

Generalized Quantization Principle in Canonical Quantum Gravity and Application to Quantum Cosmology

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In this paper, a generalized quantization principle for the gravitational field in canonical quantum gravity, especially with respect to quantum geometrodynamics is considered. This assumption can be interpreted as a transfer from the generalized uncertainty principle in quantum mechanics, which is postulated as generalization of the Heisenberg algebra to introduce a minimal length, to a corresponding quantization principle concerning the quantities of quantum gravity. According to this presupposition there have to be given generalized representations of the operators referring to the observables in the canonical approach of a quantum description of general relativity. This also leads to generalized constraints for the states and thus to a generalized WheelerDeWitt equation determining a new dynamical behaviour. As a special manifestation of this modified canonical theory of quantum gravity, quantum cosmology is explored. The generalized cosmological WheelerDeWitt equation corresponding to the application of the generalized quantization principle to the cosmological degree of freedom is solved by using Sommerfelds polynomial method.

I. INTRODUCTION

The search for a quantum theory of gravity is one of the most interesting if not the most interesting research topic in contemporary fundamental physics. A very important approach to a quantum description of general relativity is canonical quantum gravity, where are postulated canonical commutation relations for the gravitational field and its canonical conjugated quantity. Especially in its manifestation as loop quantum gravity [1],[2] based on the new formulation of canonical general relativity given in [3],[4] it is considered as a very promising candidate for a quantum theory of gravity. Another important concept in the context of quantum gravity is the so called generalized uncertainty principle, which refers to quantum mechanics and consists in the assumption of a generalization of the usual Heisenbergian commutator between position and momentum. The generalized uncertainty principle, which is closely related to the concept of noncommutative geometry, is a well-known concept and introduces the existence of a fundamental minimal length to quantum mechanics and quantum field theory. This kind of a generalized uncertainty relation and the corresponding introduction of a minimal length has been explored in [5],[6],[7],[8],[9],[10],[11] for the first time. Usually the generalized uncertainty principle is interpreted as an effective description of consequences of a fundamental quantum theory of gravity, since the existence of a minimal length in the Planck regime is assumed to be an implication of a quantum description of the gravitational field. However, there exists also the possibility to consider the generalized uncertainty principle as a fundamental description of nature by itself. If this would be the case, then one should assume that not only the variables of quantum mechanics, the position and momentum of a particle, obey such a generalized uncertainty principle, but also other variables have to be quantized by using a generalized quantization rule. In the common description of quantum field theory and canonical quantum gravity the usual Heisenbergian commutation relation between the position and momentum operator is transferred to the field and its canonical conjugated quantity. If there is postulated a generalized uncertainty principle in quantum mechanics as fundamental description of nature, then the transfer to quantum field theory and quantum gravity should be performed as well. In [12] there has been treated the quantization of fields based on the generalized uncertainty principle. Although the relation of the generalized uncertainty principle and the corresponding minimal length to gravity has been studied in many papers [13],[14],[15],[16],[17],[18],[19],[20],[21],[22],[23],[24],[25],[26],[27],[28],[29],[30],[31],[32],[33],[34],[35],[36],[37],[38],[39],[40],[41], it seems that a modification of the quantization principle of the gravitational field itself has been omitted so far. In [42] and [43] has already been studied the generalized uncertainty principle with respect to the gravitational degrees of freedom of special scenarios of a black hole and cosmology. But in these papers this transfer of the generalized uncertainty principle to gravitational degrees of freedom seems not to be considered as a special manifestation of a fundamental general description of canonical quantum gravity. At least there is given no general formulation. In

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the present paper is hold the assumption that a generalized uncertainty principle with respect to the gravitational field could represent a fundamental description of nature and it is considered within the description of quantum geometrodynamics. But of course it can analogously be transferred to the quantization procedure within loop quantum gravity. The extended quantum description of the gravitational field is applied to quantum cosmology as special manifestation of the general theory. In [44] has been considered the gauge description of general relativity under incorporation of a generalized uncertainty principle for quantum mechanics. In this description the generalized uncertainty principle already influences the fundamental description of gravity decisively. The theory has been considered as a kind of semiclassical description. In the theory of this paper the generalized uncertainty principle now refers to the fundamental quantum description of general relativity itself. A generalized quantization principle for the gravitational field can under certain conditions imply nontrivial commutation relations for the components of the gravitational field in the same way as a generalized uncertainty principle in quantum mechanics can imply noncommutative geometry. Such nontrivial commutation relations for the components of the gravitational field have been explored already in [45].

The paper is structured as follows: First there is given a short repetition of the generalized uncertainty principle in usual quantum mechanics. Then the generalized uncertainty principle is transferred to the gravitational field and accordingly it is introduced a generalized quantization principle for the variables of canonical quantum gravity. There are considered two cases: In the first case there is assumed that the commutator between the variables of canonical quantum gravity depends on the canonical conjugated quantity and in the second case there is assumed that the commutator depends on the gravitational field itself. There are given the corresponding representations of the operators and the corresponding generalized uncertainty relations. After this there are derived the generalized constraints on the quantum state of the gravitational field and thus there are obtained the generalized Wheeler-DeWitt equations. For the first case of the generalized quantization principle, which commutator depends on the canonical conjugated quantity to the gravitational field, is considered the application to the generalized dynamics of quantum cosmology. The corresponding Wheeler-DeWitt equation is obtained by applying the generalized quantization principle for the gravitational field to the cosmological degree of freedom, which is contained in the scale factor under presupposition of a Friedmann Lemaitre universe. It is found a solution by using Sommerfelds polynomial approach, which yields a recursion condition for the coefficients of the polynomial presupposed as solution ansatz for the quantum state depending on the cosmological scale factor by inserting a general polynomial to the generalized cosmological Wheeler-DeWitt equation. The obtained recursion condition maintains that the corresponding state converges, which means that it is normalizable and can thus be considered as physical solution.

