Cyclic Topology in Complex Networks

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We propose a cyclic coefficient R which represents the cyclic characteristics of complex networks. If the network forms a perfect tree-like structure then R becomes zero. The larger value of R represents that the network is more cyclic. We measure the cyclic coefficients and the distributions of the local cyclic coefficient for both various real networks and the representative network models and characterize the cyclic structures of them.

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During a recent few years, complex networks have received considerable attentions [1]. They appear in a variety of system such as biological [2, 3, 4], social [5, 6, 7, 8], informational [9, 10], and economic [11] systems. Such complex networks are characterized by some topological and geometrical properties such as small world, high degree of clustering, and scale-free topology. The smallworld property denotes that the average path length Lwhich is the average shortest path length between vertex pairs in a network, is small. It grows logarithmically with the network size N. The clustering structure in a network is measured by the clustering coefficient C which is defined as the fraction of pairs between the neighbors of a vertex that are the neighbors of each other. The high degree of clustering indicates that if vertices A and B are linked to vertex C then A and B are also likely to be linked to each other. These two properties were realized by small-world network (SWN) model [12] in which randomly selected vertex pairs are linked by short-cuts. The scale-free (SF) topology reflects that the degree distribution P(k) follows a power law, $P(k) \sim k^{-\gamma}$, where degree k is the number of edges incident upon a given vertex and γ is the degree exponent. An evolving model introduced by Barabasi-Albert (BA) [13] well illustrated the SF property. Such network is called the SF network.

Recently, many efforts have been done to elucidate the structural properties of complex networks. The hierarchical structure appears in some real networks and has been clarified by power-law behavior of the clustering coefficient C(k) as a function of the degree k[14, 15, 16, 17, 18, 19]. This indicates that the networks are fundamentally modular and it is the origin of the high degree of clustering of complex networks. Also, it was recently found that many real networks include statistically significant subnetworks, so-called motifs, in their structures [20, 21, 22].

Especially, the recent studies for the topological properties of complex networks have attracted much atten-

tion to the loop (cyclic) structure. The presence of loops has some effect on the delivery of information, transport process, epidemic spreading behavior [23], and etc. With respect to a tree-like topology, loops provide more paths along which the information or virus can propagate. A cycle of order k can be defined as a closed loop of kedges. That is, graphically a triangle is a cycle of order 3, while a rectangular is a cycle of order 4. A tree which can not form a closed loop can be regarded as a cycle of infinite order. The clustering coefficient counts the triangle structure only. Meanwhile there are many cycles of higher order which is larger than 3 in complex networks so that it is necessary to investigate the cycles of higher order for the characterization of the cyclic structure. Some previous studies [24, 25, 26, 27] in which cycles of order 4 or 5 were considered, are good trials to explain the loop structure of higher order.

In this paper, we survey the cyclic topology in complex networks by introducing a new quantity R which characterizes the degree of circulation in the systems. We consider the cycles of all order starting from three up to infinity to define the quantity R. By monitoring the values of R and its distribution, the cyclic topology of the networks is analyzed for both several real networks from technological to social systems and the network models such as SWN and BA models.

We introduce a new quantity R to measure how cyclic is a network and call it the cyclic coefficient. At first, the local cyclic coefficient r_i for a vertex i is defined as the average of the reciprocal of the size of loops which are formed by a vertex i and its two neighboring vertices, i.e.,

$$r_i = \frac{2}{k_i(k_i - 1)} \sum_{< lm >} \frac{1}{S_{lm}^i}$$
(1)

where k_i is the degree of a vertex i and $\langle lm \rangle$ is all the pairs of the neighbors of the vertex i. S_{lm}^i is the smallest size of the closed path that pass through a vertex i and its two neighbor vertices l and m. If vertices l and mis directly connected to each other than the vertices i, l, and m form a triangle. It is a cycle of order 3 and S_{lm}^i has a value three which is the smallest value of S.

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FIG. 1: (a) A typical example for the cyclic coefficient. The cyclic coefficient of filled circle is obtained by $r_{\bullet} = 0.13$. The two sample networks with same network size N = 25 and different cyclic coefficient are shown in (b) and (c) where R = 0 and R = 0.29, respectively.

