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Abstract

Estimation of an asset price process under the physical measure can be regarded as the first step of the calibration problem, hence is of practical importance. In this article, supposing that a log-price process is expressed by a possibly skewed stable driven model and that a high-frequency dataset over a fixed period is available, we provide practical procedures of estimating the dominating parameters. Especially, the scale parameter may be time-varying and possibly random as long as it is independent of the driving skewed stable Lévy process. By means of the scaling property and realized bipower variations, it is possible to estimate the index and positivity (skewness) parameters without specific knowledge of the scale process. When the target scale parameter is constant, our estimators are asymptotically normally distributed, the rate of convergence being $\sqrt{n}$. When the scale is actually time-varying, we focus on estimation of the integrated scale, which is an analogue to the integrated volatility in the Brownian-semimartingale framework. In this case we show that estimation of the integrated scale exhibits a kind of asymptotic singularity with respect to the unknown index parameter, with the rate of convergence being the slower $\sqrt{n}/\log n$.

Keywords: High-frequency sampling, Parameter estimation, Skewed stable Lévy process.

1 Introduction

Nowadays there exists a vast amount of option-pricing theories for many kinds of underlying asset price processes, which depend on either finite- or infinite-dimensional unknown parameters. In order to implement the theories in practice, we are often forced to calibrate (estimate) the model in question. Typically, we are first given an underlying asset price process whose law is governed by a physical measure (real world), and then construct a risk-neutral measure under which a price formula is provided through a change of measure. To implement this procedure in practice, the first step is to estimate the structure of the underlying asset price process based on observed return data.

In this article, we address the estimation problem for a class of asset price models driven by a possibly skewed stable Lévy process. Specifically, we provide simple recipes for estimating the parameters governing the law of the log-price process $X = \log S$, where S denotes a univariate asset price process: recall that for a semimartingale X without continuous local martingale part it follows from Itô's formula that

$$dS_t = S_{t-}\{dX_t + (e^{\Delta X_t} - 1 - \Delta X_t)\}$$

with some positive initial variable S_0, where $\Delta X_t := X_t - X_{t-}$ denotes the jump of X at time t. We model X as a stochastic integral of a positive process σ independent of the integrator Z, a skewed stable Lévy process with finite mean. Our model includes the so-called geometric stable Lévy process, where σ is constant. Undoubtedly, Lévy processes, which form *the* continuous-time counterpart of discrete-time random walks, serve as a building block for continuous-time modelling of financial data. We refer the reader to, among others, Bertoin [5] and Sato [16] for systematic accounts of Lévy processes. Recently, Miyahara and Moriwaki [15] (see also Fujiwara and Miyahara [9]) introduced an option-pricing model based on the geometric stable Lévy process and the minimal entropy martingale measure, and shown its usability to, e.g., reproduce the volatility smile/smirk properties.

Our estimation procedure utilizes empirical sign statistics and realized multipower variations (MPV for short), and its implementation is pretty simple and requires no hard numerical optimization, hence preferable in practice. Using MPVs essentially amounts to the classical method of moments with possibly random targets. Some authors have studied asymptotic behaviors concerning MPVs for estimating integrated-σ quantities: Barndorff-Nielsen and Shephard [4, Section 6] for centered and symmetric stable Z, Woerner [18, 20] for general Z admitting a symmetric Lévy density near the origin. The independence between σ and Z was crucial in these papers. On the other hand, Corcuera et al. [7] treated realized power variation for general strictly stable Z with σ not necessarily independent of Z.

Concerning joint-estimation of the stable Lévy processes based on high-frequency data, Masuda [13] considered a joint estimation of the index, scale, and location parameters in case of symmetric Lévy density. There it was shown that the sample median based estimator of the location combined with a variant of the central limit theorem led to full-joint estimators, which are asymptotically normal with finite and nondegenerate asymptotic covariance matrices. In particular, the sample median based estimator turned out to be rate-efficient. Our model setup in this article does not contain the drift parameter (presupposed to be zero), but instead allows possible skewness.

This article is organized as follows. Our model setup and objectives are described in Section 2. Section 3 presents our estimation procedures. Small simulation results are reported in Section 4. Concluding remarks are given in Section 5.

2 Setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, P)$ be an underlying probability space, which is supposed to be rich enough to carry all the random variables and processes appearing below, and to make all the random processes adapted. We denoted by E the expectation operator. For convenience, we start with describing some basic facts concerning the stable distributions and stable Lévy processes.

Denote by $S_\alpha(\rho, \sigma)$ the possibly skewed stable distribution without drift, the characteristic function of which is given by

$$u \mapsto \exp\left\{ -\sigma|u|^\alpha \left(1 - i\mathrm{sgn}(u)\tan\{\alpha\pi(\rho - 1/2)\}\right)\right\}, \quad u \in \mathbb{R}. \tag{1}$$

The dominating parameters α, ρ, and σ correspond to:

- the *stable-index parameter* $\alpha \in (1, 2)$;
- the *positivity parameter* ρ fulfilling that $1 - 1/\alpha < \rho < 1/\alpha$; and
- the *scale parameter* $\sigma > 0$.

We here rule out the "infinite-mean" case (i.e. $\alpha \in (0, 1]$), and also the case of "one-sided jumps" (i.e. either $\rho = 1 - 1/\alpha$ or $1/\alpha$) from our scope; in many cases, this restriction is non-fatal for realistic modelling in finance.

Let ζ stand for a random variable such that $\mathcal{L}(\zeta) = S_\alpha(\rho, \sigma)$. Here and in the sequel, for a random variable ξ we denote its law by $\mathcal{L}(\xi)$. The name of "positivity parameter" of ρ comes from the fact that $P[\zeta \geq 0] = \rho$; trivially, the symmetric case corresponds to $\rho = 1/2$. Note that the positivity parameter of $\mathcal{L}(c\zeta)$ is again ρ whatever $c > 0$ is. For future reference, we mention the closed-form expressions of absolute and signed-absolute moments (cf. Kuruoğlu [11]): for any $r \in (-1, \alpha)$ and $r' \in (-2, -1) \cup (-1, \alpha)$,

$$E[|\zeta|^r] = \frac{\Gamma(1 - r/\alpha)}{\Gamma(1 - r)} \frac{\cos(r\xi/\alpha)}{\cos(r\pi/2)} \frac{\sigma^{r/\alpha}}{|\cos(\xi)|^{r/\alpha}}, \tag{2}$$

$$E[|\zeta|^{r'} \mathrm{sgn}(\zeta)] = \frac{\Gamma(1 - r'/\alpha)}{\Gamma(1 - r')} \frac{\sin(r'\xi/\alpha)}{\sin(r'\pi/2)} \frac{\sigma^{r'/\alpha}}{|\cos(\xi)|^{r'/\alpha}}, \tag{3}$$

where we wrote

$$\xi = \alpha\pi(\rho - 1/2)$$

and the symbol $\mathrm{sgn}(u)$ expresses $1, 0, -1$ according as $u > 0, = 0, < 0$, respectively. We write

$$\mu_r = \sigma^{-r/\alpha} E[|\zeta|^r] \quad \text{and} \quad \nu_{r'} = \sigma^{-r'/\alpha} E[|\zeta|^{r'} \mathrm{sgn}(\zeta)],$$

the rth absolute and r'th signed-absolute moments associated with $S_\alpha(\rho, 1)$, respectively.

The most familiar parametrization of the stable distribution is, instead of (1),

$$u \mapsto \exp\left\{ - (\sigma|u|)^\alpha \left(1 - i\beta \,\mathrm{sgn}(u) \tan\frac{\alpha\pi}{2}\right)\right\},$$

where the skewness parameter fulfils $\beta \in (-1, 1)$, the symmetric case corresponding to $\beta = 0$; as such, ρ and β have the one-to-one relation

$$\tan\left\{\alpha\pi\left(\rho - \frac{1}{2}\right)\right\} = \beta \tan\frac{\alpha\pi}{2}.$$

Also, regarding as ρ as a function of β (for any fixed $\alpha \in (1, 2)$), it can be seen that ρ is monotonically decreasing on $(-1, 1)$. Hence $\rho - 1/2$ and β have opposite signs for $\alpha \in (1, 2)$, which is not the case for $\alpha \in (0, 1)$; Figure 1 illustrates this point, where also included just for comparison is the case of $\alpha = 0.8$. Interested readers can consult Zolotarev [21] for more details concerning one-dimensional stable distributions; see also Borak *et al.* [6].

