$$g(\theta^a \otimes \theta^b) = g^{ab} \tag{4.5}$$

where from (4.3) the g^{ab} must be real constants. We shall choose $g^{ab} = \delta^{ab}$. From the structure relations

$$d\theta^1 = -r^{-1}\theta^1\theta^2, \qquad d\theta^2 = 0$$

one concludes that the torsion-free metric connection has Gaussian curvature K given by $K = -r^{-2}$.

The derivations e_a dual to the 1-forms θ^a are defined by

$$e_1 x = r^{-1} y, \quad e_1 y = 0,$$

 $e_2 x = 0, \qquad e_2 y = -r^{-1} y$

In terms of them the Laplace operator Δ_h can be written [6] as

$$-\Delta_h \phi = e_1^2 \phi + e_2^2 \phi + r^{-1} e_2 \phi, \qquad \phi \in \mathcal{A}_h.$$

$$(4.6)$$

First we recall the calculation of the propagator in the commutative case. In the commutative limit Δ_h tends to the ordinary Laplace operator on the Lobachevsky plane:

$$\lim_{h \to 0} \Delta_h = \tilde{\Delta} = -r^{-2} \tilde{y}^2 (\partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2).$$
(4.7)

We have here introduced (\tilde{x}, \tilde{y}) as the commutative limits of the operators (x, y). The spectrum of Δ_h in the commutative limit is given by [41] the eigenvalue equation

$$\tilde{\Delta}\phi(\tilde{x},\tilde{y}) = \lambda_{k,\kappa}\phi(\tilde{x},\tilde{y}).$$
(4.8)

By the separation of variables $\phi(\tilde{x}, \tilde{y}) = f(\tilde{x})g(\tilde{y})$ we find the differential equations

$$\partial_{\tilde{x}}^2 f(\tilde{x}) = -k^2 f(\tilde{x}), \tag{4.9}$$

$$\tilde{y}^2 \partial_{\tilde{y}}^2 g(\tilde{y}) = (k^2 \tilde{y}^2 - r^2 \lambda_{k,\kappa}) g(\tilde{y})$$
(4.10)

where $k \in \mathbb{R}$. If we define $\kappa^2 = r^2 \lambda_{k,\kappa} - 1/4$ then

$$r^2\lambda_{k,\kappa} = \kappa^2 + \frac{1}{4} + r^2\mu^2.$$

The eigenvalues $\lambda_{k,\kappa}$ do not in fact depend on k and are infinitely degenerate. If we set then $z = ik\tilde{y}$ and $g(\tilde{y}) = \sqrt{z}J(z)$, Equation (4.10) becomes the Bessel equation

$$J''(z) + \frac{1}{z}J'(z) + (1 + \frac{\kappa^2}{z^2})J(z) = 0.$$
(4.11)

A normalized set of eigenfunctions for the Laplace operator is given by

$$\phi_{k,\kappa}(\tilde{x},\tilde{y}) = e^{ik\tilde{x}}\pi^{-3/2}\sqrt{\kappa}\sinh\pi\kappa\sqrt{\tilde{y}}K_{i\kappa}(|k|\tilde{y})$$
(4.12)

with $\kappa > 0$ and $k \neq 0$. The case $\kappa < 0$ can be excluded since

$$K_{-\nu}(|k|\tilde{y}) = K_{\nu}(|k|\tilde{y}).$$

The case k = 0 is also excluded since when $\tilde{y} \to 0$

$$K_{i\kappa}(|k|\tilde{y}) \to \frac{1}{2}\Gamma(i\kappa)\left(\frac{2}{|k|\tilde{y}}\right)^{i\kappa} + \frac{1}{2}\Gamma(-i\kappa)\left(\frac{2}{|k|\tilde{y}}\right)^{-i\kappa}.$$
(4.13)

If we set $\tilde{x}^i = (\tilde{x}, \tilde{y})$ the completeness relation can be written as

$$\delta^{(2)}(\tilde{x}^i - \tilde{x}^{i\prime}) = \int_{-\infty}^{+\infty} \int_0^\infty \phi_{k,\kappa}(\tilde{x}, \tilde{y}) \phi_{k,\kappa}^*(\tilde{x}', \tilde{y}') dk d\kappa$$
(4.14)

and the propagator is given by

$$G(\tilde{x}^i, \tilde{x}^{i\prime}) = \int_{-\infty}^{+\infty} \int_0^\infty \frac{\phi_{k,\kappa}(\tilde{x}, \tilde{y})\phi_{k,\kappa}^*(\tilde{x}', \tilde{y}')}{\kappa^2 + \frac{1}{4} + r^2\mu^2} dkd\kappa.$$
(4.15)

The value of a tadpole diagram created by a source $J \rightarrow 0$ is given by the quantity

$$I_L(\tilde{x}^i) = \lim_{\tilde{x}^{i\prime} \to \tilde{x}^i} G(\tilde{x}^i, \tilde{x}^{i\prime})$$

Because of the homogeneity of the space in fact I_L cannot vary from point to point; in ordinary field theory it is infinite.

Several interesting problems have been considered and solved [8, 25, 33, 14, 37, 19] on the Lobachevsky plane. In particular the spectrum of the Laplace operator has been found [41]. Recently [6] moreover the spectrum of the noncommutative operator (4.6) has been calculated.

