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The Effect of Wall Motions on the Governing Equations of Contained Fluids

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The equations of motion for an acoustic fluid enclosed in a moving or flexible container are studied. It is shown that the determination of the reference state must account for the surface-integrated effect of the wall motions. The governing equation of transient motions about this state in the displacement potential does not generally reduce to the classical wave equation unless special adjustments are made. The results are relevant to finite elements formulations based on the displacement potential.

1 Problem Description

The results presented in this Note were obtained in the course of a wider study by Felippa and Ohayon (1989) of variational methods for transient motions and vibrations of acoustic fluids held in flexible and/or moving containers. These partial results merit special attention on two counts. First, they extend the classical dynamic equations of acoustic fluids to include wall motions as well as the static limit in a consistent manner. Second, they are relevant to finite element implementations that have not accounted for the correction terms described herein. Computations that are particularly affected by these corrections involve liquid masses subject to prescribe dynamic motions, such as tanks and reservoirs under seismic ground motion excitations, and rocket fuel tanks under launch conditions.

The general problem is as follows. A container (the structure) is totally or partly filled with a compressible, homogeneous liquid or gas (the fluid). Although the container is generally flexible, the rigid-but-moving container case is not excluded. The fluid is modeled as an acoustic medium (the linearly compressible generalization of an ideal fluid). We consider dynamic motions about a static reference state, which will be determined as part of the study.

If the container is rigid and fixed, the reference state is well known: the static equilibrium solution in which the pressure is equal to the hydrostatic pressure, and the displacements may be taken as zero. The acoustic motions about this position are governed by the homogeneous wave equation in the displacement potential. But if the container walls move, we show that a correction term that depends on the mean boundary motion appears. The reference state is affected, and the resulting transient vibration problem is no longer given by the classical wave equation unless special adjustments are made.

A boundary integral term representing the mean container motion was introduced by Aganovic (1981) for the surface wave problem of an incompressible fluid posed in terms of the velocity potential. Ohayon (1987) considered similar terms in the displacement potential formulation. The general forms presented in this Note for a compressible fluid are believed to be new.

2 The Acoustic Fluid

The three-dimensional volume occupied by the fluid is denoted by V. This volume is assumed to be simply connected. The fluid boundary S consists generally of two portions

$$S{:}S_d \cup S_p. \tag{1}$$

S_d is the interface with the container at which the normal displacement d_n is prescribed (or found as part of the coupled fluid structure problem) whereas S_p is the "free surface" at which the pressure p is prescribed (or found as part of the "slosh" problem). If the fluid is fully enclosed by the container, as is necessarily the case for a gas, then S_p is missing and $S \equiv S_d$. The domain is referred to a Cartesian coordinate system (x_1, x_2, x_3) grouped in vector $\mathbf{x}$.

The fluid is under a body force field $\mathbf{b}$ which is assumed to be the gradient of a *time-independent* potential $\beta(\mathbf{x})$, i.e., $\mathbf{b} = \nabla\beta$. All displacements are taken to be infinitesimal and thus the fluid density ρ is invariant.

We consider three states or configurations: *original*, from which displacements, pressures and forces are measured, *current*, where the fluid is in dynamic equilibrium at time t, and *reference*, which is obtained in the static equilibrium limit of slow motions. *Transient* motions are the difference between current and reference states. It should be noted that in many situations the original configuration is not physically attainable. Table 1 summarizes the notation used in relation to these states.

Field Equations. The governing equations of the acoustic fluid are the momentum, state, and continuity equations. They are stated as follows for the current configuration, and specialized to the reference configuration later. The momentum (balance) equation expresses Newton's second law for a fluid particle:

$$\rho\ddot{\mathbf{d}}^t = -\nabla p^t + \mathbf{b} = -\nabla p^t + \nabla\beta. \tag{2}$$

The continuity equation may be combined with the linearized equation of state to produce the *constitutive equation* that expresses the small compressibility of a liquid:

$$p^t = -K\nabla\mathbf{d}^t = -\rho c^2 \nabla\mathbf{d}^t, \tag{3}$$

where K is the bulk modulus and $c = \sqrt{K/\rho}$ the fluid sound speed. If the fluid is incompressible, $K, c \rightarrow \infty$. This relation is also applicable to nonlinear elastic fluids such as gases undergoing small excursions from the reference state, if the constitutive equation is linearized there so that $K = \rho_0(dp/d\rho)_0$.

