

Absorption of mass and angular momentum by a black hole: Time-domain formalisms for gravitational perturbations, and the small-hole/slow-motion approximation

Eric Poisson

Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1

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The first objective of this work is to obtain practical prescriptions to calculate the absorption of mass and angular momentum by a black hole when external processes produce gravitational radiation. These prescriptions are formulated in the time domain (in contrast with the frequency-domain formalism of Teukolsky and Press) within the framework of black-hole perturbation theory. Two such prescriptions are presented. The first is based on the Teukolsky equation and it applies to general (rotating) black holes. The second is based on the Regge-Wheeler and Zerilli equations and it applies to nonrotating black holes. The second objective of this work is to apply the time-domain absorption formalisms to situations in which the black hole is either small or slowly moving; the mass of the black hole is then assumed to be much smaller than the radius of curvature of the external spacetime in which the hole moves. In the context of this small-hole/slow-motion approximation, the equations of black-hole perturbation theory can be solved analytically, and explicit expressions can be obtained for the absorption of mass and angular momentum. The changes in the black-hole parameters can then be understood in terms of an interaction between the tidal gravitational fields supplied by the external universe and the hole's tidally-induced mass and current quadrupole moments. For a nonrotating black hole the quadrupole moments are proportional to the rate of change of the tidal fields on the hole's world line. For a rotating black hole they are proportional to the tidal fields themselves. When placed in identical environments, a rotating black hole absorbs more energy and angular momentum than a nonrotating black hole.

I. INTRODUCTION AND SUMMARY

A. Goals and motivations

The work described in this article is concerned with the absorption of energy and angular momentum by a black hole when physical processes in its exterior produce gravitational radiation. It is assumed throughout that the rates of change of mass and angular momentum are sufficiently low that they can be calculated within the framework of first-order perturbation theory, in which the black hole differs only slightly from a stationary and axisymmetric Kerr hole.

The first goal of this work is to obtain practical prescriptions to calculate the black-hole absorption, and to modernize the tools fashioned in the early seventies by Teukolsky and Press [1]. An essential aspect of the new prescriptions is that they present the absorption formulae in the time domain instead of the frequency domain; they presuppose that in accordance with current trends, the equations of black-hole perturbation theory have been solved as partial differential equations in the time domain instead of ordinary differential equations in the frequency domain. Two such prescriptions are presented here: the first is based on the Teukolsky equation [2] and it applies to general (rotating) black holes, while the second is based on the Regge-Wheeler [3] and Zerilli [4] equations and it applies to nonrotating black holes.

That the Teukolsky equation can be separated in all of its variables is surely one of its most important properties. To a large extent, it is this property that has permitted progress during the continuing exploration of physical processes taking place in black-hole spacetimes (see, for

example, the book by Frolov and Novikov, Ref. [5]). But it has to be acknowledged that the historical importance of the separation property has diminished in recent years, as a number of time-domain integrators of the Teukolsky equation have been developed [6, 7, 8, 9, 10] and put to use in various applications. The numerical task of solving the Teukolsky equation in the time domain is still challenging: after decomposition into azimuthal modes one must solve for a function of time and two spatial coordinates. But time-domain methods appear now to be at least competitive with frequency-domain methods, with which one must solve for a number of radial and angular functions, the number increasing as the spectrum of relevant frequencies becomes wider. And it appears likely that the future will witness an increasing dominance of time-domain methods over frequency-domain methods.

While the superiority of time-domain methods is still to be proved in the case of the Teukolsky equation, it has clearly been established [11, 12, 13, 14, 15] in the context of the Regge-Wheeler [3] and Zerilli [4] equations, which determine the metric perturbations of a nonrotating black hole. In these cases the angular dependence of the perturbation variables can be completely separated, and the integrator faces the relatively simple task of solving for a function of two variables (time and a radial coordinate). Simple, but powerful, numerical methods have been devised [13] for such problems, and these can even handle, without approximations, a singular source term contributed by a point particle. The time-domain Regge-Wheeler and Zerilli equations are thus very easy to integrate, and there is now little reason to go back to a frequency-domain formulation.

The new popularity of time-domain methods to solve

the equations of black-hole perturbation theory calls for new prescriptions to calculate the black-hole absorption of energy and angular momentum. The only recipe currently available is the formalism of Teukolsky and Press [1], which is based on the frequency-domain formulation of the Teukolsky equation [2]. This formalism is not well adapted to time-domain calculations, and in this work I provide the required translation of the Teukolsky-Press recipe to the time domain. Another limitation of the Teukolsky-Press formalism is that although it can be applied without difficulty to a nonrotating black hole, this requires the use of the Teukolsky equation instead of the more practical Regge-Wheeler and Zerilli equations. Another objective of this work is therefore to relate the absorption of mass and angular momentum by a Schwarzschild black hole to the time-domain solutions to these equations.

In effect, this work is about providing practical time-domain formulae for the fluxes of mass and angular momentum across a perturbed black-hole horizon. For a nonrotating black hole these formulae are based on the Regge-Wheeler and Zerilli equations, which govern the behavior of the metric perturbations. For a rotating black hole the formulae are based instead on the Teukolsky equation, which determines the perturbations of the Weyl curvature tensor.

The second goal of this work is to apply the time-domain absorption formalisms to physical situations in which the black hole can be considered to be either small or slowly moving. In the context of this small-hole/slow-motion approximation (which I will describe in Sec. I E below), the equations of black-hole perturbation theory can be solved analytically, and explicit expressions can be obtained for the absorption of mass and angular momentum. While many results have been obtained along those lines in the past [16, 17, 18, 19], they were all restricted to various special cases; the results presented here consolidate and generalize these previous works.

The absorption of mass and angular momentum by a black hole is generally very small. In particular, the effect is likely to be too small to be observed in a gravitational-wave signal that would be measured by ground-based detectors such as LIGO, VIRGO, and GEO600. For example, Alvi [18] has calculated that for binary systems involving holes with masses ranging from 5 to 50 solar masses, black-hole absorption is truly negligible: It contributes only a small fraction of a wave cycle during the signal's sweep through the detector's frequency band. For this type of source the tools developed in this paper are not needed.

In some circumstances, however, the black-hole absorption is a significant effect that should not be neglected [20]. In particular, it is likely to be observed in gravitational-wave signals that would be measured by a space-based detector such as LISA. For example, Martel [14] has shown that during a close encounter between a massive black hole and a compact body (of a much smaller mass), up to approximately five percent of the

total radiated energy is absorbed by the black hole, the rest being transported out to infinity. Hughes [21] has calculated that when the massive hole is rapidly rotating, the absorption has the effect of slowing down the inspiral of the orbiting body, thereby increasing the duration of the gravitational-wave signal. For example, a $1 M_{\odot}$ compact body on a slightly inclined, circular orbit around a $10^6 M_{\odot}$ black hole of near-maximum spin would spend approximately two years in the LISA frequency band before its final plunge into the hole; Hughes shows that the black-hole absorption contributes approximately 20 days (and 10^4 wave cycles) to these two years. For this kind of situation the absorption is important, and the tools developed in this paper will be useful.

B. Perturbative methods

A natural starting point for the calculation of black-hole absorption would be the definition of a dynamical mass $M(v)$ and angular momentum $J(v)$ on a cross section $v = \text{constant}$ of an evolving event horizon; here v is a suitable advanced-time coordinate on the horizon. Armed with such definitions, one would differentiate with respect to v to obtain $\dot{M}(v)$ and $\dot{J}(v)$, and seek to express the right-hand sides in terms of standard perturbation variables. Such an approach to black-hole absorption has recently been pursued by a number of workers [22, 23, 24, 25], and the resulting (inequivalent) formalisms can be formulated exactly in fully nonlinear general relativity. These formalisms are based not on the event horizon, but instead on the hole's *trapping horizon*, a generally spacelike hypersurface foliated by marginally trapped surfaces; and in the Ashtekar-Krishnan formalism [22, 23] the definitions for $M(v)$ and $J(v)$ come from the Hamiltonian formulation of general relativity. These formalisms are interesting (and useful in the context of numerical relativity) because they are fully general, and because they involve a hypersurface (the trapping horizon) whose intersection with a given Cauchy slice is easy to identify; the event horizon, on the other hand, can be identified only once the future history of the spacetime is completely known.

The approach adopted here to calculate the black-hole absorption is not the one described in the preceding paragraph; it is based instead on black-hole perturbation theory, and it assumes that the evolving black hole is only slightly different from a stationary and axisymmetric Kerr hole. Because the analysis is restricted to first-order perturbation theory, it is possible to proceed without the specification of a mass function $M(v)$ and an angular-momentum function $J(v)$, so long as only the *long-term* changes in mass and angular momentum need to be calculated. In this long-term view one imagines that the black hole starts in an initial stationary state characterized by the parameters (M, J) , is perturbed for a time Δv by some external process, and then returns to another stationary state characterized by the param-

eters $(M + \delta M, J + \delta J)$. One then defines the *averaged rates of change* of mass and angular momentum by $\langle \dot{M} \rangle = (\delta M)/(\Delta v)$ and $\langle \dot{J} \rangle = (\delta J)/(\Delta v)$, and one manipulates the equations of black-hole perturbation theory to calculate these quantities. This is what I set out to do in this work. The perturbative techniques demand that $\delta M \ll M$, $\delta J \ll J$, and the long-term view demands that $\Delta v \gg M$. While the price to pay is a substantial loss of generality with respect to an exact formulation, the perturbative-long-view approach adopted in this paper allows one to proceed without having to choose a specification of $M(v)$ and $J(v)$, with the derived benefit that the final results are robust with respect to a change of definitions. Another benefit is that the approach is based on the event horizon (the true boundary of the black-hole region) instead of the trapping horizon; while locating the event horizon in a nonlinear situation can be difficult, this is not a problem in the perturbative-long-view approach.

The mathematical techniques required for the description of a perturbed event horizon go back to the pioneering work of Hawking and Hartle [26], and these form the basis of this work (they also formed the basis of the Teukolsky-Press prescription [1]). These techniques are reviewed in Secs. II, III, and IV of the paper. I begin in Sec. II with a description of the unperturbed horizon of a stationary Kerr black hole. In Sec. III, I consider the dynamics of a general evolving horizon, and in Sec. IV, I specialize the discussion to event horizons that are perturbed versions of the Kerr horizon. The equations that govern the behavior of the horizon's null generators are given in Sec. III and IV in a form that closely resembles the treatment provided by Price and Thorne [27], and in Chapter VI of the book by Thorne, Price, and Macdonald [28]. Although these equations are well known, I derive them *ab initio* in order to clearly identify the simplifying assumptions that are incorporated along the way; in particular, the averaging procedure involved in calculating $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ is explained fully in Secs. IV B and C.

The main ideas behind the Hawking-Hartle techniques [26] are as follows. The intrinsic geometry of an evolving event horizon is described by a two-dimensional metric tensor γ_{AB} , which depends on the advanced-time coordinate v as well as angular coordinates θ^A ($A = 2, 3$). The metric is degenerate (and explicitly two-dimensional) because the horizon is a null hypersurface; v is a parameter on the horizon's null generators, and θ^A are generator labels that stay constant as the generators move. The evolution of γ_{AB} is determined by the behavior of the generators, which is described in terms of an expansion scalar Θ and a shear tensor σ_{AB} . The evolution of the shear is driven by the spacetime's Weyl curvature, and the evolution of the expansion is driven by the square of the shear tensor. The evolution equations can all be integrated (assuming that the Weyl curvature is specified), and the solution for $\langle \Theta \rangle$ determines $\langle \dot{A} \rangle$, the (averaged) rate of change of the black hole's surface area. The final

step is to use the first law of black-hole mechanics (see, for example, Chapter 12 of Ref. [29], or Chapter 5 of Ref. [30]) to relate this to $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$. The end result is Eqs. (4.22)–(4.24) in Sec. IV C, which express $\langle \dot{M} \rangle$, $\langle \dot{J} \rangle$, and $\langle \dot{A} \rangle$ in terms of σ_{AB} contracted with derivatives of γ_{AB} , integrated over a cross section of the event horizon. These equations are not new: they were first presented by Thorne, Price, and Macdonald [28], but the derivation provided here is substantially different from theirs. And while these equations are not themselves very practical, they form an excellent starting point for the development of practical formalisms.

C. Curvature formalism

The development of a time-domain formalism to calculate $\langle \dot{M} \rangle$, $\langle \dot{J} \rangle$, and $\langle \dot{A} \rangle$ in terms of standard curvature variables is undertaken in Sec. V. The end result of this reformulation of Eqs. (4.22)–(4.24) is the following prescription (Sec. V D) to calculate the black-hole absorption.

First, define a Teukolsky function $\Psi \equiv -\psi_0(\text{HH})$ as in Eq. (5.6), in terms of a null-tetrad decomposition of the perturbed Weyl tensor. The label “HH” indicates that the Weyl tensor is decomposed in the Hartle-Hawking null tetrad [26], which is well behaved on the future horizon of the Kerr spacetime.

Second, decompose the Teukolsky function in terms of azimuthal modes proportional to $e^{im\psi}$,

$$\Psi(v, r, \theta, \psi) = \sum_{m=-\infty}^{\infty} \Psi^m(v, r, \theta) e^{im\psi}, \quad (1.1)$$

where m is an integer. Because the Kerr spacetime is axially symmetric, each mode $\Psi^m(v, r, \theta)$ evolves independently. Note that the coordinates (v, r, θ, ψ) are ingoing Kerr coordinates (Sec. II A), and that they are well behaved on the event horizon.