Consider now the noncommutative case. It is to be noticed that although the classical Lobachevsky plane is invariant under the reflection $\tilde{x} \to -\tilde{x}$ this is no longer the case when $h \neq 0$. In the algebra \mathcal{A}_h any monomial $\phi(x, y)$ in x and y can be factorized. Therefore one can formally separate the variables in the eigenvalue problem as before and the eigenvalue equation can be decomposed into two differential equations. The equations for the factor f(x) are given by

$$e_1^2 f(x) = -r^{-2} L_+^2 y^2 f(x),$$

$$e_1^2 f(x) = -r^{-2} L_-^2 f(x) y^2$$
(4.16)

where $L_{\pm} \in \mathbb{R}$. Since the commutation relations $[y, e_2]$ and $[\tilde{y}, \tilde{y}\partial_{\tilde{y}}]$ are of the same form, the differential equation for g(y) has the same form as that of (4.10) even though the algebra has changed:

$$(e_2^2 + r^{-1}e_2)g(y) = r^{-2}(L_{\pm}^2y^2 - \lambda_{k,\kappa})g(y).$$

Consider the function

$$L(z) = \frac{e^z - 1}{z} k$$

It is related to the generating functional of the Bernoulli numbers, which appears in one derivation of the general BaCH formula. For any $k \in \mathbb{R}$ let e^{ikx} be defined as a formal power series in the element x; formally e^{ikx} is a unitary element of \mathcal{A}_h . Then from the action of e_1 on x it follows that

$$e_1 e^{ikx} = ir^{-1}L(2hk)ye^{ikx} = -ir^{-1}L(-2hk)e^{ikx}y.$$
(4.17)

The solution of Equation (4.16) is given therefore by

$$f(x) = e^{ikx}, \qquad L_{\pm} = \pm L(\pm 2hk).$$
 (4.18)

A family of formal solutions of the eigenvalue equation on the quantum Lobachevsky plane which tend to normalized functions in the commutative limit is given for $k \neq 0$, $\kappa > 0$ by

$$\phi_{k,\kappa}(x,y) = \pi^{-3/2} \sqrt{\kappa \sinh \pi \kappa} \sqrt{y} K_{i\kappa}(|L|y) e^{ikx}.$$
(4.19)

We have here introduced the quantity

$$L = L_+(2hk).$$

It plays the role of the linear momentum associated to x. The quantity $L_{-}(2hk)$ is the linear momentum associated to -x. Although |k| remains invariant under the map $k \to -k$ this is not the case for |L|, a fact which is a manifestation of the breaking of parity by the commutation relations. Because of the transposition rule

$$e^{ikx}K(y) = K(e^{2hk}y)e^{ikx} (4.20)$$

the expression for the eigenvectors can also be written with y after x. The 1-particle Hilbert space \mathcal{H} is the space generated by the elements $\phi_{k,\kappa}(x,y)$. The elements W and G can be written then

$$W(x^{\mu}; x^{\nu\prime}) = \int_{-\infty}^{+\infty} \int_{0}^{\infty} \phi_{k,\kappa}(x, y) \otimes \phi_{k,\kappa}^{*}(x', y') dk d\kappa,$$
$$G(x^{\mu}; x^{\nu\prime}) = r^{-2} \int_{-\infty}^{+\infty} \int_{0}^{\infty} \lambda_{k,\kappa}^{-1} \phi_{k,\kappa}(x, y) \otimes \phi_{k,\kappa}^{*}(x', y') dk d\kappa$$

To proceed we must introduce a partial trace on the algebra \mathcal{A}_h which respects the $SL_h(2,\mathbb{R})$ invariance. This trace is a complex-valued linear form on \mathcal{A}_h which is in some sense translation invariant, and in the limit $h \to 0$ agrees with the undeformed integral. In the classical case, translation invariance is equivalent to Stokes' theorem. Since we have an $SL_h(2,\mathbb{R})$ -invariant calculus, it is natural to define a trace of an element of the algebra as the integral of the dual 1-form. For any $f \in \mathcal{A}_h$ we set

$$\mathrm{Tr}(f) = \int f \theta^1 \theta^2$$

where the volume 2-form,

$$\theta^1 \theta^2 = r^2 y^{-1} dx y^{-1} dy = r^2 y^{-2} dx dy,$$

is invariant under the coaction of $SL_h(2)$. We determine the integral 'over x' in turn by requiring that Stokes' theorem

$$\int d\alpha = 0. \tag{4.21}$$

hold for any 1-form α . We write $\alpha = \alpha_x dx + \alpha_y dy$. In particular if $\alpha_x = 0$ and $\alpha_y = f(x)g(y)$ then from (4.21) we find that

$$\int df(x)g(y)y^{-2}dy = \int df(x)\int_0^\infty g(y)y^{-2}dy = 0.$$
(4.22)

for any integrable function g(y). To analyze this we notice that (x + 2ih)dx = dxx from which we deduce that

$$d(x^{n}) = \left((x+2ih)^{n-1} + x(x+2ih)^{n-2} + \dots + x^{n-1} \right) dx$$

= $x^{n-1} \left(\sum_{k=0}^{n-1} (1+2ihx^{-1}) \right) dx$
= $\frac{1}{2ih} x^{n} \left((1+2ihx^{-1})^{n} - 1 \right) dx$
= $\frac{1}{2ih} \left((x+2ih)^{n} - x^{n} \right) dx.$ (4.23)

We conclude that in general

$$df(x) = \frac{1}{2ih} \Big(f(x+2ih) - f(x) \Big) dx,$$
(4.24)

which is a finite-difference operator. Therefore

$$\int \left(f(x+2ih) - f(x) \right) dx = 0.$$

It follows then that

$$e^{-2hk}\int e^{ikx}dx = \int e^{ikx}dx$$

and therefore it is consistent to set

$$\operatorname{Tr}_1(e^{ikx}) = 2\pi\delta(k). \tag{4.25}$$

Furthermore, in the representation given below we can identify

$$\operatorname{Tr}_2(f(y)) = \int_0^\infty f(y) y^{-2} dy.$$
 (4.26)

Since dy satisfies the commutation relation ydy = dyy of an ordinary de Rham form on the undeformed Lobachevsky space we can suppose that (4.26) holds in any case. The trace can be factorized then in the form

$$\operatorname{Tr}(e^{ikx}f(y)) = \operatorname{Tr}_1(e^{ikx})\operatorname{Tr}_2(f(y)).$$

and so, just as in the commutative case, we can set

$$\operatorname{Tr}(e^{ikx}f(y)) = 2\pi\delta(k)\operatorname{Tr}_2(f(y)).$$

If x has a representation with a periodic spectrum then k takes discrete values and the right-hand side of this equation must be replaced by $2\pi\delta_{k0}$. We note that for an arbitrary element $f(x, y) \in \mathcal{A}_h$ we have

$$\operatorname{Tr}(e^{ikx}f(x,y)) = \operatorname{Tr}(f(x,e^{2hk}y)e^{ikx}).$$

In general then

$$\operatorname{Tr}(fg) \neq \operatorname{Tr}(gf).$$

The 'trace' defines a state which is not a trace state.