II. GENERALIZED QUANTIZATION PRINCIPLE FOR THE GRAVITATIONAL FIELD

A. Generalized Uncertainty Principle in Quantum Mechanics

Usually a generalized uncertainty principle is assumed with respect to the variables of quantum mechanics. The Heisenbergian commutation relation, $[\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu$, is extended in such a way that the commutator depends on the position or on the momentum,

$$[\hat{x}^\mu, \hat{p}_\nu] = i \{ \delta_\nu^\mu [1 + \lambda a(\hat{x}, \hat{p})] + \kappa b_\nu^\mu(\hat{x}, \hat{p}) \}, \quad (1)$$

where $a(\hat{x}, \hat{p})$ and $b_\nu^\mu(\hat{x}, \hat{p})$ describe two arbitrary functions depending on the position and the momentum and λ and κ are constants. For the special case of

$$[\hat{x}^\mu, \hat{p}_\nu] = i \{ \delta_\nu^\mu [1 + \lambda \hat{p}^\rho \hat{p}_\rho] + 2\kappa \hat{p}^\mu \hat{p}_\nu \}, \quad \kappa = \lambda, \quad (2)$$

this leads to the following generalized uncertainty relation:

$$\begin{aligned} \Delta x^\mu \Delta p_\mu &\geq \langle \psi | [\hat{x}^\mu, \hat{p}_\mu] | \psi \rangle = \frac{1}{2} [1 + 3\lambda \Delta p^\mu \Delta p_\mu + 3\lambda \langle \hat{p}^\mu \rangle \langle \hat{p}_\mu \rangle] \quad \text{if } \mu = \nu, \\ \Delta x^\mu \Delta p_\nu &\geq \langle \psi | [\hat{x}^\mu, \hat{p}_\nu] | \psi \rangle = \lambda \langle \hat{p}^\mu \rangle \langle \hat{p}_\nu \rangle \quad \text{if } \mu \neq \nu. \end{aligned} \quad (3)$$

The special case (2) for the generalized uncertainty principle leads to a quite simple representation of the position and momentum operator in position space, where has to be performed a series expansion and which is given by

$$\hat{x}^\mu |\psi(x)\rangle = x^\mu |\psi(x)\rangle \quad , \quad \hat{p}_\mu |\psi(x)\rangle = -i(1 - \lambda \partial^\rho \partial_\rho) \partial_\mu |\psi(x)\rangle + \mathcal{O}(\lambda^2), \quad (4)$$

whereas its corresponding representation in momentum space is given by

$$\hat{x}^\mu |\psi(p)\rangle = [i(1 + \lambda p^\rho p_\rho) \partial_p^\mu + 2i\lambda p^\mu p_\rho \partial_p^\rho] |\psi(p)\rangle \quad , \quad \hat{p}_\mu |\psi(p)\rangle = p_\mu |\psi(p)\rangle. \quad (5)$$

In this paper the generalized uncertainty principle (1) is considered as a fundamental description of nature and accordingly it should be transferred to the variables of canonical quantum gravity, what is done with respect to the formulation of quantum geometrodynamics in this paper.

B. Transfer to a Generalized Quantization Principle for the Gravitational Field

If a generalized uncertainty principle with respect to quantum mechanics (1) is not interpreted as an indirect consequence of a fundamental quantum theory of gravity and thus as an effective description but as a fundamental property of quantum theory, then one should expect that it has to be transferred to field quantization. Usually the Heisenbergian commutation relation of usual quantum mechanics, $[\hat{x}_\mu, \hat{p}^\nu] = i\delta_\mu^\nu$, is considered as fundamental postulate to obtain operators and quantum states from the quantities of classical mechanics. This quantization principle is transferred to the quantum description of fields leading to the commutation relation: $[\hat{\phi}(x), \hat{\pi}_\phi(y)] = i\delta(x-y)$, if $\hat{\phi}(x)$ denotes a scalar field and $\hat{\pi}_\phi(x)$ denotes its canonical conjugated quantity. If (1) is now postulated as fundamental commutation relation of quantum mechanics and it is transferred to field quantization, this leads to: $[\hat{\phi}(x), \hat{\pi}_\phi(y)] = i\delta(x-y) [1 + \lambda A(\hat{\phi}, \hat{\pi}_\phi)]$. But if the generalized quantization principle is transferred to fields then one should expect that it also holds for the special case of the gravitational field. To obtain the analogue quantization rule for the gravitational field one has of course to refer to the canonical quantum description of general relativity. In the canonical formulation of general relativity space-time is foliated into a spacelike hypersurface and a time coordinate leading to the following representation of the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} N_a N^a - N^2 & N_b \\ N_c & h_{ab} \end{pmatrix}, \quad (6)$$

where h_{ab} denotes the three metric on the spacelike hypersurface, N denotes the lapse function and N^a is the shift vector. If there is used this representation of the metric tensor, then the Einstein Hilbert action reads as follows [46]:

$$\begin{aligned} S_{EH} &= \int_{\mathcal{M}} dt d^3x \mathcal{L}_g = \frac{1}{16\pi G} \int_{\mathcal{M}} dt d^3x N \left(G^{abcd} K_{ab} K_{cd} + \sqrt{h} [R_h - 2\Lambda] \right) \\ &= \frac{1}{16\pi G} \int_{\mathcal{M}} dt d^3x \left(\pi^{ab} \dot{h}_{ab} - N\mathcal{H} - N^a \mathcal{H}_a \right), \end{aligned} \quad (7)$$

where \mathcal{H} denotes the part of the Hamiltonian density referring to the space-time direction being orthogonal to the hypersurface, \mathcal{H}_a denotes the part of the Hamiltonian density referring to the hypersurface, K_{ab} denotes the extrinsic curvature, which can be expressed as

$$K_{ab} = \frac{1}{2N} \left(\dot{h}_{ab} - D_a N_b - D_b N_a \right), \quad (8)$$

π^{ab} denotes the canonical conjugated quantity to h_{ab} and is of the following shape:

$$\pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{h}_{ab}} = \frac{\sqrt{h}}{16\pi G} (K^{ab} - K h^{ab}), \quad (9)$$

and G_{abcd} is defined according to

$$G_{abcd} = \frac{1}{2\sqrt{h}} (h_{ac} h_{bd} + h_{ad} h_{bc} - h_{ab} h_{cd}). \quad (10)$$

