Lorentz invariance with an invariant energy scale

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We propose a modification of special relativity in which a physical energy, which may be the Planck energy, joins the speed of light as an invariant, in spite of a complete relativity of inertial frames and agreement with Einstein's theory at low energies. This is accomplished by a non-linear modification of the action of the Lorentz group on momentum space, generated by adding a dilatation to each boost in such a way that the Planck energy remains invariant. The associated algebra has unmodified structure constants, and we highlight the similarities between the group action found and a transformation previously proposed by Fock. We also discuss the resulting modifications of field theory and suggest a modification of the equivalence principle which determines how the new theory is embedded in general relativity.

A simple paradox confronts us as we seek the quantum theory of gravity. The combination of gravity (G), the quantum ($\hbar$) and relativity (c) gives rise to the Planck length, $l_P = \sqrt{\hbar G/c^3}$ or its inverse, the Planck energy E_P. These scales mark thresholds beyond which the old description of spacetime breaks down and qualitatively new phenomena are expected to appear. Thanks to the progress made by several different approaches to quantum gravity we have predictions for these new phenomena, which include discrete spatial and causal structure, discrete spectra for physical observables such as area and volume [1] and the appearance of string rather than local excitations.

However, the new theory is expected to agree with special relativity when the gravitational field is weak or absent, and in experiments probing the nature of spacetime at energy scales much smaller than E_P. This gives rise immediately to a simple question: *in whose reference frame are l_P and E_P the thresholds for new phenomena?* For suppose that there is a physical length scale which measures the size of spatial structures in quantum spacetimes such as the discrete area and volume predicted by loop quantum gravity. Then if this scale is l_P in one inertial reference frame, special relativity suggests it may be different in another observer's frame: a straightforward implication of the Lorentz-Fitzgerald contraction.

There are several different possible answers to these questions. One is that Lorentz invariance, (both global and local) is only an approximate symmetry, which is broken at the Planck scale. This has been advocated by a number of physicists, and there have been some claims that Lorentz symmetry breaking could be observable in the near future (or may even already have been observed) in cosmic ray spectra [2] and gamma ray bursts [3,4,6]. However it is troubling to contemplate giving up the principles behind Lorentz invariance, which are the relativity of inertial frames and the equivalence principle. Does incorporating the Planck scale into physics mean that in the end there are preferred states of rest and motion?

In this letter we show that the answer is no. It is in fact possible to modify the action of the Lorentz group on physical measurements so that a given energy scale, which we will take to be the Planck energy, is left invariant. That is, we can have the complete relativity of inertial frames and at the same time have all observers agree that the scale on which a transition from classical to quantum spacetime takes place is the Planck scale, which is the same in every reference frame. At the same time, the familiar and well tested actions of the boosts are maintained at large distance or low energy scales. This is achieved not by a quantum deformation of the Lorentz or Poincare group, but by a modification of the action of the Lorentz group acting on momentum space. The action is defined to be non-linear in general, but to reduce to the usual linear action at energies much below the Planck scale. The non-linearities are chosen so that the Planck energy becomes an invariant. The speed of light is still meaningful, and is still an invariant.

A similar proposal was made by Fock [7], motivated by the search of the general symmetry group preserving relativity without assuming the constancy of c. However in that case the action of the transformations are modified at large distances rather than large momentum. One can understand our proposal as an application of the Fock-Lorentz symmetry to momentum space. The fact that we may preserve the invariance of the speed of light, if we wish, is an added bonus of our approach.

Our argument is based upon four basic principles. First we assume *the relativity of inertial frames*: when gravitational effects can be neglected, all observers in free, inertial motion are equivalent. This means there is no preferred state of motion, so velocity is a purely relative quantity. Secondly we assume *the equivalence principle*: under the effect of gravity freely falling observers are all equivalent to each other and are equivalent to inertial observers. We then introduce a new principle: *the observer independence of the Planck energy:* all observers agree that there is an invariant energy scale, which we take to be the Planck scale E_P. This will lead to novelties, but we finally impose *the correspondence principle:* at energy scales much smaller than E_P conventional special and general relativity are true, that is they hold to first order in the ratio of energy scales to E_P.

The first and fourth principle tell us that there is a transformation group that converts measurements made by one inertial observer to measurements made by another. For energy scales much smaller than E_P this action should reduce to the ordinary Lorentz group. Thus we expect that the Lorentz group should be replaced by a deformed or modified group, acting on momentum space. As in ordinary special relativity that group must be a six parameter extension of the spatial rotations group - three parameters for rotations and three for boosts. However, the only six parameter group that has these characteristics is the Lorentz group itself. But we know that the usual linear action of the Lorentz group on momentum space does not fix any energy scale, as required by our third principle. The only possibility then is that the symmetry group is the ordinary Lorentz group, but it acts *non-linearly* on the momentum space. That non-linear action should involve the Planck energy in some way that ensures that the Planck energy is preserved. One way to do this is to combine each boost with a dilatation. The dilatation must be chosen so as to bring one energy scale back to the value it had before the boost transformation. We show how to do this first for the Lorentz algebra, then for the Lorentz group.

