# Isospin particle on $S^{2}$ with arbitrary number of supersymmetries 

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#### Abstract

We study the supersymmetric quantum mechanics of an isospin particle in the background of spherically symmetric Yang-Mills gauge field. We show that on $S^{2}$ the number of supersymmetries can be made arbitrarily large for a specific choice of the spherically symmetric $S U(2)$ gauge field. However, the symmetry algebra containing the supercharges becomes nonlinear if the number of fermions is greater than two. We present the exact energy spectra and eigenfunctions, which can be written as the product of monopole harmonics and a certain isospin state. We also find that the supersymmetry is spontaneously broken if the number of supersymmetries is even.


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The study of supersymmetric quantum mechanics of charged particle in the background of magnetic monopole [1] revealed several interesting aspects the system [2, 3, 4, 5]. Especially, the system is known to possess a hidden supersymmetry commuting with the original supersymmetry which squares to the Hamiltonian of the system. This feature is present both in the case [3] of a charged particle in the background of the Dirac monopole and the case [4] of an isospin particle in the Wu-Yang monopole [6] background. In Ref. [3], it was shown that the hidden supersymmetry can be identified with the usual supersymmetry of a charged particle if it is restricted on a sphere. This led to the manifest extended supersymmetric formulation of charged particle on $S^{2}$ in the Dirac magnetic monopole background. In Ref. [5] the complete energy spectra and the corresponding eigenfunctions were obtained and the issue of the invariance of the ground energy states under the supersymmetry transformation were discussed, and it was shown that spontaneous breaking of supersymmetry occurred for certain values of the monopole charge. Furthermore, it was shown that the system on $S^{2}$ admits $N=4$ real supersymmetries in contrast to the $N=1$ and $N=2$ supersymmetries on $R^{3}[2]$. It turns out that the supersymmetry generators in that case form $S U(2 \mid 1)$ [7] superalgebra rather than the standard supersymmetry algebra in which the superchages square to the Hamiltonian of the system. In this context it would be interesting to add more fermionic degrees of freedom and check the consequent superalgebra.

In this paper, we study the supersymmetric quantum mechanics of an isospin particle on $S^{2}$ in the background of the Wu-Yang monopole [6], a spherically symmetric solution to the sourceless $S U(2)$ Yang-Mills equations. In particular, we address the issue of whether the number of internal supersymmetries can be enlarged. The action principle for the supersymmetric isospin particles was given in Refs. [8] and [4], which generalized bosonic Wong's equation [9]. This action has the properties that the system is invariant under the simultaneous rotations of isospace and ordinary space and under the $N=1$ real supersymmetry transformation. We first formulate the supersymmetric isospin particle system in the $C P(1)$ model approach [5]. It is shown that by introducing $\mathcal{N}$ complex fermion degrees of freedom with a quartic fermion interaction term and considering the system on a sphere in the background of Wu-Yang monopole gauge, the number of complex supersymmetries $\mathcal{N}$ can be made arbitrary. With an appropriate choice of ordering the

[^0]quantized Hamiltonian can be made invariant under the supersymmetry transformations. However, the symmetry algebra including the supercharges becomes nonlinear when the number of fermions is greater than two. We find that the algebra becomes the usual supersymmetry algebra when $\mathcal{N}=1$. If $\mathcal{N}=2$, the symmetry algebra can be identified with $S U(2 \mid 1)$ superalgebra. For general $\mathcal{N}>2$ the algebra takes the nonlinear form. We then consider the energy eigenvalue problem and obtain the energy spectra and the eigenstates. It turns out that the energy eigenfunctions are given by the product of monopole harmonics [10] and certain isospin states. (See Eq. (27) below.) By looking at the supersymmetric structure of the ground state we find that the supersymmetry is spontaneously broken depending on the number of $\mathcal{N}$ : it is unbroken when $\mathcal{N}$ is odd whereas half of the supersymmetries are broken for even $\mathcal{N}$.

In order to proceed, let us consider an isospin particle in $R^{3}$ in the background of the Wu-Yang monopole gauge field [6],

$$
A_{i}^{a}=-\epsilon^{a}{ }_{i j} \frac{x^{j}}{g r^{2}}, \quad A_{0}^{a}=0
$$

the field strength of which is given by

$$
F_{i j}^{a}=\partial_{i} A_{j}^{a}-\partial_{j} A_{i}^{a}-g \epsilon^{a b c} A_{i}^{b} A_{j}^{c}=\epsilon_{i j k} \frac{x^{a} x_{k}}{g r^{4}} .
$$

The $N=1$ supersymmetric Lagrangian [8], [4] can be written as

$$
\begin{equation*}
L=\frac{1}{2} \dot{x}_{i}^{2}+\frac{i}{2} \psi_{i} \dot{\psi}_{i}+i \bar{\theta} \dot{\theta}-g \dot{x}_{i} A_{i}^{a} I^{a}-g S_{i} B_{i}^{a} I^{a} \tag{1}
\end{equation*}
$$

where $I^{a}=\frac{1}{2} \bar{\theta} \sigma^{a} \theta$, and $S_{i}:=-\frac{i}{2} \epsilon_{i j k} \psi_{j} \psi_{k}$ denotes the spin. Its Hamiltonian is given by

$$
H=\frac{1}{2} \dot{x}_{i}^{2}+g S_{i} B_{i}^{a} I^{a}=\frac{1}{2}\left(p_{i}-i g A_{i}^{a} I^{a}\right)^{2}+g S_{i} B_{i}^{a} I^{a}
$$

