

Gauge Theories under Incorporation of a Generalized Uncertainty Principle

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There is considered an extension of gauge theories according to the assumption of a generalized uncertainty principle which implies a minimal length scale. A modification of the usual uncertainty principle implies an extended shape of matter field equations like the Dirac equation. If there is postulated invariance of such a generalized field equation under local gauge transformations, the usual covariant derivative containing the gauge potential has to be replaced by a generalized covariant derivative. This leads to a generalized interaction between the matter field and the gauge field as well as to an additional self-interaction of the gauge field. Since the existence of a minimal length scale seems to be a necessary assumption of any consistent quantum theory of gravity, the gauge principle is a constitutive ingredient of the standard model, and even gravity can be described as gauge theory of local translations or Lorentz transformations, the presented extension of gauge theories appears as a very important consideration.

I. INTRODUCTION

It is widely assumed that a consistent formulation of a quantum theory of gravity implies the existence of a minimal length scale in nature, which is usually expected to be directly connected to the Planck length. In accordance with this, all present approaches of a formulation of a quantum theory of gravity contain a minimal length arising in a natural way. The introduction of a minimal length scale to the description of nature is directly related to the assumption of a generalized uncertainty principle in quantum mechanics [1],[2],[3], where the usual Heisenbergian commutation relation between the position and the momentum operator is extended in such a way that it depends on the position or on the momentum operator. If it depends on the momentum operator, this implies that there exists not only a minimal uncertainty between positions and momenta but also of position itself, which represents nothing else than a minimal length. According to such a generalized uncertainty principle, there arise new representations of the position operator in momentum space and the momentum operator in position space, respectively. If the generalized momentum operator represented in position space is inserted to quantum theoretical field equations, this leads to a modification of the dynamics depending on the scale, which is directly related to the corresponding parameter introduced by the modification of the uncertainty principle. There have already been considered various implications of such a generalized description of nature to certain aspects of quantum mechanics and quantum field theory as they are investigated in [2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15],[16] as well as to many topics related to gravity as they are treated in [17],[18],[19],[20],[21],[22],[23],[24],[25],[26],[27],[28],[29],[30],[31],[32],[33],[34],[35],[36],[37],[38],[39]. A very important topic with respect to the formulation of quantum field theories with a generalized uncertainty principle is the implication concerning the gauge principle on which all interaction theories of the standard model as well as gravity, which can be considered to be the gauge theory of local translations or Lorentz rotations, are based on. The issue of gauge invariance in the context of quantum field theories with a generalized uncertainty principle has already been considered in [8] for example. In the common description of gauge theories the usual derivative is replaced by a covariant derivative by introducing a gauge potential which transforms in a specific way to maintain local gauge invariance of a field equation under a certain symmetry group. Since the position representation of the momentum operator obeying a generalized uncertainty principle contains additional derivative terms, which also have to be replaced by covariant derivatives to maintain local gauge invariance, there are included further interaction terms. These additional terms imply a self-interaction of the gauge field even for Abelian gauge theories like electromagnetism and imply a more complicated interaction between the matter field and the gauge field leading to extended vertices in the corresponding quantum field theories. There arises the intricacy that a generalized uncertainty principle leads to an action of infinite order in derivatives [13] implying an infinite series of interaction terms. Therefore it has to be considered a series expansion. In [8] a series expansion in the product of the gauge coupling constant and the squared Planck length has been made. In the present paper an expansion in the modification parameter corresponding to the

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squared Planck length will be used instead. This implies that already a calculation to the first order yields additional interaction terms of the gauge field. Because of the fact that the gauge principle is, on the one hand, a constitutive ingredient of the standard model and as such determines the structure of its interactions and, on the other hand, a quantum description of gravity implies the existence of a minimal length, it seems to be very important to investigate gauge theories under incorporation of a generalized uncertainty principle. The consideration of gauge theories implies in this context that a generalized uncertainty principle leads inevitably to a modification not only of the behaviour of free fields but also of the structure of the interactions. Since also gravity can be considered as gauge theory, the generalized uncertainty principle also has an influence on the dynamics of gravity. If it is described as gauge theory of local Lorentz rotations, the introduction of the minimal length leads to generalized classical Einstein field equations and a generalized equivalence principle. This extended gauge theoretic description of gravity could be seen as a kind of semiclassical approximation to a quantum theory of gravity.

The paper is structured as follows: First, a short repetition of the generalized uncertainty principle, which is formulated in a way including the time component, is given. Then the modification of the usual formulation of gauge theories as it is used in the standard model is considered according to a generalized uncertainty principle. After showing how local invariance under a certain symmetry group can be maintained in the presence of a generalized momentum operator and the corresponding modified free field equation, the special cases of electromagnetism corresponding to a $U(1)$ gauge theory and gravity corresponding to a $SO(3,1)$ gauge theory in the chosen formulation are explored. From the extended actions for the gauge field and the matter field with the additional interaction terms, on the one hand, and the expressions for free quantum fields according to the corresponding extension of momentum eigenstates, on the other hand, both calculated in an approximation to the first order in the modification parameter, the Feynman rules of the modified theory of quantum electrodynamics are given. Subsequently, the consequences for the gauge description of gravity and the corresponding changing of the dynamics of the gravitational field and matter fields in general relativity are investigated.

II. GENERALIZED UNCERTAINTY PRINCIPLE AND MODIFIED DIRAC EQUATION

A short repetition of the introduction of a minimal length to quantum theory by postulating a generalized uncertainty principle between positions and momenta will first be given. A generalized uncertainty principle which implies a minimal length but no minimal momentum and which does therefore not spoil translation invariance has the following general shape

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu (1 + \beta f(\hat{p})) + i\beta g_\nu^\mu(\hat{p}), \quad (1)$$

where $f(\hat{p})$ and $g_\nu^\mu(\hat{p})$ are general functions depending on the four-momentum and β denotes the parameter determining the strength of the modification of the uncertainty principle that is usually assumed to be connected to the Planck scale. Here is assumed the relativistic case where the time coordinate is included which leads also to a minimal time scale Δt_0 corresponding to the minimal length scale Δx_0 . In the following consideration the special assumption that $f(\hat{p}) = \hat{p}^\rho \hat{p}_\rho$ and $g_\nu^\mu(\hat{p}) = 2\hat{p}^\mu \hat{p}_\nu$ will be made, meaning that the following special commutation relation between the position and the momentum operator is valid:

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta_\nu^\mu [1 + \beta \hat{p}^\rho \hat{p}_\rho] + 2i\beta \hat{p}^\mu \hat{p}_\nu. \quad (2)$$

This commutation relation corresponds to a generalized uncertainty relation that reads

$$\Delta x^\mu \Delta p_\mu \geq \frac{1}{2} (1 + \beta \Delta p^\rho \Delta p_\rho + \beta \langle p^\rho \rangle \langle p_\rho \rangle) + i (\beta \Delta p^\mu \Delta p_\mu + \beta \langle p^\mu \rangle \langle p_\mu \rangle) = \frac{1}{2} (1 + 3\beta \Delta p^\mu \Delta p_\mu + 3\beta \langle p^\rho \rangle \langle p_\rho \rangle). \quad (3)$$

If one considers the minimal uncertainty for the index of the position operator μ equal to the index of the momentum operator ν and solves the resulting equation for Δp^μ , one obtains

$$\Delta p^\mu = \frac{\Delta x^\mu}{3\beta} \pm \sqrt{\left(\frac{\Delta x^\mu}{3\beta}\right)^2 - \frac{1}{3\beta} - \langle p^\rho \rangle \langle p_\rho \rangle}. \quad (4)$$

By setting Δp^μ equal to zero the minimal position uncertainty in Minkowski space is obtained

$$l_s = \Delta x_{min}^\mu = \sqrt{3\beta} \sqrt{1 + 3\beta \langle p^\rho \rangle \langle p_\rho \rangle} \quad (5)$$

leading to the smallest length for $\mu = 1, \dots, 3$ and the smallest time for $\mu = 0$

$$l_s = \sqrt{3\beta} \quad , \quad t_s = \sqrt{3\beta}, \quad (6)$$

if it is assumed that $\hbar = c = 1$ as it is done throughout this paper. According to the modified uncertainty principle (2), generalized representations of the position operator in momentum space and the momentum operator in position space have to be given. Since on the right hand side of (2) there just appears the momentum, the representation in momentum space can be given directly as

$$\hat{p}_\mu = p_\mu \quad , \quad \hat{x}^\mu = i(1 + \beta p^\rho p_\rho) \frac{\partial}{\partial p_\mu} + i2\beta p^\mu p_\rho \frac{\partial}{\partial p_\rho}. \quad (7)$$

To yield the representation in position space, a series expansion has to be considered, since in this case the momentum operator becomes nontrivial instead of the position operator and this nontrivial expression depends on the momentum leading to an iterative definition. To the first order in β the operators look as follows:

$$\hat{x}^\mu = x^\mu \quad , \quad \hat{p}_\mu = -i(1 - \beta \partial^\rho \partial_\rho) \partial_\mu + \mathcal{O}(\beta^2). \quad (8)$$

If the momentum operator in (8) is inserted to the Dirac equation, it reads accordingly

$$[i(1 - \beta \partial^\rho \partial_\rho) \gamma^\mu \partial_\mu - m] \psi = 0. \quad (9)$$