Momentum space $\mathcal{M}$ is the four dimensional vector space consisting of momentum vectors p_a. The ordinary Lorentz generators act as

$$L_{ab} = p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a} \tag{1}$$

where we assume a metric signature $(+,-,-,-)$ and that all generators are antihermitian (also $a,b,c, = 0,1,2,3$, $i,j,k = 1,2,3$, and $c = 1$). In addition the dilatation generator $D = p_a \frac{\partial}{\partial p_a}$ acts on momentum space as $D \circ p_a = p_a$. We may consider now the modified algebra, generated by the usual rotations $J^i \equiv \epsilon^{ijk} L_{ij} = \epsilon^{ijk} M_{jk}$ and a modified generator of boosts,

$$K^i \equiv L_0{}^i + l_P p^i D \equiv M_0{}^i. \tag{2}$$

We note that despite the modification, J^i and K^i satisfy precisely the ordinary Lorentz algebra:

$$[J^i, K^j] = \epsilon^{ijk} K_k;\ [K^i, K^j] = \epsilon^{ijk} J_k \tag{3}$$

(with $[J^i, J^j] = \epsilon^{ijk} J_k$ trivially preserved). However the action on momentum space has become non-linear due to the term in p^i in (2). The new action can be considered to be a non-standard, and non-linear embedding of the Lorentz group in the conformal group.

To exponentiate the new action we note that

$$K^i = U^{-1}(p_0) L_0{}^i U(p_0) \tag{4}$$

where the energy dependent transformation $U(p_0)$ is given by $U(p_0) \equiv \exp(l_P p_0 D)$. The non-linear representation is then generated by $U(p_0)$ and we have

$$U(p_0) \circ p_a = \frac{p_a}{1 - l_P p_0} \tag{5}$$

We note that $U(p_0)$ is not unitary, so this is not a unitary equivalence. We also note that $U(p_0)$ is singular at $p_0 = l_P^{-1}$, a property which signals the emergence of a new invariant.

The non-linear representation of the Lorentz group is then given by

$$W[\omega_{ab}] = U^{-1}(p_0) e^{\omega^{ab} L_{ab}} U(p_0) = e^{\omega^{ab} M(p_0)_{ab}} \tag{6}$$

In evaluating this expression, note that D acts on everything to the right, and p_0 always means the time component of the vector immediately to the right. Using these rules, one finds that the boosts in the z direction are now given by:

$$p_0' = \frac{\gamma\left(p_0 - v p_z\right)}{1 + l_P(\gamma - 1)p_0 - l_P \gamma v p_z} \tag{7}$$

$$p_z' = \frac{\gamma\left(p_z - v p_0\right)}{1 + l_P(\gamma - 1)p_0 - l_P \gamma v p_z} \tag{8}$$

$$p_x' = \frac{p_x}{1 + l_P(\gamma - 1)p_0 - l_P \gamma v p_z} \tag{9}$$

$$p_y' = \frac{p_y}{1 + l_P(\gamma - 1)p_0 - l_P \gamma v p_z} \tag{10}$$

which reduces to the usual transformations for small $|p_\mu|$.

This transformation is identical with a transformation introduced by Fock [7] but applied to momentum space. Fock's transformation is obtained from the one above replacing p with x, and therefore its generators also satisfy the standard commutators (3). However non-linearity means that the group action in spatial and momentum space are radically different. Indeed Fock's transformation (defined in x space) reduces to Lorentz at *small* distances (so that it defines a *large* invariant Planck length). On the contrary, our transformation (defined in p space) becomes Lorentz for *small* energies and momenta (and defines a *large* invariant Planck energy, as we shall see) - the property we are looking for. Also Fock's transformation contains a varying speed of light [7,8], whereas, as we shall see, our proposal does not.

It is not hard to see that the Planck energy is preserved by the modified action of the Lorentz group. For example, boosts in the z direction with velocity v take $(E_p, 0, 0, 0) \to (E_p, -vE_p, 0, 0)$. From the group property we can also deduce (and then check) the following. Suppose we observe a particle in our frame with energy momentum, $(E_p, P, 0, 0)$ with $P/E_p < 1$. Then a boost in the $-\hat{z}$ direction with $v = P/E_p$ will bring us to the Planck mass particle's rest frame with $(E_p, P, 0, 0) \to (E_p, 0, 0, 0)$. Furthermore the 4-momenta of photons with Planck energy E_p traveling in the z direction are preserved under boosts in the z direction, because $(E_p, E_p, 0, 0) \to (E_p, E_p, 0, 0)$.