Equations (4.26) and (4.25) are all the properties of the trace which we shall need. Using them and the explicit expression (4.19) for the basis we find the orthogonality conditions

$$\operatorname{Tr}(\phi_{k,\kappa}^*(x,y)\phi_{k',\kappa'}(x,y)) = \delta(k-k')\delta(\kappa-\kappa').$$

In order to use the general formalism we must first decide how to introduce the annihilation and creation operators. One possibility for this is to introduce generators ξ and η which satisfy the canonical commutation relations $[\xi, \eta] = 2ih$. One can then express x and y as

$$x = \xi \eta - ih, \qquad y = \xi. \tag{4.27}$$

In the notation of (2.8) this yields

$$[\bar{\xi},\bar{\eta}] = ih, \qquad [\delta\xi,\delta\eta] = ih \tag{4.28}$$

and the condition (2.5) is satisfied. If we define

$$\Lambda = e^{ix}, \qquad q = e^{-2h}$$

we find the relation $y\Lambda = q\Lambda y$, which defines the quantum space \mathbb{R}_q^1 . Because of the isotropy of the Lobachevsky plane the Laplace operator is essentially reducible to that of a 1-dimensional manifold. The extra dimension manifests itself as a difference in the multiplicity of the eigenvalues. There is a certain formal analogy between the solutions given here and the solutions [5] to the Laplace operator in the quantum space \mathbb{R}_q^1 .

If we express the eigenvectors in terms of the new generators we find

$$\phi_{k,\kappa}(\xi,\eta) \left| \bar{p} \right\rangle = \pi^{-3/2} \sqrt{\kappa \sinh \pi \kappa} \sqrt{\xi} K_{i\kappa}^*(|L|\xi) e^{ik(\xi\eta - ih)} \left| \bar{p} \right\rangle \tag{4.29}$$

and therefore

$$\langle \vec{p}' | \phi_{k,\kappa}(x,y) \phi_{k,\kappa}^*(x',y') | \vec{p} \rangle = \pi^{-3} \kappa \sinh(\pi\kappa) \langle \vec{p}' | K_{i\kappa}(|L|\xi) \sqrt{\xi} e^{ik\xi\eta} e^{-ik\xi'\eta'} \sqrt{\xi'} K_{i\kappa}^*(|L|\xi') | \vec{p} \rangle.$$

However as an added complication now the eigenvector is no longer factorized as previously into a function of ξ times a function of η . Since we have supposed that x and x' commute we can write

$$e^{ik\xi\eta}e^{-ik\xi'\eta'} = e^{ik(\xi\eta - \xi'\eta')} = e^{-2ik(\bar{\xi}\delta\eta + \bar{\eta}\delta\xi)} = e^{-ik((\bar{\eta} - i\bar{\xi})a + (\bar{\eta} + i\bar{\xi})a^*)}.$$

We have here introduced the annihilation operator a and its adjoint such that

$$\delta\xi = \frac{1}{2}(a+a^*), \qquad \delta\eta = \frac{1}{2i}(a-a^*), \qquad [a,a^*] = 2h.$$

We cannot use the simple BaCH formula since

$$[(\bar{\eta} - i\bar{\xi})a, (\bar{\eta} + i\bar{\xi})a^*] = 2h\left(aa^* + (\bar{\eta} + i\bar{\xi})(\bar{\eta} - i\bar{\xi})\right)$$

does not commute with $(\bar{\eta} - i\bar{\xi})a$ and $(\bar{\eta} + i\bar{\xi})a^*$. In fact these three operators form a basis of the Lie algebra of $SL(2, \mathbb{C})$. The $(\bar{\eta} + i\bar{\xi})$ is essentially the extra annihilation operator b introduced in the previous section and we have thus a tensor product of two harmonicoscillator representations. It would seem that the propagator is impossible to calculate using the decomposition (4.28).

If we use x and y as generators and follow the prescription of Section 2 we find that using an ordinary tensor product

$$[\bar{x},\bar{y}] = -ih\bar{y}, \qquad [\delta x,\delta y] = -ih\bar{y} \tag{4.30}$$

as in the previous section but the condition (2.5) is not satisfied:

$$[\bar{x}, \delta x] = 0, \qquad [\bar{x}, \delta y] = -ih\delta y, [\bar{y}, \delta x] = ih\delta y, \qquad [\bar{y}, \delta y] = 0.$$

$$(4.31)$$

This means that \bar{x} acts on \mathcal{F} as well as \mathcal{D} in the product (2.6). This point can be improved upon by a change of generators. First we note that the algebra generated by (\bar{x}, \bar{y}) can be identified with the algebra \mathcal{A}_h and that the differential calculus $(\bar{\Omega}^*(\mathcal{A}_h), \bar{d})$ defined by the relations (4.31),

$$\begin{split} & [\bar{x}, \bar{d}\bar{x}] = 0, \qquad \quad [\bar{x}, \bar{d}\bar{y}] = -ih\bar{d}\bar{y}, \\ & [\bar{y}, \bar{d}\bar{x}] = ih\bar{d}\bar{y}, \qquad \quad [\bar{y}, \bar{d}\bar{y}] = 0 \end{split}$$

is the same as the original $(\Omega^*(\mathcal{A}_h), d)$. In fact one finds that the frame $(\bar{\theta}^1, \bar{\theta}^2)$ defined by

$$\bar{\theta}^1 = r\bar{d}\bar{x} - r\bar{x}\bar{y}^{-1}\bar{d}\bar{y}, \qquad \bar{\theta}^2 = r\bar{y}^{-1}\bar{d}\bar{y}$$

satisfies the same relations as the frame (4.2). In the commutative limit the new frame is the old one expressed in the new coordinates given by the involution

$$\tilde{\bar{x}} = \phi(\tilde{x}) = \tilde{x}\tilde{y}^{-1}, \qquad \tilde{\bar{y}} = \phi(\tilde{y}) = \tilde{y}^{-1},$$

General covariance would seem to suggest then that one introduce an annihilation operator such that

$$\delta x = \frac{1}{2}(a+a^*) + \frac{1}{2i}\bar{x}(a-a^*), \qquad \delta y = \frac{1}{2i}\bar{y}(a-a^*), \qquad [a,a^*] = h. \tag{4.32}$$

If one did this one would find Equations (4.31) to be equivalent to the conditions

$$[\bar{x}, a] = 0, \qquad [\bar{y}, a] = 0$$

but that the second of the commutation relations (4.30) cannot be satisfied. This is to be expected since differential forms naturally satisfy anticommutation relations. The expressions (4.32) come from the identification

$$\theta^1 = \frac{1}{2}(a+a^*), \qquad \theta^2 = \frac{1}{2i}(a-a^*)$$

and the relations satisfied by the frame would imply the relations $a^2 = 0$, $[a, a^*]_+ = 0$.