The reason why we have chosen the parametrization (1) is that, as is expected from Figure 1, estimation performance of β based on the empirical sign is destabilized for α close to 2. That is to say, a "small" change of the empirical-sign quantity (see Section 3.1.1) leads to a "big" diremption of the estimate of β from the true value; this point can be seen from Figure 1, where the curve is gentler for α closer to 2.

Denote by $Z = (Z_t)_{t\in[0,1]}$ a univariate Lévy process starting from the origin such that

$$\mathcal{L}(Z_t) = S_\alpha(\rho, t), \quad t \in [0, 1]. \tag{4}$$

The image measure of the process Z is completely characterized by the two parameter α and ρ. Figure 2 shows two simulated sample paths of Z.

For the stable Lévy processes, the (tail-)index α also corresponds to the Blumenthal-Getoor activity index (see, e.g., Sato [16, p.362]). In view of (4), we see that the time parameter t directly serves as the scale in the parametrization (1).

The process Z itself does not accommodate the scale parametrization. Now we introduce a possibly time-varying scale process. Let $\sigma = (\sigma_t)_{t\in[0,1]}$ be a positive càdlàg process (right-continuous and having left-hand side limits) independent of Z, such that

$$P\left[\int_0^1 \sigma_s^2 ds < \infty\right] = 1. \tag{5}$$

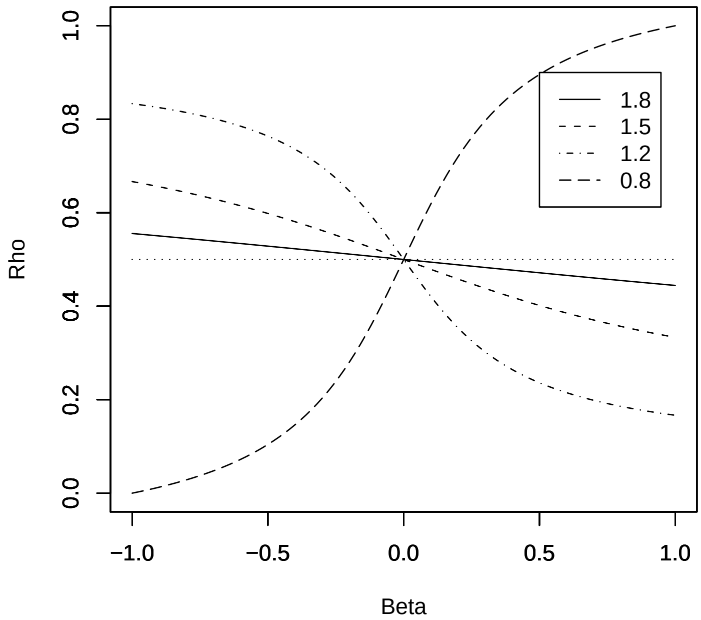

Figure 1: Plots of ρ as a function of β for the values $\alpha = 0.8, 1.2, 1.5$, and 1.8.

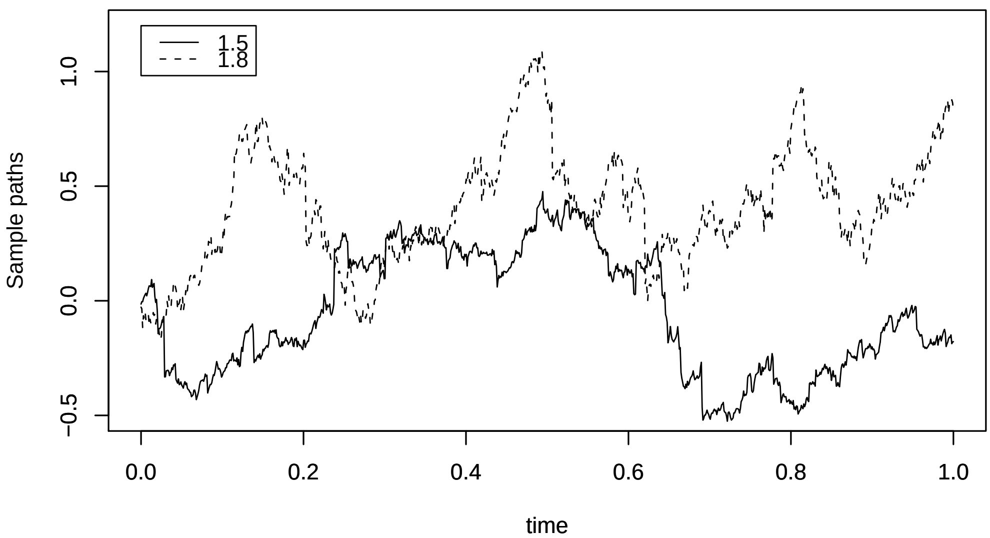

Figure 2: Two simulated sample paths of Z of (4) for $\alpha = 1.5$ and 1.8, with $\beta = -0.5$ and $\sigma_t \equiv 1$; although we drew solid and dashed lines for clarity, they are actually of pure-jump in theory.

Then we consider the process $X = (X_t)_{t\in[0,1]}$ given by

$$X_t = \int_0^t \sigma_{s-} dZ_s$$

as a model of a univariate log-price process under physical measure; without loss of generality, we have set $X_0 = 0$. The condition (5) is sufficient in order to make the stochastic integral well-defined; see, e.g., Applebaum [1, Section 4.3.3] for a general account of stochastic integration. Additionally, for a technical reason, we impose the following structure on σ^α (the αth power of σ) which is borrowed from Barndorff-Nielsen *et al.* [3, Hypothesis (H1)] (see also Barndorff-Nielsen *et al.* [2]):

$$\begin{aligned}\sigma_t^\alpha &= \sigma_0^\alpha + \int_0^t a_s ds + \int_0^t b_{s-} dw_s \\ &\quad + \int_0^t \int h \circ c(s-,z)(\mu-\nu)(ds,dz) + \int_0^t \int (c - h\circ c)(s-,z)\mu(ds,dz).\end{aligned}$$

Here the ingredients are as follows: w is a standard Wiener process; μ is a Poisson random measure having the intensity measure $\nu(ds,dz) = dsF(dz)$, where F is a σ-finite measure on $(0,\infty)\times\mathbb{R}$; a and b are real-valued càdlàg processes; $c : \Omega\times[0,\infty)\times\mathbb{R} \to \mathbb{R}$ is a càdlàg process satisfying that (i) $c(s,z) = c(\omega; s,z)$ is $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R})$-measurable for each s, and that (ii) $\sup_{\omega\in\Omega, s<S_k(\omega)} |\delta(\omega; s,z)| \le \psi_k(z)$ for some nonrandom functions $\psi_k(z)$ fulfilling that $\int_{\mathbb{R}}\{1\wedge\psi_k(z)^2\}F(dz) < \infty$ and stopping times S_k such that $S_k \to \infty$ a.s.; finally, h is a continuous function on $\mathbb{R}$ with compact support such that $h(x) = x$ near the origin.

Such σs constitute a broad class of the so-called Itô's semimartingales, including diffusions with jumps.

Remark 2.1. *Extending the present time period* $[0,1]$ *to* $[0,\infty)$*, we may equivalently set* $X_t = Z_{\int_0^t \sigma_s^\alpha ds}$ *provided that the "clock" process* $\int_0^t \sigma_s^\alpha ds \to \infty$ *a.s. for* $t \to \infty$*. This time-change representation is known to be inherent in the case of stable-Lévy integrators among general Lévy ones; see Kallsen and Shiryaev [10] for details.*

Remark 2.2. *We have set the target period is* $[0,1]$ *from the very beginning. However, this point is a matter of no importance: enlarging the length of the period is reflected in making* $\int_0^1 \sigma_s^\alpha ds$ *larger through* σ*.*

Suppose that we have a discrete-time data with sampling mesh $1/n$ over the target period $[0,1]$, where n denotes the sample size; namely, we observe the sequence of log prices

$$X_{1/n}, X_{2/n}, \dots, X_{(n-1)/n}, X_1.$$

The log-price model described above is governed by the parameter $(\rho, \alpha, \sigma.)$ unknown to observers. Nevertheless, note that $(\rho, \alpha, \sigma.)$ is possibly infinite-dimensional. Hopefully we will be able to estimate σ_t for each $t \in [0,1]$, but this is beyond the scope of this article; to the best of author's knowledge, no such result has been obtained in the non-Gaussian stable driven case. Instead, we are going to confine our objective to the following:

(A) estimation of (ρ, α, σ) when $\sigma_t \equiv \sigma$ for a positive constant σ.

