ON A FORM OF THE ELIMINANT OF TWO QUANTICS

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1. I wish to prove, in the first place, that the eliminant of two equations of the form

$$\sum_1^{n+1} \frac{A_s}{x-r_s} = 0, \qquad \sum_1^{n+1} \frac{B_s}{x-r_s} = 0$$

is given by the symmetrical determinant of the nth order

$$\begin{vmatrix} \Sigma(1, s), & -(2, 1), & -(3, 1), & \dots, & -(n, 1) \\ -(1, 2), & \Sigma(2, s), & -(3, 2), & \dots, & -(n, 2) \\ -(1, 3), & -(2, 3), & \Sigma(3, s), & \dots, & -(n, 3) \\ \dots & \dots & \dots & \dots & \dots \\ -(1, n), & -(2, n), & -(3, n), & \dots, & \Sigma(n, s) \end{vmatrix},$$

multiplied by the discriminant of $\prod_{s=1}^{s=n+1} (x-r_s)$, where

$$(s, t) = (t, s) = (A_s B_t - A_t B_s)/(r_s - r_t),$$

and $\sum_s (t, s) = (t, 1) + (t, 2) + \dots + (t, t-1) + (t, t+1) + \dots + (t, n+1),$

the term (t, t) being omitted.

This determinant is symmetrical with respect to all the numbers $1, 2, \dots, n+1$, and may be written as the sum of $(n+1)^{n-1}$ terms, each term of the sum having a positive sign and being the product of n different quantities (rs). Each of the numbers $1, 2, 3, \dots, (n+1)$ occurs at least once in each term, and the sum may be written down by taking n of the numbers, say $2, 3, \dots, (n+1)$, and distributing them, one in each bracket, and then filling in the other number in each bracket in all possible ways, so that 1 shall occur at least once, and that in any one term no two brackets shall occur which contain the same pair of numbers.

This result, which is of practical value as a simple expression for the eliminant of two equations of a particular form, is a particular case of a more general theorem, that, if $r_1, r_2, \ldots, r_n$ and $a_1, \ldots, a_n$ are two sets of arbitrary quantities, and we write (a_s, r_t) for $[\phi(a_s)\psi(r_t)-\phi(r_t)\psi(a_s)]/(a_s-r_t)$, where $\phi(x)$ and $\psi(x)$ are two quantics, each of the nth degree, then the eliminant of $\phi(x)$ and $\psi(x)$ is given by

$$\begin{vmatrix} (a_1, r_1), & (a_1, r_2), & (a_1, r_3), & \ldots, & (a_1, r_n) \\ (a_2, r_1), & (a_2, r_2), & (a_2, r_3), & \ldots, & (a_2, r_n) \\ (a_3, r_1), & (a_3, r_2), & (a_3, r_3), & \ldots, & (a_3, r_n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (a_n, r_1), & (a_n, r_2), & (a_n, r_3), & \ldots, & (a_n, r_n) \end{vmatrix},$$

divided by the product of differences

$$\Pi(r_s-r_t)\,\Pi'(a_s-a_t).$$

The eliminant in the form first given is, in fact, the result of putting $a_s = r_s$, for all values of s, in this last expression.

For, writing $$f(x) = \Pi(x-r_s),$$

and
$$\frac{\phi(x)}{f(x)} = \Sigma \frac{\phi(r_s)}{f'(r_s)} \frac{1}{x-r_s} = \Sigma \frac{A_s}{x-r_s},$$

$$\frac{\psi(x)}{f(x)} = \Sigma \frac{\psi(r_s)}{f'(r_s)} \frac{1}{x-r_s} = \Sigma \frac{B_s}{x-r_s},$$

we get for a term of the principal diagonal, $\Sigma(1, s)$ for example,

$$\Sigma(1, s) = \frac{\phi(r_1)}{f'(r_1)} \sum_2^{n+1} \frac{\psi(r_s)}{f'(r_s)(r_1-r_s)} - \frac{\psi(r_1)}{f'(r_1)} \sum_2^{n+1} \frac{\phi(r_s)}{f'(r_s)(r_1-r_s)}$$

$$= \frac{\phi(r_1)}{f'(r_1)} \left\{ \frac{\psi'(r_1)}{f'(r_1)} - \tfrac{1}{2} \frac{f''(r_1)\,\psi(r_1)}{[f'(r_1)]^2} \right\} - \frac{\psi(r_1)}{f'(r_1)} \left\{ \frac{\phi'(r_1)}{f'(r_1)} - \tfrac{1}{2} \frac{f''(r_1)\,\phi(r_1)}{[f'(r_1)]^2} \right\}$$

$$= \frac{\phi(r_1)\,\psi'(r_1) - \phi'(r_1)\,\psi(r_1)}{[f'(r_1)]^2},$$

which is the limit of $-(1, s)$ when r_s is made equal to r_1.