One can express ϕ as the commutative limit of the change to new generators given by

$$\bar{x}' = xy^{-1} - ih, \qquad \bar{y}' = y^{-1}.$$

This transformation is closely related to that given by (4.27). Under the change of parameter $h \to -2h$ we can identify $\bar{x}' = \eta$ and $\bar{y}' = \xi^{-1}$. The (\bar{x}', \bar{y}') satisfy the same commutation relation as the (\bar{x}, \bar{y}) except for a change in sign of h. We have here defined the differential calculus directly in terms of the algebra; in particular, we have deduced the module structure of the 1-forms from the commutation relation. This was possible since both the algebra and the differential calculus are defined in terms of the same R-matrix [1].

A more promising decomposition uses the generator w formally defined by the equation $y = e^{-w}$. Using it the commutation relation (4.1) becomes

$$[x,w] = 2ih$$

and in the commutative limit $r\tilde{w}$ is the geodesic distance along the \tilde{y} -axis. Following the prescription of Section 2 we find that, using an ordinary tensor product,

$$[\bar{x}, \bar{w}] = ih, \qquad [\delta x, \delta w] = ih$$

and that the condition (2.5) is now satisfied.

The Equation (4.29) becomes

$$\phi_{k,\kappa}^*(x',y') |\bar{p}\rangle = \pi^{-3/2} e^{-ik\bar{x}} e^{-ik\delta x} \sqrt{\kappa \sinh \pi \kappa} \sqrt{y'} K_{i\kappa}^*(|L|y') |\bar{p}\rangle$$

and therefore

$$\langle \bar{p}' | \phi_{k,\kappa}(x,y) \phi_{k,\kappa}^*(x',y') | \bar{p} \rangle = \pi^{-3} \kappa \sinh(\pi\kappa) \langle \bar{p}' | K_{i\kappa}(|L|y) \sqrt{y} e^{-2ik\delta x} \sqrt{y'} K_{i\kappa}^*(|L|y') | \bar{p} \rangle.$$

As above we introduce the annihilation operator a and its adjoint such that

$$\delta x = \frac{1}{2}(a+a^*), \qquad \delta w = \frac{1}{2i}(a-a^*), \qquad [a,a^*] = 2h.$$

Using again the BaCH formula we find that

$$\langle \bar{p}' | \phi_{k,\kappa}(x,y) \phi_{k,\kappa}^*(x',y') | \bar{p} \rangle = \pi^{-3}\kappa \sinh(\pi\kappa) e^{-hk^2} \langle \bar{p}' | K_{i\kappa}(|L|y) \sqrt{y} e^{-ika^*} e^{-ika} \sqrt{y'} K_{i\kappa}^*(|L|y') | \bar{p} \rangle$$

Using the transposition rules

$$we^{ika^*} = e^{ika^*}(w - hk), \qquad e^{ika}w' = (w' - hk)e^{ika}$$

we conclude therefore that

$$\begin{split} \langle \bar{p} | G(x,y;x',y') | \bar{p} \rangle &= \\ \pi^{-3} \int_{-\infty}^{+\infty} \int_{0}^{\infty} \frac{\kappa \sinh(\pi \kappa) e^{-hk^2}}{\kappa^2 + \frac{1}{4} + r^2 \mu^2} \langle \bar{p} | K_{i\kappa}(|L|e^{-hk}y) \times \\ e^{-hk} e^{-\bar{w}} K_{i\kappa}^*(|L|e^{-hk}y') | \bar{p} \rangle d\kappa dk. \end{split}$$

This can be expressed as an integral over positive values of k:

$$\begin{aligned} \langle \bar{p} | G(x,y;x',y') | \bar{p} \rangle &= \\ & 2\pi^{-3} \int_0^{+\infty} \int_0^\infty \frac{\kappa \sinh(\pi\kappa) e^{-hk^2} \cosh(hk)}{\kappa^2 + \frac{1}{4} + r^2 \mu^2} \times \\ & \langle \bar{p} | K_{i\kappa}(h^{-1} \sinh(hk)y) e^{-\bar{w}} K^*_{i\kappa}(h^{-1} \sinh(hk)y') | \bar{p} \rangle d\kappa dk \end{aligned}$$

The integral can be simplified by introducing the integration variable

 $hl = \sinh(hk)e^{-\bar{w}}.$

It becomes then

$$\langle \bar{p} | G(x,y;x',y') | \bar{p} \rangle = 2\pi^{-3} \int_0^\infty \int_0^\infty \frac{\kappa \sinh(\pi\kappa)}{\kappa^2 + \frac{1}{4} + r^2\mu^2} F(\kappa,l) d\kappa dl.$$

where

$$F(\kappa, l) = \langle \bar{p}, 0 | K_{i\kappa}(le^{+\delta w})e^{-h^{-1}\operatorname{arcsinh}^2(hle^{\bar{w}})}K_{i\kappa}^*(le^{-\delta w}) | \bar{p}, 0 \rangle.$$
(4.33)

This function is not manifestly independent of the state p, that is, of the value of \bar{w} . We can write

$$F(\kappa, l) = G(l) H(\kappa, l)$$

where

$$H(\kappa, l) = {}_{F} \langle 0 | K_{i\kappa}(le^{+\delta w}) K_{i\kappa}^{*}(le^{-\delta w}) | 0 \rangle_{F}.$$

is manifestly independent of \bar{w} but

$$G(l) = \frac{1}{D\langle \bar{p} | \bar{p} \rangle_D} \langle \bar{p} | e^{-h^{-1} \operatorname{arcsinh}^2(h l e^{\bar{w}})} | \bar{p} \rangle_D$$
(4.34)

is not.