(B) estimation of $(\rho, \alpha, \int_0^1 \sigma_s^\alpha ds)$ when $\sigma.$ is actually time-varying (possibly random).

Our goal is to provide an explicit recipe of interval estimation based on the available high-frequency data (i.e., for $n \to \infty$). To this end we are going to derive asymptotic (mixed) normality with specific asymptotic covariance matrix as well as rate of convergence.

Of course the case (A) is formally included in the case (B), however, we need a separate argument to consider the latter. In both cases, we first construct a simple estimator of (α, ρ)

with leaving $\sigma.$ unknown. Then, using the estimates, we provide a estimator of σ or $\int_0^1 \sigma_s^\alpha ds$. Estimation of integrated quantities such as $\int_0^1 \sigma_s^\alpha ds$ is already known to be possible in the light of recently developed theory of MPV for pure-jump processes; see Woerner [20] and references therein. However, to implement the procedure, as a matter of fact we need estimates of α and ρ beforehand. We can avoid this inconvenience since, in our estimation procedure, an estimator of (ρ, α) is first provided without using information of σ. This is a great advantage of our estimation procedure.

According to the scaling property of the strictly stable distributions and the independence between σ and Z, we have

$$\mathcal{L}(X_1|\sigma) = S_\alpha\left(\rho, \int_0^1 \sigma_s^\alpha ds\right) \tag{6}$$

in the case (B). It seems natural to target at the integrated scale $\int_0^1 \sigma_s^\alpha ds$; the major estimation target in the familiar Brownian semimartingale framework (e.g., Barndorff-Nielsen *et al.* [2, 3] as well as their references) is the integrated volatility $\int_0^1 \sigma_s^2 ds$. The author expects that the pricing strategy of Miyahara and Moriwaki [15] for the geometric stable Lévy process remains valid even for the cases of time-varying scale, as long as the option in question is of European type, in which only an expectation of the "terminal" variable (namely, X_1 in our framework) is concerned: this is just because, as specified in (6), the $\mathcal{L}(X_1|\sigma.)$ is exactly stable.

3 Description of estimation procedure

3.1 Preliminaries

Write the increments of successive observations as

$$\Delta_i X = X_{i/n} - X_{(i-1)/n}, \quad i \leq n.$$

Conditional on the process σ, the random variables $\Delta_i X$ are mutually independent and for each $n \in \mathbb{N}$ and $i \leq n$

$$\mathcal{L}(\Delta_i X|\sigma) = S_\alpha\left(\rho, \int_{(i-1)/n}^{i/n} \sigma_s^\alpha ds\right).$$

Before proceeding let us remind two fundamental facts, which are several times used in the sequel without notice.

- Since we are concerned here with the weak property, we may set

  $$\Delta_i X = (\bar{\sigma}_i/n)^{1/\alpha}\zeta_i \quad \text{a.s.},$$

  where $\bar{\sigma}_i := n\int_{(i-1)/n}^{i/n} \sigma_s^\alpha ds$ and (ζ_i) is an i.i.d. sequence with common law $S_\alpha(\rho, 1)$.

- Let Λ_n be a sequence of essentially bounded functionals on the product space of the path spaces of Z and σ, and let $\lambda_n(\sigma) := \int \Lambda_n(\sigma, z)P^Z(dz)$, where P^ξ denotes the image measure of a variable ξ. Suppose $\lambda_n(\sigma) \to^p \lambda_0(\sigma)$ for some functional λ_0 on the path spaces of σ, where $\to^p$ denotes the convergence in probability. In view of the independence between Z and σ, a disintegration argument gives $\lambda_n(\sigma) = E[\Lambda_n(\sigma, Z)|\sigma]$ a.s., moreover, the boundedness of $\{\lambda_n(\sigma)\}_{n\in\mathbb{N}}$ yields convergence of moments, namely, $E[\Lambda_n(\sigma, Z)] = \int \lambda_n(\sigma)P^\sigma(d\sigma) \to \int \lambda_0(\sigma)P^\sigma(d\sigma)$. That is to say, we may actually treat σ a nonrandom process in the process of deriving weak limit theorems. In particular, if some functionals $S_n(\sigma_0, Z)$ with fixed σ_0 are asymptotically centered normal with covariance matrix $V(\sigma_0)$, then it automatically follows that the limit distribution of $S_n(\sigma, Z)$ has the characteristic function $u \mapsto \int \exp\{-u^\top V(\sigma)u/2\}P^\sigma(d\sigma)$, a mixed normal if σ is random.

These are trivial, but crucial in our study.[1]

As mentioned before, first we construct concrete estimators of ρ and α in this order without any further knowledge of the scale process σ. (Section 3.2), and then, using the estimates of ρ and α so obtained, we give estimators of the remaining σ or $\int_0^1 \sigma_s^\alpha ds$ according as the cases (A) or (B) (Sections 3.3 and 3.4). For later use, in the rest of this subsection we give some background information on the empirical-sign statistics and MPVs.

3.1.1 Expression of empirical-sign statistics

Let $H_n := n^{-1}\sum_{i=1}^n \mathrm{sgn}(\Delta_i X)$, then $H_n = \frac{1}{n}\sum_{i=1}^n \mathrm{sgn}(\zeta_i) \to^p E[\mathrm{sgn}(\zeta_1)] = 2\rho - 1$. Hence

$$\hat{\rho}_n := \frac{1}{2}(H_n + 1) \tag{7}$$

serves as a consistent estimator of ρ. Since

$$\sqrt{n}(\hat{\rho}_n - \rho) = \sum_{i=1}^n \frac{1}{2\sqrt{n}}\{\mathrm{sgn}(\zeta_i) - (2\rho - 1)\}, \tag{8}$$

we easily deduce the asymptotic normality $\sqrt{n}(\rho_n - \rho) \to^d \mathcal{N}_1(0, \rho(1-\rho))$, where the symbol $\to^d$ stands for the weak convergence. It is nice that the asymptotic variance only depends on ρ as it directly enables us to provide a confidence interval of $\hat{\rho}_n$. Despite of its simplicity, it exhibits unexpectedly good finite-sample performances; see Section 4.

Perhaps the simplest possible estimator of ρ is not (7) but $n^{-1}\sum_{i=1}^n I(\Delta_i X \geq 0)$, where $I(A)$ denotes the indicator function of an event A. The reason why we chose (7) is that, thanks to (3), it leads to an explicit asymptotic covariance between the estimator of the remaining parameters. Moreover, the asymptotic variance of $n^{-1}\sum_{i=1}^n I(\Delta_i X \geq 0)$ is $\rho(1-\rho)$, which is the same as that of (7). See Section 3.2 for details.

Remark 3.1. *There are other possible ways to construct an estimate of ρ, for example, the method of moments based on $E[|\zeta|^q]$ together with $E[\zeta^{\langle q \rangle}]$, where $\mathcal{L}(\zeta) = S_\alpha(\rho, 1)$ (see Kuruoğlu [11]). However, in this case the asymptotic variance of the resulting estimator must depend on the true value of α.*

Remark 3.2. *It may be expected that there is no other Lévy process than the stable one, for which we can consistently estimate the "degree of skewness" in such a simple way. For instance, the familiar generalized hyperbolic Lévy process has the skewness parameter, but it can be consistently estimated only when we target at long-term asymptotics; see, e.g., Woerner [19].*

3.1.2 Expression of normalized MPV

Fix an $m \in \mathbb{N}$, and let $r = (r_l)_{l=1}^m$ be such that $r_l \geq 0$, $r_+ := \sum_{l=1}^m r_l > 0$, and $\max_{l \leq m} r_l < \alpha/2$. Then we define the rth MPV as

$$M_n(r) := \frac{1}{n}\sum_{i=1}^{n-m+1}\prod_{l=1}^m |n^{1/\alpha}\Delta_{i+l-1}X|^{r_l}. \tag{9}$$

(We should note that this quantity does depend on the unknown α if $r_+ = \alpha$.) By the equivalent expression of $(\Delta_i X)$, we may replace "$|n^{1/\alpha}\Delta_{i+l-1}X|^{r_l}$" in the right-hand side of (9) by "$\bar{\sigma}_{i+l-1}^{r_l/\alpha}|\zeta_{i+l-1}|^{r_l}$". Let

$$\sigma_q^* := \int_0^1 \sigma_s^q ds$$

[1] Moreover, if necessary in the proof, we may suppose that $(\sigma_t)_{t\in[0,1]}$ is bounded from above and bounded away from zero without loss of generality: this follows from the localization arguments as in Barndorff-Nielsen *et al.* [3, Section 3].

for $q > 0$ and $\mu(r) := \prod_{l=1}^m \mu_{r_l}$. Here we prepare a first-order stochastic expansion useful for our goal.