III. THE GAUGE PRINCIPLE UNDER INCORPORATION OF A GENERALIZED UNCERTAINTY PRINCIPLE

The modification of the formulation of gauge theories according to the generalization of the uncertainty principle (2) and the corresponding modification of a free matter field equation like the Dirac equation (9) will now be considered. It should be emphasized again that in contrast to [8], where is already formulated a Dirac equation and a field strength tensor containing the generalized uncertainty principle and being gauge invariant, in the present paper an expansion in the modification parameter will be used implying self-interaction terms of the gauge field in a calculation to the first order instead of an expansion in the product of the gauge coupling constant and the squared Planck length. As in the usual case, one starts with the action of the free matter field, which reads in the scenario of this paper as follows:

$$\mathcal{S} = \int d^4x \bar{\psi} [i(1 - \beta \partial^\rho \partial_\rho) \gamma^\mu \partial_\mu - m] \psi \quad (10)$$

being invariant under certain global symmetry transformations. But in this case there exist additional derivative expressions. Since the additional derivatives also act on the local unitary transformation operator $U(x)$, they have to be replaced by covariant derivatives too:

$$(\mathbf{1} - \beta \partial^\rho \partial_\rho) \partial_\mu \rightarrow (\mathbf{1} - \beta D^\rho D_\rho) D_\mu, \quad (11)$$

leading to a transition $\mathcal{S} \rightarrow \mathcal{S}_m$ to the following generalized matter action containing the coupling of the matter field to the gauge field:

$$\mathcal{S}_m = \int d^4x \bar{\psi} [i(\mathbf{1} - \beta D^\rho D_\rho) \gamma^\mu D_\mu - m] \psi. \quad (12)$$

This action is invariant under global gauge transformations, since the additional term introduced by the generalized uncertainty principle transforms as follows:

$$i\beta\bar{\psi}D^\rho D_\rho\gamma^\mu D_\mu\psi \rightarrow i\beta\bar{\psi}U^\dagger(x)U(x)D^\rho U^\dagger(x)U(x)D_\rho U^\dagger(x)\gamma^\mu U(x)D_\mu U^\dagger(x)U(x)\psi = i\beta\bar{\psi}D^\rho D_\rho\gamma^\mu D_\mu\psi. \quad (13)$$

In (13) it has been used that $U^\dagger(x)U(x) = \mathbf{1}$ and that $U(x)$ either commutes with γ^μ , if $U(x)$ does not refer to a space-time transformation, or γ^μ transforms according to $\gamma^\mu \rightarrow U(x)\gamma^\mu U^\dagger(x)$. Therefore it makes sense to define a generalized covariant derivative in accordance with (11)

$$\mathcal{D}_\mu \equiv (\mathbf{1} - \beta D^\rho D_\rho) D_\mu = [\mathbf{1} - \beta(\partial^\rho \mathbf{1} + iA^\rho)(\partial_\rho \mathbf{1} + iA_\rho)](\partial_\mu \mathbf{1} + iA_\mu), \quad (14)$$

which behaves because of (13) like the usual covariant derivative under a local gauge transformation

$$\mathcal{D}_\mu \rightarrow U^\dagger(x)\mathcal{D}_\mu U(x). \quad (15)$$

Accordingly, a generalized field strength tensor for the gauge field also has to be defined, which is done by replacing the usual covariant derivative by the new covariant derivative

$$\mathcal{F}_{\mu\nu} = -i[\mathcal{D}_\mu, \mathcal{D}_\nu]. \quad (16)$$

This generalized definition of a field strength tensor is necessary because the action for the gauge field which is built from (16) has to correspond to a field equation where the generalized uncertainty principle is contained. It will be shown later that replacing the usual momentum operator by the generalized momentum operator in the free field equation of a vector field leads, in the approximation where self-interaction terms are neglected, to the same equation as variation of the action that is built from (16). Of course, the generalized field strength tensor is still invariant under local gauge transformations. The generalization of the gauge principle according to (10),(11),(12),(13),(14),(15),(16) holds for all gauge theories independent of the special symmetry group that is considered. In the special case of a Yang-Mills theory there can be built the following generalized action of the gauge field $\mathcal{S}_g = \frac{1}{4} \int d^4x \text{tr} [\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}]$ by using (16) and thus the complete action of the modified Yang-Mills gauge theory reads

$$\mathcal{S}_c = \int d^4x \left(\bar{\psi} (i\gamma^\mu \mathcal{D}_\mu - m) \psi - \frac{1}{4} \text{tr} [\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] \right). \quad (17)$$

Thus there has been obtained a generalized action for a Yang-Mills gauge theory in the presence of a generalized uncertainty principle. This action will be explored explicitly for the simplest case of the $U(1)$ gauge group in the next section, and the implications for the corresponding theory of quantum electrodynamics will be considered.

IV. MODIFIED QUANTUM ELECTRODYNAMICS

In this section the extended quantum field theoretic description arising from the generalization of gauge theories according to the generalization of the uncertainty principle will be considered as it has been described above. Specifically, the special case of the $U(1)$ gauge group corresponding to quantum electrodynamics will be regarded. The actions for the gauge field and the matter field in terms of the gauge potential A_μ will be calculated and the corresponding Feynman rules will be determined. Similar considerations referring to the other series expansion can be found in [11].

A. Calculation of the Generalized Action

If the matter sector (12) of the generalized action (17) is now calculated to the first order in the modification parameter β for the special case of a $U(1)$ gauge group, one obtains

$$\begin{aligned} \mathcal{S}_m &= \int d^4x \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu - m) \psi = \int d^4x \bar{\psi} [i(1 - \beta D^\rho D_\rho) \gamma^\mu D_\mu - m] \psi \\ &= \int d^4x \bar{\psi} \{i\gamma^\mu [1 - \beta(\partial^\rho + iA^\rho)(\partial_\rho + iA_\rho)](\partial_\mu + iA_\mu) - m\} \psi \\ &= \int d^4x \bar{\psi} \{i\gamma^\mu [\partial_\mu + iA_\mu - \beta(\partial^\rho \partial_\rho \partial_\mu + i\partial^\rho A_\rho \partial_\mu + 2iA_\rho \partial^\rho \partial_\mu - A^\rho A_\rho \partial_\mu + i\partial^\rho \partial_\rho A_\mu \\ &\quad + 2i\partial_\rho A_\mu \partial^\rho + iA_\mu \partial^\rho \partial_\rho - \partial^\rho A_\rho A_\mu - 2A_\rho \partial^\rho A_\mu - 2A_\rho A_\mu \partial^\rho - iA^\rho A_\rho A_\mu)] - m\} \psi. \end{aligned} \quad (18)$$

This action, of course, describes a much more complicated interaction structure between the matter field and the gauge field than the usual action giving rise to new vertices where more than one gauge boson interact with the matter field. If the gauge sector is to be calculated explicitly, the shape of the generalized field strength tensor $\mathcal{F}_{\mu\nu}$ defined in (16) first has to be calculated. For the case of electromagnetism, inserting (14) to (16) yields

$$\begin{aligned}
\mathcal{F}_{\mu\nu} &= -i [D_\mu, D_\nu] = -i [(1 - \beta D^\rho D_\rho) D_\mu, (1 - \beta D^\rho D_\rho) D_\nu] \\
&= -i \left\{ (1 - \beta D^\rho D_\rho)^2 [D_\mu, D_\nu] + (1 - \beta D^\rho D_\rho) [D_\mu, (1 - \beta D^\rho D_\rho)] D_\nu \right. \\
&\quad \left. + (1 - \beta D^\rho D_\rho) [(1 - \beta D^\rho D_\rho), D_\nu] D_\mu + [(1 - \beta D^\rho D_\rho), (1 - \beta D^\rho D_\rho)] D_\mu D_\nu \right\} \\
&= -i \left\{ (1 - \beta D^\rho D_\rho)^2 [D_\mu, D_\nu] - (1 - \beta D^\rho D_\rho) \beta [D_\mu, D^\rho D_\rho] D_\nu - (1 - \beta D^\rho D_\rho) \beta [D^\rho D_\rho, D_\nu] D_\mu \right\} \\
&= -i \left\{ (1 - \beta D^\rho D_\rho)^2 [D_\mu, D_\nu] - (1 - \beta D^\rho D_\rho) \beta D^\rho [D_\mu, D_\rho] D_\nu - (1 - \beta D^\rho D_\rho) \beta [D_\mu, D^\rho] D_\rho D_\nu \right. \\
&\quad \left. + (1 - \beta D^\rho D_\rho) \beta D^\rho [D_\nu, D_\rho] D_\mu + (1 - \beta D^\rho D_\rho) \beta [D_\nu, D^\rho] D_\rho D_\mu \right\} \\
&= (1 - \beta D^\rho D_\rho) [(1 - \beta D^\rho D_\rho) F_{\mu\nu} - \beta (D^\rho F_{\mu\rho} D_\nu - D^\rho F_{\nu\rho} D_\mu) - \beta (F_{\mu\rho} D^\rho D_\nu - F_{\nu\rho} D^\rho D_\mu)]. \tag{19}
\end{aligned}$$

This means that the generalized field strength tensor $\mathcal{F}_{\mu\nu}$ expressed by the usual field strength tensor $F_{\mu\nu}$ and the usual covariant derivative D_μ to the first order in β reads

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} - 2\beta D^\rho D_\rho F_{\mu\nu} - \beta (D^\rho F_{\mu\rho} D_\nu - D^\rho F_{\nu\rho} D_\mu) - \beta (F_{\mu\rho} D^\rho D_\nu - F_{\nu\rho} D^\rho D_\mu) + \mathcal{O}(\beta^2). \tag{20}$$