In an attempt to clarify this we consider an explicit representation of the algebra. On the Hilbert space $L^2(\mathbb{R}, d\alpha)$ one has the representation given on smooth functions by

$$(\bar{x}f)(\alpha) = ih\partial_{\alpha}f(\alpha), \qquad (\bar{w}f)(\alpha) = \alpha f(\alpha).$$

A convenient basis is given by $|p\rangle = e^{ip\alpha/h}$. We find then the expression

$$e^{\bar{w}} \left| \bar{p} \right\rangle = e^{\alpha} \left| \bar{p} \right\rangle$$

and the function (4.34) can be written as

$$G(l) = \frac{1}{D\langle \bar{p} \mid \bar{p} \rangle_D} \langle \bar{p} \mid e^{-h^{-1} \operatorname{arcsinh}^2(hle^{\bar{w}})} \mid p \rangle_D = \lim_{\alpha \to \infty} \frac{1}{2\alpha_0} \int_{-\alpha_0}^{+\alpha_0} e^{-h^{-1} \operatorname{arcsinh}^2(hle^{\alpha})} d\alpha = \frac{1}{2}.$$

This is certainly independent of α but depends in the choice of basis; the states $|p\rangle$ are planewave states and the \bar{w} 'coordinate' is 'smeared out' over the entire line. Another choice of representation is obtained by interchanging \bar{x} and \bar{w} . That is, with

$$(\bar{x}f)(\alpha) = \alpha f(\alpha), \qquad (\bar{w}f)(\alpha) = ih\partial_{\alpha}f(\alpha).$$

In this representation \bar{w} is diagonal and p is an eigenvalue, a measure of the geodesic distance along the \bar{y} -axis. It leads to

$$G(l) = e^{-h^{-1}\operatorname{arcsinh}^2(hle^p)},$$

which definitely depends on p. Because of the discussion that led to Equation (4.32) we shall argue below that the results are only valid at the point p = 0 on the \bar{y} -axis. This would imply that

$$G(l) = e^{-h^{-1}\operatorname{arcsinh}^2(hl)}.$$

We found in the previous case that the result depended on our choice of representation of the δq^{μ} -algebra; we find here that it depends also on the representation of the \bar{q}^{μ} .

We set $k = 2hr^2$ and we define as previously $I_L(k\mu^2)$ by the equation

$$\langle \bar{p} | G(x, y; x', y') | \bar{p} \rangle = I_L(\hbar \mu^2) \langle \bar{p} | \bar{p} \rangle.$$

We have then

$$I_L(\hbar\mu^2) = 2\pi^{-3} \int_0^\infty \int_0^\infty \frac{\kappa \sinh(\pi\kappa)}{\kappa^2 + \frac{1}{4} + r^2\mu^2} G(l)H(\kappa, l)d\kappa dl.$$
(4.35)

We shall leave the evaluation of H to a future publication. The integral $I_L(\hbar\mu^2)$ can be estimated however to leading order from the fact that the uncertainty relations, as encoded in the commutation relation between a and its adjoint, imply that $_F\langle 0| \delta w |0\rangle_F \gtrsim h$. From this we can deduce that

$$I_L(\hbar\mu^2) \simeq -\frac{1}{4\pi} \log(\hbar\mu^2) + \cdots.$$

All the interesting information is in the sub-dominant terms, which appear in the difference

$$\Delta \langle T_{00} \rangle_0 = \frac{1}{2} \mu^3 (I_L(\hbar \mu^2) - I(\hbar \mu^2)),$$

in the energy density with and without the curvature.

On the fuzzy sphere [28] of radius r the laplacian has n distinct eigenvectors f_s with associated eigenvalues $\omega_s^2 = s(s+1)r^{-2}$ of multiplicity 2s + 1. Let $\{|i\rangle\}$ be a basis of coherent states and define $I_S(k\mu^2)$ by the equation

$$\langle i, i | G(x^a; x^{a\prime}) | i, i \rangle = 4\pi r^2 I_S(\hbar \mu^2) \langle i | i \rangle^2.$$

Because of the properties of coherent states $I_S(\hbar\mu^2)$ will be independent of the state. As in the case of the plane we write $f_s(x^a) = f_s(\bar{x}^a - \delta x^a)$ and $f_s(x^{a\prime}) = f_s(\bar{x}^a + \delta x^a)$. If $|i\rangle$ is the state concentrated on the north pole of the sphere then for large n we can write the commutation relations as $([x, y] - \hbar) |i\rangle = 0$ and identify the sphere with the tangent plane. Comparing the two cases one finds that for large n

$$I_S(\hbar\mu^2) \simeq \frac{1}{8\pi} \sum_{s=0}^{n-1} \frac{2s+1}{s(s+1)+r^2\mu^2} = \frac{1}{4\pi} \sum_{s=1/2}^{n-1/2} \frac{s}{s^2 - \frac{1}{4} + r^2\mu^2}$$
(4.36)

from which we deduce that

$$I_S(\hbar\mu^2) \simeq I\Big(\hbar(\mu^2 - \frac{1}{4r^2})\Big).$$

We have here used the relation $4\pi r^2 \simeq 2\pi k n$ between the area of the sphere and the area of the fundamental cell. We find therefore, when $k\mu^2 \to 0$ and $r\mu \to \infty$, that

$$I_S(\hbar\mu^2) - I(\hbar\mu^2) \sim \frac{1}{32\pi r^2\mu^2}.$$

It is tempting to use the difference

$$\Delta \langle T_{00} \rangle_0 \sim \frac{1}{64\pi} \frac{\mu}{r^2}$$

of $\langle T_{00} \rangle_0$ as a source in the gravitational field equations. We shall return to a similar calculation in dimension 4 below.