If there does not exist any paths that connect vertices l and m except for the path through the vertex i, then the vertices i, l, and m form a tree. In this case, there does not exist any loop pathing through the three vertices i, l, and m. It is the cycle of infinite order and the value of S_{lm}^i becomes infinity. For an example as shown in Fig. 1 (a) the local cyclic coefficient r_{\bullet} of a vertex \bullet is given as $r_{\bullet} = 0.13$ with $S_{12}^{\bullet} = 3$, $S_{23}^{\bullet} = 4$, $S_{13}^{\bullet} = 5$, and $S_{14}^{\bullet} = S_{24}^{\bullet} = S_{34}^{\bullet} = \infty$.

The cyclic coefficient R is the average of r_i over all the vertices, $R = \langle r_i \rangle$ which has a value between zero and 1/3. R=0 means that a network has a perfect treelike structure in which no loops is formed. Meanwhile if all the neighbor pairs are connected to each other i.e., the clustering coefficient becomes C = 1, and R=1/3. Figure 1 (b) and (c) show two examples with R=0 and R=0.29 for N = 25, respectively. Thus the larger is the cyclic coefficient R, the more cyclic is the architecture of the network. The cyclic coefficient R could be a good quantity to identify the degree of circulation in a complex network.

In order to characterize the cyclic topology in real networks we have measured the cyclic coefficient R for several real networks [28] appearing in biological, technological, and social systems. In the measurement, we excluded the isolated vertices and focused on the entirely connected part of the network.

First, we consider the protein network [3] which is composed of 1458 proteins. It has 1948 identified direct physical interactions. The proteins and the direct interactions are considered as vertices and edges, respectively. Figure 2 (a) shows the histogram of the distribution P(r) of the local cyclic coefficient. About 60% of the total vertices have r = 0 and P(r) has small value for all the range $0 < r \leq 1/3$, resulting in small value $R \approx 0.06$, which indicates that there are very little loops and the network constitutes a tree-like structure. Thus a tree-like topology of the protein network pictured in the reference [3] is well quantified by our cyclic coefficient.

Second, the physical internet network [9] at the interdomain (Autonomous System(AS)) level is considered. Each domain, composed of hundreds of routers and com-



FIG. 2: The probability distribution of local cyclic coefficient for four real networks, (a) protein network, (b) Internet network, (c) math co-authorship network, and (d) movie actor collaboration network.

puters, acts as a vertex and an edge is drawn between two domains if there is at least one route that connects them. The network at the AS level, as of 15th September 1999 is composed of both 5746 ASs and 11017 edges. $R \approx 0.16$ is obtained in this network. From the distribution of the local cyclic coefficient r (Fig. 2 (b)), we found that the most vertices have a value among r = 0, r = 0.25, and r = 1/3. That is, the vertices with a tree structure are dominant (r = 0) and the most of the rest form loops of small size (3 or 4).

Third, we consider the network of scientific collaborations in the field of mathematics published in the period 1991-1998 [6], in which the vertices are the scientists. They are connected if they write a paper together. The total number of vertices and edges are 57516 and 143778 respectively. The value of cyclic coefficient is $R \approx 0.19$. Figure 2 (c) shows the probability distribution of r. It has the first peak at r = 0.33, which indicates that cycles of order 3 dominate in the networks. distinguishing from the protein and internet networks where the treelike structure dominates.

Finally, the movie actor collaboration network [5] is constituted of 9865 vertices and 273412 edges. The vertices are the actors and two vertices are connected if the corresponding actors have acted in the same movie together. Figure 2 (d) shows that the distribution of r has a maximum value at r = 0.33, which reflects the high degree of clustering in social networks. Meanwhile the vertices with r = 0 almost do not exist in contrast to the case of the other networks. This explains that the movie actor network is more cyclic with large value of R = 0.29.

From the results of the above four examples, we have found that both the well clustered parts and non clusterd parts coexist in the real networks especially in the math co-authorship network. The probability distribution P(r) is not uniform. Instead, there are a few peaks at certain values of r such as r = 0 or r = 1/3. It means that most of the vertices have triangle structure



FIG. 3: The cyclic coefficient R(p) (circle) and the clustering coefficient C(p)(square) for the SW network model. The data is normalized by the R(0) and C(0) which are 0.283 and 0.5 respectively for a regular network. The distribution of local cyclic coefficient for the random network (p = 1) is shown in the inset.

or tree structure with the neighbor vertices. That is, the neighbor vertices in the well clustered parts have high connections each other while in the other parts there are no clustering at all. Thus by measuring the distribution of local cyclic coefficient we can understand the details of cyclic structure in the complex networks.