If we go further and take the limit when all the a's and all the r's are made equal to infinity (or zero), we get finally Bezout's determinant.

2. I take then two equations of the nth degree

$$\sum_1^{n+1} \frac{A_s}{x-r_s} = 0, \tag{A}$$

$$\sum_1^{n+1} \frac{B_s}{x-r_s} = 0. \tag{B}$$

Then $$(s, t) = (t, s) = (A_s B_t - A_t B_s)/(r_s - r_t).$$

Also I write $D(r_1, r_2, \ldots, r_{n+1})$ for the discriminant of $\prod_1^{n+1}(x-r_s)=0$, and $E(r_1, r_2, \ldots, r_{n+1})$ for the eliminant of the equations (A) and (B).

If we increase the roots of each equation by r_1, and write $\rho_s \equiv r_s - r_1$, we have

$$\frac{A_1}{x}+\frac{A_2}{x-\rho_2}+\frac{A_3}{x-\rho_3}+\ldots=0,$$

$$\frac{B_1}{x}+\frac{B_2}{x-\rho_2}+\frac{B_3}{x-\rho_3}+\ldots=0.$$

If these equations be multiplied up, and arranged in ascending powers of x, we get

$$\rho_2\rho_3\ldots\rho_{n+1}A_1 - x\left[A_1\beta+A_2\rho_3\rho_4\ldots\rho_{n+1}+A_3\rho_2\rho_4\ldots\rho_{n+1}+\ldots\right]+\ldots=0,$$

$$\rho_2\rho_3\ldots\rho_{n+1}B_1 - x\left[B_1\beta+B_2\rho_3\rho_4\ldots\rho_{n+1}+B_3\rho_2\rho_4\ldots\rho_{n+1}+\ldots\right]+\ldots=0,$$

where β is the sum of the products of $\rho_2, \rho_3, \ldots, \rho_{n+1}$, $n-1$ at a time.

Again, it is a well known and immediate deduction from Bezout's expression for the eliminant as a determinant,* that for two equations of the forms

$$a-a_1x+\ldots=0,$$

$$b-b_1x+\ldots=0,$$

the terms of the eliminant which do not contain either a^2, ab, or b^2 are (ab_1-a_1b), multiplied by the eliminant of the equations, of the next lower degree, obtained by putting $a=0$, $b=0$, and dividing by x.

Thus, taking our equations in the form just obtained, (ab_1-a_1b) becomes

$$\rho_2^2\rho_3^2\rho_4^2\ldots\rho_{n+1}^2\sum_{s=2}^{s=n+1}(A_1B_s-B_1A_s)/\rho_s,$$

or $$(r_2-r_1)^2(r_3-r_1)^2\ldots(r_{n+1}-r_1)^2\,\Sigma(A_1B_s-B_1A_s)/(r_1-r_s).$$

3. It is obvious that each term in the eliminant must contain either A_s or B_s for every value of s, since, when $A_s=0$, $B_s=0$, the two equations

$$\sum_1^{n+1}A_s/(x-r_s)=0, \qquad \sum_1^{n+1}B_s/(x-r_s)=0,$$

have a common root $x=r_s$.

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* Salmon, *Higher Algebra*, p. 82, § 85.

Also for each term in the eliminant there must be at least two values of s, say t is one, such that A_t or B_t occurs to the first power only, that is to say, that the term does not contain A_t^2, A_tB_t or B_t^2. For, if the term contained A_s^2, A_sB_s or B_s^2 for all but one, that is, for n values of s, its degree would be $2n+1$, at least, in the A's and B's together, and the eliminant is only of degree n in the A's and B's respectively.