Clearly these transformations do not preserve the usual quadratic invariant on momentum space. But there is a modified invariant, obtained from $U(\eta^{ab} p_a p_b)$, which is:

$$||p||^2 \equiv \frac{\eta^{ab} p_a p_b}{(1 - l_P p_0)^2} \tag{11}$$

This invariant is infinite for the new invariant energy scale of the theory $E = l_P^{-1}$, and it's not quadratic for energies close or above $E = l_P^{-1}$. This signals the expected collapse in this regime of the concept of metric (i.e. a quadratic invariant).

It is also evident from (11) that the symmetry of positive and negative values of the energy is broken. The formalism may be defined with l_P equal to minus the Planck length, in which case the invariant diverges for energy $E = -E_p$. The two theories with the two signs of l_P are physically distinct; and we know of no theoretical consideration which fixes the sign of l_P. Even though in what follows we shall assume $E_P > 0$, we will also briefly consider how conclusions change if $E_P < 0$, so that both the sign and magnitude of l_P may be determined experimentally, from effects that we shall now discuss.

We start by considering massive particles. These have a positive invariant $||p||^2 > 0$ which may be identified with the square of the mass $||p||^2 = m_0^2 c^4$. Considering the rest frame we therefore obtain a modified relation between energy and mass:

$$E_0 = \frac{m_0 c^2}{1 + \frac{m_0 c^2}{E_p}} \tag{12}$$

In a general frame we find that m transforms in the usual way $m = \gamma m_0$, however:

$$E = \frac{m}{1 + \frac{m}{E_p}} \tag{13}$$

$$p = \frac{mv}{1 + \frac{m}{E_p}} \tag{14}$$

It is at once obvious that the energy of a particle can never equal or exceed E_p, even though its mass may be as large as wanted. Asymptotically a particle may have $E = E_p$ if it has infinite rest mass. Its energy and momentum are then frame independent, in agreement with the postulates of the theory. Notice that if $E_P > 0$ the energy of a particle is smaller than the usual $E = mc^2$; however if $E_P < 0$ its energy is larger than mc^2 and in fact diverges for Planck mass particles.

All these remarks apply to fundamental particles, not macroscopic sets of them. The latter may have masses larger than E_P, but if they are made of particles with $E \ll E_P$ they do not feel the transformations (7) because these, being non-linear, are not additive.

The modified invariant for photons still has the property: $||p||^2 = 0$ and so $E = p_0 = |p_i|$. Consider a photon moving in the z direction, so that $E = |p_i| = p_z$, and consider a boost in the z direction as above. We thus obtain the Doppler shift formula

$$E' = \frac{E\gamma(1 - v)}{1 + (\gamma(1 - v) - 1) l_P E} \tag{15}$$

This can be rewritten as

$$\frac{1}{E'} - \frac{1}{E_P} = \frac{1}{\gamma(1 - v)} \left(\frac{1}{E} - \frac{1}{E_P} \right) \tag{16}$$

showing how $E = E_p = 1/l_P$ is invariant - so the Planck energy and momentum for photons is frame independent. Furthermore super and sub Planckian energies never get mixed via Doppler shift, as $\gamma(1 - v) > 0$ and the sign of both sides of eqn. (16) must be the same. It is impossible to blueshift a sub-Planckian photon up to E_p, or redshift a super-Planckian photon down to E_p. Closer inspection reveals an abnormality: super-Planckian photons redshift if the source moves towards the observer, blueshift otherwise. It is impossible to redshift them below E_p whatever the speed of a source towards us. If the source moves away from us there is a recession speed for which $E' = \infty$ and beyond which $E' < 0$. (These remarks apply to $E_P > 0$ only).

Using the equivalence principle we can now derive a formula for the first order gravitational redshift. In the non-relativistic regime the Doppler shift is $\frac{\Delta E}{E} = v(1 - l_P E)$ showing a decrease in the Doppler shift as the photon approaches the Planck energy. Using the equivalence principle this translates into a similar modification for the gravitational shift

$$\frac{\Delta E}{E} = \Delta\phi(1 - l_P E) \tag{17}$$

The Pound Rebbka experiment is of course not sensitive enough for detecting this new effect, but ultra high energy cosmic rays (UHECR) might not be.

In future work we shall examine the effects of our proposal for fields of all spins, including gravity. Here we merely outline how our approach leads to modifications in field theory, considering the case of a scalar field. Up till now we considered the modification of the Lorentz transformations on momentum space. When applied to field theory, the derivatives of a field should transform as momentum, as they correspond to physical frequencies and wavelengths. Thus, under a change of inertial observers we have $(\partial_a \phi) \rightarrow (\partial_a \phi)' = W(\partial_0 \phi)_a{}^b (\partial_b \phi)$ and we see that the transformation is non-linear, with $l_P \partial_0 \phi$ playing the role of p_0.