The quantization of the gravitational field,

$$h_{ab} \rightarrow \hat{h}_{ab} \quad , \quad \pi^{ab} \rightarrow \hat{\pi}^{ab} \quad , \quad (11)$$

is performed in analogy to quantum mechanics by postulating commutation relations, in this case between the three metric h_{ab} and its canonical conjugated quantity π^{ab} . If there is presupposed a generalized quantization principle as fundamental property of quantum theory, then with respect to the quantization of the gravitational field this means that instead of postulating the usual commutation relation [46],[47],[48],

$$[\hat{h}_{ab}(x), \hat{\pi}^{cd}(y)] = \frac{i}{2} \delta(x-y) [\delta_a^c \delta_b^d + \delta_a^d \delta_b^c] \quad , \quad (12)$$

one has to postulate an extended commutation relation for the gravitational field:

$$[\hat{h}_{ab}(x), \hat{\pi}^{cd}(y)] = i\delta(x-y) \left\{ \frac{1}{2} [\delta_a^c \delta_b^d + \delta_a^d \delta_b^c] \left[1 + \lambda A(\hat{h}, \hat{\pi}) \right] + \kappa B_{ab}^{cd}(\hat{h}, \hat{\pi}) \right\} \quad , \quad (13)$$

where $A(\hat{h}, \hat{\pi})$ and $B_{ab}^{cd}(\hat{h}, \hat{\pi})$ denote two arbitrary functions depending on the three metric and its canonical conjugated quantity and λ and κ still denote constants. This quantization principle (13) is obtained from the analogous generalized quantization principle in quantum mechanics (1). The analogy between the generalization of the quantization principle of general relativity (13) and the generalization of the quantization principle in quantum mechanics (1) is represented in the tabular below. In principle there are conceivable various concrete manifestations of the generalized quantization principle of the gravitational field (13) leading to different kinematics of the corresponding quantum theory of gravity. In the following two subsections are explored two special manifestations. This means that the usual quantum description of the gravitational field in quantum geometrodynamics is generalized by postulating (13) and presupposing two different pairs of special generalization functions $A(\hat{h}, \hat{\pi})$ and $B_{ab}^{cd}(\hat{h}, \hat{\pi})$.

	quantum mechanics	quantum gravity
usual quantization principle	$[\hat{x}_\mu, \hat{p}^\nu] = i\delta_\mu^\nu$	$[\hat{h}_{ab}(x), \hat{\pi}^{cd}(y)] = i\delta(x,y)\delta_{(a}^c\delta_{b)}^d$
generalized quantization principle	$[\hat{x}_\mu, \hat{p}^\nu] = i \left\{ \delta_\mu^\nu [1 + \lambda A(\hat{x}, \hat{p})] + \kappa B_\mu^\nu(\hat{x}, \hat{p}) \right\}$	$[\hat{h}_{ab}(x), \hat{\pi}^{cd}(y)] = i\delta(x,y) \left\{ \delta_{(a}^c\delta_{b)}^d [1 + \lambda A(\hat{h}, \hat{\pi})] + \kappa B_{ab}^{cd}(\hat{h}, \hat{\pi}) \right\}$

III. REPRESENTATION OF OPERATORS AND CORRESPONDING WHEELER-DEWITT EQUATIONS IN TWO SPECIAL CASES

A. Generalized Quantization Principle Depending on Canonical Conjugated Quantity

As first special manifestation of (13) shall be considered a generalized quantization principle, where the commutator between the three metric and its canonical conjugated quantity depends on the canonical conjugated quantity. This case is the transfer of a generalized uncertainty principle in quantum mechanics leading to a minimal length. Especially it is presupposed the following pair of functions: $A(\hat{h}, \hat{\pi}) = \hat{\pi}^{mn}\hat{\pi}_{mn}$ and $B_{ab}^{cd}(\hat{h}, \hat{\pi}) = 2\hat{\pi}^{ab}\hat{\pi}_{cd}$, leading to the following shape of the postulated commutator:

$$[\hat{h}_{ab}(x), \hat{\pi}^{cd}(y)] = i\delta(x-y) \left[\frac{1}{2} (\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) (1 + \lambda \hat{\pi}^{mn} \hat{\pi}_{mn}) + 2\lambda \hat{\pi}^{ab} \hat{\pi}_{cd} \right] \quad , \quad (14)$$

where has been set: $\kappa = \lambda$. The parameter λ determining the influence of the additional terms of the generalized quantization principle is assumed to be proportional to the squared Planck length in case of quantum mechanics. This leads to a corresponding minimal length, which is equal to the Planck length. To maintain the analogy of the generalized quantization principle in case of general relativity, one should assume that the parameter λ is also directly related to the squared Planck length. This means that it is assumed to be very small and accordingly leads to the

possibility to consider a series expansion to the first order. This generalized commutation relation (14) is the analogue to (2) and can be realized by the following representation of the operators \hat{h}_{ab} and $\hat{\pi}^{ab}$ in the three metric space:

$$\begin{aligned}\hat{h}_{ab}|\Psi[h]\rangle &= h_{ab}|\Psi[h]\rangle, \\ \hat{\pi}^{ab}|\Psi[h]\rangle &= -i \left(1 - \lambda \frac{\delta}{\delta h_{mn}} \frac{\delta}{\delta h^{mn}} \right) \frac{\delta}{\delta h_{ab}} |\Psi[h]\rangle + \mathcal{O}(\lambda^2),\end{aligned}\tag{15}$$

which is analogous to (4) and where has accordingly to be used a series expansion of the operator $\hat{\pi}^{ab}$ in λ . In the representation referring to the canonical conjugated quantity $\hat{\pi}^{ab}$ commutation relation (14) can be realized as follows:

$$\begin{aligned}\hat{h}_{ab}|\Psi[\pi]\rangle &= i \left[(1 + \lambda \pi^{mn} \pi_{mn}) \frac{\delta}{\delta \pi^{ab}} + 2 \lambda \pi_{ab} \pi^{mn} \frac{\delta}{\delta \pi^{mn}} \right] |\Psi[\pi]\rangle, \\ \hat{\pi}^{ab}|\Psi[\pi]\rangle &= \pi^{ab}|\Psi[\pi]\rangle,\end{aligned}\tag{16}$$

which is analogous to (5). From (16) one can derive that the several components of the three metric fulfil nontrivial commutation relations as well, if there is presupposed (14), which are given by

$$[\hat{h}_{ab}, \hat{h}_{cd}] = 4\lambda^2 \pi^{mn} \pi_{mn} \left(\pi_{cd} \frac{\delta}{\delta \pi^{ab}} - \pi_{ab} \frac{\delta}{\delta \pi^{cd}} \right).\tag{17}$$