Thus, from the necessary symmetry in respect to the sets of quantities A_s, B_s, r_s, the eliminant is completely determined when the terms containing A_1 or B_1 to the first power only are known.

When $n=1$, we have at once

$$E(r_1, r_2) = (r_1-r_2)(A_1B_2-A_2B_1) = D(r_1, r_2)(1, 2).$$

Thus the terms in $E(r_1, r_2, r_3)$ which contain A_1 or B_1 to the first power, are

$$(r_2-r_1)^2(r_3-r_1)^2\{(1, 2)+(1, 3)\}(r_2-r_3)^2(2, 3),$$

and we get

$$E(r_1, r_2, r_3) = D(r_1, r_2, r_3)\{(1, 2)(1, 3)+(2, 1)(2, 3)+(3, 1)(3, 2)\}.$$

From this again, the terms in $E(r_1, r_2, r_3, r_4)$, of the first power in A_1 or B_1, will be

$$D(r_1, r_2, r_3, r_4)\{(1, 2)+(1, 3)+(1, 4)\}\{(2, 3)(2, 4)+(3, 2)(3, 4)+(4, 2)(4, 3)\},$$

and the complete expression is

$$\begin{aligned} E(r_1, r_2, r_3, r_4)/D(r_1, r_2, r_3, r_4) \\ = (1, 2)(1, 3)(1, 4)+(2, 1)(2, 3)(1, 4)+(3, 1)(3, 2)(1, 4) \\ +(1, 2)(1, 3)(2, 4)+(2, 1)(2, 3)(2, 4)+(3, 1)(3, 2)(2, 4) \\ +(1, 2)(1, 3)(3, 4)+(2, 1)(2, 3)(3, 4)+(3, 1)(3, 2)(3, 4) \\ +(4, 1)(4, 2)(1, 3)+(4, 1)(4, 2)(2, 3)+(4, 1)(4, 3)(1, 2)+(4, 1)(4, 3)(3, 2) \\ +(4, 2)(4, 3)(2, 1)+(4, 2)(4, 3)(3, 1)+(4, 1)(4, 2)(4, 3). \end{aligned}$$

It is seen, on inspection, that the rules given in § 1, for writing down the eliminant as a sum of terms, apply to $E(r_1, r_2, r_3)$ and $E(r_1, r_2, r_3, r_4)$, and therefore by induction must apply universally. For, if they apply to

$E(r_2, r_3, \ldots, r_{n+1})$, they must apply to the terms in $E(r_1, r_2, \ldots, r_{n+1})$, which only contain the number 1 once, since these terms are

$$\{(1, 2)+(1, 3)+\ldots+(1, n+1)\}\, E(r_2, r_3, \ldots, r_{n+1}),$$

and so, by symmetry, must apply to all the terms of $E(r_1, r_2, \ldots, r_{n+1})$.

4. Again $E(r_1, r_2, r_3)/D(r_1, r_2, r_3)$, which is

$$(1, 2)(1, 3)+(2, 1)(2, 3)+(3, 1)(3, 2),$$

may obviously be written in the form

$$\begin{vmatrix} (1, 2)+(1, 3), & -(2, 1) \\ -(1, 2), & (2, 1)+(2, 3) \end{vmatrix},$$

and then, again by induction,

$$\begin{vmatrix} \Sigma(1, s), & -(2, 1), & -(3, 1), & \ldots, & -(n, 1) \\ -(1, 2), & \Sigma(2, s), & -(3, 2), & \ldots, & -(n, 2) \\ -(1, 3), & -(2, 3), & \Sigma(3, s), & \ldots, & -(n, 3) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -(1, n), & -(2, n), & -(3, n), & \ldots, & \Sigma(n, s) \end{vmatrix}$$

may be identified with $E(r_1, r_2, \ldots, r_{n+1})/D(r_1, r_2, \ldots, r_{n+1})$.

For, by adding all the other rows to the last row, and then all the other columns to the last column, the determinant is easily seen to be symmetrical with regard to all the numbers $1, 2, 3, \ldots, (n+1)$. Also the terms in it which contain 1 once only, are $(1, 2)+(1, 3)+\ldots+(1, n+1)$ multiplied by the corresponding determinant got by erasing the first row and column. Then since the expression is right for $E(r_1, r_2, r_3)/D(r_1, r_2, r_3)$, it is also right for $E(r_1, r_2, r_3, r_4)/D(r_1, r_2, r_3, r_4)$, and so by induction for all values of n.