Observe that

$$\sqrt{n}\{M_n(r) - \mu(r)\sigma^*_{r_+}\} = \sum_{i=1}^{n-m+1} \frac{1}{\sqrt{n}}\chi'_{ni}(r) + R_n(r),$$

where

$$\chi'_{ni}(r) := \left(\prod_{l=1}^m \bar{\sigma}_{i+l-1}^{r_l/\alpha}\right)\left(\prod_{l=1}^m |\zeta_{i+l-1}|^{r_l} - \mu(r)\right),$$

$$R_n(r) := \mu(r)\left\{\sum_{i=1}^{n-m+1} \frac{1}{\sqrt{n}}\left(\prod_{l=1}^m \bar{\sigma}_{i+l-1}^{r_l/\alpha} - \sigma_{(i-1)/n}^{r_+}\right) + \sum_{i=1}^{n-m+1} \sqrt{n}\int_{(i-1)/n}^{i/n} (\sigma_{(i-1)/n}^{r_+} - \sigma_s^{r_+})ds\right\} + O\left(\frac{1}{\sqrt{n}}\right).$$

From the same argument as in Woerner [20] together with Barndorff-Nielsen *et al.* [3] (see also Masuda [14]), we can deduce that $R_n(r) \to^p 0$. Similarly, straightforward but rather messy computations lead to

$$\sum_{i=1}^{n-m+1} \frac{1}{\sqrt{n}}\chi'_{ni}(r) = \sum_{i=m}^{n} \frac{1}{\sqrt{n}}\chi_{ni}(r) + o_p(1),$$

where

$$\chi_{ni}(r) := \left(\prod_{l=1}^m \bar{\sigma}_{i-m+l}^{r_l/\alpha}\right)\sum_{q=1}^m \left(\prod_{l=1}^{q-1} |\zeta_{i+l-q}|^{r_l}\right)\left(\prod_{l=q+1}^m \mu_{r_l}\right)(|\zeta_i|^{r_q} - \mu_{r_q}).$$

In summary, we have

$$\sqrt{n}\{M_n(r) - \mu(r)\sigma^*_{r_+}\} = \sum_{i=m}^{n} \frac{1}{\sqrt{n}}\chi_{ni}(r) + o_p(1). \tag{10}$$

3.1.3 A basic limit result

Building on the arguments above, we now derive a basic distributional result.

Let $r = (r_l)_{l=1}^m$ be as before, and also let $r' = (r'_l)_{l=1}^m$ be another vector fulfilling the same conditions as r. In what follows we set

$$r_+ = r'_+ = p \tag{11}$$

for some $p > 0$; this setting is enough for both (A) and (B). We here derive the limit distribution (normal conditional on σ) of the random vectors

$$S_n(r, r') := \sqrt{n}\begin{pmatrix} H_n & - & (2\rho - 1) \\ M_n(r) & - & \mu(r)\sigma^*_p \\ M_n(r') & - & \mu(r')\sigma^*_p \end{pmatrix},$$

which serves as a basic tool for our purpose.

In view of (8) and (10), it follows that $S_n(r, r')$ admits the stochastic expansion

$$S_n(r, r') = \sum_{i=m}^{n} \frac{1}{\sqrt{n}}\begin{pmatrix} \mathrm{sgn}(\zeta_i) - (2\rho - 1) \\ \chi_{ni}(r) \\ \chi_{ni}(r') \end{pmatrix} + o_p(1) =: \sum_{i=m}^{n} \frac{1}{\sqrt{n}}\gamma_{ni} + o_p(1).$$

For the leading term $\sum_{i=m}^n \frac{1}{\sqrt{n}}\gamma_{ni}$, we can apply a central limit theorem either for finite-order dependent arrays or for martingale difference arrays. Here we formally use the latter, where the underlying filtration may be taken as $\{\mathcal{G}_{ni}\}_{i\le n}$ with $\mathcal{G}_{ni} := \sigma(\zeta_j : j \le i)$; recall that we are now regarding σ a nonrandom process. The Lindeberg condition readily follows from the condition

$$\max_{l\le m}(r_l \vee r_l') < \frac{\alpha}{2},$$

hence it suffices to compute the quadratic variation. Therefore we are left to finding the limits in probability of $n^{-1}\sum_{i=m}^n E[\gamma_{ni}\gamma_{ni}^\top|\mathcal{G}_{n,i-1}]$. After lengthy computation, it turns out that, under the regularity conditions imposed on σ,

$$\frac{1}{n}\sum_{i=m}^n E\big[\gamma_{ni}\gamma_{ni}^\top\big|\mathcal{G}_{n,i-1}\big] \to^p \Sigma(\rho,\alpha,\sigma.) := \begin{pmatrix} 4\rho(1-\rho) & A(r)\sigma^*_{r_+} & A(r')\sigma^*_{r'_+} \\ & B(r,r)\sigma^*_{2r_+} & B(r,r')\sigma^*_{r_++r'_+} \\ \text{sym.} & & B(r',r')\sigma^*_{2r'_+} \end{pmatrix},$$

where we conveniently wrote

$$\begin{aligned} A(r) &= \sum_{q=1}^m \Big(\prod_{1\le l\le m, l\ne q}\mu_{r_l}\Big)\{\nu_{r_q} - (2\rho-1)\mu_{r_q}\}, \\ B(r,r') &= \prod_{1=1}^m \mu_{r_l+r_l'} - (2m-1)\prod_{1=1}^m \mu_{r_l}\mu_{r_l'} \\ &\quad + \sum_{q=1}^{m-1}\Big\{\Big(\prod_{l=1}^{m-q}\mu_{r_l'}\Big)\Big(\prod_{l=m-q+1}^m \mu_{r_l'+r_{l-m+q}}\Big)\Big(\prod_{l=q+1}^m \mu_{r_l}\Big) \\ &\qquad + \Big(\prod_{l=1}^{m-q}\mu_{r_l}\Big)\Big(\prod_{l=m-q+1}^m \mu_{r_l+r'_{l-m+q}}\Big)\Big(\prod_{l=q+1}^m \mu_{r_l'}\Big)\Big\}, \end{aligned}$$

with obvious analogues $A(r')$ and $B(r,r)$, and $B(r',r')$. Thus we arrive at

$$S_n(r,r') \to^d \mathcal{N}_3\big(0, \Sigma(\rho,\alpha,\sigma.)\big), \tag{12}$$

which implies that the limit distribution of $S_n(r,r')$ is a normal scale mixture conditional on σ with conditional covariance matrix $\Sigma(\rho,\alpha,\sigma.)$. Here we note that $\Sigma(\rho,\alpha,\sigma.)$ depends on the process $\sigma.$ only through the integrated quantities $\sigma^*_{r_+}$, $\sigma^*_{r'_+}$, $\sigma^*_{2r_+}$, $\sigma^*_{2r'_+}$, and $\sigma^*_{r_++r'_+}$.

Having the basic convergence (12) in hand, we now turn to our main objectives, (A) and (B) mentioned in Section 2.