Third, integrate the Teukolsky equation [2] for each relevant mode $\Psi^m(v, r, \theta)$, and evaluate the result at $r = r_+ \equiv M + \sqrt{M^2 - a^2}$, the position of the unperturbed horizon; $a \equiv J/M$ is the specific angular momentum of the Kerr black hole.

Fourth, calculate the integrated curvatures

$$\Phi_+^m(v, \theta) = e^{\kappa v} \int_v^{\infty} e^{-(\kappa - im\Omega_H)v'} \Psi^m(v', r_+, \theta) dv' \quad (1.2)$$

and

$$\Phi_-^m(v, \theta) = \int_{-\infty}^v e^{im\Omega_H v'} \Psi^m(v', r_+, \theta) dv', \quad (1.3)$$

where $\kappa = (r_+ - M)/(r_+^2 + a^2)$ is the surface gravity of the Kerr horizon, and $\Omega_H = a/(r_+^2 + a^2)$ its angular velocity. Notice that Φ_+^m at advanced time v depends on the behavior of Ψ^m at *later times*; this is a consequence of the teleological nature of the event horizon.

Fifth, and finally, insert the integrated curvatures and their complex conjugates (indicated with an overbar) into the flux formulae

$$\langle \dot{M} \rangle = \frac{r_+^2 + a^2}{4\kappa} \sum_{m=-\infty}^{\infty} \left[2\kappa \int \langle |\Phi_+^m|^2 \rangle \sin \theta d\theta - im\Omega_H \int \langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \rangle \sin \theta d\theta \right], \quad (1.4)$$

$$\langle \dot{J} \rangle = -\frac{r_+^2 + a^2}{4\kappa} \sum_{m=-\infty}^{\infty} (im) \times \int \langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \rangle \sin \theta d\theta, \quad (1.5)$$

and

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \frac{1}{2} (r_+^2 + a^2) \sum_{m=-\infty}^{\infty} \int \langle |\Phi_+^m|^2 \rangle \sin \theta d\theta. \quad (1.6)$$

These equations reduce to those of Teukolsky and Press [1] when $\Psi^m(v, r, \theta)$ is a pure mode of frequency ω , $\Psi^m \propto e^{-i\omega v}$; this is established in Sec. V C. Equations (1.4)–(1.6) are therefore the time-domain equivalent to the standard frequency-domain prescription.

D. Metric formalism

The curvature formalism of the preceding subsection applies to a general rotating black hole, and the special case of a nonrotating hole can be handled simply by setting $a = 0$. But in this case it is often desirable to work with metric perturbations instead of curvature perturbations, and it becomes useful to present the flux formulae in terms of $\Psi_{\text{RW}}^{lm}(v, r)$ and $\Psi_{\text{ZM}}^{lm}(v, r)$, the standard Regge-Wheeler and Zerilli-Moncrief functions, instead of the Teukolsky function $\Psi^m(v, r, \theta)$. Here the decomposition into modes involves spherical-harmonic functions of degree l and azimuthal number m .

The development of a time-domain formalism to calculate $\langle \dot{M} \rangle$, $\langle \dot{J} \rangle$, and $\langle \dot{A} \rangle$ in terms of standard metric variables is undertaken in Sec. VII, after laying some important foundations in Sec. VI. The end result of this reformulation of Eqs. (4.22)–(4.24) is the following prescription (Sec. VII C) to calculate the absorption of mass and angular momentum by a Schwarzschild black hole.

First, integrate the Regge-Wheeler equation [3] for $\Psi_{\text{RW}}^{lm}(v, r)$, which describes the odd-parity sector of the metric perturbations. This gauge-invariant function is defined in subsection 3 of the Appendix.

Second, integrate the Zerilli equation [4] for $\Psi_{\text{ZM}}^{lm}(v, r)$, which describes the even-parity sector of the metric perturbations. This gauge-invariant function is defined in subsection 4 of the Appendix.

Third, and finally, evaluate the Regge-Wheeler and Zerilli-Moncrief functions at $r = r_+ \equiv 2M$ and insert

them into the flux formulae

$$\langle \dot{M} \rangle = \frac{1}{64\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^l (l-1)l(l+1)(l+2) \times \left\langle 4|\Psi_{\text{RW}}^{lm}(v, r_+)|^2 + |\dot{\Psi}_{\text{ZM}}^{lm}(v, r_+)|^2 \right\rangle \quad (1.7)$$

and

$$\langle \dot{J} \rangle = \frac{1}{64\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^l (l-1)l(l+1)(l+2)(im) \times \left\langle 4\Psi_{\text{RW}}^{lm}(v, r_+) \int^v \bar{\Psi}_{\text{RW}}^{lm}(v', r_+) dv' + \dot{\Psi}_{\text{ZM}}^{lm}(v, r_+) \bar{\Psi}_{\text{ZM}}^{lm}(v, r_+) \right\rangle. \quad (1.8)$$

Except for the substitution ($v \rightarrow u, r_+ \rightarrow \infty$), these formulae are identical to Eqs. (A.26) and (A.27) which give the rates at which energy and angular momentum are transported to future null infinity. Note that for a nonrotating black hole, the first law of black-hole mechanics reduces to $(\kappa/8\pi)\langle \dot{A} \rangle = \langle \dot{M} \rangle$.

The flux formulae of Eqs. (1.7), (1.8) were first presented and used by Martel [14] in his numerical exploration of gravitational-wave processes associated with the motion of a small-mass body in the field of a Schwarzschild black hole. Although he arrived at the correct results, the derivation of Eqs. (1.7) and (1.8) presented by Martel is flawed, and the analysis presented in Sec. VII puts them on a firm footing. Martel's derivation incorporates both a conceptual and a computational error, the latter compensating for the former. Martel based his derivation of Eq. (1.7) and (1.8) on Isaacson's effective stress-energy tensor for gravitational waves [31, 32], incorrectly assuming that Isaacson's high-frequency description is always applicable near the event horizon of a black hole. This story is related more fully in Sec. VII C, and its proper telling requires the connection between $\langle \dot{M} \rangle$, $\langle \dot{J} \rangle$ and the Isaacson stress-energy tensor established in Sec. VI C. The limitations of the high-frequency description become especially clear in view of this connection.

E. Small-hole/slow-motion approximation

A concrete evaluation of the flux formulae would typically require the numerical integration of the Teukolsky equation, or the Regge-Wheeler and Zerilli equations; an illustration is provided by Martel's recent work [14]. But in some circumstances it is possible to solve these equations analytically, and to obtain approximate expressions for $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$. I carry out such calculations in Secs. VIII and IX, in the context of a small-hole/slow-motion approximation that I now describe.

Consider a situation in which the black hole is immersed in an external universe whose radius of curvature \mathcal{R} is such that $M/\mathcal{R} \ll 1$. For example, suppose that the

black hole is moving on a circular orbit of radius b in the gravitational field of another body of mass M_{ext} . Then \mathcal{R}^{-1} is of the order of the hole's angular velocity, and we have

$$\frac{M}{\mathcal{R}} \sim \frac{M}{M + M_{\text{ext}}} V^3, \quad V = \sqrt{\frac{M + M_{\text{ext}}}{b}},$$

where V is the hole's orbital velocity. One way to make this ratio small is to let $M/M_{\text{ext}} \ll 1$; then M/\mathcal{R} will be small irrespective of the magnitude of V . This is the *small-hole approximation*, which allows the small black hole to move at relativistic speeds in the strong gravitational field of the external body. Another way is to let $V \ll 1$; then M/\mathcal{R} will be small for all mass ratios. This is the *slow-motion approximation*, which allows the slowly-moving black hole to have a mass comparable to (or even much larger than) M_{ext} . These two limiting approximations are special cases of the fundamental requirement that M/\mathcal{R} be small; I call this the small-hole/slow-motion (SH/SM) approximation.

When viewed on the large scale \mathcal{R} , the black hole occupies a very small region of the actual spacetime, and this region can be idealized as a world line γ in the external spacetime. Let u^α be the (normalized) tangent vector to this world line, and call this the four-velocity of the black hole in the external spacetime. Assume that the Ricci tensor of the external spacetime vanishes on γ , so that the black hole's neighborhood will be empty of matter. The curvature of the external spacetime in this neighborhood is then described entirely by the Weyl tensor. This can be decomposed into its electric and magnetic components (see, for example, Ref. [33]), respectively

$$\mathcal{E}_{\alpha\beta} = C_{\mu\alpha\nu\beta} u^\mu u^\nu \quad (1.9)$$

and

$$\mathcal{B}_{\alpha\beta} = \frac{1}{2} u^\mu \varepsilon_{\mu\alpha}{}^{\gamma\delta} C_{\gamma\delta\beta\nu} u^\nu, \quad (1.10)$$

where the Levi-Civita tensor $\varepsilon_{\mu\alpha\nu\beta}$ and the Weyl tensor $C_{\mu\alpha\nu\beta}$ are evaluated on the world line γ . The tensors $\mathcal{E}_{\alpha\beta}$ and $\mathcal{B}_{\alpha\beta}$ are orthogonal to u^α , and they are both symmetric and tracefree; they comprise all ten independent components of the Weyl tensor. These tensors represent the tidal gravitational fields that are supplied by the external universe, and these act on the black hole so as to produce a tidal distortion. This distortion, in turn, gives rise to a change of mass and angular momentum that can be computed with the formalisms described in the preceding subsections.

In Sec. VIII, I calculate $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ for a Schwarzschild black hole moving in an external universe, to leading order in a SH/SM approximation. The results are

$$\langle \dot{M} \rangle = \frac{16M^6}{45} \langle \dot{\mathcal{E}}_{\alpha\beta} \dot{\mathcal{E}}^{\alpha\beta} + \dot{\mathcal{B}}_{\alpha\beta} \dot{\mathcal{B}}^{\alpha\beta} \rangle \quad (1.11)$$

and

$$\langle \dot{J} \rangle = -\frac{32M^6}{45} u^\mu \varepsilon_{\mu\alpha\gamma\delta} \langle \dot{\mathcal{E}}_{\alpha\beta} \mathcal{E}^{\beta\gamma} + \dot{\mathcal{B}}_{\alpha\beta} \mathcal{B}^{\beta\gamma} \rangle s^\delta, \quad (1.12)$$

where s^α is a unit vector, orthogonal to u^α , that gives the direction of the vector $\langle \dot{J} \rangle = \langle \dot{J} \rangle s^\alpha$ (a more precise definition is found in Sec. VIII F), and $\dot{\mathcal{E}}_{\alpha\beta} \equiv \mathcal{E}_{\alpha\beta;\mu} u^\mu$, $\dot{\mathcal{B}}_{\alpha\beta} \equiv \mathcal{B}_{\alpha\beta;\mu} u^\mu$ are the proper-time derivative of the tidal gravitational fields. From these expressions we infer that $\langle \dot{M} \rangle$ scales as M^6/\mathcal{R}^6 , while $\langle \dot{J} \rangle$ scales as M^6/\mathcal{R}^5 . In Sec. VIII G, I show that the change in mass and angular momentum can be understood in terms of a coupling between the tidal fields and the hole's induced mass and current quadrupole moments, which are given by $M_{\alpha\beta} = \frac{32}{45} M^6 \dot{\mathcal{E}}_{\alpha\beta}$ and $J_{\alpha\beta} = \frac{8}{15} M^6 \dot{\mathcal{B}}_{\alpha\beta}$, respectively. As illustrative examples, Eq. (1.11) and (1.12) are evaluated in two different limits in the case of circular binary motion: In Sec. VIII H, I calculate $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ for a slowly-moving binary system consisting of bodies of comparable masses (one being the black hole); and in Sec. VIII I, I take the mass ratio to be small ($M/M_{\text{ext}} \ll 1$) but allow the black hole to move rapidly in the strong gravitational field of the external body.

In Sec. IX, I calculate $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ for a Kerr black hole moving in an external universe. I again work to leading order in a SH/SM approximation, but the statement of the approximation must now be refined to $M/\mathcal{R} \ll \chi$, where $\chi \equiv a/M \equiv J/M^2$ is the dimensionless rotational parameter of the black hole. The results, which were obtained previously by D'Eath [19], are

$$\langle \dot{M} \rangle = O(M^5/\mathcal{R}^5) \quad (1.13)$$

and

$$\begin{aligned} \langle \dot{J} \rangle = & -\frac{2}{45} M^5 \chi \left[8(1 + 3\chi^2) \langle E_1 + B_1 \rangle \right. \\ & - 3(4 + 17\chi^2) \langle E_2 + B_2 \rangle \\ & \left. + 15\chi^2 \langle E_3 + B_3 \rangle \right]. \end{aligned} \quad (1.14)$$

where $E_1 = \mathcal{E}_{\alpha\beta} \mathcal{E}^{\alpha\beta}$, $E_2 = \mathcal{E}_{\alpha\beta} s^\beta \mathcal{E}^\alpha_{\gamma} s^\gamma$, $E_3 = (\mathcal{E}_{\alpha\beta} s^\alpha s^\beta)^2$, and $B_1 = \mathcal{B}_{\alpha\beta} \mathcal{B}^{\alpha\beta}$, $B_2 = \mathcal{B}_{\alpha\beta} s^\beta \mathcal{B}^\alpha_{\gamma} s^\gamma$, $B_3 = (\mathcal{B}_{\alpha\beta} s^\alpha s^\beta)^2$. The leading-order calculations carried out in Sec. IX are not sufficient to determine $\langle \dot{M} \rangle$, but they indicate that $\langle \dot{J} \rangle$ scales as M^5/\mathcal{R}^4 . This result can also be understood in terms of a coupling between the tidal fields and the hole's induced mass and current quadrupole moments; here, as I show in Sec. IX E, the relationship between $M_{\alpha\beta}$ and $\mathcal{E}_{\alpha\beta}$, and the relationship between $J_{\alpha\beta}$ and $\mathcal{B}_{\alpha\beta}$, do not involve a time derivative (as they do in the case of a Schwarzschild black hole). Three illustrative applications of Eq. (1.14) are worked out: In Sec. IX G, I examine a Kerr black hole in circular motion in a slowly-moving binary system; in Sec. IX H, I consider instead the case of a small hole in relativistic circular motion; and in Sec. IX I, the Kerr black hole is placed in a static tidal gravitational field.