5. I have found the number of terms in the eliminant when expressed as a sum of products of the quantities (r, s) as follows.

Disregarding altogether the factor $D(r_1, \ldots, r_{n+1})$, suppose the terms of $E(r_1, \ldots, r_{n+1})$ arranged in sets, according as the term contains two of the numbers $1, 2, 3, \ldots, n+1$, once only, or three numbers once only, and so on, and put

$$E(r_1, \ldots, r_{n+1}) = S_2+S_3+S_4+\ldots+S_n.$$

Then we shall also have

$$\sum_{s=1}^{s=n+1}\left\{\sum_{t=1}^{t=n+1}(t,s)\,E(r_1,\ldots,r_{s-1},r_{s+1},\ldots,r_n)\right\}=2S_2+3S_3+\ldots+nS_n.$$

Similarly, $(t\neq s',\ t'\neq s)$,

$$\sum_{s'=1}^{s'=n+1}\sum_{s=1}^{s=n+1}\left\{\sum_{t=1}^{t=n+1}\sum_{t'=1}^{t'=n+1}(t,s)(t',s')\,E(r_1,\ldots,r_{s-1},r_{s+1},\ldots,r_{s'-1},r_{s'+1},\ldots,r_n)\right\}$$
$$=S_2+\frac{3\,.\,2}{1\,.\,2}S_3+\frac{4\,.\,3}{1\,.\,2}S_4+\ldots+\frac{n(n-1)}{1\,.\,2}S_n.$$

A similar sum obtained by taking the product of three brackets $(t,s)(t',s')(t'',s'')$, into the eliminant of the equations obtained by crossing out s, s', s'' is equal to

$$S_3+\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}S_4+\frac{5\,.\,4\,.\,3}{1\,.\,2\,.\,3}S_5+\ldots+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}S_n,$$

and we may continue the series until we finally arrive at $n!\,S_n/n!$ on the right-hand side, and $\sum\limits_{s=1}^{s=n+1}\prod\limits_{t=1}^{t=n+1}(s,t)$, $(t\neq s)$, on the left.

From these equations we eliminate S_2, S_3, ..., by taking the left-hand sides with alternately positive and negative signs, and arrive at an identity which may be written

$$E_{n+1}-\sum_s\sum_t(s,t)\,E_n[s]+\sum_{ss'}\sum_{tt'}(s,t)(s',t')\,E_{n-1}[s,s']-\ldots=0,$$

where, for example, $E_{n-1}[s,s']$ means the eliminant of the equations obtained by striking out the terms with s or s' as suffix.

Now, write N_{n+1} for the number of terms in E_{n+1}, N_n for the number in E_n, and so on.

In $\sum\limits_s\sum\limits_t(s,t)$ we have $n+1$ choices for s and n for t, so that the number of terms is $(n+1)n$. In $\sum\limits_{ss'}\sum\limits_{tt'}(s,t)(s',t')$ we have $\frac{1}{2}n(n+1)$ pairs s, s', and for each pair $n-1$ choices for t or t', so that the number of terms is $\frac{1}{2}n(n+1)(n-1)^2$.

Determining in this way the number of terms in each constituent, we have, since all the terms in the eliminants have positive signs, the relation

$$N_{n+1}-(n+1)nN_n+\frac{(n+1)n}{1\,.\,2}(n-1)^2N_{n-1}$$
$$-\frac{(n+1)n(n-1)}{1\,.\,2\,.\,3}(n-2)^3N_{n-2}+\ldots+(-1)^n(n+1)N_1=0,$$

where N_1 is to be taken as 1. Now we know that $N_2 = 1$, $N_3 = 3$, $N_4 = 16$, and the above relation is satisfied by putting

$$N_{n+1} = (n+1)^{n-1},$$

becoming in this case

$$(n+1)^{n-1}-(n+1)\,n^{n-1}+\frac{(n+1)\,n}{1.2}(n-1)^{n-1}-\ldots = 0.$$

Therefore the number of terms in the eliminant is $(n+1)^{n-1}$.

