Universality of Entropy Scaling in 1D Gap-less Models

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We consider critical models in one dimension. We study the ground state in thermodynamic limit [infinite lattice]. Following Bennett, Bernstein, Popescu, and Schumacher, we use the entropy of a sub-system as a measure of entanglement. We calculate the entropy of a part of the ground state. At zero temperature it describes entanglement of this part with the rest of the ground state. We obtain an explicit formula for the entropy of the subsystem at low temperature. At zero temperature we reproduce a logarithmic formula of Holzhey, Larsen and Wilczek. Our derivation is based on the second law of thermodynamics. The entropy of a subsystem is calculated explicitly for Bose gas with delta interaction, the Hubbard model and spin chains with arbitrary value of spin.

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Entanglement is an important resource for quantum computation [1, 2, 3, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 57, 58, 59]. Recently entanglement was studied in detail in ground state of different physical models [6, 36, 37, 38, 39, 40, 41, 42, 56].

Conformal field theory [7] is useful for the description of low temperature behavior of gap-less models in one space and one time dimensions. Critical models are classified by a central charge c of corresponding Virasoro algebra. A definition of the central charge is given the the Appendix. Conformal field theory is closely related to Luttinger liquid [8] approach. We are interested in specific entropy s [entropy per unit length]. Let us start with specific heat C = Tds/dT. Low temperature behavior was obtained in [9, 10]:

$$C = \frac{\pi T c k_B^2}{3\hbar v}.$$

Here c is a central charge and v is Fermi velocity. We put both Plank and Boltzmann constants equal to $k_B = \hbar = 1$. Later in the paper we shall use:

$$C = \frac{\pi Tc}{3v}$$
, as $T \to 0$. (1)

We are more interested in specific entropy s. We can integrate the equation and fix the integration constant from the third law of thermodynamics (s = 0 at T = 0). So for specific entropy we have the same low temperature behavior:

$$\mathbf{s} = \frac{\pi T c}{3v}, \quad \text{as} \quad T \to 0. \tag{2}$$

For quantum spin chains this formula agrees with [11] . To formulate the problem precisely let us consider Bose

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$$H = \int_{-\infty}^{\infty} dx \left[\partial \psi_x^{\dagger} \partial \psi_x + g \psi^{\dagger} \psi^{\dagger} \psi \psi \right]. \tag{3}$$

Here ψ is a canonical Bose field and g > 0 is a coupling constant. The model was solved in [12]. Physics of the model is described in a book [13]. First let us consider the model at zero temperature and in the infinite volume. The ground state is unique $|gs\rangle$. We consider gas with positive density. We are interested in the entropy S(x) of the part of the gas present on a space interval (0, x). Formally we can define it by means of the density matrix

$$\rho = \operatorname{tr}_{\infty} (|gs\rangle\langle gs|) \tag{4}$$

Here we traced out the 'external' degrees of freedom, they describe the gas on the rest of the ground state: on the unification of the intervals $(-\infty,0)$ and (x,∞) . The density matrix ρ describes gas on the interval (0,x). Now we can calculate von Neumann entropy S(x) of the part of the gas on the interval (0,x):

$$S(x) = -\operatorname{tr}_x \rho \ln \rho \tag{5}$$

Here we are taking the trace with respect to the degrees of freedom representing the part of the gas on the interval (0,x). In the major text books it is shown that the laws of thermodynamics can be derived from statistical mechanics, see for example [14, 15]. Second law of thermodynamics states that the entropy is extensive parameter: the entropy of a subsystem S(x) is proportional to the system size x:

$$S(x) = \mathbf{s}x \quad \text{at} \quad T > 0 \tag{6}$$

Thermodynamics is applicable to the subsystem of macroscopical size, meaning large x. Here specific entropy \mathbf{s} depends on the temperature. For small temperatures the dependence simplifies, see (2):

$$S(x) = \frac{\pi Tc}{3v}x, \quad x > \frac{1}{T}.$$
 (7)

gas with delta interaction. The Hamiltonian of the model is: e^{∞}

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Let us try to find out how S(x) depends on x for zero temperature. It is some functions of the size:

$$S(x) = f(x), \text{ at } T = 0.$$
 (8)

Now let us apply the ideas of conformal field theory, see [7, 9, 10] and also Chapter XVIII of [13]. We can arrive to small temperatures from zero temperature by conformal mapping $\exp [2\pi Tz/v]$. It maps the whole complex plane of z without the origin to a strip of the width 1/T. This replaces zero temperature by positive temperature T. The conformal mapping results in a replacement of variable x by $[v/\pi T] \sinh[\pi Tx/v]$. So the entropy of the subsystem at temperature T is given by the formula:

$$S(x) = f\left(\frac{v}{\pi T}\sinh\left[\frac{\pi Tx}{v}\right]\right), \quad \text{at} \quad T > 0$$
 (9)

In order to match this to formula (7) we have to consider asymptotic of large x. The formula simplifies:

$$S(x) = f\left(\exp\left[\frac{\pi T(x - x_0)}{v}\right]\right), \quad Tx \to \infty.$$
 (10)

Here $\pi T x_0/v = -\ln(v/2\pi T)$.