and the total angular momentum becomes

$$
\begin{align*}
J_{i} & =\epsilon_{i j k} x_{j}\left(\dot{x}_{k}-g A_{k}^{a} I^{a}\right)+I_{i}+S_{i} \\
& =\epsilon_{i j k} x_{j} \dot{x}_{k}+\left(\hat{x}_{k} I_{k}\right) \hat{x}_{i}+S_{i}, \tag{2}
\end{align*}
$$

where in the second line the explicit form of the gauge field was used. This system is known [3, 4] to possess a hidden supersymmetry commuting with the original supersymmetry. In Ref. [3], it is also shown that when the particle is restricted to the sphere of constant radius they together form $N=2$ supersymmetries. Therefore, in terms of the local coordinates of $S^{2}$ one can write down a manifestly $N=2$ supersymmetric Lagrangian. In this paper, however, we will use $C P(1)$ model type of approach expressing the Lagrangian in terms of the $S^{3}$ variable $z=\left(z_{1}, z_{2}\right)$ satisfying $\bar{z} \cdot z=1$ and the complex spinor $\psi=\left(\psi_{1}, \psi_{2}\right)$ satisfying $\bar{z} \cdot \psi=0$ and its conjugation relation, where we have used the notation $A \cdot B=A_{1} B_{1}+A_{2} B_{2}$. The dynamics on $S^{2}$ is recovered by imposing the local phase symmetry. The Lagrangian can be written as

$$
\begin{equation*}
L=2\left|D_{t} z\right|^{2}+i \bar{\psi} \cdot D_{t} \psi+i \bar{\theta} \dot{\theta}-i\left(\bar{z} \sigma_{i} D_{t} z-D_{t} \bar{z} \sigma_{i} z\right) I_{i} \tag{3}
\end{equation*}
$$

where $D_{t}$ defined by $D_{t} z \equiv \dot{z}-i a z$ with $a$ given by

$$
a=-\frac{i}{2}(\bar{z} \cdot \dot{z}-\dot{\bar{z}} \cdot z)-\frac{1}{2} \bar{\psi} \cdot \psi
$$

is a covariant derivative with respect to the $U(1)$ phase transformation. Consequently, the Lagrangian describes a particle on unit $S^{2}$ although it is written in $S^{3}$ variables. One can show that, using the unit radial vector $x_{i}=\bar{z} \sigma_{i} z$ defining $S^{2}$, the last term of Eq. (3) can be written as

$$
\epsilon_{k i j} \dot{x}_{i} x_{j} I_{k}+\bar{\psi} \cdot \psi x_{i} I_{i}
$$

which is the same as the interaction term of Eq. (1) evaluated on the unit $S^{2}$ in the background of the Wu-Yang monopole, if $-(\bar{\psi} \cdot \psi)$ is identified as the classical analogue of the radial component ${ }^{1}$ of the spin $S_{i}$. With the identification

$$
\begin{equation*}
\psi_{i}=\frac{1}{\sqrt{2}}\left(\bar{z} \sigma_{i} \psi+\bar{\psi} \sigma_{i} z\right) \tag{4}
\end{equation*}
$$

[^1]one can further show that the first two terms of the Eq. (3) become the kinetic part of the isospin particle on $S^{2}$. In this formalism, supercharges can be written as
$$
Q=2 \dot{\bar{z}} \cdot \psi, \quad \bar{Q}=2 \bar{\psi} \cdot \dot{z}
$$

From these, two real combinations can be obtained, one of which is the ordinary supercharge restricted to the sphere and the other one is the so-called hidden supercharge.

We now consider a many fermion generalization of this model. The Lagrangian is given by

$$
\begin{equation*}
L=L_{1}+L_{2} \tag{5}
\end{equation*}
$$

with

$$
\begin{aligned}
L_{1} & =2\left|D_{t} z\right|^{2}+\frac{i}{2}\left(\bar{\psi}_{\alpha} \cdot D_{t} \psi_{\alpha}-D_{t} \bar{\psi}_{\alpha} \cdot \psi_{\alpha}\right)-\frac{1}{2}\left(\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\right)^{2} \\
L_{2} & =\frac{i}{2}(\bar{\theta} \dot{\theta}-\dot{\bar{\theta}} \theta)-i\left(\bar{z} \sigma_{i} D_{t} z-D_{t} \bar{z} \sigma_{i} z\right) I_{i}
\end{aligned}
$$

where the fermionic variable $\psi_{\alpha}=\left(\psi_{\alpha 1}, \psi_{\alpha 2}\right)$ has additional flavor index $\alpha=1, \cdots, \mathcal{N}$ and satisfies

$$
\begin{equation*}
\bar{z} \cdot \psi_{\alpha}=\bar{\psi}_{\alpha} \cdot z=0, \tag{6}
\end{equation*}
$$

Covariant derivative $D_{t}$ is defined, as before, by $D_{t} z \equiv \dot{z}-i a z$, but $a$ given by

$$
\begin{equation*}
a=-\frac{i}{2}(\bar{z} \cdot \dot{z}-\dot{\bar{z}} \cdot z)-\frac{1}{2} \bar{\psi}_{\alpha} \cdot \psi_{\alpha} \tag{7}
\end{equation*}
$$

Besides the trivial summation over the flavor indices this Lagrangian differs from the previous one by quartic fermionic interaction term.