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Table 1 Notation for fluid states

Quantities	Domain	Original	Reference	Current	Transient
Displacements	V	$\mathbf{0}$	$\mathbf{d}^0$	$\mathbf{d}^t$	$\mathbf{d}=\mathbf{d}^t-\mathbf{d}^0$
Velocities	V	$\mathbf{0}$	$\dot{\mathbf{d}}^0$	$\dot{\mathbf{d}}^t$	$\dot{\mathbf{d}}=\dot{\mathbf{d}}^t-\dot{\mathbf{d}}^0$
Boundary displacements*	S	0	d_n^0	d_n^t	$d_n=d_n^t-d_n^0$
Displacement potential	V	$\mathbf{0}$	ψ^0	ψ^t	$\psi=\psi^t-\psi^0$
Pressures	V	0	p^0	p^t	$p=p^t-p^0$
Body forces	V	$\mathbf{0}$	$\mathbf{b}=\nabla\beta$	$\mathbf{b}=\nabla\beta$	
Density	V	ρ	ρ	ρ	

*Positive along outward normal

The boundary conditions are

$$d_n^t=\bar{d}_n^t \text{ on } S_d,\ p^t=\bar{p}^t \text{ on } S_p, \tag{4}$$

where $\bar{d}_n^t$ is either prescribed or comes from the solution of an auxiliary problem as in fluid-structure interaction, and $\bar{p}$ may be either prescribed or a function of d_n and $\mathbf{b}$, as in the surface wave ("slosh") problem.

3 The Reference State

Taking the curl of both sides of (2) yields

$$\mathbf{curl}\ \ddot{\mathbf{d}}^t=\mathbf{0}. \tag{5}$$

The general integral of this equation for a simply connected domain is

$$\mathbf{d}^t=\nabla\psi^t+\mathbf{a}+\mathbf{b}t, \tag{6}$$

where $\psi^t=\psi^t(\mathbf{x},t)$ is the displacement potential, $\mathbf{a}=\mathbf{a}(\mathbf{x})$ and $\mathbf{b}=\mathbf{b}(\mathbf{x})$ are time-independent vector functions, and t denotes the time. If accelerationless motions (for example, rigid body motions) are precluded by the boundary conditions, $\mathbf{a}$ and $\mathbf{b}$ vanish. Replacing $\ddot{\mathbf{d}}^t=\nabla\ddot{\psi}^t$ into the momentum equation (2) we get

$$\nabla p^t=-\rho\nabla\ddot{\psi}^t+\nabla\beta, \tag{7}$$

which spatially integrated gives

$$p^t=-\rho\ddot{\psi}^t+\beta+C(t), \tag{8}$$

where the scalar $C(t)$ is not spatially dependent. Next, integrate the constitutive equation (3) over V and apply the divergence theorem to $\nabla\mathbf{d}$:

$$(p^t)_V+(\rho c^2\nabla\mathbf{d}^t)_V=(p^t)_V+[\rho c^2 d_n^t]_S=0. \tag{9}$$

Inserting p^t from (8) into equation (9) furnishes a condition on $C(t)$, giving

$$C(t)=-\frac{\rho c^2}{v}[d_n^t]_S+\frac{\rho}{v}(\ddot{\psi}^t)V-\frac{1}{v}(\beta)V$$

$$=-\frac{\rho c^2}{v}[d_n^t]_S+\rho\bar{\ddot{\psi}}^t-\bar{\beta}, \tag{10}$$

where $v=(1)_V$ is the fluid volume and $\bar{f}=(f)_V/v$ denotes the volume average of a function f defined over V. Substituting $C(t)$ into (9) we get

$$p^t=-(\rho(\ddot{\psi}^t-\bar{\ddot{\psi}}^t)+(\beta-\bar{\beta})-\frac{\rho c^2}{v}[d_n^t]_S. \tag{11}$$

In the static limit the inertia terms may be neglected and we recover the reference solution

$$p^0=(\beta-\bar{\beta})-\frac{\rho c^2}{v}[d_n^0]_S. \tag{12}$$

For an incompressible fluid, $[d_n]_S=0$ but $c\to\infty$; thus it would be incorrect to conclude that $p^0=\beta-\bar{\beta}$. To illustrate this point, consider a rigid cylindrical container of cross-section area A, filled with liquid up to height $H=H_1+H_2$. The origin of the Cartesian system (x_1, x_2, x_3) is placed at H_2 below the free surface, with $x_1\equiv x$ upwards and normal to that surface (see Fig. 1). The body force is the gravity field $\mathbf{b}=(-\rho g,0,0)$; thus $\beta=-\rho gx+B$, B being an arbitrary constant.

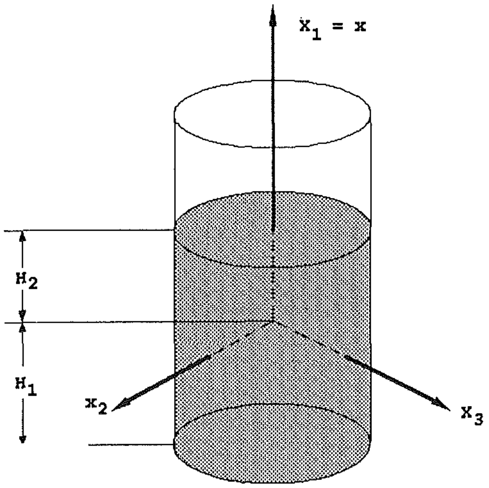

Fig. 1 Cylindrical fluid container in gravity field

In passing from the original configuration under zero body force to the reference configuration under gravity, the free surface moves downwards by the amount