6. This expression for the eliminant as a determinant, once obtained, may be easily verified in a different manner.

Returning to the equations

$$\sum_1^{n+1} \frac{A_s}{x-r_s} = 0, \qquad \sum_1^{n+1} \frac{B_s}{x-r_s} = 0,$$

multiply the first by $B_1/(x-r_1)$, and the second by $A_1/(x-r_1)$, and subtract.

The term $(A_sB_1-A_1B_s)/(x-r_s)(x-r_1)$ on the left-hand side of the resulting equation may be written

$$\frac{(A_sB_1-A_1B_s)}{r_s-r_1}\left\{\frac{1}{x-r_s}-\frac{1}{x-r_1}\right\},$$

and we obviously get

$$-\frac{1}{x-r_1}\sum_{s=2}^{s=n+1}(1,\,s)+\sum_{s=2}^{s=n+1}\frac{(1,\,s)}{x-r_s} = 0.$$

There will be $n-1$ similar equations got by replacing 1 by 2, 3, ..., n, and we may add to these the equation

$$\Sigma\, A_s/(x-r_s) = 0,$$

and eliminate the $(n+1)$ quantities $1/(x-r_s)$. The result is

$$\begin{vmatrix} \Sigma(1,\,s), & -(2,\,1), & -(3,\,1), & \ldots, & -(n+1,\,1) \\ -(1,\,2), & \Sigma(2,\,s), & -(3,\,2), & \ldots, & -(n+1,\,2) \\ -(1,\,3), & -(2,\,3), & \Sigma(3,\,s), & \ldots, & -(n+1,\,3) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_1, & A_2, & A_3, & \ldots, & A_{n+1} \end{vmatrix} = 0.$$

By adding all the other columns to the last column, all its constituents become zero except the last, which is ΣA_s, and the result is (ΣA_s) multiplied by the determinant of the nth order already obtained (§ 4). That by this method of elimination, the extraneous factor ΣA_s is introduced, is easily seen, if we notice that all the equations of the type

$$\Sigma\{(1, s)/(x-r_s)\}-\Sigma(1, s)/(x-r_1)=0,$$

have a common root $x=\infty$, since the sum of the coefficients is zero.

7. By what is really the same method we obtain the eliminant in the form of the determinant whose elements are (a_s, r_t) (§ 1).

If $\phi(x)$, $\psi(x)$ have a common root x, this is also a root of

$$[\phi(x)\,\psi(a)-\phi(a)\,\psi(x)]/(x-a).^*$$

Putting

$$f(x)\equiv\prod_{s=1}^{s=n}(x-r_s),$$

we have, by partial fractions,

$$\frac{\phi(x)\,\psi(a)-\phi(a)\,\psi(x)}{f(x)(x-a)}=\sum_{s=1}^{s=n}\frac{(a, r_s)}{f'(r_s)(x-r_s)},$$

and now putting the left-hand side equal to zero, and giving to a the n values, $a_1, a_2, \ldots, a_n$, we have n equations linear in $1/f'(r_s)(x-r_s)$, and the determinant is at once obtained.

It is not difficult to show directly that this determinant, which I will call D, whose constituents are (a_s, r_t) is equal to $\Pi(r_s-r_t)\,\Pi(a_s-a_t)\,B$, where B is Bezout's determinant.

If we write $f(a, r)$ for (a, r), it is obvious, by expanding $f(a_s, r_t)$ in ascending powers of (a_s-a) by Taylor's theorem and applying the formula for the multiplication of two determinants, that D is equal to the product of two determinants, the rows (or columns) of which are respectively,

$$f(a, r_s),\quad \frac{\partial f(a, r_s)}{\partial a},\quad \frac{1}{2!}\frac{\partial^2 f(a, r_s)}{\partial a^2},\quad \ldots,\quad \frac{1}{(n-1)!}\frac{\partial^{n-1} f(a, r_s)}{\partial a^{n-1}},$$

and

$$1,\quad a_s-a,\quad (a_s-a)^2,\quad \ldots,\quad (a_s-a)^{n-1},$$

the latter determinant being $\Pi(a_s-a_t)$.