The corresponding action of the gauge field sector reads accordingly

$$\begin{aligned}
\mathcal{S}_g &= \frac{1}{4} \int d^4x \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \\
&= \frac{1}{4} \int d^4x \{ F_{\mu\nu} F^{\mu\nu} - \beta [2F_{\mu\nu} (D^\rho D_\rho F^{\mu\nu}) + F_{\mu\nu} (F^{\mu\rho} D_\rho D^\nu - F^{\nu\rho} D_\rho D^\mu) + F_{\mu\nu} (D_\rho F^{\mu\rho} D^\nu - D_\rho F^{\nu\rho} D^\mu) \\
&\quad + 2(D^\rho D_\rho F_{\mu\nu}) F^{\mu\nu} + (F_{\mu\rho} D^\rho D_\nu - F_{\nu\rho} D^\rho D_\mu) F^{\mu\nu} + (D^\rho F_{\mu\rho} D_\nu - D^\rho F_{\nu\rho} D_\mu) F^{\mu\nu}] \} + \mathcal{O}(\beta^2) \\
&= \frac{1}{4} \int d^4x [F_{\mu\nu} F^{\mu\nu} - 4\beta F_{\mu\nu} D^\rho D_\rho F^{\mu\nu} - 4\beta F_{\mu\nu} F^{\mu\rho} D_\rho D^\nu - 4\beta F_{\mu\nu} D_\rho F^{\mu\rho} D^\nu] + \mathcal{O}(\beta^2). \tag{21}
\end{aligned}$$

Inserting the expression for the usual covariant derivative yields for the generalized action of the gauge field to the first order in β

$$\begin{aligned}
\mathcal{S}_g &= \frac{1}{4} \int d^4x [F_{\mu\nu} F^{\mu\nu} - \beta (4F_{\mu\nu} \partial_\rho \partial^\rho F^{\mu\nu} + 4i \partial_\rho A^\rho F_{\mu\nu} F^{\mu\nu} + 8i F_{\mu\nu} A_\rho \partial^\rho F^{\mu\nu} \\
&\quad + 8i \partial^\rho A_\nu F_{\mu\rho} F^{\mu\nu} + 4i A^\rho F_{\mu\rho} \partial_\nu F^{\mu\nu} - 8A^\rho F_{\mu\rho} A_\nu F^{\mu\nu} - 4A^\rho A_\rho F_{\mu\nu} F^{\mu\nu})] \tag{22}
\end{aligned}$$

and expressing the field strength tensor by the gauge potential finally leads to

$$\begin{aligned}
\mathcal{S}_g &= \frac{1}{4} \int d^4x [2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu + \beta (-8\partial_\mu A_\nu \partial_\rho \partial^\rho \partial^\mu A^\nu + 8\partial_\mu A_\nu \partial_\rho \partial^\rho \partial^\nu A^\mu \\
&\quad - 8i \partial_\rho A^\rho \partial_\mu A_\nu \partial^\mu A^\nu + 8i \partial_\rho A^\rho \partial_\mu A_\nu \partial^\nu A^\mu - 16i \partial_\mu A_\nu A_\rho \partial^\rho \partial^\mu A^\nu + 16i \partial_\mu A_\nu A_\rho \partial^\rho \partial^\nu A^\mu \\
&\quad - 8i \partial^\rho A_\nu \partial_\mu A_\rho \partial^\mu A^\nu + 8i \partial^\rho A_\nu \partial_\mu A_\rho \partial^\nu A^\mu + 8i \partial^\rho A_\nu \partial_\rho A_\mu \partial^\mu A^\nu - 8i \partial^\rho A_\nu \partial_\rho A_\mu \partial^\nu A^\mu \\
&\quad - 4i A^\rho \partial_\mu A_\rho \partial_\nu \partial^\mu A^\nu + 4i A^\rho \partial_\mu A_\rho \partial_\nu \partial^\nu A^\mu + 4i A^\rho \partial_\rho A_\mu \partial_\nu \partial^\mu A^\nu - 4i A^\rho \partial_\rho A_\mu \partial_\nu \partial^\nu A^\mu \\
&\quad + 8A^\rho \partial_\mu A_\rho A_\nu \partial^\mu A^\nu - 8A^\rho \partial_\mu A_\rho A_\nu \partial^\nu A^\mu - 8A^\rho \partial_\rho A_\mu A_\nu \partial^\mu A^\nu + 8A^\rho \partial_\rho A_\mu A_\nu \partial^\nu A^\mu \\
&\quad + 8A^\rho A_\rho \partial_\mu A_\nu \partial^\mu A^\nu - 8A^\rho A_\rho \partial_\mu A_\nu \partial^\nu A^\mu)]. \tag{23}
\end{aligned}$$

If just the terms without self-interaction are considered, one obtains as field equation for the free electromagnetic potential

$$\partial_\mu F^{\mu\nu} - 2\beta \partial_\rho \partial^\rho \partial_\mu F^{\mu\nu} = 0. \tag{24}$$

For Lorentz gauge $\partial_\mu A^\mu = 0$ this means

$$\partial_\mu \partial^\mu A^\nu - 2\beta \partial_\rho \partial^\rho \partial_\mu \partial^\mu A^\nu = 0. \quad (25)$$

This is exactly the equation that is obtained if the wave equation of a massless vector field $\partial_\mu \partial^\mu A^\nu = 0$ is generalized according to (2) and (8), which means that the definition of the generalized field strength tensor yields the correct free field limit.

B. Propagators

In the usual setting of quantum field theory, fields are expanded in terms of momentum eigenstates corresponding to plane waves. If the fields are quantized, the coefficients of plane wave modes become operators that play the role of creation and annihilation operators of particles. Because of the modified commutation relation between position and momentum operators (2), the representation of the momentum operator in position space is changed (8) and therefore also the shape of the momentum eigenstates is modified, which leads to a different expression for a quantum field. The modified momentum eigenstates $|p\rangle$ fulfill the equation

$$\hat{p}_\mu |p\rangle = p_\mu |p\rangle \quad (26)$$

where p_μ describes the eigenvalue to the eigenstate $|p\rangle$. The momentum eigenstates have the following shape:

$$|p\rangle = \exp [i(1 - \beta p^\rho p_\rho) p_\mu x^\mu] + \mathcal{O}(\exp(\beta^2)), \quad (27)$$

if they are expressed to the first order in β , since if there are only taken terms to the first order one obtains

$$\hat{p}_\mu \exp [i(1 - \beta p^\rho p_\rho) p_\nu x^\nu] = -i(1 - \beta \partial^\sigma \partial_\sigma) \partial_\mu \exp [i(1 - \beta p^\rho p_\rho) p_\nu x^\nu] = p_\mu \exp [i(1 - \beta p^\rho p_\rho) p_\nu x^\nu]. \quad (28)$$

If one now considers quantum field theory, the quantum fields have to be expanded by using these generalized momentum eigenstates. It will be considered the case of a scalar field, which can be assigned to the cases of a spinor field and a vector field as they appear in quantum electrodynamics easily at the end

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 2p_0}} \left[a(\mathbf{p}) e^{i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} + a^\dagger(\mathbf{p}) e^{-i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} \right]. \quad (29)$$

The commutation relations for the creation and annihilation operators have to remain the same since the Fock space structure referring to many particles is not changed by the generalized uncertainty relation which means $[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = \delta^3(\mathbf{p} - \mathbf{p}')$. The propagator is now obtained as usual by building the expectation value of the time ordered product of the generalized expression of the quantum field (29) at two different space-time points with respect to the vacuum state $|0\rangle$ of the quantum field

$$G(x - y) = \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle \quad (30)$$

where T denotes the time ordering operator. If one uses (29) in (30), one obtains the expression for the propagator in position space

$$G(x - y) = i \int \frac{d^4 p}{(2\pi)^4} \frac{e^{i(1 - \beta p^2) p(x - y)}}{p^2 - m^2 + i\epsilon}. \quad (31)$$

One has to build the inner product with the momentum eigenstates to obtain the corresponding propagator in momentum space

$$G(p) = i \int \frac{d^4 z}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \frac{e^{i(1 - \beta p'^2) p'z - i(1 - \beta p^2) pz}}{p'^2 - m^2 + i\epsilon}. \quad (32)$$

To perform the integral it is useful to define k_μ as $k_\mu = (1 - \beta p^\rho p_\rho) p_\mu$ and to substitute p_μ by k_μ leading to

$$\begin{aligned} G(k) &= i \int \frac{d^4 z}{(2\pi)^4} \frac{d^4 k'}{(2\pi)^4 J(p(k'))} \frac{e^{ik'z - ikz}}{(p^2(k') - m^2 + i\epsilon)} = i \int \frac{d^4 k'}{(2\pi)^4 J(p(k'))} \frac{\delta(k' - k)}{(p^2(k') - m^2 + i\epsilon)} \\ &= \frac{1}{J(p(k))} \frac{i}{(p^2(k) - m^2 + i\epsilon)} \end{aligned} \quad (33)$$

where $J(p)$ has been defined as the Jacobian determinant of the transformation between k_μ and p_μ

$$J(p) = \det \left[\frac{\partial k_\mu}{\partial p_\nu} \right] = (1 - \beta p^\rho p_\rho)^4 - 2\beta(1 - \beta p^\rho p_\rho)^3 p^\sigma p_\sigma + 4\beta^2(1 - \beta p^\rho p_\rho)^2 (-p_0^2 p_1^2 + p_1^2 p_2^2 + p_2^2 p_3^2 - p_3^2 p_0^2). \quad (34)$$

Resubstituting k_μ with p_μ yields the propagator in momentum space

$$i\Delta(p) = G(p) = \frac{1}{J(p)} \frac{i}{p^2 - m^2 + i\epsilon}. \quad (35)$$

