The Generalized Uncertainty Principle and Corrections to the Cardy-Verlinde Formula in $SAdS_5$ Black Holes

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Abstract

In this letter, we investigate a possible modification to the temperature and entropy of 5-dimensional Schwarzschild anti de Sitter black holes due to incorporating stringy corrections to the modified uncertainty principle. Then we subsequently argue for corrections to the Cardy-Verlinde formula in order to account for the corrected entropy. Then we show, one can taking into account the generalized uncertainty principle corrections of the Cardy-Verlinde entropy formula by just redefining the Virasoro operator L_0 and the central charge c.

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1 Introduction

It is commonly believed that any valid theory of quantum gravity must necessary incorporate the Bekenestein-Hawking definition of black hole entropy [1, 2] into its conceptual framework [3]. However, the microscopic origin of this entropy remains an enigma for two reasons. First of all although the various counting methods have pointed to the expected semi-classical result, there is still a lack of recognition as to what degrees of freedom are truly being counted. This ambiguity can be attributed to most of these methods being based on dualities with simpler theories, thus obscuring the physical interpretation from the perspective of the black hole in question. Secondly, the vast and varied number of successful counting techniques only serve to cloud up an already fuzzy picture.

The Cardy-Verlinde formula proposed by Verlinde [4], relates the entropy of a certain CFT with its total energy and its Casimir energy in arbitrary dimensions. Using the AdS_d/CFT_{d-1} [5] and dS_d/CFT_{d-1} correspondences [6], this formula has been shown to hold exactly for different black holes (see for example [7]-[15]).

Black hole thermodynamic quantities depend on the Hawking temperature via the usual thermodynamic relations. The Hawking temperature undergoes corrections from many sources: the quantum corrections, the self-gravitational corrections, and the corrections due to the generalized uncertainty principle.

It has been known for some time that the quantum effect (a quantum correction to the microcanonical entropy due to the correction to the number of microstates, and another correction due to the thermal fluctuation around equilibrium state) result in logarithmic corrections to the black hole entropy [16]-[36].

Concerning the Hawking effect [37] much work has been done using a fixed background during the emission process. The idea of Keski-Vakkuri, Kraus and Wilczek (KKW) [38]-[40] is to view the black hole background as dynamical by treating the Hawking radiation as a tunnelling process. The energy conservation is the key to this description. The total (ADM) mass is kept fixed while the mass of the black hole under consideration decreases due to the emitted radiation. The effect of this modification gives rise to additional terms in the formulae concerning the known results for black holes [41]-[43]; a nonthermal partner to the thermal spectrum of the Hawking radiation shows up.

The generalized uncertainty principle corrections are not tied down to any specific model of quantum gravity; these corrections can be derived using arguments from string theory [45] as well as other approaches to quantum gravity [46].

Previous studies of the corrected Cardy-Verlinde formula in AdS/CFT or dS/CFT context have neglected the corrections due to the generalized uncertainty principle [48] In the present paper, we take into account corrections to the Cardy-Verlinde entropy formula of the five-dimensional SAdS black hole that arise due to the generalized uncertainty principle. In section 2 we review the connection between uncertainty principle and thermodynamic quantities, then we drive the corrections to these quantities due to the generalized uncertainty principle [49]. In section 3 we consider the Cardy-Verlinde formula of a 5-dimensional Schwarzschild anti de Sitter black hole, then we obtain the generalized uncertainty principle corrections to this entropy formula.