Again, applying the same method and expanding in ascending powers

* Salmon, *loc. cit.*, p. 83, § 87, Cayley's "Statement of Bezout's method."

of $(r_s - r)$, the first of these determinants becomes $\Pi(r_s - r_t)$ multiplied by the determinant whose constituents are $\dfrac{1}{s!\,t!}\dfrac{\partial^{s+t} f(a, r)}{\partial a^s \partial r^t}$. It remains to show, therefore, that this last determinant is equal to B; it is, in fact, identical with it when $a = 0$, $r = 0$. For, if

$$\phi(x) = A_0 + A_1 x + \ldots + A_r x^r + \ldots + A_n x^n,$$

$$\psi(x) = B_0 + B_1 x + \ldots + B_r x^r + \ldots + B_n x^n,$$

$$\begin{vmatrix} \phi(a), & \phi(r) \\ \psi(a), & \psi(r) \end{vmatrix} = \sum_{p,\,q} \begin{vmatrix} A_p, & A_q \\ B_p, & B_q \end{vmatrix} \begin{vmatrix} a^p, & a^q \\ r^p, & r^q \end{vmatrix};$$

and therefore, assuming $p > q$,

$$f(a, r) = \Sigma \begin{vmatrix} A_p, & A_q \\ B_p, & B_q \end{vmatrix} (a^{p-1} r_q + a^{p-2} r^{q-1} + \ldots + a^{q-1} r^{p-2} + a^q r^{p-1}).$$

Thus $\left[\dfrac{1}{s!\,t!}\dfrac{\partial^{s+t} f(a, r)}{\partial a^s \partial r^t}\right]_{a=0,\ r=0}$, which is the coefficient of $a^s r^t$ in $f(a, r)$, is equal to $\Sigma(A_p B_q - A_q B_p)$, where $p+q = s+t+1$, and, taking $p > q$, $q \not> s$ or t, $p-1 \not< s$ or t. Thus we have finally Bezout's determinant, viz.,

$$\begin{vmatrix} (A_0, B_1), & (A_0, B_2), & (A_0, B_3), & \ldots \\ (A_0, B_2), & (A_0, B_3) + (A_1, B_2), & (A_0, B_4) + (A_1, B_3), & \ldots \\ (A_0, B_3), & (A_0, B_4) + (A_1, B_3), & (A_0, B_5) + (A_1, B_4) + (A_2, B_3), & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

8. There remains the consideration of the modifications to be introduced when the degrees of the given equations are different, say n and $n-m$.

Consider first the determinant whose constituents are (r, s), the equations being given in the forms

$$\Sigma A_s/(x - r_s) = 0, \tag{A}$$

$$\Sigma B_s/(x - r_s) = 0; \tag{B}$$

and suppose that the first m coefficients of (B), when it is multiplied up and arranged in descending powers of x, vanish identically. The equation (B) may be considered as having m infinite roots, and the expression for the eliminant, as already found, will have an extraneous factor $(\Sigma A_s)^m$.

If, however, $B_1 = B_2 = B_3 = \ldots = B_m = 0$, we have

$$(s, t) = 0, \text{ if } s < m+1, \; t < m+1,$$

$$(s, t) = A_s B_t/(r_s - r_t), \text{ if } s < m+1, \; t > m.$$

Then A_s can be divided out from either the sth row or the sth column, where $s < m+1$, and we have left the pure eliminant, in the form of a determinant, whose first principal minor of the mth order consists of a principal diagonal only, all the other elements being zero.

Next consider the determinant whose constituents are (a_s, r_t). To find the eliminant of $\phi'(x)$, $\psi(x)$, where $\psi(x)$ is of degree n, and $\phi'(x)$ of degree $n-m$, let us write

$$\phi(x) = \phi'(x) \prod_{t=1}^{t=m} (x - r_t).$$

Then $$\phi(r_t) = 0 \quad (t = 1, 2, \ldots, m);$$

and therefore $$(a_s, r_t) = \phi(a_s)\psi(r_t)/(a_s - r_t) \quad (t < m+1),$$

so that the first m columns of the determinant have each a common factor $\psi(r_t)$, which can be divided out, leaving as a result the pure eliminant.