To obtain the corresponding propagator for a spinor field and a vector field, the expressions which are analogue to (29) are considered

$$\begin{aligned} \psi(\mathbf{x}, t) &= \sum_{\pm s} \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{p_0}} \left[b(\mathbf{p}, s) u(\mathbf{p}, s) e^{i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} + d^\dagger(\mathbf{p}, s) v(\mathbf{p}, s) e^{-i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} \right], \\ \psi^\dagger(\mathbf{x}, t) &= \sum_{\pm s} \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{p_0}} \left[b^\dagger(\mathbf{p}, s) u^\dagger(\mathbf{p}, s) e^{-i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} + d(\mathbf{p}, s) v^\dagger(\mathbf{p}, s) e^{i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} \right] \end{aligned} \quad (36)$$

with $\{b(\mathbf{p}, s), b^\dagger(\mathbf{p}', s')\} = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ss'}$, $\{d(\mathbf{p}, s), d^\dagger(\mathbf{p}', s')\} = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ss'}$ and

$$A_\mu(\mathbf{x}, t) = \sum_{\lambda=1}^2 \int \frac{d^3 p}{\sqrt{(2\pi)^3} 2p_0} \left[a(\mathbf{p}, \lambda) \epsilon_\mu(\mathbf{p}, \lambda) e^{i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} + a^\dagger(\mathbf{p}, \lambda) \epsilon_\mu(\mathbf{p}, \lambda) e^{-i(1 - \beta p^\rho p_\rho) p_\mu x^\mu} \right] \quad (37)$$

with $[a(\mathbf{p}, \lambda), a^\dagger(\mathbf{p}', \lambda')] = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\lambda\lambda'}$. Besides there are needed as usual the following relations for the summation over the spin states:

$$\sum_{\pm s} u_a(\mathbf{p}, s) \bar{u}_b(\mathbf{p}, s) = \left(\frac{\not{p} + m}{2m} \right)_{ab}, \quad \sum_{\pm s} v_a(\mathbf{p}, s) \bar{v}_b(\mathbf{p}, s) = \left(\frac{\not{p} - m}{2m} \right)_{ab}, \quad \sum_{\lambda=1}^2 \epsilon_\mu(\mathbf{p}, \lambda) \epsilon_\nu(\mathbf{p}, \lambda) = -g_{\mu\nu}, \quad (38)$$

where the last equation referring to the polarization vector of the electromagnetic field only holds in Feynman gauge. A calculation that is analogue to the derivation of the propagator of the scalar field (35) yields for the propagator of a spinor field describing an electron

$$i\Delta_{ab}(p) = \frac{1}{J(p)} \left(\frac{i}{\not{p} - m + i\epsilon} \right)_{ab} \quad (39)$$



and for the propagator of a vector field describing a photon

$$i\Delta_{\mu\nu}(p) = \frac{1}{J(p)} \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}, \quad (40)$$

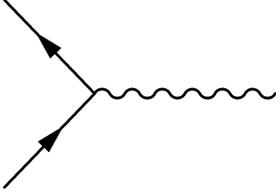


if Feynman gauge is used as it is presupposed.

C. Vertices

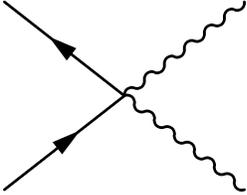
To extract the vertices arising from the generalized gauge theory of quantum electrodynamics, the interaction terms within the actions (18) and (23) have to be considered. Because of the additional terms there arise additional vertices where two and three photons interact with the matter field and there also arise vertices between photons, since the generalization of gauge theories causes self-interaction terms of the gauge field even in the Abelian case. In the following notation p_μ denotes the four-momentum of a fermion running into the vertex and the k_μ denote the four-momenta of the bosons, if they appear in the expression of the respective vertex. Since the bosons are indistinguishable, any of the k_μ can be identified with any boson. The vertex of the interaction between the matter field and the electromagnetic field referring to one photon to the first order in β reads

$$v^\lambda = -\gamma^\mu [\delta_\mu^\lambda + \beta (k^\lambda p_\mu + 2p^\lambda p_\mu + \delta_\mu^\lambda k^\rho k_\rho + 2\delta_\mu^\lambda k^\rho p_\rho + \delta_\mu^\lambda p^\rho p_\rho)]. \quad (41)$$



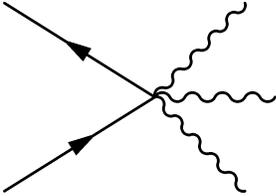
In this extended case from (18) it can be seen that there also exists a vertex where two photons interact with the matter particle. This vertex reads

$$v^{\lambda\rho} = -\beta\gamma^\mu (\delta^{\lambda\rho} p_\mu + \delta_\mu^\lambda k_1^\rho + 2\delta_\mu^\lambda k_2^\rho + 2\delta_\mu^\lambda p^\rho). \quad (42)$$



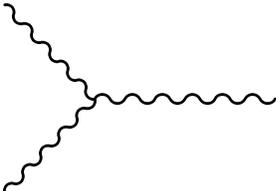
Finally there also exists a vertex with three photons interacting with the matter particle looking as follows:

$$v^{\mu\lambda\rho} = -\beta\gamma^\mu \delta^{\lambda\rho}. \quad (43)$$



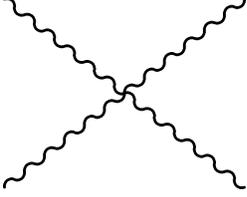
There are two vertices corresponding to a self-interaction of the electromagnetic field. One vertex describes the interaction of three photons with each other looking as follows:

$$\begin{aligned} v^{\mu\nu\lambda} = & \beta (2k_1^\mu k_2^\rho k_3^\lambda \delta^{\nu\lambda} - 2k_1^\mu k_2^\lambda k_3^\nu + 4k_{1\rho} k_3^\rho k_3^\nu \delta^{\mu\lambda} - 4k_1^\lambda k_3^\nu k_3^\mu \\ & + 2k_1^\nu k_2^\rho k_3^\lambda \delta^{\mu\lambda} - 2k_1^\nu k_2^\lambda k_3^\mu - 2k_{1\rho} k_2^\rho k_3^\nu \delta^{\mu\lambda} + 2k_{1\rho} k_2^\rho k_3^\mu \delta^{\nu\lambda} \\ & + k_{2\rho} k_3^\rho k_3^\lambda \delta^{\mu\nu} - k_2^\lambda k_{3\rho} k_3^\rho \delta^{\mu\nu} - k_2^\mu k_3^\nu k_3^\lambda + k_2^\mu k_{3\rho} k_3^\rho \delta^{\nu\lambda}) \end{aligned} \quad (44)$$



and one describes the interaction of four photons, which reads

$$v^{\mu\nu\lambda\rho} = \beta \left(-2k_{2\sigma}k_4^\sigma \delta^{\mu\nu} \delta^{\lambda\rho} + 2k_2^\rho k_4^\lambda \delta^{\mu\nu} + 2k_2^\mu k_4^\nu \delta^{\lambda\rho} - 2k_2^\mu k_4^\lambda \delta^{\nu\rho} - 2k_{3\sigma}k_4^\sigma \delta^{\mu\nu} \delta^{\lambda\rho} + 2k_3^\rho k_4^\lambda \delta^{\mu\nu} \right). \quad (45)$$



V. GENERALIZED GAUGE PRINCIPLE IN GENERAL RELATIVITY

Since general relativity can also be formulated as a gauge theory, the generalization of the gauge principle should also have an influence on the description of gravity. It is important that in such a consideration, where gravity is modified according to a generalized uncertainty principle, the emergence of a minimal length in the description of quantum field theories cannot be interpreted as an approximative treatment of properties arising from a quantum description of gravity on a fundamental level. That means that the minimal length is not induced by an underlying quantum theory of gravity, but the causal connection is the other way round. The description of gravity, even on a classical level, is influenced by the assumption of a generalized uncertainty principle. Therefore the generalized uncertainty principle should be interpreted as a fundamental description of nature in this context or one has at least to assume that it has another origin. In [40] and [41] the possibility is argued that a minimal length could indeed arise from completely different scenarios being independent of the usual considerations regarded as an effective description of the consequences of a quantum theory of gravity. Concerning the gauge theoretic description of general relativity there can be used the translation group as well as the Lorentz group $SO(3, 1)$ as gauge group. The first possibility leads to a formulation of general relativity where the torsion as a decisive quantity plays the central role [42],[43],[44] but which is equivalent to usual general relativity. Here it will be considered the $SO(3, 1)$ gauge theory [45],[43] and it will be generalized to the case of matter field equations obeying a generalized uncertainty principle. This means that there is postulated invariance of the matter action under local Lorentz transformations

$$x^m \rightarrow \Lambda_n^m(x)x^n, \quad \psi \rightarrow U(x)\psi = \exp\left(\frac{i}{2}\Lambda^{ab}(x)\Sigma_{ab}\right)\psi, \quad (46)$$

where the Σ_{ab} describe the generators of the Lorentz group represented within the Dirac spinor space $\Sigma_{ab} = -\frac{i}{4}[\gamma_a, \gamma_b]$, fulfilling the commutation relation $[\Sigma_{ab}, \Sigma_{cd}] = \eta_{bc}\Sigma_{ad} - \eta_{ac}\Sigma_{bd} + \eta_{bd}\Sigma_{ca} - \eta_{ad}\Sigma_{cb}$, where η_{mn} describes the flat Minkowski metric. Within the usual formulation one has to introduce the following covariant derivative:

$$D_m = e_m^\mu \left(\partial_\mu \mathbf{1} + \frac{i}{2} \omega_\mu^{ab} \Sigma_{ab} \right) = e_m^\mu D_\mu, \quad (47)$$

where the spin connection ω_μ^{ab} is related to the tetrad e_m^μ according to

$$\omega_\mu^{ab} = 2e^{\nu a} \partial_\mu e_\nu^b - 2e^{\nu b} \partial_\mu e_\nu^a - 2e^{\nu a} \partial_\nu e_\mu^b + 2e^{\nu b} \partial_\nu e_\mu^a + e_{\mu c} e^{\nu a} e^{\sigma b} \partial_\sigma e_\nu^c - e_{\mu c} e^{\nu a} e^{\sigma b} \partial_\nu e_\sigma^c \quad (48)$$