The main results of Sec. VIII, Eqs. (1.11) and (1.12), hold to leading order in $M/\mathcal{R} \ll 1$, and they reveal that for a Schwarzschild black hole, $\langle \dot{M} \rangle = O(M^6/\mathcal{R}^6)$ and $\langle \dot{J} \rangle = O(M^6/\mathcal{R}^5)$. On the other hand, the main results

of Sec. IX, Eqs. (1.13) and (1.14), hold to leading order in $M/\mathcal{R} \ll \chi$, and they reveal that for a Kerr black hole, $\langle \dot{M} \rangle = O(M^5/\mathcal{R}^5)$ and $\langle \dot{J} \rangle = O(M^5/\mathcal{R}^4)$. The scalings are thus very different, and the condition $M/\mathcal{R} \ll \chi$ implies that the Schwarzschild results cannot straightforwardly be obtained from the Kerr results in a limit $\chi \rightarrow 0$. These scalings indicate that when a rotating and a non-rotating black hole are placed in identical environments, the rotating hole will absorb larger quantities of energy and angular momentum. The agent responsible for this enhanced absorption is evidently the hole's rotation, and some insight into this matter is offered in Sec. IX F.

This concludes the summary of the work presented in this article.

F. Organization of the paper

The rest of the paper contains the derivations of the results summarized previously, as well as additional results and details. I begin in Sec. II with a description of the event horizon of an unperturbed Kerr black hole, and I derive a number of results that will be used in later sections of the paper. A description of a general evolving horizon is presented in Sec. III. This discussion is specialized, in Sec. IV, to the case of a perturbation of the Kerr horizon, and I derive the flux formulae of Eqs. (4.22)–(4.24).

These formulae are translated into a practical curvature formalism in Sec. V; this was summarized in Sec. I C. In Sec. VI they are translated into a metric formalism that applies to both rotating and nonrotating black holes; in that section I introduce “preferred” and “radiation” gauges for the metric perturbations, and I establish a connection between the fluxes and Isaacson's effective stress-energy tensor for gravitational waves [31, 32]. The metric formalism is specialized to a Schwarzschild black hole in Sec. VII; this was summarized in Sec. I D.

The small-hole/slow-motion approximation is worked out in the last two sections of the paper, first for a Schwarzschild black hole immersed in an external universe (Sec. VIII), and next for a Kerr black hole (Sec. IX). These results were summarized in Sec. I E.

The Appendix contains a brief summary of the theory of metric perturbations of a Schwarzschild spacetime. The material contained in this Appendix is well known, but it is convenient to record the main results there because they are referred to on many occasions in the article's main body.

Throughout this work I use geometrized units in which $c = G = 1$, and I adopt the conventions of Misner, Thorne, and Wheeler [34].

II. KINEMATICS OF THE KERR HORIZON

To prepare the way for the discussion of dynamical event horizons in the next two sections, in this section

I cover the kinematics of a stationary event horizon described by the Kerr metric. I shall introduce a parametric description of the horizon's null generators, and derive from this an intrinsic description of the horizon. Part of this discussion will be devoted to the construction of null tetrads on the horizon, and a description of the (well-known) algebraic structure of the Weyl tensor.

A. Kerr metric

Throughout this work the Kerr metric will be written in terms of ingoing Kerr coordinates (v, r, θ, ψ) , so that its form will be regular on the event horizon. It is given by (see, for example, Box 33.2 of Ref. [34], or Sec. 5.3 of Ref. [30])

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dv^2 + 2 dv dr - \frac{4Mar \sin^2 \theta}{\rho^2} dv d\psi - 2a \sin^2 \theta dr d\psi + \frac{\Sigma \sin^2 \theta}{\rho^2} d\psi^2 + \rho^2 d\theta^2, \quad (2.1)$$

where M is the black-hole mass, $J \equiv Ma$ its angular momentum, $\rho^2 = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$, and $\Sigma = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$. The transformation from the more usual Boyer-Lindquist coordinates $(t_{\text{BL}}, r_{\text{BL}}, \theta_{\text{BL}}, \phi_{\text{BL}})$ is given by $v = t_{\text{BL}} + \int (r^2 + a^2) \Delta^{-1} dr$, $r = r_{\text{BL}}$, $\theta = \theta_{\text{BL}}$, and $\psi = \phi_{\text{BL}} + a \int \Delta^{-1} dr$; the Kerr coordinates are sometimes denoted $(\tilde{V}, r, \theta, \tilde{\phi})$, as is done in Ref. [34]. The event horizon is situated at the largest root of Δ , at $r = r_+ \equiv M + \sqrt{M^2 - a^2}$.

The Kerr spacetime admits the Killing vectors $t^\alpha = \partial x^\alpha / \partial v$ and $\phi^\alpha = \partial x^\alpha / \partial \psi$. The vector

$$k^\alpha = t^\alpha + \Omega_{\text{H}} \phi^\alpha, \quad (2.2)$$

with

$$\Omega_{\text{H}} = \frac{a}{r_+^2 + a^2} = \frac{a}{2Mr_+}, \quad (2.3)$$

is also a Killing vector, and it is null on the event horizon; it is tangent to the horizon's null generators. The quantity Ω_{H} is the angular velocity of the black hole.

B. Parametric description of the horizon

We wish to introduce a system of coordinates (v, θ^A) on the horizon, adopting v as a parameter on the generators, and θ^A ($A = 2, 3$) as generator labels that stay constant as the generators move. Because $k^\alpha = (1, 0, 0, \Omega_{\text{H}})$ in the spacetime coordinates (v, r, θ, ψ) , we have that θ is constant on each generator, and it can therefore be chosen as one of the comoving coordinates. On the other hand, $d\psi/dv = \Omega_{\text{H}}$ and ψ increases linearly as the generators wrap around the event horizon; a suitable choice of comoving coordinate is therefore $\phi = \psi - \Omega_{\text{H}} v$, which stays constant. Our horizon coordinates are therefore

$$(v, \theta^A) = (v, \theta, \phi = \psi - \Omega_{\text{H}} v). \quad (2.4)$$

It is important not to confuse the horizon coordinate ϕ with the Boyer-Lindquist coordinate ϕ_{BL} ; these are not equal.

The horizon generators can now be described by parametric equations of the form $x^\alpha = z^\alpha(v, \theta^A)$, in which z^α gives the spacetime-coordinate positions of the generators in terms of the intrinsic horizon coordinates. Explicitly, the parametric description is $v = v$, $r = r_+$, $\theta = \theta$, and $\psi = \phi + \Omega_{\text{H}}v$. The vectors

$$k^\alpha = \frac{\partial z^\alpha}{\partial v}, \quad e_A^\alpha = \frac{\partial z^\alpha}{\partial \theta^A} \quad (2.5)$$

are tangent to the horizon; k^α is tangent to each generator while e_A^α points in the directions transverse to the generators. In the spacetime coordinates (v, r, θ, ψ) we have $k^\alpha = (1, 0, 0, \Omega_{\text{H}})$ as before, $e_\theta^\alpha = (0, 0, 1, 0)$, and $e_\phi^\alpha = (0, 0, 0, 1)$. Because the coordinates θ^A are comoving, the transverse vectors e_A^α are Lie transported along the generators, and they therefore satisfy

$$e_{A;\beta}^\alpha k^\beta = k^\alpha e_A^\beta. \quad (2.6)$$

They are also Lie transported along one another, so that $e_{A;\beta}^\alpha e_B^\beta = e_{B;\beta}^\alpha e_A^\beta$.

The basis vectors also satisfy

$$k_\alpha k^\alpha = 0 = k_\alpha e_A^\alpha. \quad (2.7)$$

The only nonvanishing inner products are

$$\gamma_{AB} = g_{\alpha\beta} e_A^\alpha e_B^\beta, \quad (2.8)$$

and these form the components of the induced metric on the horizon. To see this, deduce from Eq. (2.5) that a displacement on the horizon is described by $dx^\alpha = k^\alpha dv + e_A^\alpha d\theta^A$ and calculate $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ for this displacement; use of Eqs. (2.7) and (2.8) returns $ds^2 = \gamma_{AB} d\theta^A d\theta^B$, with the interpretation that γ_{AB} is indeed the induced metric. Notice that the horizon metric is degenerate, and explicitly two-dimensional in the comoving coordinates. The nonvanishing components of the horizon metric are $\gamma_{\theta\theta} = r_+^2 + a^2 \cos^2 \theta$, $\gamma_{\phi\phi} = (r_+^2 + a^2)^2 \sin^2 \theta / (r_+^2 + a^2 \cos^2 \theta)$, and

$$\sqrt{\gamma} = (r_+^2 + a^2) \sin \theta \quad (2.9)$$

is the square root of the metric determinant.

The vector basis on the horizon can be completed with another null vector N^α that satisfies

$$N_\alpha N^\alpha = 0 = N_\alpha e_A^\alpha, \quad N_\alpha k^\alpha = -1. \quad (2.10)$$

These conditions determine the vector uniquely, and we find

$$N_\alpha dx^\alpha = -dv + \frac{a^2 \sin^2 \theta}{2(r_+^2 + a^2)} dr. \quad (2.11)$$

The four basis vectors give us completeness relations for the inverse metric evaluated on the event horizon,

$$g^{\alpha\beta} = -k^\alpha N^\beta - N^\alpha k^\beta + \gamma^{AB} e_A^\alpha e_B^\beta, \quad (2.12)$$

where γ^{AB} is the inverse of γ_{AB} . In the sequel we will use the horizon metric and its inverse to lower and raise upper-case Latin indices. We will also introduce a two-dimensional connection Γ_{BC}^A compatible with γ_{AB} , and denote covariant differentiation in this connection with a vertical stroke; for example, $\gamma_{AB|C} \equiv 0$.

C. Horizon connections

The tangential derivatives of the basis vectors are given by

$$k^\alpha{}_{;\beta} k^\beta = \kappa k^\alpha, \quad (2.13)$$

$$k^\alpha{}_{;\beta} e_A^\beta = \omega_A k^\alpha = e_{A;\beta}^\alpha k^\beta, \quad (2.14)$$

$$e_{A;\beta}^\alpha e_B^\beta = p_{AB} k^\alpha + \Gamma_{AB}^C e_C^\alpha = e_{B;\beta}^\alpha e_A^\beta, \quad (2.15)$$

where κ , ω_A , p_{AB} , and Γ_{AB}^C are the horizon connections. The surface gravity $\kappa = -N_\alpha k^\alpha{}_{;\beta} k^\beta$ of a Kerr black hole is given by

$$\kappa = \frac{r_+ - M}{r_+^2 + a^2} = \frac{\sqrt{M^2 - a^2}}{2Mr_+}, \quad (2.16)$$

and Eq. (2.13) states that the vector k^α satisfies the geodesic equation, but that the generator parameter v is not an affine parameter. Explicit expressions for ω_A and p_{AB} will not be needed; the two-dimensional connection Γ_{AB}^C can easily be computed from γ_{AB} .

The 2-vector $\phi^A = (0, 1)$ is a Killing vector of the horizon's intrinsic geometry, and it therefore satisfies Killing's equation, $\phi_{(A|B)} = 0$. This vector is related to the spacetime Killing vector ϕ^α by the relation $\phi^\alpha = \phi^A e_A^\alpha$. The 2-tensor

$$c_{AB} \equiv -\phi_{A|B} = -\phi_{\alpha;\beta} e_A^\alpha e_B^\beta \quad (2.17)$$

will be needed in Sec. VI of the paper. This tensor is anti-symmetric by virtue of Killing's equation; its only nonvanishing components are $c_{\theta\phi} = (r_+^2 + a^2)^3 \sin \theta \cos \theta / (r_+^2 + a^2 \cos^2 \theta)^2 = -c_{\phi\theta}$.

The vector k^α introduced in Eq. (2.5) is defined on the horizon only, but Eq. (2.2) provides an extension away from the horizon. The extended vector field is null on the horizon only, but it is everywhere a Killing vector; it satisfies

$$k_{\alpha;\beta} = -\kappa(k_\alpha N_\beta - N_\alpha k_\beta) + k_\alpha \omega_\beta - \omega_\alpha k_\beta \quad (2.18)$$

on the horizon, where $\omega^\alpha \equiv \omega^A e_A^\alpha$.