9. By a method similar to that used in § 7, we may also find the eliminant of two quantics $\phi(x)$, $\psi(x)$ of degrees n and m respectively, as a determinant of order $n+m$, which is equivalent to the result obtained by Sylvester's dialytic method. Put

$$f(x) = \prod_{s=1}^{s=n} (x - a_s), \qquad F(x) = \prod_{s=1}^{s=m} (x - b_s).$$

Then $$\frac{\phi(x)}{f(x)(x-b_t)} = \frac{\phi(b_t)}{f(b_t)} \frac{1}{x-b_t} + \sum_{s=1}^{s=n} \frac{\phi(a_s)}{f'(a_s)(a_s - b_t)} \frac{1}{x-a_s},$$

and $$\frac{\psi(x)}{F(x)(x-a_t)} = \frac{\psi(a_t)}{F(a_t)} \frac{1}{x-a_t} + \sum_{s=1}^{s=m} \frac{\psi(b_s)}{F'(b_s)(b_s - a_t)} \frac{1}{x-b_s}.$$

Putting $\phi(x) = 0$ and $\psi(x) = 0$, we have $m+n$ equations linear in the $m+n$ quantities, $1/(x-a_s)$, $1/(x-b_s)$, and the eliminant of $\phi(x)$ and $\psi(x)$

appears as the determinant

$$\begin{vmatrix}
\dfrac{\phi(b_1)}{f(b_1)}, & 0, & \dots, & 0, & \dfrac{\phi(a_1)}{f'(a_1)(a_1-b_1)}, & \dfrac{\phi(a_2)}{f'(a_2)(a_2-b_1)}, & \dots, & \dfrac{\phi(a_n)}{f'(a_n)(a_n-b_1)} \\
0, & \dfrac{\phi(b_2)}{f(b_2)}, & \dots, & 0, & \dfrac{\phi(a_1)}{f'(a_1)(a_1-b_2)}, & \dfrac{\phi(a_2)}{f'(a_2)(a_2-b_2)}, & \dots, & \dfrac{\phi(a_n)}{f'(a_n)(a_n-b_2)} \\
\dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\
\dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\
0, & 0, & \dots, & \dfrac{\phi(b_m)}{f(b_m)}, & \dfrac{\phi(a_1)}{f'(a_1)(a_1-b_m)}, & \dfrac{\phi(a_2)}{f'(a_2)(a_2-b_m)}, & \dots, & \dfrac{\phi(a_n)}{f'(a_n)(a_n-b_m)} \\
\dfrac{\psi(b_1)}{F'(b_1)(b_1-a_1)}, & \dfrac{\psi(b_2)}{F'(b_2)(b_2-a_1)}, & \dots, & \dfrac{\psi(b_m)}{F'(b_m)(b_m-a_1)}, & \dfrac{\psi(a_1)}{F(a_1)}, & 0, & \dots, & 0 \\
\dfrac{\psi(b_1)}{F'(b_1)(b_1-a_2)}, & \dfrac{\psi(b_2)}{F'(b_2)(b_2-a_2)}, & \dots, & \dfrac{\psi(b_m)}{F'(b_m)(b_m-a_2)}, & 0, & \dfrac{\psi(a_2)}{F(a_2)}, & \dots, & 0 \\
\dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\
\dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots \\
\dfrac{\psi(b_1)}{F'(b_1)(b_1-a_n)}, & \dfrac{\psi(b_2)}{F'(b_2)(b_2-a_n)}, & \dots, & \dfrac{\psi(b_m)}{F'(b_m)(b_m-a_n)}, & 0, & 0, & \dots, & \dfrac{\psi(a_n)}{F(a_n)}
\end{vmatrix}$$

multiplied by the eliminant of $f(x)$ and $F(x)$.

Note on § 5.

It has been pointed out to me by a referee that, when we know that the eliminant is the sum of products of (r, s), each product having a positive sign, the number of terms is at once found by writing 1 for (r, s) throughout the determinant of § 4. In this way we get

$$\begin{vmatrix} n, & -1, & -1, & \dots \\ -1, & n, & -1, & \dots \\ -1, & -1, & n, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} n, & -n-1, & -n-1, & \dots \\ -1, & n+1, & 0, & \dots \\ -1, & 0, & n+1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0, & \dots \\ -1, & n+1, & 0, & \dots \\ -1, & 0, & n+1, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = (n+1)^{n-1}.$$