and the tetrad e_μ^m is related to the general metric $g_{\mu\nu}$ as usual $g_{\mu\nu} = e_\mu^m e_\nu^n \eta_{mn}$. Note that latin indices denote as usual flat Minkowski coordinates, whereas greek indices denote curved coordinates. The covariant derivative (47) transforms according to

$$D_m \rightarrow \Lambda_m^n(x) U(x) D_n U^\dagger(x), \quad (49)$$

since $e_m^\mu \rightarrow \Lambda_m^n(x) e_n^\mu$ and $\omega_\mu \rightarrow U^\dagger(x) \omega_\mu U(x) - U^\dagger(x) \partial_\mu U(x)$. This means that in the generalized case one obtains the following matter action being invariant under a local $SO(3, 1)$ gauge transformation consisting of (46) and the corresponding transformation (49):

$$\mathcal{S}_m = \int d^4x \sqrt{-g} \bar{\psi} [i\gamma^m e_m^\mu (\mathbf{1} - \beta D^\rho D_\rho) D_\mu - m] \psi = \int d^4x e \bar{\psi} [i\gamma^m e_m^\mu \mathcal{D}_\mu - m] \psi, \quad (50)$$

where $g = \det [g_{\mu\nu}]$, $e = \det [e_\mu^m]$, and there is introduced the generalized covariant derivative according to (14):

$$\mathcal{D}_m = e_m^\mu (\mathbf{1} - \beta D^\rho D_\rho) D_\mu. \quad (51)$$

Because of the generalized coupling of the gauge field to the matter field, (50) contains a generalized equivalence principle. If the generalized field strength tensor corresponding to the generalized covariant derivative (51) is built, one obtains

$$\begin{aligned} \mathcal{F}_{mn} &= [\mathcal{D}_m, \mathcal{D}_n] = [e_m^\mu \mathcal{D}_\mu, e_n^\nu \mathcal{D}_\nu] = e_m^\mu (\mathcal{D}_\mu e_n^\nu) \mathcal{D}_\nu - e_n^\nu (\mathcal{D}_\nu e_m^\mu) \mathcal{D}_\mu + e_m^\mu e_n^\nu \mathcal{D}_\mu \mathcal{D}_\nu - e_n^\nu e_m^\mu \mathcal{D}_\nu \mathcal{D}_\mu \\ &= e_m^\mu e_n^\nu \mathcal{T}_{\mu\nu}^p D_p + e_m^\mu e_n^\nu \frac{i}{2} \mathcal{R}_{\mu\nu}^{ab} \Sigma_{ab} = \mathcal{T}_{mn}^p D_p + \frac{i}{2} \mathcal{R}_{mn}^{ab} \Sigma_{ab}, \end{aligned} \quad (52)$$

where $\mathcal{T}_{\mu\nu}^p$ and $\mathcal{R}_{\mu\nu}^{ab}$ are the generalized quantities corresponding to the usual torsion tensor and the usual Riemann curvature tensor. If the torsion is assumed to vanish $\mathcal{T}_{mn}^p = 0$, there remains the generalized Riemann tensor $\mathcal{R}_{\mu\nu}^{ab}$ reading in analogy to (20) as follows:

$$\mathcal{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} - 2\beta D^\rho D_\rho R_{\mu\nu}^{ab} - \beta (D^\rho R_{\mu\rho}^{ab} D_\nu - D^\rho R_{\nu\rho}^{ab} D_\mu) - \beta (R_{\mu\rho}^{ab} D^\rho D_\nu - R_{\nu\rho}^{ab} D^\rho D_\mu) + \mathcal{O}(\beta^2), \quad (53)$$

where $R_{\mu\nu}^{ab}$ is the usual Riemann tensor expressed with two Minkowski indices reading

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - \omega_\nu^{ac} \omega_\mu^{cb}. \quad (54)$$

A. Generalized Einstein Hilbert Action and Corresponding Field Equation

From (53) the generalization of the Einstein Hilbert action has to be built. Therefore the generalized Ricci scalar has to be built according to

$$\mathcal{R} = e_a^\mu e_b^\nu \mathcal{R}_{\mu\nu}^{ab} = e_a^\mu e_b^\nu [R_{\mu\nu}^{ab} - 2\beta D^\rho D_\rho R_{\mu\nu}^{ab} - \beta (D^\rho R_{\mu\rho}^{ab} D_\nu - D^\rho R_{\nu\rho}^{ab} D_\mu) - \beta (R_{\mu\rho}^{ab} D^\rho D_\nu - R_{\nu\rho}^{ab} D^\rho D_\mu)]. \quad (55)$$

Since $R_{\mu\nu}^{ab} = -R_{\mu\nu}^{ba}$ and $D_\mu e_\nu^a = 0$, if the torsion vanishes, this expression can be transformed to

$$\mathcal{R} = R - 2\beta D^\rho D_\rho R - 2\beta D^\rho e_b^\nu R_\rho^b D_\nu - 2\beta e_b^\nu R_\rho^b D^\rho D_\nu. \quad (56)$$

This leads to the following generalized Einstein Hilbert action:

$$\mathcal{S}_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \mathcal{R} = \frac{1}{2\kappa} \int d^4x e [R - 2\beta D^\rho D_\rho R - 2\beta D^\rho e_b^\nu R_\rho^b D_\nu - 2\beta e_b^\nu R_\rho^b D^\rho D_\nu] \quad (57)$$

where $\kappa = 8\pi G$ and G denotes the gravitational constant. To obtain the corresponding free field equations for the gravitational field, the variation of the generalized Einstein Hilbert action (57) has to be built with respect to the tetrad e_a^μ . If R_β is defined according to $\mathcal{R} \equiv R + R_\beta + \mathcal{O}(\beta^2)$, then the variation can be expressed as

$$\delta \mathcal{S}_{EH} = \frac{1}{2\kappa} \int d^4x \delta [e \mathcal{R}] = \frac{1}{2\kappa} \int d^4x \delta [eR + eR_\beta] = \frac{1}{2\kappa} \int d^4x [\delta (eR) + \delta (eR_\beta)]. \quad (58)$$

The first term yields, of course, the usual Einstein equation, whereas the second term yields the corrections according to the generalized uncertainty principle. To determine the contribution of the correction term, there has to be calculated the variation $\delta [eR_\beta]$

$$\begin{aligned}
\frac{1}{2\kappa} \int d^4x \delta [eR_\beta] &= -\frac{\beta}{\kappa} \int d^4x \delta [eD^\rho D_\rho e_a^\mu e_b^\nu R_{\mu\nu}^{ab} + eD^\rho e_a^\mu e_b^\nu R_{\mu\rho}^{ab} D_\nu + ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D^\rho D_\nu], \\
&= -\frac{\beta}{\kappa} \int d^4x \delta [D^\rho D_\rho ee_a^\mu e_b^\nu R_{\mu\nu}^{ab} + D^\rho ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D_\nu + ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D^\rho D_\nu], \\
&= -\frac{\beta}{\kappa} \int d^4x \partial^\rho \delta [D_\rho ee_a^\mu e_b^\nu R_{\mu\nu}^{ab} + ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D_\nu] - \frac{\beta}{\kappa} \int d^4x \delta [ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D^\rho D_\nu], \\
&= \underbrace{-\frac{\beta}{\kappa} \oint_{\partial V} d^3x \delta [D_\rho ee_a^\mu e_b^\nu R_{\mu\nu}^{ab} + ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D_\nu]}_{=0} - \frac{\beta}{\kappa} \int d^4x \delta [ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D^\rho D_\nu], \\
&= -\frac{\beta}{\kappa} \int d^4x \delta [ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D^\rho D_\nu], \tag{59}
\end{aligned}$$

where Stokes theorem $\int_V dA = \int_{\partial V} A$ has been used in the fourth step and the fact that the variation vanishes at infinity in the fifth step makes the surface term vanishing $-\frac{\beta}{\kappa} \oint_{\partial V} d^3x \delta [D_\rho ee_a^\mu e_b^\nu R_{\mu\nu}^{ab} + ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D_\nu] = 0$. Variation of the remaining term yields

$$-\frac{\beta}{\kappa} \int d^4x \delta [ee_a^\mu e_b^\nu R_{\mu\rho}^{ab} D^\rho D_\nu] = -\frac{\beta}{\kappa} \int d^4x [-ee_\mu^a \delta e_a^\mu R D^\rho D_\nu + e\delta e_a^\mu e_b^\nu R_{\mu\rho}^{ab} D^\rho D_\nu + e\delta e_a^\mu R_\rho^a D^\rho D_\mu + ee_b^\nu e_c^\lambda \delta (R_{\lambda\rho}^{cb} D^\rho D_\nu)], \tag{60}$$

if it is used that $\delta e = -ee_\mu^a \delta e_a^\mu$. This means that the variation of the last term has to be calculated with respect to the tetrad e_μ^a . First, the expression that is varied $R_{\lambda\rho}^{cb} D^\rho D_\nu$ has to be expressed in terms of the connection ω_μ^{ab}

$$\begin{aligned}
R_{\lambda\rho}^{bc} D^\rho D_\nu &= (\partial_\lambda \omega_\rho^{bc} \partial^\rho \omega_\nu^{ef} - \partial_\rho \omega_\lambda^{bc} \partial^\rho \omega_\nu^{ef} + \partial_\lambda \omega_\rho^{bc} \omega^{\rho eg} \omega_\nu^{gf} - \partial_\rho \omega_\lambda^{bc} \omega^{\rho eg} \omega_\nu^{gf} \\
&\quad + \omega_\lambda^{bd} \omega_\rho^{dc} \partial^\rho \omega_\nu^{ef} - \omega_\rho^{bd} \omega_\lambda^{dc} \partial^\rho \omega_\nu^{ef} + \omega_\lambda^{bd} \omega_\rho^{dc} \omega^{\rho eg} \omega_\nu^{gf} - \omega_\rho^{bd} \omega_\lambda^{dc} \omega^{\rho eg} \omega_\nu^{gf}) \Sigma_{ef}. \tag{61}
\end{aligned}$$