D. Null tetrads

The transverse vectors e_A^α can be combined into complex vectors $e^\alpha = e^A e_A^\alpha$ that satisfy

$$e_\alpha e^\alpha = 0 = \bar{e}_\alpha \bar{e}^\alpha, \quad e_\alpha \bar{e}^\alpha = 1, \quad (2.19)$$

with an overbar indicating complex conjugation. In terms of the complex coefficients e^A , these relations read

$$\gamma_{AB}e^Ae^B = 0 = \gamma_{AB}\bar{e}^A\bar{e}^B, \quad \gamma_{AB}e^A\bar{e}^B = 1, \quad (2.20)$$

and these produce the completeness relations $\gamma^{AB} = e^A\bar{e}^B + \bar{e}^Ae^B$. Substituting this into Eq. (2.12) yields

$$g^{\alpha\beta} = -k^\alpha N^\beta - N^\alpha k^\beta + e^\alpha \bar{e}^\beta + \bar{e}^\alpha e^\beta. \quad (2.21)$$

A particular (and traditional) choice of coefficients e^A that achieves these properties is

$$e^\theta = \frac{1}{\sqrt{2}} \frac{r_+ - ia \cos \theta}{r_+^2 + a^2 \cos^2 \theta}, \quad e^\phi = \frac{i}{\sqrt{2}} \frac{r_+ - ia \cos \theta}{(r_+^2 + a^2) \sin \theta}. \quad (2.22)$$

Note that the inversion formula is $e_A^\alpha = \bar{e}_A e^\alpha + e_A \bar{e}^\alpha$, where the upper-case Latin index was lowered with the horizon metric γ_{AB} .

The basis $(k^\alpha, N^\alpha, e^\alpha, \bar{e}^\alpha)$ is a null tetrad on the horizon, and it can be used to decompose various tensors, as is customary in the Newman-Penrose formalism (see, for example, the presentation of Ref. [35]). This tetrad, however, is not adapted to the algebraic structure of $C_{\alpha\gamma\beta\delta}$, the Weyl tensor of the Kerr spacetime. With this tetrad we would find that the Weyl scalars ψ_0 and ψ_1 vanish, but that ψ_3 and ψ_4 do not. (These quantities will be introduced below.) This is remedied by a null rotation (a rotation of class I in the language of Chandrasekhar [35]) to a new tetrad $(k^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ given by

$$n^\alpha = N^\alpha + |A|^2 k^\alpha + \bar{A} e^\alpha + A \bar{e}^\alpha, \quad m^\alpha = e^\alpha + A k^\alpha, \quad (2.23)$$

where

$$A = \frac{i}{\sqrt{2}} a \sin \theta \frac{r_+ - ia \cos \theta}{r_+^2 + a^2 \cos^2 \theta}. \quad (2.24)$$

In the spacetime coordinates (v, r, θ, ψ) , the components of the new transverse vectors are

$$m^\alpha = \frac{1}{\sqrt{2}} \frac{r_+ - ia \cos \theta}{r_+^2 + a^2 \cos^2 \theta} \left(ia \sin \theta, 0, 1, \frac{i}{\sin \theta} \right). \quad (2.25)$$

Notice that this rotation leaves the vector k^α unchanged.

The tetrad $(k^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ introduced here is the *Hartle-Hawking null tetrad* of the Kerr spacetime [26] (as it was defined by Teukolsky [2] and Teukolsky and Press [1]) restricted to the event horizon. The Hartle-Hawking tetrad is related to the more standard *Kinnersley null tetrad* [36] by a rescaling of the vectors k^α and n^α (a rotation of class III in the language of Chandrasekhar [35]):

$$k^\alpha(\text{HH}) = \frac{\Delta}{2(r^2 + a^2)} k^\alpha(\text{K}) \quad (2.26)$$

and

$$n^\alpha(\text{HH}) = \frac{2(r^2 + a^2)}{\Delta} n^\alpha(\text{K}). \quad (2.27)$$

Notice that while the Hartle-Hawking tetrad is defined globally in the Kerr spacetime, the tetrad $(k^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ is defined on the horizon only; the extension of k^α to $k^\alpha(\text{HH})$ is different from the extension of Eq. (2.2). Notice also that whereas the Hartle-Hawking tetrad is well behaved on the event horizon, the Kinnersley tetrad is not.

In the tetrad $(k^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ the horizon Weyl scalars are defined by (see, for example, Ref. [35])

$$\psi_0 = -C_{\alpha\gamma\beta\delta} k^\alpha m^\gamma k^\beta m^\delta, \quad (2.28)$$

$$\psi_1 = -C_{\alpha\gamma\beta\delta} k^\alpha n^\gamma k^\beta m^\delta, \quad (2.29)$$

$$\psi_2 = -C_{\alpha\gamma\beta\delta} k^\alpha m^\gamma \bar{m}^\beta n^\delta, \quad (2.30)$$

$$\psi_3 = -C_{\alpha\gamma\beta\delta} k^\alpha n^\gamma \bar{m}^\beta n^\delta, \quad (2.31)$$

$$\psi_4 = -C_{\alpha\gamma\beta\delta} n^\alpha \bar{m}^\gamma n^\beta \bar{m}^\delta, \quad (2.32)$$

where $C_{\alpha\gamma\beta\delta}$ is the Weyl tensor of the Kerr spacetime evaluated on the event horizon. We have that $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$, a property that is true globally with the Hartle-Hawking or Kinnersley tetrads, and

$$\psi_2 = \frac{Mr_+(r_+^2 - 3a^2 \cos^2 \theta)}{(r_+^2 + a^2 \cos^2 \theta)^3} + \frac{iMa \cos \theta (3r_+^2 - a^2 \cos^2 \theta)}{(r_+^2 + a^2 \cos^2 \theta)^3} \quad (2.33)$$

is the only nonvanishing horizon Weyl scalar.

E. Weyl identities

For future reference I record here a number of identities satisfied by the Weyl tensor of the Kerr spacetime:

$$C_{\alpha\gamma\beta\delta} k^\gamma e_A^\beta e_A^\delta = 0, \quad (2.34)$$

$$C_{\alpha\gamma\beta\delta} (e_A^\alpha e_B^\beta + e_B^\alpha e_A^\beta) k^\gamma = -2\text{Re}(\psi_2) \gamma_{AB} k_\delta, \quad (2.35)$$

$$C_{\alpha\gamma\beta\delta} k^\gamma k^\delta = -2\text{Re}(\psi_2) k_\alpha k_\beta. \quad (2.36)$$

As a consequence of Eqs. (2.34), (2.35), and (2.23) we also obtain

$$C_{\alpha\gamma\beta\delta} k^\gamma m^\beta k^\delta = 0 = C_{\alpha\gamma\beta\delta} m^\alpha m^\beta k^\gamma. \quad (2.37)$$

These identities are formulated on the horizon only.

III. DYNAMICS OF AN EVOLVING HORIZON

In this section I generalize the preceding discussion to a nonstationary event horizon. The presentation is patterned after Price and Thorne [27] and Chapter VI of the book by Thorne, Price, and Macdonald [28].

A. Comoving coordinates and vector basis

As in Sec. II we take (v, θ^A) as our system of intrinsic coordinates on the horizon, with v now promoted to an

arbitrary parameter on the null generators, and θ^A still denoting constant generator labels. The horizon is still described by the parametric equations $x^\alpha = z^\alpha(v, \theta^A)$, but the coordinate positions of the dynamical horizon may be displaced with respect to those of a stationary Kerr horizon.

The vectors $k^\alpha = \partial z^\alpha / \partial v$ and $e_A^\alpha = \partial z^\alpha / \partial \theta^A$ form a partial basis on the horizon; as before k^α is tangent to the generators, e_A^α is transverse to them, and $k_\alpha k^\alpha = k_\alpha e_A^\alpha = 0$. The nonvanishing inner products

$$\gamma_{AB}(v, \theta^A) = g_{\alpha\beta} e_A^\alpha e_B^\beta \quad (3.1)$$

give the components of the induced metric on the horizon. The basis is completed by another null vector N^α , which is orthogonal to e_A^α and normalized by the condition $N_\alpha k^\alpha = -1$. The completeness relations of Eq. (2.12) still apply.

The vectors k^α and e_A^α are all Lie transported along one another, so that $e_{A;\beta}^\alpha k^\beta = k^\alpha{}_{;\beta} e_A^\beta$ and $e_{A;\beta}^\alpha e_B^\beta = e_{B;\beta}^\alpha e_A^\beta$.

B. Generator kinematics

The tangent vector k^α satisfies the geodesic equation in its generalized form

$$k^\alpha{}_{;\beta} k^\beta = \kappa k^\alpha, \quad (3.2)$$

where κ is the evolving surface gravity of the event horizon, defined with respect to our choice of parameterization v .

The transverse derivatives of the tangent vector can be decomposed as

$$k^\alpha{}_{;\beta} e_A^\beta = \omega_A k^\alpha + B_A^B e_B^\alpha = e_{A;\beta}^\alpha k^\beta, \quad (3.3)$$

for some 2-vector $\omega_A(v, \theta^A)$ and 2-tensor $B_{AB}(v, \theta^A)$; this generalizes Eq. (2.14). It is easy to show that the right-hand side of Eq. (3.3) cannot include a term proportional to N^α ; this follows from the fact that k^α is a null vector field. And it can be established that B_{AB} is a symmetric tensor, because the congruence of null geodesics to which k^α is tangent is necessarily hypersurface orthogonal (see, for example, Sec. 9.2 of Ref. [29], or Sec. 2.4 of Ref. [30]). The tensor $B_{AB} = k_{\alpha;\beta} e_A^\alpha e_B^\beta$ can be decomposed into its irreducible parts,

$$B_{AB} = \frac{1}{2} \Theta \gamma_{AB} + \sigma_{AB}, \quad (3.4)$$

thereby defining the *expansion scalar* $\Theta = \gamma^{AB} B_{AB}$ and the *shear tensor* $\sigma_{AB} = B_{AB} - \frac{1}{2} \Theta \gamma_{AB}$. Notice that the expansion is the trace of B_{AB} , while the shear is the tracefree part of this tensor.

C. Generator dynamics

Evolution equations can easily be derived for γ_{AB} , Θ , and σ_{AB} , and these will form the basis of the discussion of perturbed event horizons in the next section.

Starting with the identity $\partial \gamma_{AB} / \partial v = (g_{\alpha\beta} e_A^\alpha e_B^\alpha)_{;\gamma} k^\gamma$ and using Eqs. (3.3) and (3.4), we quickly arrive at an evolution equation for the horizon metric,

$$\frac{\partial \gamma_{AB}}{\partial v} = \Theta \gamma_{AB} + 2\sigma_{AB}. \quad (3.5)$$

From this it follows that $\partial \gamma^{AB} / \partial v = -\Theta \gamma^{AB} - 2\sigma^{AB}$ and

$$\Theta = \frac{1}{\sqrt{\gamma}} \frac{\partial \sqrt{\gamma}}{\partial v}, \quad (3.6)$$

where γ is the metric determinant.

To derive evolution equations for the expansion and shear we follow the usual route that leads to Raychaudhuri's equation (see, for example, Sec. 9.2 of Ref. [29], or Sec. 2.4 of Ref. [30]). Starting with the identity $\partial B_{AB} / \partial v = (k_{\alpha;\beta} e_A^\alpha e_B^\alpha)_{;\gamma} k^\gamma$ and using Eqs. (3.2), (3.3), as well as Ricci's identity, we arrive at $\partial B_{AB} / \partial v = \kappa B_{AB} + B_A^C B_{CB} - R_{\alpha\gamma\beta\delta} e_A^\alpha k^\gamma e_B^\beta k^\delta$. Taking the trace of this equation, using the fact that a symmetric-tracefree tensor automatically satisfies $\sigma^A_C \sigma^C_B = \frac{1}{2} (\sigma_{CD} \sigma^{CD}) \delta^A_B$, produces Raychaudhuri's equation,

$$\frac{\partial \Theta}{\partial v} = \kappa \Theta - \frac{1}{2} \Theta^2 - \sigma_{AB} \sigma^{AB} - 8\pi \rho, \quad (3.7)$$

where $\rho \equiv (R_{\alpha\beta} / 8\pi) k^\alpha k^\beta = T_{\alpha\beta} k^\alpha k^\beta$ after using the Einstein field equations. The tracefree part of the equation reduces to

$$\frac{\partial \sigma^A_B}{\partial v} = (\kappa - \Theta) \sigma^A_B - C^A_B, \quad (3.8)$$

where

$$C_{AB} \equiv C_{\alpha\gamma\beta\delta} e_A^\alpha k^\gamma e_B^\beta k^\delta \quad (3.9)$$

are tangential components of the Weyl tensor. In these equations all upper-case Latin indices are lowered and raised with γ_{AB} and γ^{AB} , respectively; the horizon metric evolves according to Eq. (3.5).

The area of any cross section $v = \text{constant}$ of the event horizon is given by $A(v) = \oint \sqrt{\gamma} d^2\theta$. Assuming that the number of generators stays constant as the horizon evolves (that is, assuming that no new generator joins the horizon at a caustic), a change of area occurs when γ , the metric determinant, varies with time. Equation (3.6) yields

$$\frac{dA}{dv} = \oint \Theta dS, \quad (3.10)$$

where $dS = \sqrt{\gamma} d^2\theta$ is an element of surface area on the horizon cross sections.

The equations derived in this section are all exact, and they apply to an event horizon that evolves dynamically. [The assumption made in the derivation of Eq. (3.10), that the number of generators must be conserved during the horizon's evolution, represents a serious restriction. In the perturbative context to be described in the next

paragraph, however, this limitation is lifted because the formation of a caustic is necessarily associated with a large, nonperturbative value of Θ .] The choice of parameter v and generator labels θ^A is completely arbitrary, and the quantities κ , γ_{AB} , Θ , and σ_{AB} all refer to this choice. For a stationary Kerr black hole, v is chosen so that k^α is given by Eq. (2.2), and the generator labels of Eq. (2.4) are adopted. This means that κ is given by Eq. (2.16), γ_{AB} by the expressions listed near Eq. (2.9), and that $\Theta = 0 = \sigma_{AB}$, as can be seen by comparing Eq. (2.14) with Eq. (3.3). Equations (3.7) and (3.8) are then consistent if $\rho = 0$, which follows because the Kerr metric is a vacuum solution to the Einstein field equations, and $C_{AB} = 0$, which follows from Eq. (2.35).