A straightforward calculation shows that $L_{1}$ and $L_{2}$ are separately invariant and the constraints are preserved under the following supersymmetric transformations:

$$
\begin{array}{lll}
\delta_{\alpha} z=\psi_{\alpha}, & \delta_{\alpha} \bar{z}=0, & \delta_{\alpha} \psi_{\beta}=0 \\
\delta_{\alpha} \bar{\psi}_{\beta}=2 i \nabla_{\alpha \beta} \bar{z}, & \delta_{\alpha} \theta=\frac{1}{2}\left(\bar{z} \sigma_{a} \psi_{\alpha}\right) \sigma_{a} \theta, & \delta_{\alpha} \bar{\theta}=-\frac{1}{2}\left(\bar{z} \sigma_{a} \psi_{\alpha}\right) \bar{\theta} \sigma_{a}
\end{array}
$$

and

$$
\begin{array}{ll}
\bar{\delta}_{\alpha} z=0, & \bar{\delta}_{\alpha} \bar{z}=\bar{\psi}_{\alpha}, \\
\bar{\delta}_{\alpha} \psi_{\beta}=2 i \nabla_{\alpha \beta} z \\
\bar{\delta}_{\alpha} \bar{\psi}_{\beta}=0, & \bar{\delta}_{\alpha} \theta=-\frac{1}{2}\left(\bar{\psi}_{\alpha} \sigma_{a} z\right) \sigma_{a} \theta, \\
\bar{\delta}_{\alpha} \bar{\theta}=\frac{1}{2}\left(\bar{\psi}_{\alpha} \sigma_{a} z\right) \bar{\theta} \sigma_{a}
\end{array}
$$

where

$$
\nabla_{\alpha \beta} \bar{z}=\delta_{\alpha \beta} D_{t} \bar{z}-\frac{i}{2}\left(\bar{\psi}_{\beta} \cdot \psi_{\alpha}-\delta_{\alpha \beta} \bar{\psi}_{\gamma} \cdot \psi_{\gamma}\right) \bar{z}
$$

One can show that $a$ defined in Eq. (7) is invariant, so that the transformations commute with $D_{t}$. Furthermore, it turns out that the constraints are also preserved.

The momentum conjugate to $z$ is given by

$$
p=2 D_{t} \bar{z}+\frac{i}{2}\left(\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\right) \bar{z}-i \bar{z} \sigma_{i}(1-z \bar{z}) I_{i}
$$

For the consistency of the time evolution with the constraint Eq. (6), the momentum should satisfy

$$
p \cdot z+\bar{z} \cdot \bar{p}=0
$$

Furthermore, the local $U(1)$ phase symmetry gives rise to the the Gauss law constraint

$$
-i(\bar{z} \cdot \bar{p}-p \cdot z)-\bar{\psi}_{\alpha} \cdot \psi_{\alpha}=0
$$

The Hamiltonian can be written as

$$
\begin{aligned}
H & =2\left|D_{t} \bar{z}\right|^{2}-\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\left(\bar{z} \sigma_{i} z\right) I_{i} \\
& =2|\dot{z}-z(\bar{z} \cdot \dot{z})|^{2}+\frac{1}{2}\left(\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\right)^{2}-\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\left(\bar{z} \sigma_{i} z\right) I_{i} \\
& =\frac{1}{2} \left\lvert\,\left(\bar{p}-z(\bar{z} \cdot \bar{p})-\left.i(1-z \bar{z}) \sigma_{i} z I_{i}\right|^{2}+\frac{1}{2}\left(\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\right)^{2}-\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\left(\bar{z} \sigma_{i} z\right) I_{i}\right.\right.
\end{aligned}
$$

To quantize this system one needs to compute the Dirac brackets. The result can be summarized as follows in the quantum commutator version ${ }^{2}$ :

$$
\begin{array}{ll}
{\left[p_{m}, z_{n}\right]=-i \delta_{m n}+\frac{i}{2} \bar{z}_{m} z_{n},} & {\left[p_{m}, \bar{z}_{n}\right]=\frac{i}{2} \bar{z}_{m} \bar{z}_{n}} \\
{\left[p_{m}, p_{n}\right]=\frac{i}{2}\left(p_{m} \bar{z}_{n}-p_{n} \bar{z}_{m}\right),} & {\left[p_{m}, \bar{p}_{n}\right]=\frac{i}{2}\left(p_{m} z_{n}-\bar{p}_{n} \bar{z}_{m}\right)-\bar{\psi}_{\alpha m} \psi_{\alpha n}-\frac{3}{2}\left(\delta_{n m}-z_{n} \bar{z}_{m}\right)} \\
{\left[\bar{\psi}_{\alpha m}, \psi_{\beta n}\right]=\delta_{\alpha \beta}\left(\delta_{n m}-z_{n} \bar{z}_{m}\right),} & {\left[p_{m}, \bar{\psi}_{\alpha n}\right]=i \bar{\psi}_{\alpha m} \bar{z}_{n}}  \tag{8}\\
{\left[I_{i}, I_{j}\right]=i \epsilon_{i j k} I_{k}} &
\end{array}
$$

where indices $m, n=(1,2)$ are explicitly written and the square bracket between two fermionic operators should be interpreted as the anticommutator.