3.2 Joint asymptotic normality

Given a $p>0$ and (r,r') (remind that we are assuming (11)), we write $\hat\theta_n := (\hat\rho_n, \hat\alpha_{p,n}, \hat\sigma^*_{p,n})$ for the random root of

$$\begin{pmatrix} H_n & - & (2\rho-1) \\ M_n(r) & - & \mu(r)\sigma^*_p \\ M_n(r') & - & \mu(r')\sigma^*_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{13}$$

For a moment we suppose that such a $\hat\theta_n$ indeed exists. We introduce the function

$$F(\rho,\alpha,s) := \begin{pmatrix} 2\rho-1 \\ \mu(r)s \\ \mu(r')s \end{pmatrix}.$$

Now let us recall (2) with $\sigma=1$. As we are assuming that $\alpha\in(1,2)$ and $1-1/\alpha<\rho<1/\alpha$, we have $\xi\in(-\pi/2,\pi/2)$, so that $\cos(\xi)>0$. Hence the quantities $\mu(r)$ and $\mu(r')$ are continuously differentiable with respect to (ρ, α). Let $D_\rho(r) := \frac{\partial}{\partial\rho}\mu(r)$ and $D_\alpha(r) := \frac{\partial}{\partial\alpha}\mu(r)$: here, the variable s is supposed to be independent of (ρ, α). Trivially,

$$\nabla F(\rho, \alpha, s) = \begin{pmatrix} 2 & 0 & 0 \\ sD_\rho(r) & sD_\alpha(r) & \mu(r) \\ sD_\rho(r') & sD_\alpha(r') & \mu(r') \end{pmatrix},$$

which is nonsingular for each $s > 0$ as soon as

$$\mu(r')D_\alpha(r) \neq \mu(r)D_\alpha(r'). \tag{14}$$

Again let us recall that we may proceed as if σ is nonrandom. The classical delta method (e.g., van der Vaart [17]) yields that, if (14) holds true, then

$$\sqrt{n}\begin{pmatrix} \hat{\rho}_n - \rho \\ \hat{\alpha}_{p,n} - \alpha \\ \hat{\sigma}^*_{p,n} - \sigma^*_p \end{pmatrix} \to^d \mathcal{N}_3(0, V(\rho, \alpha, \sigma.)), \tag{15}$$

where

$$V(\rho, \alpha, \sigma.) := \{\nabla F(\rho, \alpha, \sigma^*_p)\}^{-1}\Sigma(\rho, \alpha, \sigma.)\{\nabla F(\rho, \alpha, \sigma^*_p)\}^{-1,\top}.$$

We see that $\Sigma(\rho, \alpha, \sigma.)$ here depends on σ only through σ^*_p and σ^*_{2p}; hence, more specifically we may write $\Sigma(\rho, \alpha, \sigma.) = \Sigma(\rho, \alpha, \sigma^*_p, \sigma^*_{2p})$, and accordingly, $V(\rho, \alpha, \sigma.) = V(\rho, \alpha, \sigma^*_p, \sigma^*_{2p})$. We should note that the function $V(\rho, \alpha, \sigma^*_p, \sigma^*_{2p})$ is fully explicit as a function of its four arguments.

Now we set $m = 2$ and consider $r = (2q, 0)$ and $r' = (q, q)$ for a $q > 0$ (hence $p = 2q$). In order to make (12) valid, we need $q < \alpha/4$: as we are assuming that $\alpha \in (1, 2)$, a naive choice is $q = 1/4$ (see Remark 3.3 below).

Let us mention the computation of the solution to (13). We already have a closed-form solution $\hat{\rho}_n$ in (7). As for $\hat{\alpha}_n$, we can conveniently utilize the second and third arguments of (13): write $\hat{\mu}(\cdot)$ for the $\mu(\cdot)$ with (ρ, α) replaced by $(\hat{\rho}_n, \hat{\alpha}_{p,n})$, and then consider the estimating equation $M_n(q, q)/M_n(2q, 0) = \hat{\mu}(q, q)/\hat{\mu}(2q)$, which can be rewritten as

$$\frac{\sum_{i=1}^{n-1} |\Delta_i X|^q |\Delta_{i+1} X|^q}{\sum_{i=1}^{n} |\Delta_i X|^{2q}} = C_1(q)C_2(q, \hat{\rho}_n)\frac{\{\Gamma(1 - q/\hat{\alpha}_{p,n})\}^2}{\Gamma(1 - 2q/\hat{\alpha}_{p,n})}, \tag{16}$$

where, having $\hat{\rho}_n$ beforehand, we can regard

$$C_1(q) := \frac{\Gamma(1 - 2q)\cos(q\pi)}{\{\Gamma(1 - q)\cos(q\pi/2)\}^2} \quad \text{and} \quad C_2(q, \hat{\rho}_n) := \frac{[\cos\{q\pi(\hat{\rho}_n - 1/2)\}]^2}{\cos\{2q\pi(\hat{\rho}_n - 1/2)\}}$$

as constants. Since the function

$$\alpha \mapsto \frac{\{\Gamma(1 - q/\alpha)\}^2}{\Gamma(1 - 2q/\alpha)} \tag{17}$$

is strictly monotone on $(1, 2)$, it is easy to search the root $\hat{\alpha}_{p,n}$. Clearly, the root does uniquely exists with probability tending to one.

Remark 3.3. *We see that the range of the function (17) becomes narrower for smaller q, so that the root $\hat{\alpha}_{p,n}$ becomes too sensitive for a small change of the sample quantity in the left-hand side of (16). This implies that the law of large numbers for the sample quantity should be in force with high degree of accuracy for smaller q.*

Thus, given a $p = 2q > 0$, we could get the estimates $\hat{\rho}_n$ and $\hat{\alpha}_{p,n}$ without special knowledge of σ, which may be time-varying and random as long as the regularity conditions on σ imposed on Section 2 hold true. It is important here that we have used the bipower variation in part; the procedure using the first and second empirical moments as in Masuda [13] is valid only when σ is constant.

The present asymptotic covariance matrix is $V(\rho,\alpha,\sigma^*_{2q},\sigma^*_{4q})$, for which we want to provide a consistent estimator. We only need to give consistent estimators of σ^*_{2q} and σ^*_{4q}; recall that we need

$$4q < \alpha$$

in order to make the distributional result (15) with $p = 2q$ valid. For instance, we can proceed as follows. First, (15) with $p = 2q$ implies that $M_n(2p,0) \to^p \mu(2q,0)\sigma^*_{2q}$. Using the estimates $(\hat{\rho}_n,\hat{\alpha}_{p,n})$ and the continuous mapping theorem, we deduce that $M_n(2q,0)/\hat{\mu}(2q,0)$ is a consistent estimator of σ^*_{2q}. We should notice the dependence of $M_n(2q,0)$ on α (recall (9)): $M_n(2q,0) = n^{2q/\alpha-1}\sum_{i=1}^n |\Delta_i X|^{2q}$. Nevertheless, as in Masuda [13, Remark 8], we see that the α can be replaced by $\hat{\alpha}_{p,n}$ since we already know that $\sqrt{n}(\hat{\alpha}_{p,n}-\alpha) = O_p(1)$. Therefore,

$$\hat{\sigma}^*_{2q,n} := \frac{n^{2q/\hat{\alpha}_{p,n}-1}}{\hat{\mu}(2q,0)}\sum_{i=1}^n |\Delta_i X|^{2q} \to^p \sigma^*_{2q}. \tag{18}$$

Once again, let us remind that $\hat{\mu}(2q,0)$ can be easily computed in view of (2) with $\sigma = 1$. By the same token, we could deduce that (still under $4q < \alpha$, of course)

$$\hat{\sigma}^*_{4q,n} := \frac{n^{4q/\hat{\alpha}_{p,n}-1}}{\hat{\mu}(2q,2q)}\sum_{i=1}^{n-1} |\Delta_i X|^{2q}|\Delta_{i+1} X|^{2q} \to^p \sigma^*_{4q}.$$

After all, $V(\hat{\rho}_n,\hat{\alpha}_{p,n},\hat{\sigma}^*_{2q,n},\hat{\sigma}^*_{4q,n})$ can serve as a desired consistent estimator.

Now we are in position to complete our main objectives (A) and (B).

3.3 Case (A): geometric skewed stable Lévy process

When $\sigma_t \equiv \sigma > 0$, our model reduces to the geometric skewed stable Lévy process. In this case we can perform a full-joint interval estimation concerning the dominating (three-dimensional) parameter (ρ,α,σ) at rate $\sqrt{n}$.