2 The generalized uncertainty principle

The metric of an SAdS black hole in 5-dimension is given by

$$ds^{2} = -\left(1 - \frac{16\pi G_{5}M}{3\Omega_{3}c^{2}r^{2}} + \frac{r^{2}}{l^{2}}\right)dt^{2} + \left(1 - \frac{16\pi G_{5}M}{3\Omega_{3}c^{2}r^{2}} + \frac{r^{2}}{l^{2}}\right)^{-1}dr^{2} + r^{2}d\Omega_{3}^{2},\tag{1}$$

where Ω_3 is the metric of the unit S^3 and G_5 is the 5-dimensional Newton's constant. Since the Hawking radiation is a quantum process, the emitted quanta must satisfy the Heisenberg uncertainty principle

$$\Delta x_i \Delta p_j \ge \hbar \delta_{ij},\tag{2}$$

where x_i and p_j , i, j = 1...4, are the spatial coordinates and momenta, respectively. By modelling a black hole as a 5-dimensional cube of size equal to twice its Schwarzschild radius r_+ , the uncertainty in the position of a Hawking particle at the emission is

$$\Delta x \approx 2r_{+} = l \sqrt{\frac{-1 + \sqrt{1 + \frac{64\pi G_5 M}{3l^2 \Omega_3 c^2}}}{2}},\tag{3}$$

Using Eq.(2), the uncertainty in the energy of the emitted particle is

$$\Delta E \approx c \Delta p \approx \frac{\hbar}{l \sqrt{\frac{-1 + \sqrt{1 + \frac{64\pi G_5 M}{3l^2 \Omega_3 c^2}}}},\tag{4}$$

The entropy, Hawking temperature and energy of black hole are as

$$S_{BH} = \frac{\Omega_3 r_+^3}{4l_p^3} \approx \frac{\Omega_3}{4l_p^3} (\frac{\pi l^2}{\hbar c})^3 T^3$$
(5)

$$T = \frac{\hbar c (4r_{+}^{2} + 2l^{2})}{4\pi l^{2}r_{+}} \approx \frac{\hbar c}{\pi l^{2}}r_{+}, \qquad r_{+} \gg l,$$
(6)

$$E = \frac{3\Omega_3 r_+^2 c^4}{16\pi G_5} \left(1 + \frac{r_+^2}{l^2}\right) \tag{7}$$

where the approximation $r_+ \gg l$ is known as the high-temperature limit. We now determine the corrections to the above results due to the generalized uncertainty principle. The general form of the generalized uncertainty principle is

$$\Delta x_i \ge \frac{\hbar}{\Delta p_i} + \alpha^2 l_{lp}^2 \frac{\Delta p_i}{\hbar},\tag{8}$$

where $l_{pl} = (\frac{\hbar G_5}{c^3})^{1/3}$ is the Planck length and α is a dimensionless constant of order one. There are many derivations of the generalized uncertainty principle, some heuristic and some more rigorous. Eq.(8) can be derived in the context of string theory [45], noncommutative quantum mechanics [46], and from minimum length consideration [47]. The exact value of α depends on the specific model. The second term in r.h.s of Eq.(8) becomes effective when momentum and length are of the order of Planck mass and of the Planck length, respectively. This limit is usually called quantum regime. Inverting Eq.(8), we obtain

$$\frac{\Delta x_i}{2\alpha^2 l_{pl}^2} \left[1 - \sqrt{1 - \frac{4\alpha^2 l_{pl}^2}{\Delta x_i^2}}\right] \le \frac{\Delta p_i}{\hbar} \le \frac{\Delta x_i}{2\alpha^2 l_{pl}^2} \left[1 + \sqrt{1 - \frac{4\alpha^2 l_{pl}^2}{\Delta x_i^2}}\right] \tag{9}$$

Now we consider the corrections to the black hole thermodynamic quantities. Setting $\Delta x = 2r_+$ and using Eq.(6) the generalized uncertainty principle-corrected Hawking temperature is

$$T' = \frac{c\hbar\alpha^2 l_{pl}^2}{2l^2\pi r_+ (1 - \sqrt{1 - \frac{\alpha^2 l_{pl}^2}{r_+^2}})}$$
(10)

Denominator Eq.(10) may be Taylor expanded around $\alpha = 0$:

$$T' = \frac{c\hbar r_+}{\pi l^2 (1 + \frac{\alpha^2 l_{pl}^2}{4r_+^2})} = \frac{c\hbar r_+}{\pi l^2} (1 - \frac{\alpha^2 l_{pl}^2}{4r_+^2}) = (1 - \frac{\alpha^2 l_{pl}^2}{4r_+^2})T.$$
 (11)