In the next section we will apply these equations to situations in which the horizon is very close to being stationary, so that it can be described as a slightly perturbed version of the Kerr horizon. The horizon coordinates (v, θ^A) will then be chosen to be “close to” the Kerr coordinates, and we will see that the ambiguities associated with this choice never explicitly enter the discussion; our final expressions will be gauge invariant. This implies that the vectors k^α , N^α , e_A^α on the perturbed horizon will be perturbed versions of the Kerr basis, and that all derived quantities, such as κ , γ_{AB} , Θ , and σ_{AB} , will be perturbed versions of the corresponding Kerr quantities.

IV. PERTURBED HORIZON

I now specialize the formalism of the preceding section to event horizons that are slightly nonstationary, those that can be considered to be perturbed versions of the Kerr horizon. I shall assume that the perturbation is caused entirely by gravitational radiation, and no matter will be allowed to cross the event horizon. The perturbation formalism described here is adapted from Price and Thorne [27] and Chapter VI of the book by Thorne, Price, and Macdonald [28]; these methods go back to the pioneering work of Hawking and Hartle [26].

A. Perturbation equations

Changing our notation with respect to the previous section, the perturbed values for the horizon metric, surface gravity, expansion, shear, and Weyl tensor will now be denoted $\hat{\gamma}_{AB}$, $\hat{\kappa}$, $\hat{\Theta}$, $\hat{\sigma}_{AB}$, and \hat{C}_{AB} , respectively; these quantities were all introduced in Sec. III. The unperturbed (Kerr) values will be denoted without a decorating caret; for example $\Theta = 0$ is the background expansion scalar, $\sigma_{AB} = 0$ the background shear tensor, and $C_{AB} = 0$ the background Weyl tensor. The only nonvanishing background quantities are the metric γ_{AB} and surface gravity κ ; these were introduced in Sec. II.

The horizon perturbation is driven by the Weyl tensor \hat{C}_{AB} , which we imagine to be a quantity of the first order

in an expansion parameter λ ; we write this as

$$\hat{C}_B^A = \lambda C_1^A{}_B + O(\lambda^2). \quad (4.1)$$

At the end of the calculation we will set $\lambda \equiv 1$ by absorbing it into the definition of the perturbations. Equation (3.8) indicates that the Weyl tensor drives a first-order perturbation in the shear, and we have

$$\hat{\sigma}_B^A = \lambda \sigma_1^A{}_B + O(\lambda^2). \quad (4.2)$$

Equation (3.7), on the other hand, shows that it is the square of the shear tensor that is driving a perturbation in the expansion (recall that we have set $\rho = 0$), and we must therefore have

$$\hat{\Theta} = \lambda^2 \Theta_2 + O(\lambda^3). \quad (4.3)$$

(A more careful treatment that incorporates a first-order term would eventually lead to $\Theta_1 = 0$ and therefore back to this assertion.) Finally, Eq. (3.5) shows that the shear produces a first-order perturbation in the metric,

$$\hat{\gamma}_{AB} = \gamma_{AB} + \lambda \gamma_{AB}^1 + O(\lambda^2), \quad (4.4)$$

and these considerations lead to the statement that the perturbed surface gravity will differ from its Kerr value by a first-order quantity: $\hat{\kappa} = \kappa + O(\lambda)$.

Substituting the expansions of Eqs. (4.1)–(4.4) into Eqs. (3.5), (3.7), and (3.8) gives us the following set of perturbation equations:

$$\frac{\partial \gamma_{AB}^1}{\partial v} = 2\sigma_{AB}^1, \quad (4.5)$$

$$\frac{\partial \Theta_2}{\partial v} = \kappa \Theta_2 - \sigma_{AB}^1 \sigma_1^{AB}, \quad (4.6)$$

$$\frac{\partial \sigma_{AB}^1}{\partial v} = \kappa \sigma_{AB}^1 - C_{AB}^1. \quad (4.7)$$

In these equations, κ is the surface gravity of the Kerr black hole, and the generator parameter v can be identified with the advanced-time coordinate of the Kerr space-time (as introduced in Sec. II); the first-order deviations of these quantities with respect to the Kerr values do not enter the perturbation equations. It should be noted that upper-case Latin indices can now be manipulated with γ_{AB} , the background horizon metric, and its inverse.

According to Eq. (3.10), a growth in the horizon area is driven by the expansion scalar, and we therefore have $\dot{A} = \lambda^2 \dot{A}_2 + O(\lambda^3)$, with

$$\dot{A}_2 \equiv \frac{dA_2}{dv} = \oint \Theta_2 dS, \quad (4.8)$$

where $dS \equiv \sqrt{\gamma} d^2\theta$ is an element of surface area on the cross sections of the *unperturbed* (Kerr) horizon.

B. Integration of the perturbation equations

We imagine a horizon that starts in an initial Kerr state, is perturbed for some time by an external process,

and then eventually returns to another Kerr state. The expansion and shear vanish in the initial state, and they must return to zero after the external process has ended; this requires the imposition of *teleological boundary conditions* (as opposed to retarded boundary conditions; see, for example, Sec. VI C 6 of Ref. [28]) on the solutions to the perturbation equations.

The teleological solution to the equation $(d/dv - \kappa)\psi = -f(v)$ is $\psi(v) = \int_v^\infty e^{-\kappa(v'-v)} f(v') dv'$. It shows that $\psi(v)$ depends on the future behavior of the driving force, but that $\psi(v)$ goes to zero after the driving force is switched off; the causal solution would depend only on the past behavior of the driving force, but it would grow exponentially after the force is switched off. If the driving force varies very slowly over a time comparable to $1/\kappa$, then the teleological solution reduces to the local expression $\psi(v) = \kappa^{-1} f(v)$, to a fractional accuracy of order $(\kappa\tau)^{-1}$, where $\tau \sim f/\dot{f}$ is the time scale over which the driving force varies. The local expression can be simply obtained by noting that in this limit, $d/dv \ll \kappa$ and the differential term can be neglected in the differential equation; a more careful derivation starts with the teleological solution and employs integration by parts.

It is not permissible to neglect the differential term in Eq. (4.7), and one must write down a proper teleological solution to this equation. To see this, suppose that the black hole is a member of a binary system, and that it moves in the field of an external body with an angular velocity Ω . As seen in the rotating frame of the generators, the Weyl tensor behaves as $C \sim e^{-i\omega v}$, where $\omega \equiv \Omega - \Omega_H$ is the *relative* angular velocity between the external field and the generators. Thus, unless the external field is nearly corotating with the black hole, ω will be of order Ω_H , which is itself of order κ , and the Weyl tensor will not vary slowly.

The exact solution to Eq. (4.7) is

$$\sigma_{AB}^1(v, \theta^A) = \int_v^\infty e^{-\kappa(v'-v)} C_{AB}^1(v', \theta^A) dv', \quad (4.9)$$

and this can be substituted into the right-hand side of Eq. (4.6). Here we shall allow ourselves some simplification. For a Weyl tensor that behaves schematically as $C \sim e^{-i\omega v}$, the shear tensor will go as $\sigma \sim e^{-i\omega v}/(\kappa + i\omega)$, and it will vary as rapidly as C . The *square* of the shear tensor, however, will contain a piece that oscillates at twice the frequency ω , and a piece that stays constant. The force driving the expansion therefore contains both a slowly-varying piece and a rapidly-varying piece. In a generic situation we expect that $\sigma_{AB}^1 \sigma_1^{AB}$ can always be decomposed into such slowly-varying and rapidly-varying pieces, and we isolate the slowly-varying component by averaging over a time scale that is long compared with κ^{-1} : $(\sigma_{AB}^1 \sigma_1^{AB})_{\text{slow}} = \langle \sigma_{AB}^1 \sigma_1^{AB} \rangle$. If we now agree to follow only the slow evolution of the expansion scalar, and to ignore its rapid fluctuations around the slowly-evolving mean, then

$$\langle \Theta_2 \rangle = \frac{1}{\kappa} \langle \sigma_{AB}^1 \sigma_1^{AB} \rangle \quad (4.10)$$

is an adequate approximate solution to Eq. (4.6). The meaning of the averaging sign should be clear: The expansion scalar, and the square of the shear tensor, are averaged over a time τ that is long compared with κ^{-1} , the black-hole time scale. The time scale τ is identified with a characteristic time associated with the growth of the black-hole area, $\tau \sim \langle A \rangle / \langle \dot{A} \rangle$. We note that it is a requirement of the perturbative treatment that $\kappa\tau \gg 1$; the simplification of Eq. (4.10) therefore represents no significant loss of generality, other than a coarse-graining over short time scales.

Equation (4.9) can also be inserted into Eq. (4.5) to calculate the metric perturbation, which is given by

$$\gamma_{AB}^1(v, \theta^A) = 2 \int_{-\infty}^v \sigma_{AB}^1(v', \theta^A) dv'. \quad (4.11)$$

After altering the order of the integrations and performing one of the integrals, we obtain

$$\gamma_{AB}^1(v, \theta^A) = \frac{2}{\kappa} \left[\int_{-\infty}^v C_{AB}^1(v', \theta^A) dv' + \int_v^\infty e^{-\kappa(v'-v)} C_{AB}^1(v', \theta^A) dv' \right]; \quad (4.12)$$

this result is exact, and it does not involve a coarse-graining over short time scales.

The averaged rate of change of the horizon area can be calculated on the basis of Eqs. (4.8) and (4.10). The result is

$$\langle \dot{A}_2 \rangle = \frac{1}{\kappa} \oint \langle \sigma_{AB}^1 \sigma_1^{AB} \rangle dS, \quad (4.13)$$

where an overdot indicates differentiation with respect to v . This result can be expressed in terms of the Weyl tensor by means of Eq. (4.9).

In the remainder of the paper we will denote C_{AB}^1 simply as C_{AB} , σ_{AB}^1 as σ_{AB} , Θ_2 as Θ , and \dot{A}_2 as \dot{A} ; because these quantities all vanish for a stationary horizon, this change of notation will not generate ambiguities. But to avoid ambiguities we will continue to denote the metric perturbation as γ_{AB}^1 .

C. Fluxes of mass and angular momentum

We shall now derive expressions for the averaged rates of change of the black-hole mass M and angular momentum J , using Eq. (4.13) as our main input.

In the case of a horizon perturbed by a matter field it can be shown (see, for example, Sec. 6.4.2 of Ref. [37]) that these rates are related by $(\kappa/8\pi)\dot{A} = \dot{M} - \Omega_H \dot{J}$, which is a statement of the first law of black-hole mechanics (see, for example, Chapter 12 of Ref. [29], or Chapter 5 of Ref. [30]). In the present case of a horizon perturbed by a purely gravitational perturbation, we shall assume that this relation holds on the average, so that

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \langle \dot{M} \rangle - \Omega_H \langle \dot{J} \rangle. \quad (4.14)$$

The averaging introduced here is the same that was involved in Eq. (4.10). If we imagine that the horizon evolves from an initial Kerr state to a final Kerr state in a time $\Delta v \sim \tau$, then a precise statement of Eq. (4.14) is $(\kappa/8\pi)(\delta A)/(\Delta v) = (\delta M)/(\Delta v) - \Omega_H(\delta J)/(\Delta v)$, where δA , δM , and δJ are the total accumulated changes in the black-hole parameters. Because these changes relate two stationary black-hole states, we have here the usual statement of the first law divided by the time interval Δv .

In the case of a horizon perturbed by a matter field it can also be shown (see, for example, Sec. 6.4.2 of Ref. [37]) that if the matter field is decomposed into modes proportional to $e^{-i\omega v} e^{im\psi}$, where ω is a continuous frequency and m an integer, then each mode contribution to the averaged rates is such that

$$\langle \dot{M} \rangle_{m,\omega} = \frac{\omega}{m} \langle \dot{J} \rangle_{m,\omega}. \quad (4.15)$$

This mode decomposition is motivated by the symmetries of the background Kerr spacetime, and Eq. (4.15) states that a mode labeled by (m, ω) carries across the horizon a quantity of energy proportional to ω and a quantity of angular momentum proportional to m . This statement is easily understood on the basis of a quantum picture, but it holds for classical matter fields as well. We shall assume that Eq. (4.15) is not restricted to matter fields, but that it holds also for gravitational perturbations. Such an assumption was made previously by Teukolsky and Press [1] in their pioneering study of horizon fluxes.

Equations (4.14) and (4.15) imply

$$\langle \dot{M} \rangle_{m,\omega} = \frac{\omega}{k} \frac{\kappa}{8\pi} \langle \dot{A} \rangle_{m,\omega}, \quad (4.16)$$

$$\langle \dot{J} \rangle_{m,\omega} = \frac{m}{k} \frac{\kappa}{8\pi} \langle \dot{A} \rangle_{m,\omega}, \quad (4.17)$$

where

$$k \equiv \omega - m\Omega_H \quad (4.18)$$

and $\langle \dot{A} \rangle_{m,\omega}$ is the mode contribution to the averaged rate of change of the horizon area. These equations will allow us to turn Eq. (4.13) into useful expressions for $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$.