In terms of the variables

$$
\begin{array}{ll}
\beta_{\alpha}=\epsilon_{m n} z_{n} \psi_{\alpha m} \equiv \epsilon z \psi_{\alpha}, & \bar{\beta}_{\alpha}=\bar{\psi}_{\alpha m} \epsilon_{m n} \bar{z}_{n} \equiv \bar{\psi}_{\alpha} \epsilon \bar{z} \\
B_{m}=p_{n} A_{n m}, & \bar{B}_{m}=A_{m n} \bar{p}_{n} \\
\tilde{U}_{B}=-i(\bar{z} \cdot \bar{p}-p \cdot z)+\bar{\psi}_{\alpha} \cdot \psi_{\alpha} &
\end{array}
$$

the above commutation relations can be simplified [5]:

$$
\begin{array}{ll}
{\left[B_{m}, z_{n}\right]=-i A_{n m},} & {\left[B_{m}, \bar{z}_{n}\right]=0} \\
{\left[B_{m}, B_{n}\right]=-i\left(\bar{z}_{m} B_{n}-\bar{z}_{n} B_{m}\right),} & {\left[\bar{B}_{m}, B_{n}\right]=\left(\tilde{U}_{B}+1\right) A_{m n}} \\
{\left[\tilde{U}_{B}, z_{m}\right]=z_{m},} & {\left[\tilde{U}_{B}, B_{m}\right]=-B_{m}}  \tag{9}\\
{\left[\bar{\beta}_{\alpha}, \beta_{\beta}\right]=\delta_{\alpha \beta},} & {\left[I^{a}, I^{b}\right]=i \epsilon_{a b c} I^{c}}
\end{array}
$$

where we have defined $A_{m n} \equiv \delta_{m n}-z_{m} \bar{z}_{n}$ for notational convenience. These should be supplemented by the adjoint relations and all other omitted commutators vanish. Besides these commutation relations the following should be satisfied:

$$
B \cdot z=0, \quad \bar{z} \cdot \bar{B}=0
$$

as operator identities. The Gauss law constraints can be written as

$$
\begin{equation*}
C_{G} \equiv \tilde{U}_{B}-2 \bar{\beta}_{\alpha} \beta_{\alpha}+\alpha_{G}=0 \tag{10}
\end{equation*}
$$

which has to be imposed on the physical states. The constant $\alpha_{G}$ is the ordering parameter that will be fixed. [See Eq. (13).]

Conversely, original variables can be recovered as follows:

$$
p=B-\frac{i}{2}\left(\tilde{U}_{B}-\bar{\beta}_{\alpha} \beta_{\alpha}\right) \bar{z}, \quad \psi_{\alpha}=\epsilon \bar{z} \beta_{\alpha}
$$

Comparing this expression with the definition of the momentum yields the following interpretation

$$
\begin{aligned}
& 2 \nabla_{t} \bar{z} \equiv 2 \dot{\bar{z}}(1-z \bar{z})=B+i \bar{z} \sigma_{i}(1-z \bar{z}) I_{i} \\
& 2 \nabla_{t} z \equiv 2(1-z \bar{z}) \dot{z}=\bar{B}-i(1-z \bar{z}) \sigma_{i} z I_{i}
\end{aligned}
$$

[^2]In quantizing the Hamiltonian there arises ordering ambiguity. One can fix one particular ordering and determine the Hamiltonian by adding the terms arising from changing the operator ordering using other requirement. In our case we will impose the invariance under the supersymmetric transformations. Therefore, we will tentatively choose the following form of the Hamiltonian:

$$
\begin{equation*}
H=2 \nabla_{t} \bar{z} \cdot \nabla_{t} z+\frac{1}{2}\left(\bar{\psi}_{\alpha} \cdot \psi_{\alpha}\right)^{2}-\bar{\psi}_{\alpha} \cdot \psi_{\alpha} I_{r}+C \bar{\psi}_{\alpha} \cdot \psi_{\alpha}+D I_{r} \tag{11}
\end{equation*}
$$

where $I_{r} \equiv\left(\bar{z} \sigma_{i} z\right) I_{i}$ denotes the radial component of the isospin and the constants $C, D$ are ordering parameters to be fixed later.

The quantum angular momentum operator satisfying the correct commutation relations turns out to be

$$
\begin{align*}
J_{i} & =-\frac{i}{2}\left(p \sigma_{i} z-\bar{z} \sigma_{i} \bar{p}\right)-\bar{z} \sigma_{i} z+\frac{1}{2} \bar{\psi}_{\alpha} \sigma_{i} \psi_{\alpha}+I_{i} \\
& =-\frac{i}{2}\left(B \sigma_{i} z-\bar{z} \sigma_{i} \bar{B}-2 i \bar{z} \sigma_{i} z\right)-\frac{1}{2} \tilde{U}_{B} \bar{z} \sigma_{i} z+I_{i} \\
& =\epsilon_{i j k} \bar{z} \sigma_{j} z\left(\nabla_{t} \bar{z} \sigma_{k} z+\bar{z} \sigma_{k} \nabla_{t} z\right)-\frac{1}{2} \tilde{U}_{B} \bar{z} \sigma_{i} z+\left(\bar{z} \sigma_{k} z I_{k}\right) \bar{z} \sigma_{i} z \tag{12}
\end{align*}
$$

The first expression is the same as the Noether charge except the ordering term which is necessary to produce the correct angular momentum commutation relations, and the third expression shows the decomposition of the angular momentum into the orbital part and the the rest. Comparing the final expression with Eq. (2) suggests identifying the second term as the spin. Indeed, using Eq. (4) one can show that, when restricted to $S^{2}$, only the radial component of the spin survives due to the constraint (6). Furthermore, with the choice of ordering constant

$$
\begin{equation*}
\alpha_{G}=\mathcal{N} \tag{13}
\end{equation*}
$$

the Gauss law constraint, Eq. (10) can be written as

$$
\begin{equation*}
C_{G}=\tilde{U}_{B}+2 \Sigma \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
\Sigma \equiv-\frac{1}{2}\left[\bar{\beta}_{\alpha}, \beta_{\alpha}\right] \tag{15}
\end{equation*}
$$

can be identified as the radial component of the total spin. ${ }^{3}$ This expression can be obtained by directly imposing the constraints on the total spin $S_{i}$ and the eigenvalues range from $-\frac{\mathcal{N}}{2}$ to $\frac{\mathcal{N}}{2}$.