If we compare (4.35) with (4.36) we see that the eigenvalues are identical except for a change in sign in the curvature term. We can therefore reasonably suppose that

$$I_L(\hbar\mu^2) \simeq I\Big(\hbar(\mu^2 + \frac{1}{4r^2})\Big).$$

If we define z = x + iy then the commutation relation (4.1) which define the algebra \mathcal{A}_h can be written in the form

$$[z,\bar{z}] = 2ih(z-\bar{z}).$$

There is a Cayley transformation

$$z' = \frac{z-i}{z+i}$$

from the Lobachevsky plane to the Poincaré disc. To compare the calculations of this section with those of the flat plane one might think that it would be simpler to use the disc but the commutation relation in terms of z' is rather complicated:

$$[z',\bar{z}'] = \frac{1}{4}(1-\bar{z}')(1-z')[z,\bar{z}](1-z')(1-\bar{z}') = -h(1-z')(1-\bar{z}'z')(1-\bar{z}') + o(h^2).$$
(4.37)

One definite advantage of the disc is that in the commutative limit there is one point $\tilde{z}' = 0$, at which the metric assumes the gaussian normal form, with the first derivatives of the components equal to zero. The Poincaré disc has also been 'quantized' by Berezin [3], with the commutation relation

$$[z', \bar{z}'] = h(1 - z'\bar{z}')(1 - \bar{z}'z').$$
(4.38)

There is no obvious relation between the two commutation relations (4.37) and (4.38).

It would seem that when studying a noncommutative version of a general manifold one first has to choose a system of coordinates which are gaussian normal at a point $\tilde{q}^{\mu} = \tilde{q}_{0}^{\mu}$. The corresponding generators of the noncommutative tensor-product algebra must be then studied in a coherent state $|0\rangle$ with $\langle 0| \bar{q}^{\mu} |0\rangle = \tilde{q}_{0}^{\mu}$. The propagator in this state is the noncommutative version of the propagator at the point $\tilde{q}^{\mu} = \tilde{q}_{0}^{\mu}$. From the above experience with the Lobachevsky plane we conclude that even in the case of a noncommutative version of a homogeneous manifold H, with therefore $I_{H}(k\mu^{2})$ a constant function, the propagator can only be calculated in a state which is localized about a classical point at which the metric is at least euclidean, if not gaussian normal. The problem is due to the fact that even in the simplest of noncommutative geometries the relation of the noncommutative structure to the metric is not well understood. This is already apparent at the commutative limit. The Poisson structure defined in this limit is in the canonical form in a system of coordinates which in general has no obvious preferred relation to the metric.

With the addition of an extra time coordinate the algebra becomes $\mathcal{A} = \mathcal{C}(\mathbb{R}) \otimes \mathcal{A}_h$ generated by the three hermitian elements (t, x, y) and their inverses. The differential calculus $\Omega^*(\mathcal{A})$ is constructed by adding to the two 'space' 1-forms θ^a the time 1-form $\theta^0 = dt$ and imposing the standard relations. In the limit $h \to 0$, \mathcal{A} becomes an algebra of time-dependent functions on the Lobachevsky plane and $\Omega^*(\mathcal{A})$ the corresponding de Rham differential calculus. The euclidean Laplace operator of a free scalar field is

$$\Delta = -\partial_t^2 + \Delta_h + \mu^2.$$

We find then the curved variant

$$I_{EL}(hr^{2}\mu^{2}) = 2\pi^{-3}r^{2}\mu^{-1}\int_{-\infty}^{+\infty}\int_{0}^{\infty}\int_{0}^{\infty}\frac{\kappa\sinh(\pi\kappa)}{r^{2}\omega^{2}+\kappa^{2}+\frac{1}{4}+r^{2}\mu^{2}}G(l)H(\kappa,l)d\kappa dld\omega$$

of $I_E(\hbar\mu^2)$. It is also possible to let h vary with time but because the curvature does not depend on the value of h there can be no dynamical evolution. Since the space is completely isotropic and homogeneous one might speculate that there is variation in h (in space and time) only in the presence of inhomogeneities and that these latter relax to yield a homogeneous space and a constant h. One would have to consider the time evolution of perturbations of the Lobachevsky metric to determine whether or not this is the case.

The cut-off effect which we have found was obtained using an ordinary tensor product. As in the flat case, and for the same reasons, one finds that the use of a braided tensor product will yield a propagator which is independent of h and which can be identified with the divergent propagator of the commutative limit [30].

5 The noncommutative flat 4-space

We define the noncommutative flat 4-space as the algebra \mathcal{A}_{k} generated by four elements $q^{\mu} = x^{\mu}$ which satisfy the commutation relations [12]

$$[x^{\mu}, x^{\nu}] = i\hbar J^{\mu\nu}$$

where $J^{\mu\nu}$ is a non-degenerate matrix of real numbers. The associated differential calculus $\Omega^*(\mathcal{A}_k)$ is defined by the relations $[x^{\mu}, dx^{\nu}] = 0$. If we introduce the derivations

$$e_{\alpha} = \operatorname{ad} \lambda_{\alpha}, \qquad \lambda_{\alpha} = \frac{1}{i\hbar} J_{\alpha\mu}^{-1} x^{\mu}$$

dual to dx^{μ} then an appropriate generalization [29] of the Laplace operator Δ with mass μ is given by

$$\Delta = \Delta_{k} + \mu^{2} = -\sum_{\alpha} e_{\alpha}^{2} + \mu^{2}.$$

For each $k \in \mathbb{R}$ we introduce the elements $u_{\mu}(k) \in \mathcal{A}_k$ defined by

$$u_{\mu}(k) = e^{ikx^{\mu}}.$$

They satisfy the commutation relations

$$u_{\mu}(k_1)u_{\nu}(k_2) = q^{J^{\mu\nu}k_1k_2\hbar}u_{\nu}(k_2)u_{\mu}(k_1), \qquad q = e^{-i}.$$

A basis for the Hilbert space \mathcal{H} is given by the eigenvectors

$$\phi_k = u_1(k_1)u_2(k_2)u_3(k_3)u_4(k_4) = e^{ik_1x^1}e^{ik_2x^2}e^{ik_3x^3}e^{ik_4x^4}, \qquad k = (k_1, k_2, k_3, k_4)$$

of Δ . The corresponding eigenvalues are $\lambda_k = k^2 + \mu^2$ where we have set $k^2 = g^{\mu\nu}k_{\mu}k_{\nu}$. The element G can be written then

$$G(x^{\mu}; x^{\mu\prime}) = \frac{1}{(2\pi)^4} \int (k^2 + \mu^2)^{-1} \phi_k \otimes \phi_k^* dk, \qquad dk = dk_1 dk_2 dk_3 k_4.$$