We keep using the framework of the last subsection. It directly follows from (15) that

$$\sqrt{n}\begin{pmatrix} \hat{\rho}_n - \rho \\ \hat{\alpha}_{p,n} - \alpha \\ (\hat{\sigma}_{p,n})^p - \sigma^p \end{pmatrix} \to^d \mathcal{N}_3(0, V(\rho,\alpha,\sigma)), \tag{19}$$

where $V(\rho,\alpha,\sigma)$ explicitly depends on the three-dimensional parameter (ρ,α,σ); recall that $p = 2q < \alpha/2$. Applying the delta method to (19) in order to convert $(\hat{\sigma}_{p,n})^p$ to $\hat{\sigma}_{p,n}$ in (19), we readily get the asymptotic normality of $\sqrt{n}(\hat{\rho}_n-\rho,\hat{\alpha}_{p,n}-\alpha,\hat{\sigma}_{p,n}-\sigma)$; we omit the details. Our first objective (A) is thus achieved.

In summary, we may proceed with the choice $q = 1/4$ (so $p = 1/2$) as follows.

1. Compute the estimate $\hat{\rho}_n$ of ρ by (7).
2. Using the $\hat{\rho}_n$, find the root $\hat{\alpha}_{1/2,n}$ of (16).
3. Using $(\hat{\rho}_n,\hat{\alpha}_{1/2,n})$ thus obtained, an estimate of σ is provided by, e.g. (recall (18)),

$$\hat{\sigma}_{1/2,n} := \left\{\frac{n^{1/(2\hat{\alpha}_{p,n})-1}}{\hat{\mu}(1/2,0)}\sum_{i=1}^n \sqrt{|\Delta_i X|}\right\}^2.$$

3.4 Case (B): time-varying scale process

Now we turn to the case (B). Again by means of the argument give in Section 3.2, it remains to construct an estimator of $\sigma^*_\alpha = \int_0^1 \sigma_s^\alpha ds$. The point here is that, different from the case (A), a direct use of (15) is sufficient to deduce the distributional result concerning estimating σ^*_α because the dependence of (r, r') on α is not allowed there. In order to utilize $M_n(r)$ with r depending on α, we need some additional arguments.

Extracting the second row of (12), we have

$$\sqrt{n}\{M_n(r) - \mu(r)\sigma^*_{r_+}\} \to^d \mathcal{N}_1\big(0, B(r,r)\sigma^*_{2r_+}\big). \tag{20}$$

In view of the condition $\max_{l\le m} r_l < \alpha/2$, we need (at least) a tripower variation for setting $r_+ = \alpha$. For simplicity, we set $m = 3$ and

$$r = r(\alpha) = \left(\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3}\right).$$

With this choice, we are going to provide an estimator of σ^*_α with specifying its rate of convergence and limiting distribution.

Let $M^*_n(\alpha) := M_n(\alpha/3, \alpha/3, \alpha/3)$. In this case the normalizing factor is $n^{r_+/\alpha-1} \equiv 1$, so that

$$M^*_n(\alpha) = \sum_{i=1}^{n-2}\prod_{l=1}^{3}|\Delta_{i+l-1}X|^{\alpha/3},$$

which is computable as soon as we have an estimate of α. We have already obtained the estimator $\hat{\alpha}_{p,n}$, hence want to use $M^*_n(\hat{\alpha}_{p,n})$. For this, we have to look at the asymptotic behavior of the gap

$$\sqrt{n}\{M^*_n(r(\alpha)) - \mu(r(\alpha))\sigma^*_\alpha\} - \sqrt{n}\{M^*_n(\hat{\alpha}_{p,n}) - \mu(r(\hat{\alpha}_{p,n}))\sigma^*_\alpha\},$$

namely, the effect of "plugging in $\hat{\alpha}_{p,n}$".

By means of Taylor's formula

$$a^x = a^y + (\log a)^y(x-y) + (\log a)^2\int_0^1 (1-u)a^{y+u(x-y)}du(x-y)^2$$

applied to the function $x \mapsto a^x$ $(x, y, a > 0)$, we get

$$\begin{aligned}
&\sqrt{n}\left\{M^*_n(\hat{\alpha}_{p,n}) - \mu\left(\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3}\right)\sigma^*_\alpha\right\}\\
&= \sqrt{n}\left\{M^*_n(\alpha) - \mu\left(\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3}\right)\sigma^*_\alpha\right\} + \frac{1}{3}\sqrt{n}(\hat{\alpha}_{p,n}-\alpha)\sum_{i=1}^{n-2} x_i^{\alpha/3}\log x_i\\
&\quad + \left\{\frac{1}{3}\sqrt{n}(\hat{\alpha}_{p,n}-\alpha)\right\}^2\frac{1}{\sqrt{n}}\sum_{i=1}^{n-2}(\log x_i)^2\int_0^1(1-u)x_i^{\{\alpha+u(\hat{\alpha}_{p,n}-\alpha)\}/3}du,
\end{aligned} \tag{21}$$

where we wrote $x_i = \prod_{l=1}^3 |\Delta_{i+l-1}X|$. We look at the right-hand side of (21) termwise. Let $y_i := \prod_{l=1}^3 |n^{1/\alpha}\Delta_{i+l-1}X|$.

- The first term is $O_p(1)$, as is evident from (20).
- Concerning the second term, we have

$$\begin{aligned}
\sum_{i=1}^{n-2} x_i^{\alpha/3}\log x_i &= \frac{1}{n}\sum_{i=1}^{n-2} y_i^{\alpha/3}\log y_j - \frac{3}{\alpha}(\log n)\frac{1}{n}\sum_{i=1}^{n-2} y_i^{\alpha/3}\\
&= O_p(1) - (\log n)\frac{3}{\alpha}\left\{\mu\left(\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3}\right)\sigma^*_\alpha + O_p\left(\frac{1}{\sqrt{n}}\right)\right\}\\
&= O_p(1) - (\log n)\frac{3}{\alpha}\mu\left(\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3}\right).
\end{aligned}$$

- Write the third term as $\{\sqrt{n}(\hat{\alpha}_{p,n}-\alpha)/3\}^2 T_n$, and let us show that $T_n = o_p(1)$. Fix any $\epsilon > 0$ and $\epsilon_0 \in (0, \alpha/2)$ in the sequel. Then

$$P[|T_n| > \epsilon] \leq P[|\hat{\alpha}_{p,n} - \alpha| > \epsilon_0] + P\big[|T_n| > \epsilon,\ |\hat{\alpha}_{p,n} - \alpha| \leq \epsilon_0\big] =: p_n' + p_n''.$$

Clearly $p_n' \to 0$ by the $\sqrt{n}$-consistency of $\hat{\alpha}_{p,n}$. As for p_n'', we first note that

$$\inf_{u\in[0,1]} \frac{1}{\alpha}\{\alpha + u(\hat{\alpha}_{p,n} - \alpha)\} \geq 1 - \frac{\epsilon_0}{\alpha} > 0$$

on the event $\{|\hat{\alpha}_{p,n} - \alpha| \leq \epsilon_0\}$. We estimate p_n'' as follows:

$$\begin{aligned}
p_n'' &= P\Bigg[|\hat{\alpha}_{p,n} - \alpha| \leq \epsilon_0, \\
&\qquad \frac{1}{\sqrt{n}} \sum_{i=1}^{n-2} (\log x_i)^2 \int_0^1 (1-u) y_i^{\{\alpha + u(\hat{\alpha}_{p,n}-\alpha)\}/3} n^{-\{\alpha+u(\hat{\alpha}_{p,n}-\alpha)\}/\alpha} du > \epsilon\Bigg] \\
&\leq P\Bigg[|\hat{\alpha}_{p,n} - \alpha| \leq \epsilon_0,\ n^{\epsilon_0/\alpha - 1/2} \frac{1}{n} \sum_{i=1}^{n-2} (\log x_i)^2 \int_0^1 (1-u) y_i^{\{\alpha + u(\hat{\alpha}_{p,n}-\alpha)\}/3} du > \epsilon\Bigg] \\
&\leq P\Bigg[n^{\epsilon_0/\alpha - 1/2} \frac{1}{n} \sum_{i=1}^{n-2} \{(\log n)^2 + (\log y_i)^2\}(1 + y_i)^{(\alpha+\epsilon_0)/3} > C\epsilon\Bigg] \\
&\leq \frac{1}{C\epsilon} n^{\epsilon_0/\alpha - 1/2} (\log n)^2 \to 0
\end{aligned}$$

for some constant $C > 0$. Here we used Markov's inequality in the last inequality; note that $(\alpha + \epsilon_0)/3 < \alpha/2$, hence the moment does exist.