Quantum mechanical supercharges $Q_{\alpha}$ and $\bar{Q}_{\alpha}$ can be readily obtained from the classical Noether charge because they have no ordering ambiguity,

$$
Q_{\alpha}=p \cdot \psi_{\alpha}+i \bar{z} \sigma_{i} \psi_{\alpha} I_{i}, \quad \bar{Q}_{\alpha}=\bar{\psi}_{\alpha} \cdot \bar{p}-i \bar{\psi}_{\alpha} \sigma_{i} z I_{i}
$$

One can further show that they can be rewritten in the following suggestive form:

$$
\begin{equation*}
Q_{\alpha}=J_{+} \beta_{\alpha}, \quad \bar{Q}_{\alpha}=J_{-} \bar{\beta}_{\alpha} \tag{16}
\end{equation*}
$$

where

$$
\begin{align*}
& J_{+} \equiv 2 \nabla_{t} \bar{z} \epsilon \bar{z}=i \bar{z} \sigma_{i} \epsilon \bar{z} J_{i} \\
& J_{-} \equiv 2 \epsilon z \nabla_{t} z=-i \epsilon z \sigma_{i} z J_{i} \tag{17}
\end{align*}
$$

Together with the radial component of the angular momentum,

$$
\begin{equation*}
J_{r}=-\frac{1}{2} \tilde{U}_{B}+I_{r}=-\frac{1}{2} C_{G}+\left(\Sigma+I_{r}\right) \tag{18}
\end{equation*}
$$

$J_{ \pm}$satisfy the usual $S U(2)$ algebra relations:

$$
\begin{equation*}
\left[J_{+}, J_{-}\right]=2 J_{r}, \quad\left[J_{r}, J_{+}\right]=J_{+}, \quad\left[J_{r}, J_{-}\right]=-J_{-} \tag{19}
\end{equation*}
$$

[^3]For quantum Hamiltonian, first observe that

$$
\begin{align*}
\frac{1}{2} J^{2} & =\frac{1}{4}\left(J_{+} J_{-}+J_{-} J_{+}\right)+\frac{1}{2} J_{r}^{2} \\
& =\frac{1}{2} J_{+} J_{-}+\frac{1}{2} J_{r}^{2}-\frac{1}{2} J_{r} \\
& =2 \nabla_{t} \bar{z} \cdot \nabla_{t} z+\frac{1}{2} \Sigma^{2}+\Sigma I_{r}+\frac{1}{2} I_{r}^{2}-\frac{1}{2} \Sigma-\frac{1}{2} I_{r} . \tag{20}
\end{align*}
$$

Without the forth term this expression would take the same form as the classical Hamiltonian Eq. (11). Therefore, from now on we will choose

$$
\begin{align*}
H & \equiv \frac{1}{2} J^{2}-\frac{1}{2} I_{r}^{2} \\
& =\frac{1}{4}\left(J_{+} J_{-}+J_{-} J_{+}\right)+I_{r} \Sigma+\frac{1}{2} \Sigma^{2} \tag{21}
\end{align*}
$$

as our Hamiltonian. It is supersymmetric and rotationally invariant, and corresponds to quantizing the classical Hamiltonian with a specific choice of the ordering parameters $C, D$ in Eq. (11). The second term represents the spin-isospin coupling and the third term may be interpreted as the spin-spin coupling which becomes trivial when $\mathcal{N}=1$, but not in general.

From Eqs. (15), (16) and (19) we find

$$
\begin{align*}
{\left[Q_{\alpha}, \bar{Q}_{\beta}\right] } & =J_{+} J_{-} \delta_{\alpha \beta}-2 J_{r} \bar{\beta}_{\beta} \beta_{\alpha} \\
& =\frac{1}{2}\left(J_{+} J_{-}+J_{-} J_{+}\right) \delta_{\alpha \beta}-2 J_{r}\left(\bar{\beta}_{\beta} \beta_{\alpha}-\frac{1}{2} \delta_{\alpha \beta}\right) \\
& =\left(2 H-2\left(\frac{\mathcal{N}-1}{\mathcal{N}}\right) I_{r} \Sigma-\left(\frac{\mathcal{N}-2}{\mathcal{N}}\right) \Sigma^{2}\right) \delta_{\alpha \beta}-2\left(I_{r}+\Sigma\right) S_{\beta \alpha} \tag{22}
\end{align*}
$$

where $S_{\beta \alpha} \equiv \bar{\beta}_{\beta} \beta_{\alpha}-\frac{1}{\mathcal{N}}\left(\bar{\beta}_{\gamma} \beta_{\gamma}\right) \delta_{\alpha \beta}$ denotes $S U(\mathcal{N})$ generator. Note that the trace part is not the Hamiltonian. In fact, the trace part alone does not commute with the supersymmetry generators. One can show that

$$
H=\frac{1}{2 \mathcal{N}}\left[Q_{\alpha}, \bar{Q}_{\alpha}\right]+\left(\frac{\mathcal{N}-1}{\mathcal{N}}\right) I_{r} \Sigma+\left(\frac{\mathcal{N}-2}{2 \mathcal{N}}\right) \Sigma^{2}
$$