In [45] has been presupposed a nontrivial noncommutativity algebra between the components of the tetrad field and there have been derived generalized dynamics for general relativity by using coherent states. (17) shows that such a structure could indeed arise from a quantum description of the gravitational field, if there is presupposed a generalized quantization principle. The uncertainty relation between the three metric and its canonical conjugated quantity corresponding to the generalized quantization principle (14) and being analogue to the case in quantum mechanics (3) is given by

$$\begin{aligned}\Delta h_{ab} \Delta \pi^{ab} &= \frac{1}{2} |\langle \Psi | [h, \pi] | \Psi \rangle| = \frac{1}{4} (1 + 3\lambda \langle \hat{\pi}^{ab} \rangle \langle \hat{\pi}^{ab} \rangle + 3\lambda \Delta \hat{\pi}_{ab} \Delta \hat{\pi}^{ab}) \quad \text{if } a = c \wedge b = d \quad \text{or } a = d \wedge b = c, \\ \Delta h_{ab} \Delta \pi^{cd} &= \frac{1}{2} |\langle \Psi | [h, \pi] | \Psi \rangle| = \lambda \langle \hat{\pi}_{ab} \rangle \langle \hat{\pi}^{cd} \rangle \quad \text{if } a \neq c \vee b \neq d \quad \text{and } a \neq d \vee b \neq c.\end{aligned}\tag{18}$$

B. Generalized Quantization Principle Depending on Three Metric

As second special manifestation of (13) shall be considered a generalized quantization principle for the gravitational field depending on the three metric. Especially, it is presupposed the following pair of functions: $A(\hat{h}, \hat{\pi}) = \hat{h}^{mn} \hat{h}_{mn}$ and $B_{ab}^{cd}(\hat{h}, \hat{\pi}) = 0$, leading to the following shape of the postulated commutator:

$$[\hat{h}_{ab}(x), \hat{\pi}^{cd}(y)] = i\delta(x - y) \left[\frac{1}{2} (\delta_a^c \delta_b^d + \delta_a^d \delta_b^c) \left(1 + \lambda \hat{h}^{mn} \hat{h}_{mn} \right) \right].\tag{19}$$

In the three metric space the quantization principle (19) can be realized by the following representation of the operators \hat{h}_{ab} and $\hat{\pi}^{ab}$:

$$\begin{aligned}\hat{h}_{ab}|\Psi[h]\rangle &= h_{ab}|\Psi[h]\rangle, \\ \hat{\pi}^{ab}|\Psi[h]\rangle &= -i (1 + \lambda h^{mn} h_{mn}) \frac{\delta}{\delta h_{ab}} |\Psi[h]\rangle.\end{aligned}\tag{20}$$

The representation referring to the canonical conjugated quantity will be omitted here. From (20) one can derive that the several components of the canonical conjugated quantity fulfil the following commutation relations, if there is presupposed (19):

$$[\hat{\pi}^{ab}, \hat{\pi}^{cd}] = 2i\lambda \left(\hat{h}^{cd} \hat{\pi}^{ab} - \hat{h}^{ab} \hat{\pi}^{cd} \right). \quad (21)$$

The generalized uncertainty relation between the three metric and its canonical conjugated quantity corresponding to (19) is given by

$$\Delta h_{ab} \Delta \pi^{cd} = \frac{1}{2} |\langle \Psi | [h^{ab}, \pi_{cd}] | \Psi \rangle| = \frac{1}{4} \left(\delta_a^c \delta_b^d + \delta_a^d \delta_b^c \right) \left(1 + \lambda \langle \hat{h}^{mn} \rangle \langle \hat{h}_{mn} \rangle + \lambda \Delta h^{mn} \Delta h_{mn} \right). \quad (22)$$

The corresponding shape of the quantum constraints and thus the Wheeler-DeWitt equations describing the dynamics of the quantum states referring to the gravitational field with respect to the two special quantization conditions is provided in the next subsection.

C. Corresponding Generalized Wheeler-DeWitt Equations

The quantum states for the gravitational field, $|\Psi[h]\rangle$, which are valid, are restricted by quantum constraints being obtained by quantizing the Hamiltonian density of general relativity. The complete Hamiltonian density of general relativity, $\mathcal{H}^g = N\mathcal{H} + N^a\mathcal{H}_a$, reads

$$\mathcal{H}^g = \pi^{ab} \dot{h}_{ab} - \mathcal{L}^g = 16\pi G N G_{abcd} \pi^{ab} \pi^{cd} - \frac{\sqrt{h}}{16\pi G} (R_h - 2\Lambda) - 2N_b D_a \pi^{ab}, \quad (23)$$

where D_a denotes the covariant derivative with respect to the gravitational field. π^{ab} and \mathcal{L}^g are defined in (7) and (9). Variation of the Einstein Hilbert action (7) with respect to N and N_a yields the classical constraints

$$\mathcal{H} = 16\pi G G_{abcd} \pi^{ab} \pi^{cd} - \frac{\sqrt{h}}{16\pi G} (R_h - 2\Lambda) = 0 \quad , \quad \mathcal{H}_a = -2D_b \pi_a^b = 0. \quad (24)$$

The first equation of (24) describes the Hamiltonian constraint and the second equation describes the diffeomorphism constraint. To obtain the quantum constraints of the first quantization condition (14) there has to be inserted the corresponding representation of the operators with respect to the three metric (15) to the constraints (24) leading to

$$\left\{ 16\pi G G_{abcd} \left[\frac{\delta}{\delta h_{ab}} \frac{\delta}{\delta h_{cd}} - 2\lambda \frac{\delta}{\delta h_{mn}} \frac{\delta}{\delta h^{mn}} \frac{\delta}{\delta h_{ab}} \frac{\delta}{\delta h_{cd}} \right] + \frac{\sqrt{h}}{16\pi G} (R_h - 2\Lambda) \right\} |\Psi[h]\rangle + \mathcal{O}(\lambda^2) = 0, \\ 2iD_b h_{ac} \left(1 - \lambda \frac{\delta}{\delta h_{mn}} \frac{\delta}{\delta h^{mn}} \right) \frac{\delta}{\delta h_{bc}} |\Psi[h]\rangle + \mathcal{O}(\lambda^2) = 0, \quad (25)$$

if there used an expansion to the first order in the parameter λ . In case of the second quantization condition there has to be performed no series expansion. Here accordingly there has to be inserted the three metric representation of the second case (20) to the constraints (24) leading to the quantum constraints

$$\left\{ 16\pi G G_{abcd} (1 + \lambda h^{kl} h_{kl}) \left[(1 + \lambda h^{mn} h_{mn}) \frac{\delta}{\delta h_{ab}} \frac{\delta}{\delta h_{cd}} + 2\lambda h^{ab} \frac{\delta}{\delta h_{cd}} \right] + \frac{\sqrt{h}}{16\pi G} (R_h - 2\Lambda) \right\} |\Psi[h]\rangle = 0, \\ 2iD_b h_{ac} (1 + \lambda h^{mn} h_{mn}) \frac{\delta}{\delta h_{bc}} |\Psi[h]\rangle = 0. \quad (26)$$