If (61) is inserted into (60), the variation of the last term with respect to the connection ω_μ^{ab} yields

$$\begin{aligned}
&-\frac{\beta}{\kappa} \int d^4x [ee_c^\nu e_b^\lambda \delta (R_{\lambda\rho}^{bc} D^\rho D_\nu)] \\
&= -\frac{\beta}{\kappa} \int d^4x [ee_c^\nu e_b^\lambda (\partial_\lambda \delta \omega_\rho^{bc} \partial^\rho \omega_\nu^{ef} + \partial_\lambda \omega_\rho^{bc} \partial^\rho \delta \omega_\nu^{ef} - \partial_\rho \delta \omega_\lambda^{bc} \partial^\rho \omega_\nu^{ef} - \partial_\rho \omega_\lambda^{bc} \partial^\rho \delta \omega_\nu^{ef} + \partial_\lambda \delta \omega_\rho^{bc} \omega^{\rho eg} \omega_\nu^{gf} \\
&\quad + \partial_\lambda \omega_\rho^{bc} \delta \omega^{\rho eg} \omega_\nu^{gf} + \partial_\lambda \omega_\rho^{bc} \omega^{\rho eg} \delta \omega_\nu^{gf} - \partial_\rho \delta \omega_\lambda^{bc} \omega^{\rho eg} \omega_\nu^{gf} - \partial_\rho \omega_\lambda^{bc} \delta \omega^{\rho eg} \omega_\nu^{gf} - \partial_\rho \omega_\lambda^{bc} \omega^{\rho eg} \delta \omega_\nu^{gf} \\
&\quad + \delta \omega_\lambda^{bd} \omega_\rho^{dc} \partial^\rho \omega_\nu^{ef} + \omega_\lambda^{bd} \delta \omega_\rho^{dc} \partial^\rho \omega_\nu^{ef} + \omega_\lambda^{bd} \omega_\rho^{dc} \partial^\rho \delta \omega_\nu^{ef} - \delta \omega_\rho^{bd} \omega_\lambda^{dc} \partial^\rho \omega_\nu^{ef} - \omega_\rho^{bd} \delta \omega_\lambda^{dc} \partial^\rho \omega_\nu^{ef} \\
&\quad - \omega_\rho^{bd} \omega_\lambda^{dc} \partial^\rho \delta \omega_\nu^{ef} + \delta \omega_\lambda^{bd} \omega_\rho^{dc} \omega^{\rho eg} \omega_\nu^{gf} + \omega_\lambda^{bd} \delta \omega_\rho^{dc} \omega^{\rho eg} \omega_\nu^{gf} + \omega_\lambda^{bd} \omega_\rho^{dc} \delta \omega^{\rho eg} \omega_\nu^{gf} + \omega_\lambda^{bd} \omega_\rho^{dc} \omega^{\rho eg} \delta \omega_\nu^{gf} \\
&\quad - \delta \omega_\rho^{bd} \omega_\lambda^{dc} \omega^{\rho eg} \omega_\nu^{gf} - \omega_\rho^{bd} \delta \omega_\lambda^{dc} \omega^{\rho eg} \omega_\nu^{gf} - \omega_\rho^{bd} \omega_\lambda^{dc} \delta \omega^{\rho eg} \omega_\nu^{gf} - \omega_\rho^{bd} \omega_\lambda^{dc} \omega^{\rho eg} \delta \omega_\nu^{gf}) \Sigma_{ef}] \\
&\equiv -\frac{\beta}{\kappa} \int d^4x [\Xi_{hi}^\kappa \delta \omega_\kappa^{hi}], \tag{62}
\end{aligned}$$

where the expression Ξ_{hi}^κ has been defined according to

$$\begin{aligned}
\Xi_{hi}^\kappa &= \left[-\delta_\rho^\kappa \delta_h^b \delta_i^c \partial_\lambda (ee_c^\nu e_b^\lambda \partial^\rho \omega_\nu^{ef}) - \delta_\nu^\kappa \delta_h^e \delta_i^f \partial^\rho (ee_c^\nu e_b^\lambda \partial_\lambda \omega_\rho^{bc}) + \delta_\lambda^\kappa \delta_h^b \delta_i^c \partial_\rho (ee_c^\nu e_b^\lambda \partial^\rho \omega_\nu^{ef}) + \delta_\nu^\kappa \delta_h^e \delta_i^f \partial^\rho (ee_c^\nu e_b^\lambda \partial_\rho \omega_\lambda^{bc}) \right. \\
&\quad - \delta_\rho^\kappa \delta_h^b \delta_i^c \partial_\lambda (ee_c^\nu e_b^\lambda \omega^{\rho eg} \omega_\nu^{gf}) + \delta^{\rho\kappa} \delta_h^e \delta_i^g ee_c^\nu e_b^\lambda \partial_\lambda \omega_\rho^{bc} \omega_\nu^{gf} + \delta_\nu^\kappa \delta_h^g \delta_i^f ee_c^\nu e_b^\lambda \partial_\lambda \omega_\rho^{bc} \omega^{\rho eg} + \delta_\lambda^\kappa \delta_h^b \delta_i^c \partial_\rho (ee_c^\nu e_b^\lambda \omega^{\rho eg} \omega_\nu^{gf}) \\
&\quad - \delta^{\rho\kappa} \delta_h^e \delta_i^g ee_c^\nu e_b^\lambda \partial_\rho \omega_\lambda^{bc} \omega_\nu^{gf} - \delta_\nu^\kappa \delta_h^g \delta_i^f ee_c^\nu e_b^\lambda \partial_\rho \omega_\lambda^{bc} \omega^{\rho eg} + \delta_\lambda^\kappa \delta_h^b \delta_i^c ee_c^\nu e_b^\lambda \omega_\rho^{dc} \partial^\rho \omega_\nu^{ef} + \delta_\rho^\kappa \delta_h^d \delta_i^e ee_c^\nu e_b^\lambda \omega_\lambda^{bd} \partial^\rho \omega_\nu^{ef} \\
&\quad - \delta_\nu^\kappa \delta_h^e \delta_i^f \partial^\rho (ee_c^\nu e_b^\lambda \omega_\rho^{bd} \omega_\lambda^{dc}) - \delta_\rho^\kappa \delta_h^b \delta_i^c ee_c^\nu e_b^\lambda \omega_\lambda^{dc} \partial^\rho \omega_\nu^{ef} - \delta_\lambda^\kappa \delta_h^d \delta_i^e ee_c^\nu e_b^\lambda \omega_\rho^{bd} \partial^\rho \omega_\nu^{ef} + \delta_\nu^\kappa \delta_h^e \delta_i^f \partial^\rho (ee_c^\nu e_b^\lambda \omega_\rho^{bd} \omega_\lambda^{dc}) \\
&\quad + \delta_\lambda^\kappa \delta_h^b \delta_i^d ee_c^\nu e_b^\lambda \omega_\rho^{dc} \omega^{\rho eg} \omega_\nu^{gf} + \delta_\rho^\kappa \delta_h^d \delta_i^e ee_c^\nu e_b^\lambda \omega_\lambda^{bd} \omega^{\rho eg} \omega_\nu^{gf} + \delta^{\rho\kappa} \delta_h^e \delta_i^g ee_c^\nu e_b^\lambda \omega_\lambda^{bd} \omega_\rho^{dc} \omega_\nu^{gf} + \delta_\nu^\kappa \delta_h^g \delta_i^f ee_c^\nu e_b^\lambda \omega_\lambda^{bd} \omega_\rho^{dc} \omega^{\rho eg} \\
&\quad \left. - \delta_\rho^\kappa \delta_h^b \delta_i^d ee_c^\nu e_b^\lambda \omega_\lambda^{dc} \omega^{\rho eg} \omega_\nu^{gf} - \delta_\lambda^\kappa \delta_h^d \delta_i^e ee_c^\nu e_b^\lambda \omega_\rho^{bd} \omega^{\rho eg} \omega_\nu^{gf} - \delta^{\rho\kappa} \delta_h^e \delta_i^g ee_c^\nu e_b^\lambda \omega_\rho^{bd} \omega_\lambda^{dc} \omega_\nu^{gf} - \delta_\nu^\kappa \delta_h^g \delta_i^f ee_c^\nu e_b^\lambda \omega_\rho^{bd} \omega_\lambda^{dc} \omega^{\rho eg} \right] \Sigma_{ef}
\end{aligned} \tag{63}$$

and used again the fact that the variation vanishes at infinity. To obtain the variation of (62) with respect to the tetrad e_μ^a , (48) has to be used. Variation of (48) with respect to e_μ^a yields

$$\begin{aligned} \delta\omega_\mu^{ab} &= 2\delta e^{\nu a}\partial_\mu e_\nu^b + 2e^{\nu a}\partial_\mu\delta e_\nu^b - 2\delta e^{\nu b}\partial_\mu e_\nu^a - 2e^{\nu b}\partial_\mu\delta e_\nu^a - 2\delta e^{\nu a}\partial_\nu e_\mu^b - 2e^{\nu a}\partial_\nu\delta e_\mu^b + 2\delta e^{\nu b}\partial_\nu e_\mu^a + 2e^{\nu b}\partial_\nu\delta e_\mu^a \\ &\quad + \delta e_{\mu c}e^{\nu a}e^{\sigma b}\partial_\sigma e_\nu^c + e_{\mu c}\delta e^{\nu a}e^{\sigma b}\partial_\sigma e_\nu^c + e_{\mu c}e^{\nu a}\delta e^{\sigma b}\partial_\sigma e_\nu^c + e_{\mu c}e^{\nu a}e^{\sigma b}\partial_\sigma\delta e_\nu^c \\ &\quad - \delta e_{\mu c}e^{\nu a}e^{\sigma b}\partial_\nu e_\sigma^c - e_{\mu c}\delta e^{\nu a}e^{\sigma b}\partial_\nu e_\sigma^c - e_{\mu c}e^{\nu a}\delta e^{\sigma b}\partial_\nu e_\sigma^c - e_{\mu c}e^{\nu a}e^{\sigma b}\partial_\nu\delta e_\sigma^c. \end{aligned} \quad (64)$$