The conserved quantities associated with the symmetries of the system are (i) the total angular momentum $\vec{J}=\vec{K}+\vec{I}$ where $\vec{K}$ is the angular momentum associated with spatial rotations of $z$ 's and $\psi_{\alpha}$ 's [5] and $\vec{I}$ is the isospin associated with isospace rotations of $\theta$ 's, (ii) the spin $\Sigma=-\bar{\beta}_{\alpha} \cdot \beta_{\alpha}+\frac{\mathcal{N}}{2}$ (or the fermion number $N_{F}=\bar{\beta}_{\alpha} \cdot \beta_{\alpha}$ ) associated with the global $U(1)$ phase symmetry of the fermion, (iii) the internal $S U(\mathcal{N})$ charge, $S^{A} \equiv S_{\alpha \beta} T_{\alpha \beta}=\bar{\beta}_{\alpha} T_{\alpha \beta}^{A} \beta_{\beta}$, associated with the fermion flavor, where $T^{A}$ are the traceless Hermitian $\mathcal{N} \times \mathcal{N}$ matrices satisfying $\left[T^{A}, T^{B}\right]=i f^{A B C} T^{C}$ and normalized by $\operatorname{tr}\left(T^{A} T^{B}\right)=\frac{1}{2} \delta^{A B}$, and (iv) the supersymmetric generators $Q_{\alpha}=J_{+} \beta_{\alpha}, \bar{Q}_{\alpha}=J_{-} \bar{\beta}_{\alpha}$.

Although it is not the usual supersymmetry algebra, there are several interesting aspects. If the number of fermion species is one, i.e., $\mathcal{N}=1$, Eq. (22) simply becomes $[Q, \bar{Q}]=2\left(H+\frac{1}{8}\right)$ which is the ordinary $\mathcal{N}=1$ SUSY algebra. If $\mathcal{N}=2$, it can be shown that $\Sigma S_{\beta \alpha}=0$ identically and the algebra reduces to

$$
\begin{equation*}
\left[Q_{\alpha}, \bar{Q}_{\beta}\right]=2 H \delta_{\alpha \beta}-2 I_{r}\left(S_{\beta \alpha}+\frac{1}{2} \Sigma \delta_{\alpha \beta}\right) \tag{23}
\end{equation*}
$$

This algebra can be cast in the form which is identical to the $S U(2 \mid 1)$ algebra. For general $\mathcal{N}$ the algebra is nonlinear.
The commutator algebra given in Eq. (9) can be concretely represented on the Hilbert space composed of certain functions on $S^{3}$ as follows:

$$
B_{m}=-i A_{n m} \frac{\partial}{\partial z_{n}}+i \bar{z}_{m}, \quad \bar{B}_{m}=-i A_{m n} \frac{\partial}{\partial \bar{z}_{n}}, \quad \tilde{U}_{B}=z_{m} \frac{\partial}{\partial z_{m}}-\bar{z}_{m} \frac{\partial}{\partial \bar{z}_{m}} .
$$

Operators $z$ and $\bar{z}$ act as multiplication, and the fermion operators and the isospin can be represented as usual in terms of matrices. From these, representations for other physical operators can be obtained. In particular, we find

$$
\begin{equation*}
J_{i}=\frac{1}{2}\left(\bar{z} \sigma_{i} \frac{\partial}{\partial \bar{z}}-\frac{\partial}{\partial z} \sigma_{i} z\right)+I_{i} \tag{24}
\end{equation*}
$$

Let us study energy eigenfunctions of the Hamiltonian (21). Since $J_{i}$ is the sum of two commuting angular momentum operators $K_{i}$ and $I_{i}$ and the isospin is easy to handle, first consider the representation theory of $K_{i}$. Besides $K^{2}, K_{3}$ there is another operator commuting with them, which is the radial component $K_{r}$ of $K_{i}$. Let $\left|k, k_{3}, k_{r}\right\rangle$ be their simultaneous eigenstates. It can be shown that $k_{r}$ behaves exactly as $k_{3}$ relative to $k$, i.e., they are integer spaced and for a given $k,-k \leq k_{r} \leq k$. In fact, as $k_{3}$ is raised and lowered by $K_{1} \pm i K_{2}, k_{r}$ is raised and lowered by $K_{ \pm}$defined as in Eq. (17). Thus, one can construct all $\left|k, k_{3}, k_{r}\right\rangle$ starting from

$$
\begin{equation*}
\left|k, k_{3}=k, k_{r}=k\right\rangle \sim \bar{z}_{1}^{2 k} \tag{25}
\end{equation*}
$$

via a successive application of the lowering operators $K_{1}-i K_{2}$ and $K_{-}$. For instance, one finds

$$
\begin{equation*}
\left|k, k_{3}=k, k_{r}\right\rangle \sim z_{2}^{k-k_{r}} \bar{z}_{1}^{k+k_{r}} \tag{26}
\end{equation*}
$$

They are in fact monopole harmonics. ${ }^{4}$
Since the Hamiltonian is made of $J^{2}$ and $I_{r}$, we need a complete set of commuting operators containing them. We can choose $\vec{J}^{2}, J_{3}, K_{r}, I^{2}, I_{r}$ and $\Sigma$. Since the Gauss law constraint (18) can be written in terms of these operators as $K_{r}=\Sigma$, the problem reduces to constructing the simultaneous eigenstates of $\vec{J}^{2}, J_{3}, I^{2}, I_{r}, \Sigma$ from the simultaneous eigenstates of $K^{2}, K_{3}, I^{2}, I_{3}, \Sigma$. This can be done in general but for the sake of simplicity we will consider $\frac{1}{2}$-isospin case. Since $I_{r}^{2}=\frac{1}{4}$ in this case, the energy spectrum of the Hamiltonian (21) is determined by $j$. Other quantum numbers represent degeneracies. Let $I_{3}=\frac{1}{2} \sigma_{3}$. Among the eigenstates of $I_{r}$ we choose the ones having $j=0$. They are given by

$$
\begin{equation*}
\left|j=0, i_{r}=+1 / 2\right\rangle=\binom{z_{1}}{z_{2}}, \quad\left|j=0, i_{r}=-1 / 2\right\rangle=\binom{-\bar{z}_{2}}{\bar{z}_{1}} \tag{27}
\end{equation*}
$$