The first equations of (25) and (26) are the corresponding generalized Wheeler-DeWitt equations to the quantization principles (14) and (19) and thus describe the generalized quantum theoretical dynamics of the gravitational field. In (25) and (26) has been used a special choice of factor ordering corresponding to the transition:

$$G_{abcd} \pi^{ab} \pi^{cd} \longrightarrow \left[\frac{1}{2\sqrt{\hat{h}}} \left(\hat{h}_{ac} \hat{h}_{bd} + \hat{h}_{ad} \hat{h}_{bc} - \hat{h}_{ab} \hat{h}_{cd} \right) \hat{\pi}^{ab} \hat{\pi}^{cd} \right]. \quad (27)$$

IV. APPLICATION TO QUANTUM COSMOLOGY

Cosmological considerations in the context of a generalized uncertainty principle in quantum mechanics have already been treated. Usual Cosmology with a minimal length has been treated in [35],[36],[37],[38],[39],[41]. Quantum cosmology under incorporation of a generalized uncertainty principle has been considered in [24]. In this section the general theory of quantum gravity with a generalized quantization principle for the gravitational field developed in the last two sections shall be applied to quantum cosmology. This means that the generalized quantization principle for the gravitational field also leads to a generalized quantization rule for the scale factor, which represents the degree of freedom of the metric in the cosmological case. A similar quantization of the cosmological degree of freedom of the gravitational field can be found in [42]. But there has been treated a completely different scenario and as already mentioned in the introduction the quantization of the cosmological degree of freedom of the gravitational field according to a generalized uncertainty principle in [42] seems not be interpreted as a special manifestation of a generalized quantization principle for the gravitational field considered as a fundamental property of quantum gravity.

A. Derivation of the Generalized Cosmological Wheeler-DeWitt Equation

In this section the generalized quantization condition for the gravitational field and the corresponding Wheeler-DeWitt equation will be applied to cosmology. Especially it will be presupposed the second special case (14), where the commutation relation between the three metric and its canonical conjugated quantity depends on the canonical conjugated quantity rather than on the three metric. The usual line element, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, of a Friedmann Lemaitre universe takes the following special form:

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2, \quad \text{with} \quad d\Omega_3^2 = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (28)$$

In this ansatz for the line element the three metric h^{ab} is completely determined by the cosmological scale factor a . If there is additionally incorporated a cosmological scalar field as matter field one can obtain the following Hamiltonian as special manifestation of the Hamiltonian of quantum geometrodynamics [46],[47]:

$$H = \frac{N}{2} \left(-\frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{a^3} - a + \frac{\Lambda a^3}{3} + m^2 a^3 \phi^2 \right), \quad (29)$$

if π_a denotes the canonical conjugated quantity to the scale factor a and π_ϕ denotes the canonical conjugated quantity to the cosmological scalar field ϕ . By integrating the two quantities π_a and π_ϕ into a two dimensional vector space endowed with a metric G_{AB} in the following way:

$$\pi_A = (\pi_a, \pi_\phi) \quad , \quad G_{AB} = \begin{pmatrix} -a & 0 \\ 0 & a^3 \end{pmatrix}, \quad (30)$$

the cosmological Hamiltonian (29) can be written as

$$H = \frac{N}{2} \left(G^{AB} \pi_A \pi_B - a + \frac{\Lambda a^3}{3} + m^2 a^3 \phi^2 \right). \quad (31)$$

If the variables a and ϕ and their canonical conjugated variables π_a and π_ϕ shall be quantized to obtain the corresponding Wheeler-DeWitt equation to the Hamiltonian (31), there has to be performed the transition

$$G^{AB} \pi_A \pi_B \rightarrow \frac{1}{\sqrt{-G}} \hat{\pi}_A \left(\sqrt{-G} G^{AB} \hat{\pi}_B \right). \quad (32)$$

The transition (32) is based on a special choice of factor-ordering in analogy to usual quantum cosmology. In the usual case this would lead to $\frac{1}{a^2} \frac{\partial}{\partial a} \left(a \frac{\partial}{\partial a} \right) - \frac{1}{a^3} \frac{\partial^2}{\partial \phi^2}$, since $\pi_a \rightarrow -i \frac{\partial}{\partial a}$ and $\pi_\phi \rightarrow -i \frac{\partial}{\partial \phi}$. In the scenario considered in this section there is presupposed the generalized quantization principle (14) concerning the gravitational field and thus there has to be performed another transition with respect to the gravitational field, which is obtained from (20). The

scalar field is however quantized in the usual way in this section, what can be considered as a kind of approximation. This leads to the following transitions of the canonical conjugated quantities:

$$\pi_a \rightarrow \hat{\pi}_a = -i(1 - \lambda \frac{\partial^2}{\partial a^2}) \frac{\partial}{\partial a} + \mathcal{O}(\lambda^2) \quad , \quad \pi_\phi \rightarrow \hat{\pi}_\phi = -i \frac{\partial}{\partial \phi}. \quad (33)$$

Using (33) together with (30) in (32) yields

$$-\frac{\pi_a^2}{a} + \frac{\pi_\phi^2}{a^3} = G^{AB} \pi_A \pi_B \rightarrow \frac{1}{a^2} \left(1 - \lambda \frac{\partial^2}{\partial a^2}\right) \frac{\partial}{\partial a} \left[a \left(1 - \lambda \frac{\partial^2}{\partial a^2}\right) \frac{\partial}{\partial a} \right] - \frac{1}{a^3} \frac{\partial^2}{\partial \phi^2} + \mathcal{O}(\lambda^2). \quad (34)$$

This leads to the following expression for the cosmological Wheeler-DeWitt equation:

$$\frac{N}{2} \left\{ \frac{1}{a^2} \left(1 - \lambda \frac{\partial^2}{\partial a^2}\right) \frac{\partial}{\partial a} \left[a \left(1 - \lambda \frac{\partial^2}{\partial a^2}\right) \frac{\partial}{\partial a} \right] - \frac{1}{a^3} \frac{\partial^2}{\partial \phi^2} + V(\phi, a) \right\} \Psi(a, \phi) + \mathcal{O}(\lambda^2) = 0, \quad (35)$$

where it has been defined

$$V(\phi, a) = -a + \frac{\Lambda a^3}{3} + m^2 a^3 \phi^2. \quad (36)$$

The cosmological Wheeler-DeWitt equation (35) can be transformed to

$$\left\{ -\frac{2\lambda}{a} \frac{\partial^4}{\partial a^4} - \frac{4\lambda}{a^2} \frac{\partial^3}{\partial a^3} + \frac{1}{a} \frac{\partial^2}{\partial a^2} + \frac{1}{a^2} \frac{\partial}{\partial a} - \frac{1}{a^3} \frac{\partial^2}{\partial \phi^2} + V(\phi, a) \right\} \Psi(a, \phi) + \mathcal{O}(\lambda^2) = 0. \quad (37)$$