In the spacetime coordinates (v, r, θ, ψ) , the mode decomposition of the metric perturbation is given by

$$\gamma_{AB}^1 = \sum_m \int d\omega \gamma_{AB}^{m,\omega}(r, \theta) e^{-i\omega v} e^{im\psi}. \quad (4.19)$$

In the horizon coordinates (v, θ, ϕ) , where $\phi = \psi - \Omega_H v$, we have instead

$$\gamma_{AB}^1 = \sum_m \int d\omega \gamma_{AB}^{m,\omega}(r_+, \theta) e^{-ikv} e^{im\phi}, \quad (4.20)$$

where we have set $r = r_+$, choosing the coordinate position of the perturbed horizon to coincide with the position of the Kerr horizon. (That this choice can always

be made is proved in Sec. VI A.) Substituting Eq. (4.20) into Eq. (4.5) we obtain

$$\sigma_{AB} = \frac{1}{2} \sum_m \int d\omega (-ik) \gamma_{AB}^{m,\omega} e^{-ikv} e^{im\phi}, \quad (4.21)$$

the mode decomposition of the shear tensor.

We now insert Eq. (4.21) into Eq. (4.13), but we do not decompose σ^{AB} into modes. This gives

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \sum_m \int d\omega \frac{-ik}{16\pi} \oint \langle \sigma^{AB} \gamma_{AB}^{m,\omega} e^{-ikv} e^{im\phi} \rangle dS,$$

and from this we can read off each mode contribution to $(\kappa/8\pi)\langle \dot{A} \rangle$. According to Eqs. (4.16) and (4.17), then, we have

$$\langle \dot{M} \rangle = \sum_m \int d\omega \frac{-i\omega}{16\pi} \oint \langle \sigma^{AB} \gamma_{AB}^{m,\omega} e^{-ikv} e^{im\phi} \rangle dS$$

and

$$\langle \dot{J} \rangle = \sum_m \int d\omega \frac{-im}{16\pi} \oint \langle \sigma^{AB} \gamma_{AB}^{m,\omega} e^{-ikv} e^{im\phi} \rangle dS.$$

The metric perturbation can now be reconstructed from its mode decomposition. It is easy to show that the factor of $-i\omega$ is generated by applying the differential operator $\partial/\partial v - \Omega_H \partial/\partial \phi$ to $\gamma_{AB}^1(v, \theta, \phi)$; this can be written as a Lie derivative in the direction of $t^\alpha = k^\alpha - \Omega_H \phi^\alpha$, the timelike Killing vector of the background Kerr spacetime. Similarly, the factor of im is generated by acting with $\partial/\partial \phi$, which is a Lie derivative in the direction of ϕ^α , the rotational Killing vector of the background Kerr spacetime.

The final expressions are

$$\langle \dot{M} \rangle = \frac{1}{16\pi} \oint \langle \sigma^{AB} \mathcal{L}_t \gamma_{AB}^1 \rangle dS, \quad (4.22)$$

$$\langle \dot{J} \rangle = -\frac{1}{16\pi} \oint \langle \sigma^{AB} \mathcal{L}_\phi \gamma_{AB}^1 \rangle dS, \quad (4.23)$$

and

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \frac{1}{16\pi} \oint \langle \sigma^{AB} \mathcal{L}_k \gamma_{AB}^1 \rangle dS, \quad (4.24)$$

where we have used $\sigma_{AB} = \frac{1}{2} \mathcal{L}_k \gamma_{AB}^1$. As we have seen, the Lie derivatives acting on the metric perturbations refer to specific directions in the background Kerr spacetime. In the horizon coordinates (v, θ, ϕ) , the Lie derivatives take the explicit form

$$\mathcal{L}_k = \left(\frac{\partial}{\partial v} \right)_{\theta,\phi}, \quad \mathcal{L}_\phi = \left(\frac{\partial}{\partial \phi} \right)_{v,\theta}, \quad (4.25)$$

and $\mathcal{L}_t = \mathcal{L}_k - \Omega_H \mathcal{L}_\phi$. On the other hand, in the spacetime coordinates (v, r, θ, ψ) they take the form

$$\mathcal{L}_t = \left(\frac{\partial}{\partial v} \right)_{r,\theta,\psi}, \quad \mathcal{L}_\phi = \left(\frac{\partial}{\partial \psi} \right)_{v,r,\theta}, \quad (4.26)$$

and $\mathcal{L}_k = \mathcal{L}_t + \Omega_H \mathcal{L}_\phi$. We also note that in Eqs. (4.22)–(4.24), the surface element $dS = \sqrt{\gamma} d^2\theta$ refers to the metric γ_{AB} of the unperturbed Kerr horizon.

Equations (4.22)–(4.24) are an excellent starting point for the development of a practical formalism to calculate the horizon fluxes, a topic we shall turn to in the next three sections. These equations are not new: they appear in Sec. VI C 11 of the book by Thorne, Price, and Macdonald [28]. The derivation presented here, however, is substantially different from theirs, and it incorporates the Teukolsky-Press assumption of Eq. (4.15); this assumption was not part of the original derivation, and their route from Eq. (4.13) to Eqs. (4.22), (4.23) is not as direct. It should be clear that while the present derivation relies on a mode decomposition of the metric perturbation, the final expressions involve differential (as opposed to algebraic) operations and are independent of the decomposition.

D. Rigid rotation

The perturbed black hole is part of a system in rigid rotation when the vector

$$\xi^\alpha = t^\alpha + \Omega \phi^\alpha \quad (4.27)$$

is a Killing vector of both the background Kerr spacetime and the perturbed spacetime; here Ω is a constant angular velocity. An example of a system in rigid rotation is when the black hole is a member of a binary system, moving with a uniform angular velocity in the field of the external body. The fact that ξ^α is a Killing vector of the perturbed spacetime means that

$$\mathcal{L}_\xi \gamma_{AB}^1 = 0. \quad (4.28)$$

If the metric perturbation is expressed in the spacetime coordinates (v, r, θ, ψ) , then Eq. (4.28) implies that its dependence on v and ψ is through the combination $\psi - \Omega v$ only; in a reference frame that is rotating at an angular velocity Ω with respect to the original inertial frame, the perturbation appears to be stationary, and the system is indeed in rigid rotation.

Equations (2.2) and (4.27) imply $k^\alpha = \xi^\alpha + (\Omega_H - \Omega)\phi^\alpha$ and $t^\alpha = \xi^\alpha - \Omega\phi^\alpha$, and by virtue of Eq. (4.28) the Lie derivatives of γ_{AB}^1 in the directions of k^α and t^α can be expressed in terms of a derivative along ϕ^α ,

$$\mathcal{L}_k \gamma_{AB}^1 = (\Omega_H - \Omega) \mathcal{L}_\phi \gamma_{AB}^1$$

and

$$\mathcal{L}_t \gamma_{AB}^1 = -\Omega \mathcal{L}_\phi \gamma_{AB}^1.$$

Substituting this into Eqs. (4.22)–(4.24), and using also $\sigma^{AB} = \frac{1}{2}(\Omega_H - \Omega) \mathcal{L}_\phi \gamma_{AB}^1$, we obtain

$$\langle \dot{M} \rangle = \Omega(\Omega - \Omega_H) \mathcal{K}, \quad (4.29)$$

$$\langle \dot{J} \rangle = (\Omega - \Omega_H) \mathcal{K}, \quad (4.30)$$

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = (\Omega - \Omega_H)^2 \mathcal{K}, \quad (4.31)$$

where

$$\mathcal{K} = \frac{1}{32\pi} \oint \langle (\mathcal{L}_\phi \gamma_1^{AB}) (\mathcal{L}_\phi \gamma_{AB}^1) \rangle dS; \quad (4.32)$$

we recall that γ_1^{AB} is obtained from γ_{AB}^1 by raising indices with γ^{AB} , the inverse background metric.

Equation (4.30) indicates that the black hole’s angular momentum will increase when $\Omega > \Omega_H$, that is, when the external rotation is faster than the rotation of the generators; otherwise the angular momentum will decrease. Equation (4.29) indicates that $\langle \dot{M} \rangle = \Omega \langle \dot{J} \rangle$ when the black hole is in rigid rotation; the sign of $\langle \dot{M} \rangle$ is tied to the sign of $\langle \dot{J} \rangle$ and the sign of the angular velocity (which is defined relative to Ω_H). Finally, Eq. (4.31) shows that the black-hole area will always increase, as is dictated by Hawking’s area theorem (see, for example, Sec. 12.2 of Ref. [29]). These equations also appear in Sec. VII B 1 of the book by Thorne, Price, and Macdonald [28].

V. CURVATURE FORMALISM

In this section I translate the flux formulae of Eqs. (4.22)–(4.24) into a more practical language that involves *curvature variables*. The most important variable in this formalism is the Weyl scalar ψ_0 , which can be obtained by solving Teukolsky’s differential equation [2].

A. Relation between C_{AB} and ψ_0

Our first task is to express C_{AB} , the Weyl tensor of the perturbed horizon, in terms of the more practical curvature variable ψ_0 . The calculation is straightforward but somewhat lengthy; it requires a number of steps.

The Weyl tensor of the perturbed spacetime is $\hat{C}_{\alpha\gamma\beta\delta} = C_{\alpha\gamma\beta\delta} + \lambda C_{\alpha\gamma\beta\delta}^1 + O(\lambda^2)$, and the basis vectors of the perturbed horizon are $\hat{k}^\alpha = k^\alpha + \lambda k_1^\alpha + O(\lambda^2)$ and $\hat{e}_A^\alpha = e_A^\alpha + \lambda e_{1A}^\alpha + O(\lambda^2)$. The Weyl tensor of the perturbed horizon is then defined by

$$\hat{C}_{AB} = \hat{C}_{\alpha\gamma\beta\delta} \hat{e}_A^\alpha \hat{k}^\gamma \hat{e}_B^\beta \hat{k}^\delta, \quad (5.1)$$

which is the same equation as Eq. (3.9). By virtue of Eq. (2.35) we have that $C_{AB} = 0$ for the Kerr horizon, and $\hat{C}_{AB} = \lambda C_{AB}^1 + O(\lambda^2)$. To comply with preceding usage we shall now omit the label “1” on the Weyl tensor and set $\lambda = 1$ after the expansion in powers of λ has been carried out.

Direct evaluation of \hat{C}_{AB} from the preceding information gives

$$C_{AB} = C_{\alpha\gamma\beta\delta}^1 e_A^\alpha k^\gamma e_B^\beta k^\delta + C_{\alpha\gamma\beta\delta} (e_A^\alpha k^\gamma e_B^\beta k_1^\delta + e_A^\alpha k_1^\gamma e_B^\beta k^\delta + e_A^\alpha k^\gamma e_{1B}^\beta k^\delta + e_{1A}^\alpha k^\gamma e_B^\beta k^\delta).$$

We will now simplify this expression, and show that C_{AB} can be expressed solely in terms of $C_{\alpha\gamma\beta\delta}^1$, the perturbation of the Weyl tensor.

The main source of simplification comes from the algebraic properties of the unperturbed Weyl tensor: The last two terms within the brackets vanish by virtue of Eq. (2.34), and the first two can be rewritten with the help of Eq. (2.35). This gives

$$C_{AB} = C_{\alpha\gamma\beta\delta}^1 e_A^\alpha k^\gamma e_B^\beta k^\delta - 2\text{Re}(\psi_2)\gamma_{AB}k_\alpha k_1^\alpha.$$

Next we decompose e_A^α in the basis of complex vectors (e^α, \bar{e}^α) introduced in Sec. II D; the relations are $e_A^\alpha = \bar{e}_A e^\alpha + e_A \bar{e}^\alpha$ with the coefficients $e_A = \gamma_{AB}e^B$ obtained from Eq. (2.22). This yields

$$\begin{aligned} C_{\alpha\gamma\beta\delta}^1 e_A^\alpha k^\gamma e_B^\beta k^\delta &= \bar{e}_A \bar{e}_B C_{\alpha\gamma\beta\delta}^1 e^\alpha k^\gamma e^\beta k^\delta \\ &+ e_A e_B C_{\alpha\gamma\beta\delta}^1 \bar{e}^\alpha k^\gamma \bar{e}^\beta k^\delta \\ &+ (\bar{e}_A e_B + e_A \bar{e}_B) C_{\alpha\gamma\beta\delta}^1 e^\alpha k^\gamma \bar{e}^\beta k^\delta \end{aligned}$$

for the first term in the previous expression for C_{AB} . The factor within brackets is γ_{AB} , and by virtue of the symmetries of the Weyl tensor and the completeness relations of Eq. (2.21), we also have $C_{\alpha\gamma\beta\delta}^1 e^\alpha k^\gamma \bar{e}^\beta k^\delta = \frac{1}{2}g^{\alpha\beta}C_{\alpha\gamma\beta\delta}^1 k^\gamma k^\delta$, where $g^{\alpha\beta}$ is the inverse of the Kerr metric. But because the perturbed Weyl tensor must be traceless in the perturbed metric, $0 = \hat{g}^{\alpha\beta}\hat{C}_{\alpha\gamma\beta\delta} = (g^{\alpha\beta} - \lambda h^{\alpha\beta})(C_{\alpha\gamma\beta\delta} + \lambda C_{\alpha\gamma\beta\delta}^1)$, where $\lambda h_{\alpha\beta} \equiv \hat{g}_{\alpha\beta} - g_{\alpha\beta}$ is the metric perturbation. We therefore have $g^{\alpha\beta}C_{\alpha\gamma\beta\delta}^1 = h^{\alpha\beta}C_{\alpha\gamma\beta\delta}$, and gathering these results we obtain

$$\begin{aligned} C_{\alpha\gamma\beta\delta}^1 e_A^\alpha k^\gamma e_B^\beta k^\delta &= \bar{e}_A \bar{e}_B C_{\alpha\gamma\beta\delta}^1 e^\alpha k^\gamma e^\beta k^\delta \\ &+ e_A e_B C_{\alpha\gamma\beta\delta}^1 \bar{e}^\alpha k^\gamma \bar{e}^\beta k^\delta \\ &+ \frac{1}{2}\gamma_{AB}h^{\alpha\beta}C_{\alpha\gamma\beta\delta}k^\gamma k^\delta. \end{aligned}$$