Piecing together these three items and (21), we arrive at the asymptotic relation:

$$\frac{\sqrt{n}}{\log n}\left\{M_n^*(\hat{\alpha}_{p,n}) - \mu\left(\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3}\right)\sigma_\alpha^*\right\} = -\frac{1}{\alpha}\mu\left(\frac{\alpha}{3}, \frac{\alpha}{3}, \frac{\alpha}{3}\right)\sigma_\alpha^* \sqrt{n}(\hat{\alpha}_{p,n} - \alpha) + O_p\left(\frac{1}{\log n}\right). \tag{22}$$

Now, recalling (2) we note that the quantity $\mu(\alpha/3, \alpha/3, \alpha/3)$ is a continuously differentiable function of (ρ, α). Write $\bar{\mu}(\rho, \alpha) = \mu(\alpha/3, \alpha/3, \alpha/3)$. In view of the $\sqrt{n}$-consistency of $(\hat{\rho}_n, \hat{\alpha}_{p,n})$ and the delta method, we obtain

$$\bar{\mu}(\rho, \alpha) = \bar{\mu}(\hat{\rho}_n, \hat{\alpha}_{p,n}) + O_p\left(\frac{1}{\sqrt{n}}\right). \tag{23}$$

Substituting (23) in (22) we end up with

$$\frac{\sqrt{n}}{\log n}\left\{\frac{M_n^*(\hat{\alpha}_{p,n})}{\bar{\mu}(\hat{\rho}_n, \hat{\alpha}_{p,n})} - \sigma_\alpha^*\right\} = -\frac{1}{\alpha}\sigma_\alpha^* \sqrt{n}(\hat{\alpha}_{p,n} - \alpha) + O_p\left(\frac{1}{\log n}\right), \tag{24}$$

which implies that

$$\hat{\sigma}_{\alpha,n}^* := \frac{M_n^*(\hat{\alpha}_{p,n})}{\bar{\mu}(\hat{\rho}_n, \hat{\alpha}_{p,n})} \tag{25}$$

serves as $(\sqrt{n}/\log n)$-consistent estimator of σ_α^*. Its asymptotic distribution is the centered normal scale mixture with limiting (possibly random) variance being

$$v(\rho, \alpha, \sigma_\alpha^*, \sigma_p^*, \sigma_{2p}^*) := \left(\frac{\sigma_\alpha^*}{\alpha}\right)^2 V_{22}(\rho, \alpha, \sigma_p^*, \sigma_{2p}^*),$$

where V_{22} denotes the $(2,2)$th entry of V; recall that p is a parameter-free constant (see Section 3.2). A consistent estimator of $v(\rho, \alpha, \sigma_\alpha^*, \sigma_p^*, \sigma_{2p}^*)$ can be constructed by plugging in the estimators of its arguments.

The stochastic expansion (24) indicates an asymptotic linear dependence of $\sqrt{n}(\hat{\alpha}_{p,n}-\alpha)$ and $(\sqrt{n}/\log n)(\hat{\sigma}^*_{\alpha,n}-\sigma^*_\alpha)$, Of course, this occurs even for constant σ, if we try to estimate (α,σ^α) instead of (α,σ). The point is that, plugging in a $\sqrt{n}$-consistent estimator of α into the index r of the MPV $M_n(r)$ slows down estimation of σ^*_α from $\sqrt{n}$ to $\sqrt{n}/(\log n)$. It is beyond the scope of this article to explore a better alternative estimator of σ^*_α.

4 Simulation experiments

Based on the discussion above, let us briefly observe finite-sample performance of our estimators. For simplicity, we here focus on nonrandom σ.

4.1 Case (A)

First, let σ is a positive constant, so that X is the geometric skewed stable Lévy process and the parameter to be estimated is (ρ,α,σ).

As a simulation design, we set $\alpha = 1.3, 1.5$, 1.7, and 1.9 with common $\beta = -0.5$ and $\sigma = 1$; hence $(\alpha,\rho) = (1.2, 0.7638)$, $(1.5, 0.5984)$, $(1.7, 0.5467)$, and $(1.9, 0.5132)$. The sample size are taken as $n = 500$, 1000, 2000, and 5000. In all cases, the tuning parameter q is set to be $1/4$, and 1000 independent sample paths of X are generated. Empirical means and empirical s.d.'s are given with the 1000 independent estimates obtained. The results are reported in Table 1. We see that estimation of (ρ,α) is, despite of its its simplicity, quite reliable. On the other hand, estimation variance of σ is relatively large compared with those of ρ and α. Nevertheless, it is clear that the bias is small. Moreover, as α gets close to 2, the performance of $\hat{\sigma}_n$ becomes better, while that of $(\hat{\rho}_n, \hat{\alpha}_{p,n})$ is seemingly unchanged.

In the unreported simulation results, we have observed that a change of q within its admissible region does not lead to a drastic change unless it is too small (see Remark 3.3).

4.2 Case (B)

Next we observe a case of time-varying but nonrandom scale. We set

$$\sigma_t^\alpha = \frac{2}{5}\left\{\cos(2\pi t) + \frac{3}{2}\right\}, \tag{26}$$

so that $\sigma^*_\alpha = 0.6$.

With the same choices of (ρ,α), q, and n as in the previous case, we obtain the result in Tables 2; the estimator of σ^*_α here is based on (25). There we can observe a quite similar tendency as in the previous case.

5 Concluding remarks

We have studied some statistical aspects in the calibration problem of a geometric skewed stable asset price models. Estimation of stable asset price models with possibly time-varying scale can be done easily by means of the simple empirical sign statistics and MPVs. Especially, we could estimate integrated scale, which is a natural quantity as in the integrate variance in the framework of Brownian semimartingales, with multistep estimating procedure: we estimate ρ, α, and σ (or σ^*_α) one by one in this order. Our simulation results say that finite-sample performance of our estimators are unexpectedly good despite of their simplicity, except for a relatively bigger variance in estimating σ (or σ^*_α).

We close with mentioning some possible future issues.

- Throughout we supposed the independence between the scale process σ and the driving skewed stable Lévy process Z. This may be disappointing as it excludes accommodating

$\alpha = 1.2$						
n	ρ		α		σ	
500	0.7627	(0.0186)	1.2026	(0.0790)	1.1021	(0.8717)
1000	0.7634	(0.0137)	1.2031	(0.0575)	1.0450	(0.4643)
2000	0.7645	(0.0096)	1.2031	(0.0437)	1.0253	(0.5102)
5000	0.7636	(0.0061)	1.2023	(0.0313)	1.0123	(0.2854)
$\alpha = 1.5$						
n	ρ		α		σ	
500	0.5988	(0.0222)	1.4929	(0.1030)	1.0751	(0.4066)
1000	0.5981	(0.0162)	1.5010	(0.0757)	1.0289	(0.2549)
2000	0.5986	(0.0106)	1.4986	(0.0564)	1.0284	(0.2355)
5000	0.5984	(0.0073)	1.4983	(0.0364)	1.0169	(0.1516)
$\alpha = 1.7$						
n	ρ		α		σ	
500	0.5476	(0.0219)	1.6810	(0.1103)	1.0633	(0.2359)
1000	0.5474	(0.0158)	1.6830	(0.0823)	1.0567	(0.1948)
2000	0.5472	(0.0113)	1.6930	(0.0625)	1.0308	(0.1611)
5000	0.5466	(0.0070)	1.6977	(0.0375)	1.0126	(0.1022)
$\alpha = 1.9$						
n	ρ		α		σ	
500	0.5129	(0.0224)	1.8553	(0.1026)	1.0821	(0.1767)
1000	0.5133	(0.0164)	1.8767	(0.0808)	1.0535	(0.1568)
2000	0.5131	(0.0109)	1.8870	(0.0579)	1.0330	(0.1111)
5000	0.5128	(0.0073)	1.8971	(0.0401)	1.0097	(0.0809)

Table 1: Estimation results for the true parameters $(\rho, \alpha, \sigma) = (0.7638, 1.2, 1)$, $(0.5984, 1.5, 1)$, $(0.5467, 1.7, 1)$, and $(0.5132, 1.9, 1)$ with the geometric stable Lévy processes. In each case, the empirical mean and standard deviation (in parenthesis) is given.