B. Solution of the Generalized Cosmological Wheeler-DeWitt Equation

Concerning the solution of the generalized cosmological Wheeler-DeWitt equation (37) it will be assumed that $m = 0$ implying that the intersection term between a and ϕ in the potential $V(a, \phi)$ vanishes: $m^2 a^3 \phi^2 = 0$. Under this precondition the two variables a and ϕ can be separated meaning that the wave function can be expressed as a product of wave functions only depending on one of the variables: $\Psi(a, \phi) = \psi(a) \varphi(\phi)$. Inserting this separation ansatz to the cosmological Wheeler-DeWitt equation derived above (37) under the condition $m = 0$ leads to

$$\frac{a^3}{\psi(a)} \left\{ -\frac{2\lambda}{a} \frac{\partial^4}{\partial a^4} - \frac{4\lambda}{a^2} \frac{\partial^3}{\partial a^3} + \frac{1}{a} \frac{\partial^2}{\partial a^2} + \frac{1}{a^2} \frac{\partial}{\partial a} + V(a) \right\} \psi(a) = \frac{1}{\varphi(\phi)} \frac{\partial^2 \varphi(\phi)}{\partial \phi^2}. \quad (38)$$

Since the two sides of equation (39) are equal, it has to be assumed that both sides are equal to a constant leading to the following system of two equations:

$$\begin{aligned} \left\{ -\frac{2\lambda}{a} \frac{\partial^4}{\partial a^4} - \frac{4\lambda}{a^2} \frac{\partial^3}{\partial a^3} + \frac{1}{a} \frac{\partial^2}{\partial a^2} + \frac{1}{a^2} \frac{\partial}{\partial a} + V(a) + \frac{c}{a^3} \right\} \psi(a) &= 0, \\ \frac{\partial^2 \varphi(\phi)}{\partial \phi^2} + c \varphi(\phi) &= 0. \end{aligned} \quad (39)$$

Remember that in this whole section the first order approximation in the parameter λ is presupposed. The solution of the equation referring to the scalar field ϕ is trivial: $\varphi(\phi) = \exp(i\sqrt{c}\phi)$. To solve the equation referring to the scale factor a there is used Sommerfelds polynomial method. Accordingly it is assumed the following ansatz, where $\psi(a)$ is expressed as a general polynomial and thus reads

$$\psi(a) = \sum_{n=0}^{\infty} \chi_n a^n, \quad (40)$$

where the χ_n are arbitrary coefficients, which are determined in the following solution procedure. The derivations of (40) are given by

$$\begin{aligned}\frac{\partial\psi(a)}{\partial a} &= \sum_{n=0}^{\infty} n\chi_n a^{n-1} \quad , \quad \frac{\partial^2\psi(a)}{\partial a^2} = \sum_{n=0}^{\infty} n(n-1)\chi_n a^{n-2}, \\ \frac{\partial^3\psi(a)}{\partial a^3} &= \sum_{n=0}^{\infty} n(n-1)(n-2)\chi_n a^{n-3} \quad , \quad \frac{\partial^4\psi(a)}{\partial a^4} = \sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)\chi_n a^{n-4}.\end{aligned}\quad (41)$$

Inserting (40) and (41) to the equation of (39) referring to a yields

$$\begin{aligned}\sum_{n=0}^{\infty} & \left[-2\lambda n(n-1)(n-2)(n-3)\chi_n a^{n-5} - 4\lambda n(n-1)(n-2)\chi_n a^{n-5} \right. \\ & \left. + n(n-1)\chi_n a^{n-3} + n\chi_n a^{n-3} - \chi_n a^{n+1} + \frac{\Lambda}{3}\chi_n a^{n+3} + c\chi_n a^{n-3} \right] = 0.\end{aligned}\quad (42)$$

This equation can be transformed to

$$\begin{aligned}\sum_{n=0}^{\infty} & \left[-2\lambda(n+5)(n+4)(n+3)(n+2)\chi_{n+5} - 4\lambda(n+5)(n+4)(n+3)\chi_{n+5} \right. \\ & \left. + (n+3)(n+2)\chi_{n+3} + (n+3)\chi_{n+3} - \chi_{n-1} + \frac{\Lambda}{3}\chi_{n-3} + c\chi_{n+3} \right] a^n = 0.\end{aligned}\quad (43)$$

Reordering of the terms yields

$$\begin{aligned}\sum_{n=0}^{\infty} & \left\{ \lambda [-2(n+5)(n+4)(n+3)(n+2) - 4(n+5)(n+4)(n+3)] \chi_{n+5} \right. \\ & \left. + [(n+3)(n+2) + (n+3) + c] \chi_{n+3} - \chi_{n-1} + \frac{\Lambda}{3}\chi_{n-3} \right\} a^n = 0.\end{aligned}\quad (44)$$

If this equation shall be fulfilled, then one is led to the following condition for the coefficients of the polynomial:

$$\chi_{n+5} = \frac{[(n+3)(n+2) + (n+3) + c] \chi_{n+3} - \chi_{n-1} + \frac{\Lambda}{3}\chi_{n-3}}{\lambda [-2(n+5)(n+4)(n+3)(n+2) - 4(n+5)(n+4)(n+3)]}.\quad (45)$$

By shifting the indices this recursion relation can be rewritten to

$$\chi_{n+2} = \frac{[n(n-1) + n + c] \chi_n - \chi_{n-4} + \frac{\Lambda}{3}\chi_{n-6}}{\lambda [-2(n+2)(n+1)n(n-1) - 4(n+2)(n+1)n]}.\quad (46)$$

It has now to be shown that the corresponding polynomial with the coefficients obeying (46) converges to maintain its normalizability, which is the precondition to be considered as physical solution. Therefore the behaviour of the relation (46) has to be explored for $n \rightarrow \infty$. To study the case $n \rightarrow \infty$ just the terms of highest order in n remain important in the nominator as well as the denominator leading to

$$\lim_{n \rightarrow \infty} \chi_{n+2} = \lim_{n \rightarrow \infty} \frac{[n(n-1) + n + c] \chi_n - \chi_{n-4} + \frac{\Lambda}{3}\chi_{n-6}}{\lambda [-2(n+2)(n+1)n(n-1) - 4(n+2)(n+1)n]} = \frac{n^2 \chi_n}{-2\lambda n^4} = \frac{\chi_n}{-2\lambda n^2}.\quad (47)$$