We may simplify this further by using Eq. (2.36), and we now have

$$\begin{aligned} C_{AB} &= \bar{e}_A \bar{e}_B C_{\alpha\gamma\beta\delta}^1 e^\alpha k^\gamma e^\beta k^\delta + e_A e_B C_{\alpha\gamma\beta\delta}^1 \bar{e}^\alpha k^\gamma \bar{e}^\beta k^\delta \\ &- 2\text{Re}(\psi_2)\gamma_{AB} \left(\frac{1}{2}h_{\alpha\beta}k^\alpha k^\beta + k_\alpha k_1^\alpha \right). \end{aligned}$$

In the last step we recognize that the vector \hat{k}^α must be null in the metric $\hat{g}_{\alpha\beta}$, so that $0 = (g_{\alpha\beta} + \lambda h_{\alpha\beta})(k^\alpha + \lambda k_1^\alpha)(k^\beta + \lambda k_1^\beta) = \lambda(2k_\alpha k_1^\alpha + h_{\alpha\beta}k^\alpha k^\beta)$. We finally arrive at the expression

$$C_{AB} = \bar{e}_A \bar{e}_B C_{\alpha\gamma\beta\delta}^1 e^\alpha k^\gamma e^\beta k^\delta + e_A e_B C_{\alpha\gamma\beta\delta}^1 \bar{e}^\alpha k^\gamma \bar{e}^\beta k^\delta, \quad (5.2)$$

which involves only the perturbed Weyl tensor and the Kerr basis vectors.

The Weyl scalar of the perturbed spacetime is defined as in Eq. (2.28),

$$-\hat{\psi}_0 = \hat{C}_{\alpha\gamma\beta\delta}\hat{k}^\alpha\hat{m}^\gamma\hat{k}^\beta\hat{m}^\delta. \quad (5.3)$$

Expansion in powers of λ gives $\hat{\psi}_0 = \lambda\psi_0^1 + O(\lambda^2)$. After dropping the label “1” and setting $\lambda = 1$, we obtain

$$\begin{aligned} -\psi_0 &= C_{\alpha\gamma\beta\delta}^1 k^\alpha m^\gamma k^\beta m^\delta + 2C_{\alpha\gamma\beta\delta}k^\alpha m^\gamma k^\beta m_1^\delta \\ &+ 2C_{\alpha\gamma\beta\delta}k^\alpha m^\gamma k_1^\beta m^\delta. \end{aligned}$$

The last two terms vanish by virtue of Eq. (2.37), and we have $-\psi_0 = C_{\alpha\gamma\beta\delta}^1 k^\alpha m^\gamma k^\beta m^\delta$. We now express m^α in terms e^α and k^α , as in Eq. (2.23). This yields

$$-\psi_0 = C_{\alpha\gamma\beta\delta}^1 k^\alpha e^\gamma k^\beta e^\delta \quad (5.4)$$

as our final expression for the Weyl scalar.

The relation between C_{AB} and ψ_0 is obtained by substituting Eq. (5.4) into Eq. (5.2). We write this as

$$C_{AB} = \bar{e}_A \bar{e}_B \Psi + e_A e_B \bar{\Psi}, \quad (5.5)$$

where

$$\Psi(v, \theta^A) \equiv -\psi_0(\text{HH}) \equiv -\frac{\Delta^2}{4(r^2 + a^2)^2}\psi_0(\text{K}) \quad (5.6)$$

is our main curvature variable. We choose, for convenience, to absorb a minus sign into the definition of Ψ . The first equality in Eq. (5.6) states that apart from this minus sign, Ψ is the Weyl scalar ψ_0 as defined with the Hartle-Hawking tetrad [2, 26], evaluated on the horizon and expressed in terms of the horizon coordinates. The second equality gives the relationship between Ψ and the Weyl scalar as defined with the Kinnersley tetrad [2, 36], evaluated in the limit $r \rightarrow r_+$ in which $\psi_0(\text{K})$ diverges as Δ^{-2} .

The Weyl scalars in either choice of tetrad, and therefore Ψ , can be obtained by solving the Teukolsky equation. Because Ψ is a scalar that vanishes in the Kerr spacetime, this quantity is gauge invariant. These two properties make $\Psi(v, \theta^A)$ an especially useful choice of variable to describe the horizon perturbation.

B. Fluxes

We obtain the shear tensor by inserting the Weyl tensor of Eq. (5.5) into Eq. (4.9). Because the transverse vectors e_A depend on θ^A only, we obtain

$$\sigma_{AB}(v, \theta^A) = \bar{e}_A \bar{e}_B \Phi_+ + e_A e_B \bar{\Phi}_+, \quad (5.7)$$

where

$$\Phi_+(v, \theta^A) = \int_v^\infty e^{-\kappa(v'-v)}\Psi(v', \theta^A) dv' \quad (5.8)$$

is the future integral of the Weyl scalar Ψ , weighted by the exponential factor $e^{-\kappa(v'-v)}$ so that only the near future contributes significantly.

The metric perturbation is obtained by substituting Eq. (5.5) into Eq. (4.12). Here we obtain

$$\gamma_{AB}^1(v, \theta^A) = \frac{2}{\kappa} \left(\bar{e}_A \bar{e}_B \Phi + e_A e_B \bar{\Phi} \right), \quad (5.9)$$

where

$$\Phi(v, \theta^A) = \Phi_-(v, \theta^A) + \Phi_+(v, \theta^A), \quad (5.10)$$

with

$$\Phi_-(v, \theta^A) = \int_{-\infty}^v \Psi(v', \theta^A) dv' \quad (5.11)$$

representing the past integral of the Weyl scalar (weighted uniformly).

Equations (5.7) and (5.9) can now be substituted into Eqs. (4.22)–(4.24) to obtain expressions for the fluxes. Because the basis vectors e_A do not depend on v nor ϕ , we have that $\mathcal{L}_k e_A = \mathcal{L}_\phi e_A = \mathcal{L}_t e_A = 0$, and the derivative operators act only on Φ_\pm . Using the properties $e_A e^A = 0$ and $e_A \bar{e}^A = 1$ of the basis vectors — recall Eq. (2.20) — we obtain, for example

$$\langle \dot{M} \rangle = \frac{1}{8\pi\kappa} \oint \langle \Phi_+ \mathcal{L}_t \bar{\Phi} + \bar{\Phi}_+ \mathcal{L}_t \Phi \rangle dS.$$

Using now Eq. (2.9), we arrive at the following expression for the mass flux:

$$\langle \dot{M} \rangle = \frac{r_+^2 + a^2}{8\pi\kappa} \oint \langle \Phi_+ \mathcal{L}_t \bar{\Phi} + \bar{\Phi}_+ \mathcal{L}_t \Phi \rangle d\Omega, \quad (5.12)$$

where $d\Omega = \sin\theta d\theta d\phi$ is an element of solid angle on the unit two-sphere. We obtain similarly

$$\langle \dot{J} \rangle = -\frac{r_+^2 + a^2}{8\pi\kappa} \oint \langle \Phi_+ \mathcal{L}_\phi \bar{\Phi} + \bar{\Phi}_+ \mathcal{L}_\phi \Phi \rangle d\Omega, \quad (5.13)$$

for the flux of angular momentum, and

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \frac{r_+^2 + a^2}{4\pi} \oint \langle |\Phi_+|^2 \rangle d\Omega \quad (5.14)$$

for the rate of increase of the horizon area.

In Eqs. (5.12)–(5.14) it is understood that the integrated-curvature fields $\Phi_\pm(v, \theta^A)$ are expressed in terms of the horizon coordinates, and that the Lie-derivative operators take the form given by Eq. (4.25). At a later stage we will remove this remaining dependence on the horizon coordinates, and express all quantities in terms of the original spacetime coordinates.

C. Pure mode; comparison with Teukolsky and Press

Suppose that the Weyl scalar of Eq. (5.6) has the form

$$\Psi(v, r_+, \theta, \psi) = \Psi^{m,\omega}(\theta) e^{-i\omega v} e^{im\psi} \quad (5.15)$$

when expressed in terms of the spacetime coordinates; this solution to the Teukolsky equation is then a pure mode of frequency ω and azimuthal number m . In terms of the horizon coordinates we have

$$\Psi(v, \theta^A) = \Psi^{m,\omega}(\theta) e^{-ikv} e^{im\phi}; \quad (5.16)$$

we recall that $\phi = \psi - \Omega_H v$, and $k = \omega - m\Omega_H$ was first introduced in Eq. (4.18). We wish to calculate the rates

of change of mass, angular momentum, and area for this pure mode, and to compare our results with those first obtained by Teukolsky and Press [1].

Substituting Eq. (5.16) into Eqs. (5.8), (5.11), and (5.10) yields

$$\Phi_+ = \frac{\Psi^{m,\omega}(\theta)}{\kappa + ik} e^{-ikv} e^{im\phi}, \quad (5.17)$$

$$\Phi_- = \frac{\Psi^{m,\omega}(\theta)}{-ik} e^{-ikv} e^{im\phi}, \quad (5.18)$$

and

$$\Phi = \frac{\kappa \Psi^{m,\omega}(\theta)}{-ik(\kappa + ik)} e^{-ikv} e^{im\phi}. \quad (5.19)$$

It should be noted that while the integral defining Φ_+ converges properly for a pure mode, the integral defining Φ_- diverges in the infinite past; this difficulty is remedied by inserting a converging factor inside the integral, for example e^{v/v_1} with $v_1 \gg k^{-1}$, to reflect the fact that the pure mode was turned on in the finite (but remote) past.

These expressions can now be substituted into Eqs. (5.12)–(5.14). When acting on a pure mode, \mathcal{L}_t produces a factor of $-i\omega$, \mathcal{L}_ϕ a factor of im , and \mathcal{L}_k a factor of $-ik$. A simple computation gives

$$\langle \dot{M} \rangle = \frac{\omega(r_+^2 + a^2)}{k(\kappa^2 + k^2)} \frac{1}{4\pi} \oint |\Psi^{m,\omega}(\theta)|^2 d\Omega, \quad (5.20)$$

$$\langle \dot{J} \rangle = \frac{m(r_+^2 + a^2)}{k(\kappa^2 + k^2)} \frac{1}{4\pi} \oint |\Psi^{m,\omega}(\theta)|^2 d\Omega, \quad (5.21)$$

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \frac{r_+^2 + a^2}{\kappa^2 + k^2} \frac{1}{4\pi} \oint |\Psi^{m,\omega}(\theta)|^2 d\Omega. \quad (5.22)$$

These relations are compatible with Eqs. (4.29)–(4.31) if we define the angular velocity of the pure mode to be $\Omega = \omega/m$; this follows from the fact that according to Eq. (5.15), Ψ depends on ψ and v only through the combination $\psi - (\omega/m)v$, so that the perturbation rotates rigidly with an angular velocity ω/m .

To compare our expressions with those of Teukolsky and Press [1] we must first reconcile the different notations. In their Eq. (4.40), Teukolsky and Press display the near-horizon behavior of $\psi_0(K)$ — this is the Weyl scalar as defined with the Kinnersley tetrad — in terms of Boyer-Lindquist coordinates $(t_{\text{BL}}, r, \theta, \phi_{\text{BL}})$. They have

$$\psi_0(K) \sim e^{-i\omega t_{\text{BL}}} e^{im\phi_{\text{BH}}} e^{-ikr^*} \Delta^{-2} {}_2S_{lm}(\theta) Y_{\text{hole}},$$

where $r^* = \int (r^2 + a^2) \Delta^{-1} dr = v - t_{\text{BL}}$, ${}_2S_{lm}(\theta)$ are the Teukolsky angular functions, and Y_{hole} is a normalization factor. It is easy to check that after a transformation to the well-behaved Kerr coordinates (v, r, θ, ψ) , and after the rescaling of Eq. (5.6), this expression becomes

$$\Psi = -\frac{e^{-im\beta(r_+)} {}_2S_{lm}(\theta) Y_{\text{hole}}}{4(r_+^2 + a^2)^2} e^{-i\omega v} e^{im\psi},$$

where $\beta(r) \equiv -a(r_+^2 + a^2)^{-1} \int (r + r_+)(r - r_-)^{-1} dr$ is a function that is well behaved at $r = r_+$. Here, $r_- \equiv M - \sqrt{M^2 - a^2}$ denotes the position of the inner horizon.