The generalized uncertainty principle-corrected Hawking temperature is smaller than the semiclassical Hawking temperature T of Eq.(6). The generalized uncertainty principle-corrected black hole entropy is

$$S'_{BH} = \frac{\Omega_3}{4l_p^3} (\frac{\pi l^2}{\hbar c})^3 T'^3 = \frac{\Omega_3}{4l_p^3} (\frac{\pi l^2}{\hbar c})^3 (1 - \frac{3\alpha^2 l_p^2}{4r_+^2}) T^3 = S_{BH} (1 - \frac{3\alpha^2 l_p^2}{4r_+^2}).$$
(12)

From Eq.(12) it follows that the corrected entropy is smaller than the semiclassical Bekenstein-Hawking.

3 Generalized Uncertainty Principle Corrections to the Cardy-Verlinde Formula

The entropy of a (1+1)-dimensional CFT is given by the well-known Cardy formula [50]

$$S = 2\pi \sqrt{\frac{c}{6}(L_0 - \frac{c}{24})},\tag{13}$$

where L_0 represent the product ER of the energy and radius, and the shift of $\frac{c}{24}$ is caused by the Casimir effect. After making the appropriate identifications for L_0 and c, the same Cardy formula is also valid for CFT in arbitrary spacetime dimensions d-1 in the form [4]

$$S_{CFT} = \frac{2\pi R}{d - 2} \sqrt{E_c (2E - E_c)},$$
(14)

the so called Cardy-Verlinde formula, where R is the radius of the system, E is the total energy and E_c is the Casimir energy, defined as

$$E_c = (d-1)E - (d-2)TS.$$
(15)

In this section we compute the generalized uncertainty principle corrections to the entropy

of a (d = 5)-dimensional Schwarzschild anti de Sitter black hole described by the Cardy-Verlinde formula Eq.(14). The Casimir energy Eq.(15) now will be modified due to the the uncertainty principle corrections as

$$E'_c = 4E' - 3T'S'.$$
 (16)

It is easily seen that

$$E'_{c}(2E' - E'_{c}) = (4\pi T' - 3T'S'_{BH})(-2\pi T' + 3T'S'_{BH})$$

$$= -8\pi^{2}T^{2} - 16\pi^{2}T\Delta T + 18\pi S_{BH}T^{2} + 38\pi TS_{BH}\Delta T + 18\pi T^{2}\Delta S - 9T^{2}S^{2}_{BH} - 18T^{2}S_{BH}\Delta S - 18TS^{2}_{BH}\Delta T.$$
(17)

We substitute the previous expression (17) in the Cardy-Verlinde formula in order that generalized uncertainty principle corrections to be considered,

$$S_{CFT}' = S_{CFT} \left[1 + \frac{T \left[-16\pi^2 \Delta T + 18\pi T \Delta S + 30\pi S_{BH} \Delta T - 18S_{BH} T \Delta S - 18S_{BH}^2 \Delta T\right]}{2E_c (2E - E_c)}\right] (18)$$

where

$$\Delta T = \frac{-\alpha^2 l_p^2}{4r_+^2} T \tag{19}$$

$$\Delta S = \frac{-3\alpha^2 l_p^2}{4r_+^2} S_{BH} \tag{20}$$

If we would like to express the modified Cardy-Verlinde entropy formula in terms of the energy and Casimir energy, it is necessary to rewrite the $T, S_{BH}, \Delta T, \Delta S$ in terms of energy as following

$$T = \frac{2\hbar}{\pi l^2} \left(\frac{\pi G_5 l^2 E}{3\Omega_3}\right)^{1/4} \tag{21}$$

$$S_{BH} = \frac{\Omega_3 c^3}{4\hbar G_5} (\frac{16\pi G_5 l^2 E}{3\Omega_3 c^4})^{3/4},$$
(22)

$$\Delta T = \frac{-\alpha^2 \hbar c^2 l_p^2}{8\pi l^2} (\frac{3\Omega_3}{\pi G_5 l^2 E})^{1/4},$$
(23)

$$\Delta S = \frac{-3\alpha^2 \Omega_3}{16l_p} \left(\frac{16\pi G_5 l^2 E}{3\Omega_3 c^4}\right)^{1/4} \tag{24}$$

As we saw in above discussion these corrections are caused by generalized uncertainty principle.