We must introduce a partial trace on \mathcal{A}_k . This can be done only through a representation. The only properties which we shall need are the identities

$$\operatorname{Tr}(u_{\mu}^{*}(k')u_{\nu}(k)) = 2\pi\delta(k'-k)g_{\mu\nu}$$

That is:

$$Tr(\phi_{k'}^*\phi_k) = (2\pi)^4 \delta^{(4)}(k'-k).$$

It is most convenient to choose a generalization of the second representation given in Section 3, the one which is reducible and non-singular in the limit $\hbar \to 0$. We represent \mathcal{A}_{\hbar} as an algebra of operators on $L^2(\mathbb{R}^4, dx)$ defined on $f(\alpha^{\lambda}) \in L^2(\mathbb{R}^4, dx)$ by

$$u_{\mu}(k)f(\alpha^{\lambda}) = e^{ik\alpha^{\mu}}f(\alpha^{\lambda} + \frac{1}{2}kJ^{\lambda\mu}k).$$

A convenient basis for $L^2(\mathbb{R}^4, dx)$ is $|p\rangle = e^{ip_\lambda \alpha^\lambda}$ with $p_\lambda \in \mathbb{R}$. We have then

$$u_{\mu}(k) |p\rangle = q^{\frac{1}{2}kJ^{\mu\nu}kp_{\nu}} |p_{1} + k\delta_{\mu 1}, p_{2} + k\delta_{\mu 2}, p_{3} + k\delta_{\mu 3}, p_{4} + k\delta_{\mu 4}\rangle.$$

The eigenvectors ϕ_k have matrix elements defined by

$$\phi_k \left| p \right\rangle = q^{\frac{1}{2}kJ^{\mu\nu}k_{\mu}p_{\nu}} \left| p + k \right\rangle.$$

The commutation relations (2.7) become in this case

$$[\bar{x}^{\mu}, \bar{x}^{\nu}] = \frac{1}{2}i\hbar J^{\mu\nu}, \qquad [\delta x^{\mu}, \delta x^{\nu}] = \frac{1}{2}i\hbar J^{\mu\nu}.$$

We introduce the operators a_1 and a_2 as previously in Section 2 and we write

$$\delta x^{\mu} = J_1^{\mu} a_1 + J_1^{\mu*} a_1^* + J_2^{\mu} a_2 + J_2^{\mu*} a_2^* \tag{5.1}$$

from which we conclude that

$$J_1^{[\mu}J_1^{\nu]*} + J_2^{[\mu}J_2^{\nu]*} = \frac{1}{2}iJ^{\mu\nu}.$$

We have therefore in the basis $|\bar{p}, k\rangle \equiv |\bar{p}\rangle_D \otimes |k\rangle_F$, with as before $|\bar{p}\rangle \equiv |\bar{p}, 0\rangle$,

$$\begin{split} u_{\mu}^{*}(k) |\bar{p}\rangle &= e^{-ikx^{\mu}} |\bar{p}\rangle = e^{-ik(\bar{x}^{\mu} - \delta x^{\mu})} |\bar{p}\rangle \\ &= e^{-ik\bar{x}^{\mu}} e^{ik\delta x^{\mu}} |\bar{p}\rangle \\ &= e^{-ik\bar{x}^{\mu}} e^{ik(J_{1}^{\mu}a + J_{1}^{\mu^{*}a^{*}})} e^{ik(J_{2}^{\mu}b + J_{2}^{\mu^{*}b^{*}})} |\bar{p}\rangle. \end{split}$$

Using the BaCH formula we find that

$$u_1^*(k_1) |\bar{p}\rangle = e^{-ik_1\bar{x}^1} e^{-\hbar(|J_1^1|^2 + |J_2^1|^2)k_1^2/2} e^{ik_1(J_1^{1*}a^* + J_2^{1*}b^*)} |\bar{p}\rangle$$

and therefore

$$\phi_k^* \left| \bar{p} \right\rangle = e^{-ik_\mu \bar{x}^\mu} e^{-\hbar K^{\mu\nu} k_\mu k_\nu / 4} e^{i\omega(k_\mu)} e^{ik_\mu (J_1^{\mu*} a^* + J_2^{\mu*} b^*)} \left| \bar{p} \right\rangle$$

The ω is an unimportant phase factor and we have introduced the diagonal tensor

$$\frac{1}{2}K^{\mu\nu} = \operatorname{diag}(|J_1^1|^2 + |J_2^1|^2, |J_1^2|^2 + |J_2^2|^2, |J_1^3|^2 + |J_2^3|^2, |J_1^4|^2 + |J_2^4|^2).$$

The expectation value of the propagator is given by the expression

$$\langle \bar{p} | G(x^{\mu}; x^{\nu}) | \bar{p} \rangle = \mu^2 I(\hbar \mu^2, K) \langle \bar{p} | \bar{p} \rangle$$

with

$$I(\hbar\mu^2, K) = \frac{1}{(2\pi)^4\mu^2} \int \frac{e^{-kK^{\mu\nu}k_{\mu}k_{\nu}}}{k^2 + \mu^2} dk.$$

We must now address the delicate question of (euclidean) Lorentz invariance. There are two attitudes one can take. One can suppose that Lorentz invariance is exact at all scales. One must then add [12] the $J^{\mu\nu}$ as six extra coordinates, minus possibly two because of two invariants which can be formed. Either one considers that there is no momentum associated to these coordinates, in which case one can take an average value over them and the problem is solved as above, or one can consider them to be ordinary coordinates like the four visible ones, in which case they would have to be 'quantized' also. If this be so the $J^{\mu\nu}$ cannot lie in the center [38]. Alternatively one can admit that the tensor $J^{\mu\nu}$ breaks Lorentz invariance on the scale of k. This manifests itself by the existence of the vectors J_1^{μ} and J_2^{μ} . However there is also an ambiguity in the choice of creation and annihilation operators, described by the symplectic group here of dimension 10. It is always possible then to choose a_1 and a_2 so that

$$K^{\mu\nu} = g^{\mu\nu}.$$

We shall suppose that this has been done. The issue of Lorentz invariance will not appear explicitly then, except to the extent that our calculations are not invariant under the symplectomorphism group. This is fortunate since we have motivated the introduction of noncommuting coordinates by the desire to maintain Lorentz invariance.