This means that for $n \rightarrow \infty$ it holds

$$\chi_{n+2} = -\frac{\chi_n}{2\lambda n^2}.\quad (48)$$

In case of the exponential function, $\exp(-x^2) = \sum_{n=0}^{\infty} \alpha_n x^n$, the coefficients behave as $\alpha_{n+2} = \frac{-2\alpha_n}{n}$ for $n \rightarrow \infty$. Thus it is obvious that the polynomial corresponding to (46) converges, is normalizable and thus can be considered as physical solution. The first coefficients the formula refers to are degrees of freedom, which have to be chosen at the beginning. Since the formula (46) relates coefficients, which exponent differs by two, the coefficients referring to even exponents are independent of the coefficients to odd exponents. For χ_0 , χ_1 and χ_2 the expression on the right hand side of (46) is not defined, since the denominator becomes zero as can be seen, if one chooses $n = -2$, $n = -1$ or $n = 0$ leading to χ_0 , χ_1 or χ_2 on the left hand side. χ_0 and χ_1 would have to be chosen even if the denominator would not become zero, since it is a precondition of (46) to be defined for χ_{n+2} that at least the coefficient χ_n is defined. The other coefficients, which take negative indices for the first terms, have to be set to zero. There will now be determined the first ten terms of the solution polynomial, if there is defined $\chi_0 = \alpha$, $\chi_1 = \beta$ and $\chi_3 = \gamma$:

$$\begin{aligned} \chi_0 &= \alpha, \quad \chi_1 = \beta, \quad \chi_2 = \gamma, \quad \chi_3 = \frac{(1+c)\beta}{-24\lambda}, \quad \chi_4 = \frac{(4+c)\gamma}{-144\lambda}, \quad \chi_5 = \frac{(9+c)(1+c)\beta}{11520\lambda^2}, \quad \chi_6 = \frac{(16+c)(4+c)\gamma - \alpha}{172800\lambda^2}, \\ \chi_7 &= \frac{(25+c)(9+c)(1+c)\beta - \beta}{-29030400\lambda^3}, \quad \chi_8 = \frac{(36+c)[(16+c)(4+c)\gamma - \alpha] - \gamma + \frac{\Lambda}{3}\alpha}{-812851200\lambda^3}, \\ \chi_9 &= \frac{(49+c)[(25+c)(9+c)(1+c)\beta - \beta] - \frac{(1+c)\beta}{-24\lambda} + \frac{\Lambda}{3}\beta}{234101145600\lambda^4}. \end{aligned} \tag{49}$$

This leads to the following solution of the generalized cosmological Wheeler-DeWitt equation (37):

$$\begin{aligned} \Psi(a, \phi) &= \left\{ \alpha + \beta a + \gamma a^2 + \frac{(1+c)\beta}{-24\lambda} a^3 + \frac{(4+c)\gamma}{-144\lambda} a^4 + \frac{(9+c)(1+c)\beta}{11520\lambda^2} a^5 + \frac{(16+c)(4+c)\gamma - \alpha}{172800\lambda^2} a^6 \right. \\ &\quad + \frac{(25+c)(9+c)(1+c)\beta - \beta}{-29030400\lambda^3} a^7 + \frac{(36+c)[(16+c)(4+c)\gamma - \alpha] - \gamma + \frac{\Lambda}{3}\alpha}{-812851200\lambda^3} a^8 \\ &\quad \left. + \frac{(49+c)[(25+c)(9+c)(1+c)\beta - \beta] - \frac{(1+c)\beta}{-24\lambda} + \frac{\Lambda}{3}\beta}{234101145600\lambda^4} a^9 + \mathcal{O}(a^{10}) \right\} \exp(i\sqrt{c}\phi). \end{aligned} \tag{50}$$

The free parameters α, β and γ as well as c represent something like initial conditions on the state $\Psi(a, \phi)$. Of course they have to be chosen in such a way that the state is normalized and thus they are restricted.

V. SUMMARY AND DISCUSSION

In this paper a generalized quantization principle for the gravitational field in the canonical approach to a quantum description of general relativity has been proposed in analogy to the generalized uncertainty principle in quantum mechanics. There have been considered two special cases: In one case the generalized quantization principle is assumed to depend on the canonical conjugated quantity to the three metric and in the other case it is assumed to depend on the three metric itself. It have been considered the corresponding representations of the operators describing the quantities of quantum geometrodynamics and the generalized Wheeler-DeWitt equations describing the dynamics for both cases. After this there has been considered the special application to quantum cosmology, where the quantization principle becomes manifest with respect to the scale factor containing the degree of freedom of the metric in a Friedmann Lemaitre universe. By using the representation of the operator of the scale factor there has been derived the cosmological Wheeler-DeWitt equation for the case of a generalized quantization principle depending on the canonical conjugated quantity. This equation has been solved by using Sommerfelds polynomial method. Accordingly there has been obtained a recursion formula for the coefficients by inserting a general polynomial depending on the cosmological scale factor to the cosmological Wheeler-DeWitt equation. From this recursion relation can be seen obviously that the corresponding polynomial converges and thus can be considered as physical solution.

If a generalized uncertainty principle in quantum mechanics or the closely related concept of noncommutative geometry is considered as fundamental description of nature, then it seems to be natural to transfer the corresponding generalized commutation relation to the quantization of quantities in other theories, especially field theories, as it is usually done concerning the usual Heisenbergian algebra. Since the quantization principle has to be applied to any classical theory to obtain the corresponding quantum theory, its generalization also has to be assumed to be valid for all special theories including general relativity. In this paper there has only been explored the basic idea of this concept and the corresponding Wheeler-DeWitt equation has been solved for one special scenario. It could be promising to

extend the considerations given in this paper concerning the complete mathematical formalism of canonical quantum gravity and other special applications. In principle the suggested generalized quantization principle for the variables within a quantum description of the gravitational field can be presupposed in any special formulation of canonical quantum gravity. Accordingly the exploration of the generalized quantization principle for the gravitational field with respect to Ashtekars variables and within loop quantum gravity based on these variables would be of great interest.