We have considered the cyclic coefficient R for two representative models of complex networks, the SWN [12] and the BA model [13]. The algorithm of the SWN model is the following: Consider a one-dimensional lattice of N vertices with periodic boundary conditions, i.e., a ring and connect each vertex to its first m neighbors. The small-world model is then created by randomly rewiring each edge of the lattice with probability p, moving one end of that edge to a new location chosen randomly from the lattice, except that self-connections and duplicate edges are created. This rewiring process introduces pNm/2 shortcuts which connect vertices that are in long-range and by varying p the transition between a regular lattice (p = 0) and random network (p = 1) [29] can be shown.

Figure 3 shows the plot of the normalized clustering coefficient C(p)/C(0) and the cyclic coefficient R(p)/R(0)as a function of the rewiring probability p with the network size N = 10000 and m = 4. The clustering coefficient stays almost unchanged for p < 0.01 and drops to almost zero at p = 1. It is the characteristics of SWN with the high degree of clustering for p < 0.01. The cyclic coefficient R(p) also keeps up the value of R(0) for p < 0.01 while decreases to a finite value at p = 1. The finite value of R(1) comes from the contribution of the loops for all orders. The inset of Fig. 3 shows the properties of the random network (p = 1). As shown in the inset, the cyclic distribution has a peak at r = 0.11 with $R \approx 0.11$. It is interesting that P(r) is almost zero for both r = 0 and r = 1/3.



FIG. 4: The distribution of local cyclic coefficient for the BA model

We also measure the cyclic coefficient for the BA model [13]. The BA model is carried out as the following: Start with a small number N_0 of vertices and no edges. At every time-step, a new vertex with $m (\leq N_0)$ edges is added where the m edges link the new vertex to mdifferent vertices already present in the system. The vertices to which the new vertex is connected are chosen with the preferential attachment rule in which the probability Π for a vertex *i* to be connected with a new vertex depends on the degree k_i of the vertex i, such that $\Pi(k_i) = k_i / \sum_j k_j$. We have obtained $R \approx 0.17$ with the network size N = 10000 for the BA model. As shown in Fig. 4, the distribution of the local cyclic coefficient in the BA model shows a poisson-like shape having a peak at r = 0.16. It represents the random nature of the local circulation in the BA networks. However, in the real networks given above the cyclic distributions do not follow the poisson-like shape and have a peak at r = 0 or r = 1/3. There is almost no mechanism to form a triangle or tree structure in the BA model, in contrast to the case of the real networks.

The network size N, mean degree $\langle k \rangle$, clustering coefficient C, cyclic coefficient R, the probability distribution with r = 0 P(0) (tree structure), and with r = 1/3P(1/3) (cyclic structure of loops with length three) are summerized in Table I for the considered networks.

In conclusion, we have evaluated the degree of circulation in complex networks by introducing the cyclic coefficient R. It inclueds the total effect of all the sizes of the loops. If the network forms a perfect tree-like structure, R becomes zero. The value of cyclic coefficient is in between zero and 1/3. The larger is the cyclic coefficient, the more cyclic becomes a network. We measured the cyclic coefficients for various real complex networks and the representative network models. For the protein network of biological system the cyclic coefficient is small which reflects that the protein network is tree-like, while for the movie actor collaboration network of social system we found that the cyclic coefficient is large and its structure is more cyclic. Also by measuring the proba-

r = 1/3 P(1/3) (cyclic structure of loops with length three).						
Network	Ν	(k)	С	R	P(0)	P(1/3)
protein interactions	1458	2.67	0.07	0.06	0.60	0.04
Internet	5746	3.83	0.24	0.16	0.38	0.19
math co-authorship	57516	5.00	0.48	0.19	0.24	0.35
movie actor collaborations	9853	54.95	0.58	0.29	0.01	0.34
random network $(p = 1)$	10000	4	0.0003	0.11	0	0
BA network	10000	6	0.006	0.17	0	0

TABLE I: For both four real networks and two network model, we summarized the various data of network size N, mean degree $\langle k \rangle$, clustering coefficient C, cyclic coefficient R, the probability distribution with r = 0 P(0) (tree structure), and with r = 1/3 P(1/3) (cyclic structure of loops with length three).

bility distribution of the local cyclic coefficient, we could classify the details of the cyclic structure of the complex networks. Thus the cyclic coefficient and its distribution help us to understand the cyclic structures of the complex networks. It is interesting to keep in surveying the cyclic coefficient for other various complex networks.