The integral $I(\hbar\mu^2) = I(\hbar\mu^2, g)$ is given by

$$I(\hbar\mu^2) = \frac{1}{(2\pi)^4\mu^2} \int \frac{e^{-\hbar k^2}}{k^2 + \mu^2} dk = \frac{1}{16\pi^2} \left(\frac{1}{\hbar\mu^2} + e^{\hbar\mu^2} \text{Ei}(-\hbar\mu^2)\right).$$

For all values of $\hbar \mu^2$ the function $I(\hbar \mu^2)$ is concave. When $\hbar \mu^2 \to 0$

$$I(\hbar\mu^2) = \frac{1}{16\pi^2} \left(\frac{1}{\hbar\mu^2} + \log(\hbar\mu^2) + \cdots \right)$$
(5.2)

and when $\hbar\mu^2 \to \infty$

$$I(\hbar\mu^2) = \frac{1}{16\pi^2(\hbar\mu^2)^2} \Big(1 - \frac{2}{\hbar\mu^2} + \cdots \Big).$$

We would like eventually to compare the expression (5.2) with a curved-space analogue in the limit of vanishing curvature; this would supply a preferred Lorentz frame in the limit.

The density of (euclidean) action of a uniform, static, free scalar field is given by

$$\Gamma = \frac{1}{2}\mu^2 \int \phi^2.$$

The quantity

$$\langle\Lambda\rangle_0 = \frac{1}{2} \hbar \mu^4 I(\hbar \mu^2)$$

can be interpreted [44] as a contribution of the scalar-field vacuum fluctuations to the cosmological constant. We would like to be able to compare this with a noncommutative 'curved-space' configuration but in dimension 4 we have thus far only been able to consider the flat case. In the absence of any other information we suppose the dominant contribution to be obtained by a substitution of the form

$$I_K(\hbar\mu^2) \simeq I\left(\hbar(\mu^2 + \alpha K)\right) \tag{5.3}$$

where α is a constant and K is some local mean curvature. We find then from (5.2) that

$$\Delta \langle \Lambda \rangle_0 \simeq -\frac{1}{32\pi^2} \alpha K \left(1 - \frac{\alpha K}{\mu^2} \right). \tag{5.4}$$

If the space-time has constant curvature K then $\Delta \langle \Lambda \rangle_0 = -3K$. Consistency requires then that

$$\alpha = \frac{32}{3}\pi^2 + o(K\mu^{-2}).$$

If one is interested in a cosmological solution of type Friedmann-Robertson-Walker then one can identify

$$k \to 8\pi G_N, \qquad K \to a(t)^{-2}$$

and Wick-rotate to real time. One must also replace (5.4) by

$$\hbar \langle \rho \rangle_0 \simeq \frac{1}{32\pi^2} \frac{\alpha}{a^2} \left(1 - \frac{\alpha}{\mu^2 a^2} \right), \qquad \langle \rho \rangle_0 \equiv \Delta \langle T_{00} \rangle_0 \tag{5.5}$$

since a solution cannot be found with a varying effective ccosmological constant. One obtains in the flat case (k = 0) the equation

$$\dot{a}^2 = \frac{\alpha}{96\pi^2} \left(1 - \frac{\alpha}{\mu^2 a^2} \right)$$

which has a bounce solution given by

$$a(t) = \sqrt{\frac{\alpha}{\mu^2} \left(1 + \frac{\mu^2 t^2}{96\pi^2}\right)}.$$

The effective pressure is negative [43]:

$$k\langle p \rangle_0 \simeq -\frac{1}{96\pi^2} \frac{\alpha}{a^2} \left(1 + \frac{\alpha}{\mu^2 a^2} \right)$$

and the strong energy condition is violated:

$$k(\langle \rho \rangle_0 + 3 \langle p \rangle_0) \simeq -\frac{1}{16\pi^2} \frac{\alpha^2}{\mu^2 a^4} < 0.$$

The energy which is the effective source of the solution is the difference between two vacuum energies and its sign depends simply on which of the two is the larger. The minimal radius is given by $\mu^2 a^2(0) = \alpha$ and at this value of t the approximation to $I_K(\bar{k}\mu^2)$ which we have used is no longer valid. This is evident from the fact that the expression (5.5) for $\langle \rho \rangle_0$ vanishes at the bounce and it should be maximum there.

Another problem which one can consider is the 'self-consistent' mass calculation [34] based on a 1-loop approximation to the Schwinger-Dyson equation. With an interaction of the form $\lambda \phi^4$ a scalar field acquires a mass μ which must satisfy the equation $\mu^2 = \lambda \mu^2 I(k\mu^2)$. That is, to leading order

 $\lambda \sim 8\pi^2 \hbar \mu^2.$

If k is identified with the square of the Planck length this would imply an interaction constant λ slightly larger than 10^{-20} . If on the other hand we require that $\lambda \sim 1$ then this would imply that $k\mu^2 \sim 1/(8\pi^2)$.

The noncommutative torus is the formal algebra generated by the u_{μ} for arbitrary fixed values of the k_{μ} . It was the first noncommutative geometry on which a Yang-Mills action was proposed [9]. Recently higher-loop contributions to the 'classical' action have been investigated [32, 24].

Acknowledgments

The authors would like to thank K. Chadan, R. Dick, H. Grosse, P. Kulish, D. Maison, G. Meissner, O. Pène, P. Prešnajder, T. Schücker, S. Theisen and G. Velo for enlightening conversations. Two of them (RH) and (JM) would also like to thank the Max-Planck-Institut für Physik in München for financial support and J. Wess for his hospitality there. The work was partially supported by the DAAD under the PROCOPE grant number PKZ 9822848. The work of one of the authors (SC) was supported by the Korean Ministry of Education (Project No. 1998-015-D00074).

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