Now, it can be shown that the simultaneous eigenfunctions of $\vec{J}^{2}, J_{3}, I^{2}, I_{r}$ and $\Sigma$ can be obtained by taking the product of three states

$$
\begin{equation*}
\left|j, j_{3}, i_{r}, \sigma\right\rangle=\left|j=0, i_{r}\right\rangle\left|j, j_{3}, \sigma+i_{r}\right\rangle|\sigma\rangle \tag{28}
\end{equation*}
$$

where $\left|j, j_{3}, \sigma+i_{r}\right\rangle$ is the monopole harmonics previously defined. We have omitted $I^{2}$ because it is constant.
We illustrate some cases. First, let us consider $\mathcal{N}=1$ case. In this case, $\sigma= \pm 1 / 2$. The ground states are characterized by $j=0$, for which $j_{3}=0$ and $j_{r}=0$. The latter condition implies that $\sigma+i_{r}=0$. Consequently, the ground states consist of the following two states with $E=-1 / 8$ :

$$
\begin{aligned}
& \left|0,0,-\frac{1}{2},+\frac{1}{2}\right\rangle=\left|0,-\frac{1}{2}\right\rangle|0,0,0\rangle\left|+\frac{1}{2}\right\rangle=\binom{-\bar{z}_{2}}{\bar{z}_{1}}\left|+\frac{1}{2}\right\rangle \\
& \left|0,0,+\frac{1}{2},-\frac{1}{2}\right\rangle=\left|0,+\frac{1}{2}\right\rangle|0,0,0\rangle\left|-\frac{1}{2}\right\rangle=\binom{z_{1}}{z_{2}}\left|-\frac{1}{2}\right\rangle
\end{aligned}
$$

Note that $i_{r}=\sigma= \pm 1 / 2$ states are not allowed. One can check that the both states are annihilated by $Q$ and $\bar{Q}$. The first exited states correspond to the value of $j=1$. In this case, $j_{3}=(1,0,-1), i_{r}= \pm 1 / 2, \sigma= \pm 1 / 2$. Therefore, there are twelve states with $E=7 / 8$ :

$$
\begin{align*}
& \left|1,(1,0,-1),+\frac{1}{2},+\frac{1}{2}\right\rangle=\left|0,+\frac{1}{2}\right\rangle|1,(1,0,-1), \quad 1\rangle\left|+\frac{1}{2}\right\rangle=\bar{z} \bar{z}\binom{z_{1}}{z_{2}}\left|+\frac{1}{2}\right\rangle \\
& \left|1,(1,0,-1),-\frac{1}{2},+\frac{1}{2}\right\rangle=\left|0,-\frac{1}{2}\right\rangle|1,(1,0,-1), 0\rangle\left|+\frac{1}{2}\right\rangle=z \bar{z}\binom{-\bar{z}_{2}}{\bar{z}_{1}}\left|+\frac{1}{2}\right\rangle, \\
& \left|1,(1,0,-1),+\frac{1}{2},-\frac{1}{2}\right\rangle=\left|0,+\frac{1}{2}\right\rangle|1,(1,0,-1), 0\rangle\left|-\frac{1}{2}\right\rangle=z \bar{z}\binom{z_{1}}{z_{2}}\left|-\frac{1}{2}\right\rangle, \\
& \left|1,(1,0,-1),-\frac{1}{2},-\frac{1}{2}\right\rangle=\left|0,-\frac{1}{2}\right\rangle|1,(1,0,-1),-1\rangle\left|-\frac{1}{2}\right\rangle=z z\binom{-\bar{z}_{2}}{\bar{z}_{1}}\left|-\frac{1}{2}\right\rangle, \tag{29}
\end{align*}
$$

where the omitted indices on $z z, z \bar{z}$ and $\bar{z} \bar{z}$ are determined depending on the values of $j_{3}$ and $\sigma+i_{r}$. For instance from Eq. (26), one can infer that in the first line of Eq. (29), each $j_{3}=(1,0,-1)$ corresponds to $\bar{z} \bar{z}=\left(\bar{z}_{1}^{2}, \bar{z}_{1} \bar{z}_{2}, \bar{z}_{2}^{2}\right)$,