$$d_n|_{x=H_2}=\eta^0=-\frac{\rho gH^2}{2K}=-\frac{gH^2}{2c^2}. \tag{13}$$

Evaluation of (12) gives

$$p^0=\rho g\left(x-\frac{1}{2}(H_2-H_1)\right)+B-B$$

$$+\frac{\rho c^2A\eta^0}{AH}=\rho g(H_2-x), \tag{14}$$

which is the correct hydrostatic pressure if $\delta\ll H$. If one passes to the incompressible limit, $c\to\infty$ and $\delta\to0$, but $c^2\delta$ remains fixed and equal to $gH^2/2$. Note that substracting $\bar{\beta}$ eliminates B. The associated displacement field is easily calculated to be

$$\mathbf{d}^0=\begin{Bmatrix} d_1^0 \\ d_2^0 \\ d_3^0 \end{Bmatrix}=\begin{Bmatrix} -\frac{g}{2c^2}[H^2-(x-H_2)^2]0 \\ 0 \\ 0 \end{Bmatrix}+\mathbf{d}_{rot}^0, \tag{15}$$

where $\mathbf{d}_{rot}^0$ is an arbitrary divergence-free rotational motion that satisfies the boundary conditions.

4 Transient Motions

Substracting the constitutive relations at the current and reference states we get

$$p=-\rho c^2\nabla^2\psi=\rho c^2 s, \tag{16}$$

where $s=-\nabla^2\psi$ is the condensation. Subtracting (12) from (11) yields

$$p = -\rho(\ddot{\psi} - \bar{\ddot{\psi}}) - \frac{\rho c^2}{v}[d_n]_S. \tag{17}$$

On equating (16) and (17) we get modified forms of the wave equation that account either for nonzero mean boundary surface motions,

$$s = \nabla^2\psi = \frac{\ddot{\psi} - \bar{\ddot{\psi}}}{c^2} + \frac{1}{v}[d_n]_S, \tag{18}$$

or nonzero mean dilatation,

$$c^2(\nabla^2\psi - \overline{\nabla^2\psi}) = c^2(\nabla^2\psi + \bar{s}) = \ddot{\psi} - \bar{\ddot{\psi}}. \tag{19}$$

The second form follows from $-v\bar{s} = [d_n]_S$, which is a consequence of the divergence theorem. For an incompressible fluid, $c \rightarrow \infty$ and $\bar{s} = [d_n]_S = 0$, and from either form one recovers the Laplace equation $\nabla^2\psi = 0$.

Adjusting the Displacement Potential. If the transient displacement potential is modified by a function of time:

$$\psi = \hat{\psi} + P(t), \tag{20}$$

wh $\hat{\psi}$ is the potential of (6)–(19), we may chose $P(t)$ so that $c^2\bar{\ddot{\psi}} = \overline{\nabla^2\psi} = -\bar{s}$ for any t. ($P(t)$ may be found by integrating $c^2\ddot{\hat{\psi}} - \nabla^2\psi$ twice in time.) We then recover the classical wave equation

$$c^2\nabla^2\psi = \ddot{\psi}. \tag{21}$$

If this adjustment has been made, $C(t)$ vanishes and (17) reduces to

$$p = -\rho\ddot{\psi}. \tag{22}$$

As an example, consider again the container of Fig. 1 in which $H_1 = 0$ for convenience. At $t \leq 0$, the container is in the reference state of rest. At $t \geq 0$, it is subjected to a prescribed constant velocity motion of the bottom surface,

$$\tilde{d}_n(t)\Big|_{x=0} = -\alpha t, \; t \geq 0, \tag{23}$$

positive upwards for $\alpha > 0$. The unknown free surface vertical displacement is $\eta(t)$, also positive upwards. As all quantities become independent of x_2 and x_3, the governing equation is one-dimensional:

$$\psi'' = \frac{\ddot{\psi} - \bar{\ddot{\psi}}}{c^2} + \frac{1}{H}(-\alpha t + \eta), \tag{24}$$

where primes denote derivatives with respect to $x_1 \equiv x$. The solution for $0 \leq t \leq H/c$ is

$$\eta = 0, \; \psi(x,t) = \begin{cases} \alpha t x + \dfrac{\alpha}{2c}x^2 + P(t), & x \leq ct, \\ \\ P(t) - \dfrac{1}{2}\alpha c t^2, & x \geq ct. \end{cases} \tag{25}$$

The adjustment condition gives $P(t) = -1/2\,\alpha c t^2$, and consequently $\psi = 0$ for $x \geq ct$. Hence, for finite c we have $\psi'' = c^2\ddot{\psi}$. If $c \rightarrow \infty$, the solution approaches the rigid body motion $\eta = \alpha t$.

5 References

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