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- M.E.J. Newman, SIAM Rev. 45, 167 (2003); R. Albert and A. L. Barabábasi Rev. Mod. phys. 74, 47 (2002); S.N. Dorogovtsev and F.F. Mendes Adv. Phys. 51, 1079 (2002).
- [2] H. Jeong, B. Tombor, R. Albert, Z.N. Oltvai, and A.-L. Barabási, Nature 407, 651 (2000).
- [3] H. Jeong, S.Mason, R. Albert, A.-L. Barabási, and Z.N. Oltvai, Nature 411, 41 (2001).
- [4] A.-L. Barabási and Z.N. Oltvai, Nature 5, 101 (2004).
- [5] M. E. J. Newmann, S. H. Strogatz, and D. J. Watts, cond-mat/0007235, (2000); A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
- [6] M.E.J. Newman, Proc. Nat. Acad. Sci. USA 98, 404 (2001).
- [7] A.-L. Barabási, H.Jeong, Z. Néda, E. Ravasz, A Schbert, T. Vicsek, cond-mat/0104162 (2001).
- [8] F. Liljeros, C.R. Edling, L.A.N. Amaral, H.E. Stanley, and Y. Averg, Nature 411, 907 (2001).
- M. Faloutsos, P. Faloutsos, and C. Faloutsos, ACM SIG-COMM '99 Comput. Commun. Rev. 29, 251 (1999); R. Pastor-Satorras, A. Vázquez, and A. Vespignani, Phys. Rev. Lett. 87 258701.
- [10] R. Albert, H. Jeong, and A.-L. Barabási, Nature 401, 130 (1999).
- [11] H.-J. Kim, Y. Lee, B. Kahng, and I.-m. Kim, J. Phys. Soc. Japan 71, 2133 (2002).
- [12] D.J. Watts and S.H. Strogatz, Nature (London) **393**, 440 (1998); D.J. Watts, Small Worlds: The Dynamics of Networks between Order and Randomness (Princeton, New York, 1999).
- [13] A.-L. Barabási and R. Albert, Science 286, 509 (1999);
 A.-L. Barabási, R. Albert, and H. Jeong, Physica A 272, 173 (19999).

- [14] E. Ravasz and A.-L. Barabási, Phys. Rev. E 67, 026112 (2003).
- [15] S.N. Dorogovtsev, A.V. Goltsev, and J.F.F. Mendes, Phys. Rev. E 65, 066122 (2002).
- [16] G. Szabo, M. Alava, and J. Kertész, cond-mat/0208551.
- [17] M.E.J. Newman, Phys. Rev. E 68, 026121 (2003).
- [18] K. Klemm and V.M. Eguiluz, Phys. Rev. E 65, 036123 (2002).
- [19] A. Vazquez, R. Pastro-Satorras, and A. Vespignani, Phys. Rev. E 65, 066130 (2002).
- [20] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon, Science 298, 824 (2002).
- [21] N. Kashtan, S. Itzkovitz, R. Milo, U. Alon, q-bio.MN/0312019.
- [22] V.P. Zhigulin, cond-mat/0311330.
- [23] T. Petermann and P. De Los Rios, cond-mat/0401434, (2004).
- [24] G. Bianconi and A. Capocci, Phys. Rev. Lett. 90, 078701 (2003).
- [25] G. Caldarelli, R. Pastor-Satorras, and A. Vespignani, cond-mat/0212026 (2002).
- [26] H. D. Rozenfeld, J. E. Kirk, E. M. Bolt, and D. ben-Avraham, cond-mat/0403536 (2004).
- [27] A. Fronczak, J. A. Holyst, M. Jedynak, and J. Sienkiewicz, Physica A **316**, 688 (2002).
- [28] The data of real networks used in this paper is publicly available at http://www.nd.edu/~networks and http://stat.kaist.ac.kr.
- [29] P. Erdós and A. Rényi, Publ. Math. Debrecen 6, 290 (1959); B. Bollobás, *Random Graphs* (Academic Press. London, 1985).