[^4]in the second line $z \bar{z}=\left(z_{2} \bar{z}_{1},\left|z_{2}\right|^{2}-\left|z_{1}\right|^{2}, z_{1} \bar{z}_{2}\right)$ and in the fourth line $z z=\left(z_{2}^{2}, z_{1} z_{2}, z_{1}^{2}\right)$. In general, the energy eigenfunctions with $j, j_{3}, i_{r}, \sigma$ are given by the states
$$
\left|0, i_{r}\right\rangle\left|j, j_{3}, \sigma+i_{r}\right\rangle|\sigma\rangle
$$
with an arbitrary integer $j, j_{3}\left(-j \leq j_{3} \leq j\right)$, and $i_{r}, \sigma$ satisfying $-j \leq \sigma+i_{r} \leq j$.
Next, let us consider $\mathcal{N}=2$ case. We have $\sigma=+1,0,0,-1$, and the corresponding four spin states will be denoted by $|+1\rangle,\left|0_{ \pm}\right\rangle,|-1\rangle$. Since $i_{r}= \pm 1 / 2$ and $j_{r}=\sigma+i_{r}=(3 / 2,1 / 2,-1 / 2,-3 / 2), j$ can be half integral. Also, $j_{r}$ should satisfy $-j \leq j_{r} \leq j$. The ground states correspond to $j=1 / 2$ states and are given by
\[

$$
\begin{aligned}
\left|0,-\frac{1}{2}\right\rangle\left|\frac{1}{2},\left(+\frac{1}{2},-\frac{1}{2}\right),+\frac{1}{2}\right\rangle|+1\rangle & =\bar{z}\binom{-\bar{z}_{2}}{\bar{z}_{1}}|+1\rangle, \\
\left|0,+\frac{1}{2}\right\rangle\left|\frac{1}{2},\left(+\frac{1}{2},-\frac{1}{2}\right),+\frac{1}{2}\right\rangle\left|0_{ \pm}\right\rangle & =\bar{z}\binom{z_{1}}{z_{2}}\left|0_{ \pm}\right\rangle \\
\left|0,-\frac{1}{2}\right\rangle\left|\frac{1}{2},\left(+\frac{1}{2},-\frac{1}{2}\right),-\frac{1}{2}\right\rangle\left|0_{ \pm}\right\rangle & =z\binom{-\bar{z}_{2}}{\bar{z}_{1}}\left|0_{ \pm}\right\rangle \\
\left|0,+\frac{1}{2}\right\rangle\left|\frac{1}{2},\left(+\frac{1}{2},-\frac{1}{2}\right),-\frac{1}{2}\right\rangle|-1\rangle & =z\binom{z_{1}}{z_{2}}|-1\rangle .
\end{aligned}
$$
\]

Altogether, there are twelve independent ground states with energy $E=1 / 4$. Again omitted indices depend on the values of $j_{3}$. In the first and second line $j_{3}=(+1 / 2,-1 / 2)$ corresponds to $\left(\bar{z}_{1}, \bar{z}_{2}\right)$, and in the last two lines $j_{3}=(+1 / 2,-1 / 2)$ correspond to $\left(z_{2}, z_{1}\right)$. It can be shown that the above ground states are invariant under the half of supersymmetry, and supersymmetry is, in some sense, spontaneously broken from $\mathcal{N}=2$ to $\mathcal{N}=1$.

The analysis can be extended to the general $\mathcal{N}$. Spin states can be obtained using the representation

$$
\begin{aligned}
\beta_{1}= & 1 \otimes 1 \otimes \cdots 1 \otimes \sigma_{-} \\
\beta_{2}= & 1 \otimes 1 \otimes \cdots \sigma_{-} \otimes \sigma_{3} \\
& \cdots \\
\beta_{\mathcal{N}}= & \sigma_{-} \otimes \sigma_{3} \otimes \cdots \otimes \sigma_{3}
\end{aligned}
$$

and the eigenvalues of the spin operator is given by $\sigma=(\mathcal{N} / 2, \mathcal{N} / 2-1, \cdots-\mathcal{N} / 2+1,-\mathcal{N} / 2)$. For $\mathcal{N}=$ odd integer, the ground states are given by $|0, \mp 1 / 2\rangle|0,0,0\rangle| \pm 1 / 2\rangle$, and the supersymmetry is unbroken. For $\mathcal{N}=$ even integer, the ground states are given by $|0, \mp 1 / 2\rangle|1 / 2,(+1 / 2,-1 / 2), \pm 1 / 2\rangle| \pm 1\rangle$ and $|0, \pm 1 / 2\rangle|1 / 2,(+1 / 2,-1 / 2), \pm 1 / 2\rangle|0\rangle$, and half of the supersymmetries is spontaneously broken.

In summary, we showed that the number of supersymmetries can be made arbitrarily large for supersymmetric isospin particles on sphere in the background of specifically chosen spherically symmetric $S U(2)$ gauge field. The supersymmetry generators form the standard $\mathcal{N}=1 \mathrm{SUSY}$ algebra for a single complex fermion, su(2|1) algebra for $\mathcal{N}=2$. But for higher $\mathcal{N}$, it become the nonlinear realization of the $s u(\mathcal{N} \mid 1)$ algebra. We also gave exact energy spectra and corresponding eigenfunctions in the case of $I=1 / 2$ and found that half of the supersymmetry is spontaneously broken if the complex number of fermion degrees of freedom is even. It would be interesting to investigate details of the nonlinear algebra Eq. (22) and extend the analysis further to general values of isospin $I$ other than $1 / 2$.

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[^1]:    ${ }^{1}$ For its quantum mechanical definition, see Eq. 15. In fact, when restricted to $S^{2}$, only the radial component of the spin survives.

[^2]:    ${ }^{2}$ In transition from Dirac brackets to quantum commutator there may arise an ordering ambiguity as noted in Ref. [5]. However, it can be absorbed if the Gauss Law constraint is appropriately ordered. So, here we choose a particular ordering.

[^3]:    ${ }^{3}$ In this paper we define $\Sigma$ to be the outward radial component of the spin operator, so it differs from that of Ref. [5] by the sign.

[^4]:    ${ }^{4}$ The monopole harmonics $Y_{q, l, m}$ of Ref. [10] corresponds to our $\left|k, k_{3}, k_{r}\right\rangle$ with $k=l, k_{3}=m, k_{r}=-q$.