The action for a scalar field must be Lorentz invariant, but in the present context this means that it should be invariant under the modified Lorentz transformations. The invariant action is then

$$S^\phi = \int d^4x \frac{1}{2} \frac{\eta^{ab} (\partial_a \phi)(\partial_b \phi)}{[1 - l_P^2 (\partial_0 \phi)]^2} + \frac{m^2}{2} \phi^2 \tag{18}$$

Because there is now no quadratic invariant, there is no linear field equation. Instead we may derive a complex non-linear field equation, as we shall do in a future publication. The surpising thing is that a single plane wave $\phi_k(x) = A \exp(-\imath k_a x^a)$ is still a solution with $\eta^{ab} k_a k_b = 0$, if $m = 0$. However the superposition principle no longer holds. It is also the case that for massive fields there no longer is an exact plane wave solution.

To extend the new theory to a modification of general relativity we must find the appropriate way to express the equivalence principle. Given that we have modified the action of the Lorentz transformations in special relativity in the momentum space we proceed by remarking the following: 1) Matter is most generally represented in general relativity in terms of fields; 2) Momenta of fields is associated with spatial derivatives; 3) In a field theory, the mathematical tangent space corresponds physically most closely to the derivatives of fields; 4) By the equivalence principle, the Lorentz group acts on components of fields referred to orthonormal frames. This leads us to a *modified equivalence principle:* the non-linear realization of the Lorentz group discussed above acts on derivatives of fields referred to orthonormal frame components. That is, in the presence of gravity the transformations proposed are defined for quantities of the form $(\partial_a \phi)|_x \equiv e^\mu_a(p)(\partial_\mu \phi)_x$ (where greek letters refer to ordinary manifold coordinates and latin letters to components in an orthonormal frame). These transform according to the non-linear realization, i.e. measurements made by two orthonormal frames, e^μ_a and e'^μ_a, of derivatives of a scalar field are related by $(\partial'_a \phi) = W(\partial_0 \phi)^b_a (\partial_b \phi)$ where $W(\partial_0 \phi)^b_a$ is defined by (6) and depends on $\partial_0 \phi|_p$ at event p in the same way that the momentum space realization (6) depended on the energy p_0.

We see that the orthonormal frame components themselves do not have well defined transformation rules under these modified transformations. We consider these abstract mathematical quantities, while our transformation rule only applies to physical measurements of momenta. Similarly, there is no new transformation rule for the manifold derivatives $\partial_\mu \phi$ as these also do not relate to measurements made by freely falling observers. The latter are described by $\partial_a \phi$ and so it is only to these, and not to their separate mathematical parts, that the new transformation rules apply. In future work we show that this prescription fully defines how to incorporate our transformation into field theory for all spins.

In summary what we have proposed here is a modification of the two basic principles of physics: the relativity of inertial frames and the equivalence principle. The modifications proposed are the simplest ones we are aware of consistent with the demand that the Planck energy be an invariant, while special relativity as formulated by Einstein hold at much lower energies. We explored implications for well known special relativistic effects, and found modified formulae for them.

We close this letter with a number of questions, to be examined in future work. Foremost: does the modified equivalence principle we have just stated lead uniquely to a consistent modification of general relativity? The fact that the algebra of the symmetry group remains the same suggests that perhaps the standard spin connection formulation of relativity is still valid. At high energies there is no longer a metric, as the invariant (11) is no longer quadratic. However the connection, taking values in the algebra, is still unmodified, and one may define curvature and the usual tools of Riemanian geometry without any trouble. We hope to return to this issue and study implications for cosmology and black hole physics [12].

One may further ask how the modified action of the Lorentz group is to be extended to spinor fields? A related issue is whether supersymmetry can be modified to be consistent with the modified action. Does this lead to mass differences between supersymmetric partners? Can string theory also be modified to be consistent with the principles described here? Could the principles proposed here be derived from the large distance limit of causal spin foam models, which incorporate discrete spatial and causal structure at the Planck scale?

We finally note that we can find many non-linear realizations of the action of the Lorentz group, by making other choices for $U(p_0)$ in eq. (4). These lead to other forms for the modified invariants and hence to different dispersion relations for massive and massless particles. It is interesting to ask to what extent these can be distinguished experimentally by data from gamma ray bursts and UHECRs? More general choices of $U(p_0)$ in eq. (4) in general lead to invariants which contain an energy dependent speed of light. Could these theories be used to implement the varying speed of light cosmology [10,11]?

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