Formulae (7) and (10) should coincide. Both represent the entropy of the subsystem for small positive temperatures. This provides an equation for f:

$$f\left(\exp\left[\frac{\pi T(x-x_0)}{v}\right]\right) = \frac{\pi Tc}{3v}(x-x_0) \tag{11}$$

This formula describes asymptotic for large x, so we added $-x_0$ to the right hand side. We are considering the region x>1/T and $x_0\sim \ln(1/T)$, so $x>>x_0$ at $T\to 0$. In order to solve the equation for f, let us introduce a new variable $y=\exp\left[\pi T(x-x_0)/v\right]$. Then the equation (11) reads:

$$f(y) = \frac{c}{3} \ln y \tag{12}$$

So we found the function f in (8). Now we know that at zero temperature entropy of the gas containing on the interval (0, x) is:

$$S(x) = \frac{c}{3} \ln x$$
, as $x \to \infty$ (13)

Let us remind that for Bose gas the central charge c=1, see [13]. Our result agrees with the third law of thermodynamics. Specific entropy is a limit of the ratio S(x)/x as $x \to \infty$. The limit is zero.

Now we can go back to our formula (9) and substitute the expression for f, which we found:

$$S(x) = \frac{c}{3} \ln \left(\frac{v}{\pi T} \sinh \left\lceil \frac{\pi T x}{v} \right\rceil \right) \tag{14}$$

This formula describes crossover between zero and large temperature. The proof presented here is universal. It is also applicable to quantum spin chains. For example to XXZ spin chain. The Hamiltonian of the model is:

$$\mathbf{H} = -\sum_{j} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z + 2h\sigma_j^z \right\}$$
(15)

It describes interaction of spins 1/2. Here σ are Pauli matrices. The model can be described by a conformal field theory with the central charge c=1 in a disordered regime: $-1 \leq \Delta < 1$ and magnetic field h is smaller then critical h_c , see [13]. For magnetic field larger then critical the ground state is a product of wave functions of individual spins. Entropy of any subsystem is zero.

The formula (13) was discovered in [4, 5]. More general formula can be found in [36, 37]. The entropy of the subsystem S(x) describes the entanglement of the subsystem and the rest of the ground state.

The formula (13) describes the entropy of large subsystem $x \to \infty$ at zero temperature, first correction to this formula is a constant term:

$$S(x) = \frac{c}{3} \ln \frac{x}{x_0}$$
, as $x \to \infty$

The constant term defines the scale x_0 . It does not depend on x, but it does depend on magnetic field and anisotropy, see [16].

So far we discusses the entropy of entanglement for critical models. For non-critical [gap-full] models asymptotic of S(x) is a constant, depending on parameters of the model. For AKLT spin chain $S(\infty) = 2$, see [43]. For XY spin chain $S(\infty)$ depends on anisotropy and magnetic field and shows singularities at phase transitions see [44, 45].

We evaluated von Neumann entropy (5). Calculation of generalized entropies [Rényi and Tsallis] can be reduced to calculation of a trace of some power α of the density matrix: $tr_x(\rho^{\alpha})$ it was done in the paper [16].

SPIN CHAINS WITH ARBITRARY SPIN

In this section we consider higher values of spins. We calculate dependence of entropy of a subsystem on the value of interacting spin \mathbf{s} . Let us consider isotropic XXX anti-ferromagnet [28]. For spin $\mathbf{s}=1/2$ we can represent the Hamiltonian as:

$$\mathbf{H}_{\frac{1}{2}} = \sum_{n} X_n,\tag{16}$$

$$X_n = \vec{S}_n \vec{S}_{n+1} = S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + S_n^z S_{n+1}^z$$
 (17)

The model is solvable by Bethe Ansatz [29]. A generalization of this model to spin s = 1 was found in [30]:

$$\mathbf{H}_{1} = \sum_{n} \left\{ X_{n} - X_{n}^{2} \right\} \tag{18}$$

It was solved by Takhatajan and Babujian, see [31, 32]. Generalization for higher spin **s** was found in [33, 34]

$$\mathbf{H}_s = \sum_n F(X_n) \tag{19}$$

The function F(X) is a polynomial of a degree 2s. It can be written as

$$F(X) = 2\sum_{l=0}^{2s} \sum_{k=l+1}^{2s} \frac{1}{k} \prod_{\substack{j=0\\j \neq l}}^{2s} \frac{X - y_j}{y_l - y_j}$$
 (20)

