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Hyperbolic Systems of Conservation Laws

The Theory of Classical and Nonclassical Shock Waves

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To my wife Claire

To Olivier, Aline, Bruno

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Part 2. SYSTEMS OF CONSERVATION LAWS

Preface

This set of lecture notes was written for a *Nachdiplom-Vorlesungen* course given at the Forschungsinstitut für Mathematik (FIM), ETH Zürich, during the Fall Semester 2000. I would like to thank the faculty of the Mathematics Department, and especially Rolf Jeltsch and Michael Struwe, for giving me such a great opportunity to deliver the lectures in a very stimulating environment. Part of this material was also taught earlier as an advanced graduate course at the Ecole Polytechnique (Palaiseau) during the years 1995–99, at the Instituto Superior Tecnico (Lisbon) in the Spring 1998, and at the University of Wisconsin (Madison) in the Fall 1998. This project started in the Summer 1995 when I gave a series of lectures at the Tata Institute of Fundamental Research (Bangalore).

One main objective in this course is to provide a self-contained presentation of the **well-posedness theory** for nonlinear hyperbolic systems of first-order partial differential equations in divergence form, also called **hyperbolic systems of conservation laws.** Such equations arise in many areas of continuum physics when fundamental balance laws are formulated (for the mass, momentum, total energy... of a fluid or solid material) and small-scale mechanisms can be neglected (which are induced by viscosity, capillarity, heat conduction, Hall effect...). Solutions to hyperbolic conservation laws exhibit singularities (shock waves), which appear in finite time even from smooth initial data. As is now well-established from pioneering works by Dafermos, Kruzkov, Lax, Liu, Oleinik, and Volpert, weak (distributional) solutions are not unique unless some **entropy condition** is imposed, in order to retain some information about the effect of "small-scales".

Relying on results obtained these last five years with several collaborators, I provide in these notes a complete account of the existence, uniqueness, and continuous dependence theory for the Cauchy problem associated with strictly hyperbolic systems with genuinely nonlinear characteristic fields. The mathematical theory of shock waves originates in Lax's foundational work. The existence theory goes back to Glimm's pioneering work, followed by major contributions by DiPerna, Liu, and others. The **uniqueness** of entropy solutions with bounded variation was established in 1997 in Bressan and LeFloch [2]. Three proofs of the **continuous dependence** property were announced in 1998 and three preprints distributed shortly thereafter; see [3,4,9]. The proof I gave in [4] was motivated by an earlier work ([6] and, in collaboration with Xin, [7]) on linear adjoint problems for nonlinear hyperbolic systems.

In this monograph I also discuss the developing theory of **nonclassical shock waves** for strictly hyperbolic systems whose characteristic fields are not genuinely nonlinear. Nonclassical shocks are fundamental in nonlinear elastodynamics and phase transition dynamics when capillarity effects are the main driving force behind their propagation. While classical shock waves are compressive, independent of small-scale regularization mechanisms, and can be characterized by an **entropy inequality**, nonclassical shocks are **undercompressive** and very sensitive to diffusive and dispersive mechanisms. Their unique selection requires a **kinetic relation**, as I call it following a terminology from material science (for hyperbolic-elliptic problems).

This book is intended to contribute and establish a *unified framework* encompassing both what I call here **classical** and **nonclassical entropy solutions.**

No familiarity with hyperbolic conservation laws is a priori assumed in this course. The well-posedness theory for classical entropy solutions of genuinely nonlinear systems is entirely covered by Chapter I (Sections 1 and 2), Chapter II (Sections 1 and 2), Chapter III (Section 1), Chapter IV (Sections 1 and 2), Chapter V (Sections 1 and 2), Chapter VI (Sections 1 and 2), Chapter VII, Chapter IX (Sections 1 and 2), and Chapter X. The other sections contain more advanced material and provide an introduction to the theory of nonclassical shock waves.

First, I want to say how grateful I am to Peter D. Lax for inviting me to New York University as a Courant Instructor during the years 1990–92 and for introducing me to many exciting mathematical people and ideas. I am particularly indebted to Constantine M. Dafermos for his warm interest to my research and his constant and very helpful encouragement over the last ten years. I also owe Robert V. Kohn for introducing me to the concept of kinetic relations in material science and encouraging me to read the preprint of the paper [1] and to write [6]. I am very grateful to Tai-Ping Liu for many discussions and his constant encouragement; his work [8] on the entropy condition and general characteristic fields was very influential on my research.

It is also a pleasure to acknowledge fruitful discussions with collaborators and colleagues during the preparation of this course, in particular from R. Abeyaratne, F. Asakura, P. Baiti, N. Bedjaoui, J. Knowles, B. Piccoli, M. Shearer, and M. Slemrod. I am particularly thankful to T. Iguchi and A. Mondoloni, who visited me as post-doc students at the Ecole Polytechnique and carefully checked the whole draft of these notes. Many thanks also to P. Goatin, M. Savelieva, and M. Thanh who pointed out misprints in several chapters.

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