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- [1] C. Rovelli, L. Smolin, Nucl. Phys. **B331** (1990) 80.
 - [2] C. Rovelli, L. Smolin, Nucl. Phys. **B442** (1995) 593-622. [gr-qc/9411005].
 - [3] A. Ashtekar, Phys. Rev. Lett. **57** (1986) 2244-2247.
 - [4] A. Ashtekar, Phys. Rev. **D36** (1987) 1587-1602.
 - [5] M. Maggiore, Phys. Lett. B **319** (1993) 83 [arXiv:hep-th/9309034].
 - [6] M. Maggiore, Phys. Lett. B **304** (1993) 65 [arXiv:hep-th/9301067].
 - [7] M. Maggiore, Phys. Rev. D **49** (1994) 5182 [arXiv:hep-th/9305163].
 - [8] A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. **D52** (1995) 1108-1118. [hep-th/9412167].
 - [9] H. Hinrichsen and A. Kempf, J. Math. Phys. **37** (1996) 2121 [arXiv:hep-th/9510144].
 - [10] A. Kempf, J. Math. Phys. **38** (1997) 1347 [arXiv:hep-th/9602085].
 - [11] A. Kempf and G. Mangano, Phys. Rev. D **55** (1997) 7909 [arXiv:hep-th/9612084].
 - [12] T. Matsuo, Y. Shibusa, Mod. Phys. Lett. **A21** (2006) 1285-1296. [hep-th/0511031].
 - [13] F. Scardigli, Phys. Lett. B **452** (1999) 39 [arXiv:hep-th/9904025].
 - [14] S. Capozziello, G. Lambiase and G. Scarpetta, Int. J. Theor. Phys. **39** (2000) 15 [arXiv:gr-qc/9910017].
 - [15] L. B. Crowell, Found. Phys. Lett. **12** (1999) 585.
 - [16] L. N. Chang, D. Minic, N. Okamura and T. Takeuchi, Phys. Rev. D **65** (2002) 125028 [arXiv:hep-th/0201017].
 - [17] W. Kim, Y. W. Kim and Y. J. Park, Phys. Rev. D **74** (2006) 104001 [arXiv:gr-qc/0605084].
 - [18] M. I. Park, Phys. Lett. B **659** (2008) 698 [arXiv:0709.2307 [hep-th]].
 - [19] F. Scardigli and R. Casadio, Int. J. Mod. Phys. D **18** (2009) 319 [arXiv:0711.3661 [hep-th]].
 - [20] W. Kim, E. J. Son and M. Yoon, JHEP **0801** (2008) 035 [arXiv:0711.0786 [gr-qc]].
 - [21] A. Bina, K. Atazadeh and S. Jalalzadeh, Int. J. Theor. Phys. **47** (2008) 1354 [arXiv:0709.3623 [gr-qc]].
 - [22] M. V. Battisti and G. Montani, Phys. Rev. D **77** (2008) 023518 [arXiv:0707.2726 [gr-qc]].
 - [23] T. Zhu, J. R. Ren and M. F. Li, Phys. Lett. B **674** (2009) 204 [arXiv:0811.0212 [hep-th]].
 - [24] M. V. Battisti and G. Montani, Int. J. Mod. Phys. A **23** (2008) 1257 [arXiv:0802.0688 [gr-qc]].
 - [25] B. Vakili, Phys. Rev. D **77** (2008) 044023 [arXiv:0801.2438 [gr-qc]].
 - [26] Y. S. Myung, Phys. Lett. B **681** (2009) 81 [arXiv:0909.2075 [hep-th]].
 - [27] Y. S. Myung, Phys. Lett. B **679** (2009) 491 [arXiv:0907.5256 [hep-th]].
 - [28] A. F. Ali, S. Das and E. C. Vagenas, Phys. Lett. B **678** (2009) 497 [arXiv:0906.5396 [hep-th]].
 - [29] A. Farmany, M. Dehghani, M. R. Setare and S. S. Mortazavi, Phys. Lett. B **682** (2009) 114.
 - [30] Z. H. Li, Phys. Rev. D **80** (2009) 084013.
 - [31] A. Bina, S. Jalalzadeh and A. Moslehi, Phys. Rev. D **81** (2010) 023528 [arXiv:1001.0861 [gr-qc]].
 - [32] W. Kim, Y. J. Park and M. Yoon, Mod. Phys. Lett. A **25** (2010) 1267 [arXiv:1003.3287 [gr-qc]].
 - [33] A. F. Ali, Class. Quant. Grav. **28** (2011) 065013. [arXiv:1101.4181 [hep-th]].
 - [34] S. Das and E. C. Vagenas, Phys. Rev. Lett. **101** (2008) 221301 [arXiv:0810.5333 [hep-th]].
 - [35] A. Ashoorioon, A. Kempf and R. B. Mann, Phys. Rev. D **71** (2005) 023503 [astro-ph/0410139].
 - [36] A. Ashoorioon and R. B. Mann, Phys. Rev. D **71** (2005) 103509 [gr-qc/0410053].
 - [37] A. Ashoorioon and R. B. Mann, Nucl. Phys. B **716** (2005) 261 [gr-qc/0411056].
 - [38] A. Ashoorioon, J. L. Hovdebo and R. B. Mann, Nucl. Phys. B **727** (2005) 63 [gr-qc/0504135].
 - [39] B. Vakili, Int. J. Mod. Phys. **D18** (2009) 1059-1071. [arXiv:0811.3481 [gr-qc]].
 - [40] W. Kim, J. J. Oh, JHEP **0801** (2008) 034. [arXiv:0709.0581 [hep-th]].
 - [41] R. Garattini and P. Nicolini, Phys. Rev. D **83** (2011) 064021 [arXiv:1006.5418 [gr-qc]].
 - [42] B. Majumder, Phys. Lett. B **699** (2011) 315 [arXiv:1104.3488 [gr-qc]].
 - [43] B. Majumder, Phys. Lett. **B701** (2011) 384-387. [arXiv:1105.5314 [gr-qc]].
 - [44] M. Kober, Phys. Rev. **D82** (2010) 085017. [arXiv:1008.0154 [physics.gen-ph]].
 - [45] M. Kober, Class. Quant. Grav. **28** (2011) 225021 [arXiv:1107.1071 [hep-th]].
 - [46] C. Kiefer, *Quantum Gravity*, Oxford University Press, 2004.
 - [47] C. Kiefer, B. Sandhoefer, [arXiv:0804.0672 [gr-qc]].
 - [48] D. Giulini, C. Kiefer, Lect. Notes Phys. **721** (2007) 131-150. [gr-qc/0611141].