Here $y_l = l(l+1)/2 - \mathbf{s}(\mathbf{s}+1)$. The model (19) also solvable by Bethe Ansatz. There is no gap in the spectrum of this Hamiltonians. The models can be described by a conformal field theory with the central charge:

$$c = \frac{3\mathbf{s}}{\mathbf{s} + 1},$$

see [35]. So the entropy of a block of x spins is:

$$S(x) = \frac{\mathbf{s}}{\mathbf{s} + 1} \ln x, \quad \text{as} \quad x \to \infty$$
 (21)

The coefficient in-front of the $\ln x$ increases from 1/3 to 1 as spin s increases from 1/2 to ∞ .

Note. It is interesting to remark that in ferromagnetic case [different sign in from of the Hamiltonian] the entropy also scales logarithmically, but the coefficient in front of the $\ln x$ is different, see [38, 39].

HUBBARD MODEL

The Hamiltonian for the Hubbard model ${\cal H}$ can be represented as

$$H = -\sum_{j} \sum_{\sigma=\uparrow,\downarrow} (c_{j,\sigma}^{\dagger} c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger} c_{j,\sigma})$$

$$+ u \sum_{j} n_{j,\uparrow} n_{j,\downarrow} - h \sum_{j} (n_{j,\uparrow} - n_{j,\downarrow})$$
(22)

Here $c_{j,\sigma}^{\dagger}$ is a canonical Fermi operator on the lattice [operator of creation of an electron] and $n_{j,\sigma} = c_{j,\sigma}^{\dagger} c_{j,\sigma}$ is an operator on number of electrons in cite j with spin σ . Cite summation in the Hamiltonian goes through the whole infinite lattice. Coupling constant u>0 and magnetic field h is below critical. At half filled band only spin degree of freedom is gap-less, so at zero temperature $S(x) = (1/3) \ln x$. Let us consider the model below half filling [less then one electron per lattice cite]. The model was solved in [17]. Detailed description of physics of the model can be found in the text-book [20]. Charge and spin separate in the model. The model is gap-less. Both charge and spin degrees of freedom can be described by Virasoro algebra with central charges $c_c = 1$, $c_s = 1$.

see [19]. Also Fermi velocities are different for spin v_s and charge v_c degrees of freedom. Both spin and charge degrees of freedom contribute to the specific entropy s. So for small temperature we have: $\mathbf{s} = \pi T/3v_s + \pi T/3v_c$. The entropy of the subsystem is proportional to the size of the system (second law):

$$S(x) = \left(\frac{\pi T}{3v_s} + \frac{\pi T}{3v_c}\right) x \tag{23}$$

Now we have to apply conformal arguments separately to spin and charge degrees of freedom. The results add up:

$$S\left(x\right) = \frac{2}{3}\ln x\tag{24}$$

This describes the entropy of electrons on the interval (0,x) in the infinite ground state at zero temperature. This is actually an asymptotic for large x. In case of the Hubbard model crossover formula [which mediate between zero and positive temperature] looks like this:

$$S(x) =$$

$$= \frac{1}{3} \ln \left(\frac{v_s}{\pi T} \sinh \left[\frac{\pi T x}{v_s} \right] \right) + \frac{1}{3} \ln \left(\frac{v_c}{\pi T} \sinh \left[\frac{\pi T x}{v_c} \right] \right)$$
(25)

The approach developed here is universal, see [60]. It is applicable to other models of strongly correlated electrons, see [18]. For example formulae (24),(25) describe entropy in t-J model below half-filling as well.

SUMMARY

In the paper we described universal properties of the entropy in one dimensional gap-less models. We studied the entropy of a subsystem. We considered scaling of the entropy in spin chains, strongly correlated electrons and interacting Bose gas. We discovered crossover formula for entropy of the subsystem, see (25). It describes entropy of a large subsystem at low temperature. We discovered a dependence of entanglement on spin s for antiferromagnet, see (21).

APPENDIX

Important characteristic of conformal field theory is a central charge. It can be defined by considering an energy-momentum tensor $T_{\mu,\nu}(z)$. Here z is complex space-time variable z=x+ivt and v is a Fermi velocity. The component of energy-momentum tensor with $\mu=\nu=z$ is denoted by $T_{z,z}=T$. Correlation function of this operator has a singularity: $\langle T(z)T(0)\rangle=c/z^4$, the coefficient c is the central charge.

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