If (64) is inserted into (62), one obtains the variation with respect to the tetrad

$$-\frac{\beta}{\kappa}\int d^4x [\Xi_{hi}^\kappa\delta\omega_\kappa^{hi}] \equiv -\frac{\beta}{\kappa}\int d^4x [\Theta_\mu^a\delta e_\mu^a], \quad (65)$$

where Θ_μ^a is defined according to

$$\begin{aligned} \Theta_\mu^a &= 2\delta_\mu^\nu\delta^{ah}\Xi_{hi}^\kappa\partial_\kappa e_\nu^i - 2\delta_{\mu\nu}\delta^{ai}\partial_\kappa(\Xi_{hi}^\kappa e^{\nu h}) - 2\delta_\mu^\nu\delta^{ai}\Xi_{hi}^\kappa\partial_\kappa e_\nu^h + 2\delta_{\mu\nu}\delta^{ah}\partial_\kappa(\Xi_{hi}^\kappa e^{\nu i}) \\ &\quad - 2\delta_\mu^\nu\delta^{ah}\Xi_{hi}^\kappa\partial_\nu e_\kappa^i + 2\delta_{\mu\kappa}\delta^{ai}\partial_\nu(\Xi_{hi}^\kappa e^{\nu h}) + 2\delta_\mu^\nu\delta^{ai}\Xi_{hi}^\kappa\partial_\nu e_\kappa^h - 2\delta_{\mu\kappa}\delta^{ah}\partial_\nu(\Xi_{hi}^\kappa e^{\nu i}) \\ &\quad + \delta_{\mu\kappa}\delta_c^a\Xi_{hi}^\kappa e^{\nu h}e^{\sigma i}\partial_\sigma e_\nu^c + \delta_\mu^\nu\delta^{ah}\Xi_{hi}^\kappa e_{\kappa c}e^{\sigma i}\partial_\sigma e_\nu^c + \delta_\mu^\sigma\delta^{ai}\Xi_{hi}^\kappa e_{\kappa c}e^{\nu h}\partial_\sigma e_\nu^c - \delta_{\mu\nu}\delta^{ac}\partial_\sigma(\Xi_{hi}^\kappa e_{\kappa c}e^{\nu h}e^{\sigma i}) \\ &\quad - \delta_{\mu\kappa}\delta_c^a\Xi_{hi}^\kappa e^{\nu h}e^{\sigma i}\partial_\nu e_\sigma^c - \delta_\mu^\nu\delta^{ah}\Xi_{hi}^\kappa e_{\kappa c}e^{\sigma i}\partial_\nu e_\sigma^c - \delta_\mu^\sigma\delta^{ai}\Xi_{hi}^\kappa e_{\kappa c}e^{\nu h}\partial_\nu e_\sigma^c + \delta_{\mu\sigma}\delta^{ac}\partial_\nu(\Xi_{hi}^\kappa e_{\kappa c}e^{\nu h}e^{\sigma i}). \end{aligned} \quad (66)$$

This means that the variation (60) now reads

$$-\frac{\beta}{\kappa}\int d^4x\delta[ee_\mu^a e_b^\nu R_{\mu\rho}^{ab}D^\rho D_\nu] = -\frac{\beta}{\kappa}\int d^4x[-ee_\mu^a\delta e_\mu^a R D^\rho D_\nu + e\delta e_\mu^a e_b^\nu R_{\mu\rho}^{ab}D^\rho D_\nu + e\delta e_\mu^a R_\rho^a D^\rho D_\mu + \Theta_\mu^a\delta e_\mu^a] \quad (67)$$

which means that variation of the complete action yields the following generalized vacuum Einstein field equation:

$$R_\mu^a - \frac{1}{2}Re_\mu^a - \beta\left[-e_\mu^a R D^\rho D_\nu + e_b^\nu R_{\mu\rho}^{ab}D^\rho D_\nu + R_\rho^a D^\rho D_\mu + \frac{1}{e}\Theta_\mu^a\right] = 0, \quad (68)$$

where Θ_μ^a is defined according to (66) and (63). If the matter action (50) of the fermionic field is included and the complete gravity action

$$\mathcal{S}_{gr} = \mathcal{S}_m + \mathcal{S}_{EH} = \int d^4x e\bar{\psi}[i\gamma^m e_m^\mu \mathcal{D}_\mu - m]\psi + \frac{1}{2\kappa}\int d^4x e\mathcal{R} \quad (69)$$

is varied with respect to the tetrad $\frac{\delta\mathcal{S}_{gr}}{\delta e_\mu^a}$, one obtains

$$R_\mu^a - \frac{1}{2}Re_\mu^a - \beta\left[-e_\mu^a R D^\rho D_\nu + e_b^\nu R_{\mu\rho}^{ab}D^\rho D_\nu + R_\rho^a D^\rho D_\mu + \frac{1}{e}\Theta_\mu^a\right] = -\kappa T_\mu^a, \quad (70)$$

where the energy momentum tensor is defined as usual as $T_\mu^a = \frac{1}{e}\frac{\delta\mathcal{S}_m}{\delta e_\mu^a}$.

B. Generalized Energy Momentum Tensor for a Dirac Field Coupled to the Gravitational Field

To obtain the concrete expression of the energy momentum tensor in case of a fermionic field that is coupled to the gravitational field according to the generalized gauge theory presented here, the expression (50) has to be varied with respect to the tetrad e_μ^a . Accordingly, the variation of the matter action with respect to the tetrad e_μ^a reads

$$\begin{aligned} \delta\mathcal{S}_m &= \delta\int d^4x e\bar{\psi}[i\gamma^m e_m^\mu \mathcal{D}_\mu - m]\psi = \delta\int d^4x e\bar{\psi}[i\gamma^m e_m^\mu(\mathbf{1} - \beta D^\rho D_\rho)D_\mu - m]\psi \\ &= \int d^4x\{-ee_\mu^m\delta e_\mu^m\bar{\psi}[i\gamma^m e_m^\mu(\mathbf{1} - \beta D^\rho D_\rho)D_\mu - m]\psi + e\bar{\psi}i\gamma^m\delta e_\mu^m(\mathbf{1} - \beta D^\rho D_\rho)D_\mu\psi \\ &\quad + e\bar{\psi}i\gamma^m e_m^\mu\delta[(\mathbf{1} - \beta D^\rho D_\rho)D_\mu]\psi\}. \end{aligned} \quad (71)$$

To calculate the last term, it has to be expressed by the connection ω_μ^{ab} and the connection has to be expressed by the tetrad e_a^μ

$$\begin{aligned}
& \int d^4x e\bar{\psi}i\gamma^m e_m^\mu \delta[(\mathbf{1} - \beta D^\rho D_\rho) D_\mu] \psi \\
&= \int d^4x e\bar{\psi}i\gamma^m e_m^\mu \delta [i\omega_\mu^{ab} - \beta (i\partial^\rho \partial_\rho \omega_\mu^{ab} + 2i\partial^\rho \omega_\mu^{ab} \partial_\rho + i\omega_\mu^{ab} \partial_\rho \partial^\rho + i\partial^\rho \omega_\rho^{ab} \partial_\mu + 2i\omega^{\rho ab} \partial_\rho \partial_\mu - \partial^\rho \omega_\rho^{ac} \omega_\mu^{cb} \\
&\quad - 2\omega_\rho^{ac} \partial^\rho \omega_\mu^{cb} - 2\omega_\rho^{ac} \omega_\mu^{cb} \partial^\rho - \omega^{\rho ac} \omega_\rho^{cb} \partial_\mu - i\omega^{\rho ac} \omega_\rho^{cd} \omega_\mu^{db})] \Sigma_{ab} \psi \\
&= \int d^4x e\bar{\psi}i\gamma^m e_m^\mu [i\delta\omega_\mu^{ab} - \beta (i\partial^\rho \partial_\rho \delta\omega_\mu^{ab} + 2i\partial^\rho \delta\omega_\mu^{ab} \partial_\rho + i\delta\omega_\mu^{ab} \partial^\rho \partial_\rho + i\partial^\rho \delta\omega_\rho^{ab} \partial_\mu + 2i\delta\omega^{\rho ab} \partial_\rho \partial_\mu \\
&\quad - \partial^\rho \delta\omega_\rho^{ac} \omega_\mu^{cb} - \partial^\rho \omega_\rho^{ac} \delta\omega_\mu^{cb} - 2\delta\omega_\rho^{ac} \partial^\rho \omega_\mu^{cb} - 2\omega_\rho^{ac} \partial^\rho \delta\omega_\mu^{cb} - 2\delta\omega_\rho^{ac} \omega_\mu^{cb} \partial^\rho - 2\omega_\rho^{ac} \delta\omega_\mu^{cb} \partial^\rho - \delta\omega^{\rho ac} \omega_\rho^{cb} \partial_\mu \\
&\quad - \omega^{\rho ac} \delta\omega_\rho^{cb} \partial_\mu - i\delta\omega^{\rho ac} \omega_\rho^{cd} \omega_\mu^{db} - i\omega^{\rho ac} \delta\omega_\rho^{cd} \omega_\mu^{db} - i\omega^{\rho ac} \omega_\rho^{cd} \delta\omega_\mu^{db})] \Sigma_{ab} \psi \\
&\equiv \int d^4x [\xi_{hi}^\kappa \delta\omega_\mu^{hi}] \tag{72}
\end{aligned}$$