Then, we can taking into account the generalized uncertainty principle corrections of the Cardy-Verlinde entropy formula by just redefining the Virasoro operator, $L_0 = ER$, and the central charge $\frac{c}{6} = \frac{(d-2)S_c}{\pi} = 2E_cR$, where S_c is the Casimir entropy

$$L'_{0} = E'R = \frac{3\Omega_{3}l^{6}\pi^{3}}{16\hbar^{4}G_{5}}T'^{4}R = (1 - \frac{\alpha^{2}l_{p}^{2}}{r_{+}^{2}})ER = (1 - \frac{\alpha^{2}l_{p}^{2}}{r_{+}^{2}})L_{0}$$
(25)

$$c' = 12E'_{c}R = 12(4E' - 3S'_{BH}T')R = 12R(1 - \frac{\alpha^{2}l_{p}^{2}}{r_{+}^{2}})E_{c} = (1 - \frac{\alpha^{2}l_{p}^{2}}{r_{+}^{2}})c$$
(26)

his redefinition includes only a multiplicative constant term and, therefore can be considered as a renormalization of the quantities entering in the Cardy formula. ¹In [17] Carlip have computed the logarithmic corrections to the Cardy formula, according to his calculations, logarithmic corrections to the density of states is as

$$\rho(\Delta) \approx \left(\frac{c}{96\Delta^3}\right)^{1/4} \exp\left(2\pi\sqrt{\frac{c\Delta}{6}}\right),\tag{27}$$

where $\Delta = L_0$, the exponential term in (27) gives the standard Cardy formula, but we have now found the lading correction, which is logarithmic. In the other hand as we saw the effect of the generalized uncertainty principle to the Cardy- Verlinde formula appear as the redefinition of the *c* and L_0 only. As Carlip have discussed in [17], the central charge *c* appearing in (27) is the full central charge of the conformal field theory. In general, *c* will consist of a classical term which appear in the Poisson bracets of the Virasoro algebra generators, plus a correction due to the quantum (here generalized uncertainty principle) effects, that can change the exponent in (27) from its classical value. Moreover we saw that L_0 take a similar correction as eq.(25), then the similar discussion about L_0 is correct. Therefore the first order corrections to the L_0 and *c* are given by

$$\Delta L_0 = L'_0 - L_0 = (E' - E)R = \frac{-\alpha^2 l_p^2}{r_+^2} L_0$$
(28)

$$\Delta c = c' - c = 12R(E'_c - E_c) = \frac{-\alpha^2 l_p^2}{r_+^2}c$$
(29)

4 Conclusion

In this paper we have examined the effects of the generalized uncertainty principle in the generalized Cardy-Verlinde formula. The general form of the generalized uncertainty principle is given by Eq.(8). Black hole thermodynamic quantities depend on the Hawking temperature via the usual thermodynamic relation. The Hawking temperature undergoes corrections from the generalized uncertainty principle as Eq.(10). Then we have obtained the corrections to the entropy of a dual conformal field theory live on boundary space as Eq.(18). Then we have considered this point that the Cardy-Verlinde formula is the outcome of a striking resemblance between the thermodynamics of CFTs with asymptotically Ads dual's and CFTs in two dimensions. After that we have obtained the corrections to the quantities entering the Cardy-Verlinde formula:Virasoro operator and the central charge.

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¹To see similar redefinition of the c and L_0 in the Cardy formula due to the quantum corrections refer to the [51]

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