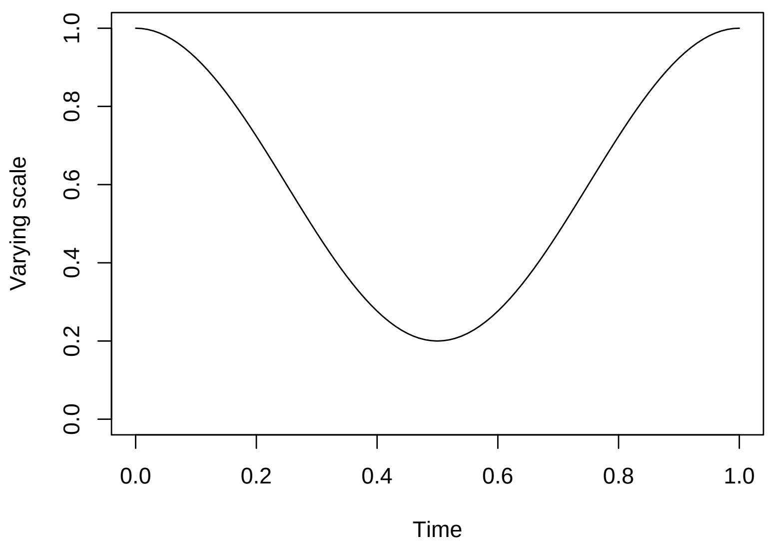

Figure 3: The plot of the function $t \mapsto \sigma_t^\alpha$ given by (26).

$\alpha = 1.2$						
n	ρ		α		σ^*_α	
500	0.7632	(0.0179)	1.1951	(0.0794)	0.6730	(0.3857)
1000	0.7636	(0.0139)	1.2042	(0.0619)	0.6274	(0.3094)
2000	0.7638	(0.0098)	1.2044	(0.0472)	0.6105	(0.2323)
5000	0.7641	(0.0059)	1.2025	(0.0305)	0.6029	(0.1521)
$\alpha = 1.5$						
n	ρ		α		σ^*_α	
500	0.5978	(0.0220)	1.4877	(0.1023)	0.6697	(0.3031)
1000	0.5981	(0.0159)	1.4908	(0.0733)	0.6551	(0.2488)
2000	0.5985	(0.0111)	1.4960	(0.0573)	0.6349	(0.2033)
5000	0.5987	(0.0069)	1.4990	(0.0376)	0.6151	(0.1414)
$\alpha = 1.7$						
n	ρ		α		σ^*_α	
500	0.5460	(0.0216)	1.6727	(0.1038)	0.6832	(0.2465)
1000	0.5465	(0.0160)	1.6801	(0.0820)	0.6714	(0.2280)
2000	0.5468	(0.0113)	1.6931	(0.0600)	0.6318	(0.1607)
5000	0.5465	(0.0071)	1.6988	(0.0393)	0.6116	(0.1135)
$\alpha = 1.9$						
n	ρ		α		σ^*_α	
500	0.5130	(0.0229)	1.8440	(0.1039)	0.7196	(0.2233)
1000	0.5131	(0.0159)	1.8703	(0.0823)	0.6762	(0.1897)
2000	0.5138	(0.0114)	1.8851	(0.0588)	0.6412	(0.1349)
5000	0.5135	(0.0068)	1.8956	(0.0411)	0.6168	(0.0998)

Table 2: Estimation results for the true parameters $(\rho, \alpha, \sigma) = (0.7638, 1.2, 0.6)$, $(\rho, \alpha, \sigma) = (0.5984, 1.5, 0.6)$, $(\rho, \alpha, \sigma) = (0.5467, 1.7, 0.6)$, and $(\rho, \alpha, \sigma) = (0.5132, 1.9, 0.6)$ with (26). In each case, the empirical mean and standard deviation (in parenthesis) is given.

the leverage effect, however, the simple constructions of our estimators (especially, $\hat{\rho}_n$) break down if they are allowed to be dependent. We may be able to deal with correlated σ and Z if we have an extension of the power-variation results obtained in Corcuera *et al.* [7] to the MPV version. To the best of author's knowledge, such an extension does not seem to have been explicitly mentioned as yet.

- Assuming that σ is indeed time-varying and possibly random, estimation of "spot" scales σ_t is an open problem. Needless to say, this is much more difficult and delicate to deal with than the integrated scale. We know several results for Brownian-semimartingale cases (see, among others, Fan and Wang [8] and Malliavin and Mancino [12]), however, yet no general result for the case of pure-jump Z.

- Finally, it might be interesting to derive an option-pricing formula for the case of time-varying scale, which seems more realistic than the mere geometric skewed stable Lévy processes.

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References

[1] Applebaum, D. (2004), *Lévy Processes and Stochastic Calculus.* Cambridge University Press, Cambridge.

[2] Barndorff-Nielsen, O. E., Graversen, S. E., Jacod, J. and Shephard, N. (2006), Limit theorems for bipower variation in financial econometrics. *Econometric Theory* **22**, 677–719.

[3] Barndorff-Nielsen, O. E., Graversen, S. E., Jacod, J., Podolskij, M. and Shephard, N. (2006), A central limit theorem for realised power and bipower variations of continuous semimartingales. *From Stochastic Calculus to Mathematical Finance*, 33–68, Springer, Berlin.

[4] Barndorff-Nielsen, O. E. and Shephard, N. (2005), Power variation and time change. *Teor. Veroyatn. Primen.* **50**, 115–130; translation in *Theory Probab. Appl.* **50** (2006), 1–15.

[5] Bertoin, J. (1996), *Lévy Processes.* Cambridge University Press.

[6] Borak, S., Härdle, W. and Weron, R. (2005), Stable distributions. *Statistical tools for finance and insurance*, 21–44, Springer.

[7] Corcuera, J. M., Nualart, D. and Woerner, J. H. C. (2007), A functional central limit theorem for the realized power variation of integrated stable processes. *Stoch. Anal. Appl.* **25**, 169–186.

[8] Fan, J. and Wang, Y. (2008), Spot volatility estimation for high-frequency data. *Stat. Interface* **1**, 279–288.

[9] Fujiwara, T. and Miyahara, Y. (2003), The minimal entropy martingale measures for geometric Lévy processes. *Finance Stoch.* **7**, 509–531.

[10] Kallsen, J. and Shiryaev, A. N. (2001), Time change representation of stochastic integrals. Teor. *Veroyatnost. i Primenen.* **46**, 579–585; translation in *Theory Probab. Appl.* **46** (2003), 522–528.

[11] Kuruoğlu, E. E. (2001), Density parameter estimation of skewed α-stable distributions. *IEEE Trans. Signal Process.* **49**, no. 10, 2192–2201.

[12] Malliavin, P. and Mancino, M. E. (2009), A Fourier transform method for nonparametric estimation of multivariate volatility. *Ann. Statist.* **37**, 1983–2010.

[13] Masuda, H. (2009), Joint estimation of discretely observed stable Lévy processes with symmetric Lévy density. *J. Japan Statist. Soc.* **39**, 1–27.

[14] Masuda, H. (2009), Estimation of second-characteristic matrix based on realized multipower variations. (Japanese) *Proc. Inst. Statist. Math.* **57**, 17–38.

[15] Miyahara, Y. and Moriwaki, N. (2009), Option pricing based on geometric stable processes and minimal entropy martingale measures. In "Recent Advances in Financial Engineering", *World Sci. Publ.*, 119–133.

[16] Sato, K. (1999), *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press.

[17] van der Vaart, A. W. (1998), *Asymptotic Statistics.* Cambridge University Press, Cambridge.

[18] Woerner, J. H. C. (2003), Purely discontinuous Lévy processes and power variation: inference for integrated volatility and the scale parameter. 2003-MF-08 Working Paper Series in Mathematical Finance, University of Oxford.

[19] Woerner, J. H. C. (2004), Estimating the skewness in discretely observed Lévy processes. *Econometric Theory* **20**, 927–942.

[20] Woerner, J. H. C. (2007), Inference in Lévy-type stochastic volatility models. *Adv. in Appl. Probab.* **39**, 531–549.

[21] Zolotarev, V. M. (1986), *One-Dimensional Stable Distributions.* American Mathematical Society, Providence, RI. [Russian original 1983]

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