where ξ_{hi}^κ has been defined according to

$$\begin{aligned}
\xi_{hi}^\kappa = & \left\{ i\delta_\mu^\kappa \delta_h^a \delta_i^b A_{ab}^\mu - \beta \left[i\delta_\mu^\kappa \delta_h^a \delta_i^b \partial^\rho \partial_\rho A_{ab}^\mu - 2i\delta_\mu^\kappa \delta_h^a \delta_i^b \partial^\rho B_{\rho ab}^\mu + i\delta_\mu^\kappa \delta_h^a \delta_i^b C_{\rho ab}^{\mu\rho} - i\delta_\rho^\kappa \delta_h^a \delta_i^b \partial^\rho B_{\mu ab}^\mu \right. \right. \\
& + 2i\delta^{\kappa\rho} \delta_h^a \delta_i^b C_{\rho\mu ab}^\mu + \delta_\rho^\kappa \delta_h^a \delta_i^c \partial^\rho (\omega_\mu^{cb} A_{ab}^\mu) - \delta_\mu^\kappa \delta_h^c \delta_i^b \partial^\rho \omega_\rho^{ac} A_{ab}^\mu - 2\delta_\rho^\kappa \delta_h^a \delta_i^c \partial^\rho \omega_\mu^{cb} A_{ab}^\mu + 2\delta_\mu^\kappa \delta_h^c \delta_i^b \partial^\rho (\omega_\rho^{ac} A_{ab}^\mu) \\
& - 2\delta_\rho^\kappa \delta_h^a \delta_i^c \omega_\mu^{cb} B_{ab}^{\mu\rho} - 2\delta_\mu^\kappa \delta_h^c \delta_i^b \omega_\rho^{ac} B_{ab}^{\mu\rho} - \delta^{\kappa\rho} \delta_h^c \delta_i^b \omega_\rho^{cb} B_{\mu ab}^\mu - \delta_\rho^\kappa \delta_h^c \delta_i^b \omega^{\rho ac} B_{\mu ab}^\mu - i\delta^{\kappa\rho} \delta_h^a \delta_i^c \omega_\rho^{cd} \omega_\mu^{db} A_{ab}^\mu \\
& \left. - i\delta_\rho^\kappa \delta_h^c \delta_i^d \omega^{\rho ac} \omega_\mu^{db} A_{ab}^\mu - i\delta_\mu^\kappa \delta_h^d \delta_i^b \omega^{\rho ac} \omega_\rho^{cd} A_{ab}^\mu \right\} \tag{73}
\end{aligned}$$

with $A_{ab}^\mu = e\bar{\psi}i\gamma^m e_m^\mu \Sigma_{ab} \psi$, $B_{ab}^{\mu\nu} = e\bar{\psi}i\gamma^m e_m^\mu \Sigma_{ab} \partial^\nu \psi$, and $C_{\rho ab}^{\mu\nu} = e\bar{\psi}i\gamma^m e_m^\mu \Sigma_{ab} \partial^\nu \partial_\rho \psi$. If (64) is inserted into (72), then one obtains

$$\int d^4x [\xi_{hi}^\kappa \delta\omega_\mu^{hi}] \equiv \int d^4x [\theta_\mu^a \delta e_a^\mu] \tag{74}$$

with θ_μ^a defined according to

$$\begin{aligned}
\theta_\mu^a = & 2\delta_\mu^\nu \delta^{ah} \xi_{hi}^\kappa \partial_\kappa e_\nu^i - 2\delta_{\mu\nu} \delta^{ai} \partial_\kappa (\xi_{hi}^\kappa e^{\nu h}) - 2\delta_\mu^\nu \delta^{ai} \xi_{hi}^\kappa \partial_\kappa e_\nu^h + 2\delta_{\mu\nu} \delta^{ah} \partial_\kappa (\xi_{hi}^\kappa e^{\nu i}) \\
& - 2\delta_\mu^\nu \delta^{ah} \xi_{hi}^\kappa \partial_\nu e_\kappa^i + 2\delta_{\mu\kappa} \delta^{ai} \partial_\nu (\xi_{hi}^\kappa e^{\nu h}) + 2\delta_\mu^\nu \delta^{ai} \xi_{hi}^\kappa \partial_\nu e_\kappa^h - 2\delta_{\mu\kappa} \delta^{ah} \partial_\nu (\xi_{hi}^\kappa e^{\nu i}) \\
& + \delta_{\mu\kappa} \delta_c^a \xi_{hi}^\kappa e^{\nu h} e^{\sigma i} \partial_\sigma e_\nu^c + \delta_\mu^\nu \delta^{ah} \xi_{hi}^\kappa e_{\kappa c} e^{\sigma i} \partial_\sigma e_\nu^c + \delta_\mu^\sigma \delta^{ai} \xi_{hi}^\kappa e_{\kappa c} e^{\nu h} \partial_\sigma e_\nu^c - \delta_{\mu\nu} \delta^{ac} \partial_\sigma (\xi_{hi}^\kappa e_{\kappa c} e^{\nu h} e^{\sigma i}) \\
& - \delta_{\mu\kappa} \delta_c^a \xi_{hi}^\kappa e^{\nu h} e^{\sigma i} \partial_\nu e_\sigma^c - \delta_\mu^\nu \delta^{ah} \xi_{hi}^\kappa e_{\kappa c} e^{\sigma i} \partial_\nu e_\sigma^c - \delta_\mu^\sigma \delta^{ai} \xi_{hi}^\kappa e_{\kappa c} e^{\nu h} \partial_\nu e_\sigma^c + \delta_{\mu\sigma} \delta^{ac} \partial_\nu (\xi_{hi}^\kappa e_{\kappa c} e^{\nu h} e^{\sigma i}). \tag{75}
\end{aligned}$$

This means that the variation of the matter action (71) reads now

$$\delta\mathcal{S}_m = \int d^4x \{ -e e_\mu^m \delta e_m^\mu \bar{\psi} [i\gamma^m e_m^\mu (\mathbf{1} - \beta D^\rho D_\rho) D_\mu - m] \psi + e\bar{\psi}i\gamma^m \delta e_m^\mu (\mathbf{1} - \beta D^\rho D_\rho) D_\mu \psi + \theta_\mu^a \delta e_a^\mu \} \tag{76}$$

leading to the following energy momentum tensor:

$$T_\mu^a = \bar{\psi} i\gamma^a (\mathbf{1} - \beta D^\rho D_\rho) D_\mu \psi - e_\mu^a \bar{\psi} [i\gamma^m e_m^\nu (\mathbf{1} - \beta D^\rho D_\rho) D_\nu - m] \psi + \theta_\mu^a, \tag{77}$$

where θ_μ^a is defined according (75) and (73).

VI. SUMMARY AND DISCUSSION

There has been considered an extension of the usual description of local gauge theories according to the implementation of a generalized uncertainty principle to quantum field theory, which causes generalized free field equations.

Since a generalized uncertainty principle leads to additional derivative terms, if the momentum operator and the field equations are represented in position space, the description of a local gauge theory becomes more intricate. The reason lies in the fact that the additional derivative terms also have to be replaced by covariant derivatives to maintain local gauge invariance and therefore there appear additional terms containing the gauge potential. These terms describe a more complicated interaction structure between the gauge field and the matter field. In accordance with this there can be defined a generalized covariant derivative from which there is built a generalized field strength tensor. If the action of the gauge field is built by using this generalized definition of the field strength tensor, the dynamics of the gauge field contains additional self-interaction terms arising from this generalization. This means that even in case of an Abelian gauge theory like electrodynamics there arises a self-interaction of the electromagnetic field. According to this, the generalized Feynman rules of quantum electrodynamics have been determined by calculating the propagators under presupposition of the generalized momentum eigenstates and extracting the vertices from the expressions of the action expressed explicitly in terms of the gauge potential. Because of the additional interaction terms not only are the usual vertices changed but also completely new vertices arise. Since general relativity can also be formulated as a gauge theory, this generalization of gauge theories also changes the structure of the gravitational interaction. The generalization of the covariant derivative in the $SO(3, 1)$ gauge description of general relativity causes a changing of the interaction of a matter field with the gravitational field, which can be seen as a modification of the equivalence principle. Further, as in the case of electrodynamics, the dynamics of the gravitational field itself is changed. The Einstein Hilbert action is built from a generalized Riemann tensor defined as the commutator of the generalized covariant derivatives in accordance with the field strength tensor in the case of electrodynamics, and the variation with respect to the tetrad field yields the generalized Einstein equations. This means that the structure of the gravitational interaction and thus the dynamics of the metric structure of space-time depend on the generalized uncertainty principle and thus on the corresponding smallest length within this consideration. Usually the generalized uncertainty principle is seen as a consequence of a quantum description of gravity. In the present paper the influence on gravity induced by a generalized uncertainty principle is considered instead, if it is assumed to yield a fundamental description of nature or to have another origin. This means that a modified description of gravity is obtained depending on the modification of the uncertainty principle independent of an underlying theory. However, the presented modification of gravity could be seen as a semiclassical approximation to a quantum theory of gravity in the sense that the existence of a minimal length is a presupposition for such a theory and not a consequence. Therefore it would be very interesting to consider the complete quantum theory of the presented generalized theory of gravity. The modifications of electrodynamics and gravity as they are investigated in this paper are in principle in accordance with the formulation of these gauge field theories on noncommutative space-time by using the star product and Seiberg Witten maps where similar modifications with additional interaction terms are obtained [46],[47],[48],[49]. This is to be expected, since the concept of noncommutative space-time, which also implies a minimal length, is closely related to the generalized uncertainty principle, which also leads to noncommuting coordinates.

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