The preceding equation relates our definition of $\Psi^{m,\omega}(\theta)$ in Eq. (5.15) with the quantities introduced by Teukolsky and Press. Inserting this into Eqs. (5.20) and (5.21) returns

$$\left\langle \frac{d^2 M}{dv d\Omega} \right\rangle = \frac{|{}_2S_{lm}(\theta)Y_{\text{hole}}|^2}{64\pi(r_+^2 + a^2)^3} \frac{\omega}{k(\kappa^2 + k^2)}$$

and

$$\left\langle \frac{d^2 J}{dv d\Omega} \right\rangle = \frac{|{}_2S_{lm}(\theta)Y_{\text{hole}}|^2}{64\pi(r_+^2 + a^2)^3} \frac{m}{k(\kappa^2 + k^2)}.$$

These are precisely the results obtained by Teukolsky and Press [1] and displayed in their Eq. (4.44). Our formalism (which is based partly on their work) is therefore consistent with theirs.

D. Decomposition into azimuthal modes

Our aim in this subsection is to derive practical flux formulae that are formulated in the time domain, in terms of fields expressed in the spacetime coordinates (v, r, θ, ψ) . We shall not, therefore, follow Teukolsky and Press [1] and decompose Ψ into modes proportional to $e^{-i\omega v} e^{im\psi}$, so as to work in the frequency domain. But we will still decompose the Weyl scalar into *azimuthal modes* proportional to $e^{im\psi}$, and write

$$\Psi(v, r_+, \theta, \psi) = \sum_m \Psi^m(v, \theta) e^{im\psi}. \quad (5.23)$$

This decomposition is motivated by the axial symmetry of the Kerr spacetime, which implies that each mode labeled by m will evolve independently. Such a decomposition is therefore likely to be involved in most attempts to integrate the Teukolsky equation numerically, in the time domain. It will also allow us to remove the remaining dependence of our flux formulae on the horizon coordinates (v, θ^A) . It should be noted that the flux formulae of Eqs. (5.12)–(5.14) do not require Ψ to be decomposed into modes; they are therefore ready to be used in situations where an azimuthal decomposition is not attempted. But the implementation of these formulae is delicate, because Φ_{\pm} must be evaluated by integrating $\Psi(v', r_+, \theta, \psi)$ along the horizon generators (integrating over dv' keeping $\phi \equiv \psi - \Omega_{\text{H}} v'$ constant). Our azimuthal decomposition will accomplish this automatically.

Substituting Eq. (5.23) into Eqs. (5.8) and (5.11) gives us the azimuthal decomposition of the integrated curvatures, which we express in terms of the horizon coordinate ϕ instead of the spacetime angle ψ :

$$\Phi_{\pm}(v, \theta^A) = \sum_m \Phi_{\pm}^m(v, \theta) e^{im\phi}, \quad (5.24)$$

where

$$\Phi_+^m(v, \theta) = e^{\kappa v} \int_v^{\infty} e^{-(\kappa - im\Omega_{\text{H}})v'} \Psi^m(v', \theta) dv' \quad (5.25)$$

and

$$\Phi_-^m(v, \theta) = \int_{-\infty}^v e^{im\Omega_{\text{H}}v'} \Psi^m(v', \theta) dv'. \quad (5.26)$$

Notice the presence of the oscillating factor $e^{im\Omega_{\text{H}}v'}$ within the integrals; this comes from the transformation between ϕ and ψ and it reflects the fact that the generators wrap around the horizon as v' is integrated forward. Notice also that Eqs. (5.25)–(5.26) are now independent of ϕ or ψ , so that it is no longer necessary to specify which is to remain constant during integration.

Equation (5.24) can now be substituted into Eqs. (5.12)–(5.14), in which $\Phi = \Phi_+ + \Phi_-$. In the horizon coordinates (v, θ, ϕ) the operator \mathcal{L}_k is a partial derivative with respect to v , \mathcal{L}_{ϕ} produces a factor of im , and $\mathcal{L}_t = \mathcal{L}_k - \Omega_{\text{H}} \mathcal{L}_{\phi}$. Simple algebra and integration over $d\phi$ give

$$\begin{aligned} \langle \dot{M} \rangle &= \frac{r_+^2 + a^2}{4\kappa} \sum_m \left[2\kappa \int \langle |\Phi_+^m|^2 \rangle \sin \theta d\theta \right. \\ &\quad \left. - im\Omega_{\text{H}} \int \langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \rangle \sin \theta d\theta \right], \quad (5.27) \end{aligned}$$

$$\begin{aligned} \langle \dot{J} \rangle &= -\frac{r_+^2 + a^2}{4\kappa} \sum_m (im) \\ &\quad \times \int \langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \rangle \sin \theta d\theta, \quad (5.28) \end{aligned}$$

and

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \frac{1}{2} (r_+^2 + a^2) \sum_m \int \langle |\Phi_+^m|^2 \rangle \sin \theta d\theta. \quad (5.29)$$

These are the final form of the flux formulae. Notice that these no longer involve the angles ϕ and ψ , and that all fields are expressed in terms of v and θ , coordinates that are shared by the spacetime and the horizon.

The steps required to compute $\langle \dot{M} \rangle$, $\langle \dot{J} \rangle$, and $\langle \dot{A} \rangle$ are therefore these (see also Sec. I C): First, solve the Teukolsky equation [2] for the functions $\Psi^m(v, \theta)$ defined by Eq. (5.23), for all relevant values of m ; recall from Eq. (5.6) that Ψ is (minus) the Weyl scalar ψ_0 in the Hartle-Hawking tetrad, evaluated at $r = r_+$. Second, compute the integrals of Eqs. (5.25) and (5.26) to obtain $\Phi_{\pm}^m(v, \theta)$. Third, and finally, substitute these values into the flux formulae of Eqs. (5.27)–(5.29), integrate over $d\theta$, and sum over m .

VI. METRIC FORMALISM FOR GENERAL BLACK HOLES

In this section I translate the flux formulae of Eqs. (4.22)–(4.24) into a more practical language that

involves *metric variables*. This translation is most useful in the context of a Schwarzschild black hole, for which the theory of metric perturbations is well developed; I shall consider this specific case in the following section. In this section I keep the discussion general, so that it applies to both rotating and nonrotating black holes.

A. Preferred gauge

We expand the metric of the perturbed black hole as

$$\hat{g}_{\alpha\beta} = g_{\alpha\beta} + \lambda h_{\alpha\beta}, \quad (6.1)$$

where $g_{\alpha\beta}$ is the metric of the unperturbed spacetime — the Kerr metric — and $\lambda h_{\alpha\beta}$ is the perturbation. (As we did previously, we keep λ for book-keeping but we set it equal to unity at the end of the calculation.) We wish first to impose a number of gauge conditions on $h_{\alpha\beta}$, which will simplify its relationship with the quantity γ_{AB}^1 introduced in Sec. IV A.

Our preferred gauge is a “horizon-locking gauge;” it has the property that the coordinate positions of the perturbed horizon are the same as those of the unperturbed (Kerr) horizon. As we shall see below, it is always possible to make this choice of gauge. In the preferred gauge the parametric description of the horizon generators is given by

$$\hat{z}^\alpha(v, \theta^A) = z^\alpha(v, \theta^A), \quad (6.2)$$

with $z^\alpha(v, \theta^A)$ giving the parametric description of the Kerr generators. This equality implies

$$\hat{k}^\alpha = k^\alpha, \quad \hat{e}_A^\alpha = e_A^\alpha, \quad (6.3)$$

so that the perturbation of the tangent vectors is identically zero: $k_1^\alpha = 0 = e_{1A}^\alpha$ in the notation of Sec. V A.

The vector k^α must be null, and orthogonal to e_A^α , in the metrics $g_{\alpha\beta}$ and $\hat{g}_{\alpha\beta}$. This observation gives rise to the three gauge conditions

$$h_{\alpha\beta} k^\alpha k^\beta = 0 = h_{\alpha\beta} k^\alpha e_A^\beta \quad (\text{preferred gauge}). \quad (6.4)$$

These equations hold on the horizon only, and we shall not need to extend them beyond the horizon. The preferred gauge is only partially determined, and the space of transformations within the preferred-gauge class is large.

For some purposes it will be convenient to supplement the gauge conditions of Eq. (6.4) with a fourth condition, $h_{\alpha\beta} k^\alpha N^\beta = 0$, so that we have the four conditions

$$h_{\alpha\beta} k^\beta = 0 \quad (\text{radiation gauge}). \quad (6.5)$$

These conditions also hold on the horizon only, and again we have a large space of gauge transformations within the radiation-gauge class. The radiation gauge of Eq. (6.5)

is similar to the one introduced by Chrzanowski (his incoming radiation gauge [38]); but it is distinct because Chrzanowski imposes Eq. (6.5) as well as $g^{\alpha\beta} h_{\alpha\beta} = 0$ globally in the Kerr spacetime [extending k^α away from the horizon as $k^\alpha(\text{HH})$, the first member of the Hartle-Hawking tetrad]. The Chrzanowski radiation gauge is therefore much more rigidly defined than (and a special case of) the radiation gauge of Eq. (6.5).

Equation (6.3) states that three of the basis vectors on the horizon are not changed by the perturbation. The fourth basis vector, \hat{N}^α , must be orthogonal to e_A^α in the perturbed metric, and it must also satisfy $\hat{g}_{\alpha\beta} \hat{N}^\alpha k^\beta = -1$. It is easy to see that these requirements imply $\hat{N}_\alpha = N_\alpha - \frac{1}{2}\lambda(h_{\beta\gamma} N^\beta N^\gamma) k_\alpha$.

We have shown that the imposition of Eq. (6.2) implies the gauge conditions of Eq. (6.4). We now examine the reversed question: Does the imposition of the preferred-gauge conditions imply that the coordinate description of the horizon is the same in the unperturbed and perturbed spacetimes? We shall show that the answer is in the affirmative.

Suppose that on the contrary, the perturbed horizon is displaced with respect to its unperturbed position. The parametric description of the generators is then

$$\hat{z}^\alpha(v, \theta^A) = z^\alpha(v, \theta^A) + \lambda \xi^\alpha(v, \theta^A), \quad (6.6)$$

where the vector $\lambda \xi^\alpha$ points from a point identified by (v, θ^A) on the unperturbed horizon to a point (carrying the same intrinsic coordinates) on the perturbed horizon. We shall show below that if Eq. (6.4) holds, then ξ^α must be tangent to the horizon; it can then be decomposed as $\xi^\alpha = a k^\alpha + a^A e_A^\alpha$ for some coefficients a and a^A . If ξ^α is tangent to the horizon, then it maps a point (v, θ^A) on one generator to another point (v', θ'_A) on another generator. (If $\theta'_A = \theta_A$ the vector links two points on the same generator; this happens when $a^A = 0$.) Because the mapping preserves the coordinate labels, this amounts to performing a transformation $(v', \theta'_A) \rightarrow (v, \theta_A)$ of the horizon’s intrinsic coordinates. This transformation can always be undone, and we conclude that ξ^α can be made to vanish whenever it is tangent to the horizon: $k_\alpha \xi^\alpha = 0 \Rightarrow \xi^\alpha = 0$. By showing that $k_\alpha \xi^\alpha = 0$ follows from Eq. (6.4) we therefore prove that Eq. (6.4) implies Eq. (6.2).

According to Eq. (6.6) the perturbed basis vectors are $\hat{k}^\alpha = k^\alpha + \lambda \xi^\alpha_{,\beta} k^\beta$ and $\hat{e}_A^\alpha = e_A^\alpha + \lambda \xi^\alpha_{,\beta} e_A^\beta$. The perturbed metric at the new horizon position is $\hat{g}_{\alpha\beta}(z + \lambda \xi) = g_{\alpha\beta}(z + \lambda \xi) + \lambda h_{\alpha\beta}(z) = g_{\alpha\beta} + \lambda(g_{\alpha\beta,\gamma} \xi^\gamma + h_{\alpha\beta})$, with all fields now evaluated at z , the position of the unperturbed horizon. Let now \hat{e}_a^α stand for any one of the vectors \hat{k}^α , \hat{e}_A^α , and e_a^α for the corresponding unperturbed vector. The statement $\hat{g}_{\alpha\beta}(z + \lambda \xi) \hat{k}^\alpha \hat{e}_a^\beta = 0$, when expanded in powers of λ , leads to

$$(\mathcal{L}_\xi g_{\alpha\beta} + h_{\alpha\beta}) k^\alpha e_a^\beta = 0.$$

The gauge conditions $h_{\alpha\beta} k^\alpha e_a^\beta = 0$ therefore imply

$k^\alpha e_a^\beta \mathcal{L}_\xi g_{\alpha\beta} = 0$, or

$$k^\alpha k^\beta \mathcal{L}_\xi g_{\alpha\beta} = 0 = k^\alpha e_A^\beta \mathcal{L}_\xi g_{\alpha\beta}. \quad (6.7)$$

These equations come with an immediate interpretation: $\mathcal{L}_\xi g_{\alpha\beta}$ is the change in $h_{\alpha\beta}$ produced by a gauge transformation generated by the vector field ξ^α ; Eq. (6.7) indicates that this transformation must preserve the gauge conditions of Eq. (6.4).

The first of Eq. (6.7) can be expressed in the form $(k^\alpha \xi_\alpha)_{;\beta} k^\beta - k^\alpha_{;\beta} k^\beta \xi_\alpha = 0$, or

$$\frac{\partial}{\partial v}(k_\alpha \xi^\alpha) = \kappa(k_\alpha \xi^\alpha) \quad (6.8)$$

after using Eq. (3.2). If we restrict ourselves to situations in which the event horizon starts in a stationary state, then $\xi^\alpha = 0$ initially, and Eq. (6.8) implies that $k_\alpha \xi^\alpha = 0$ at all times. We therefore have the statement that ξ^α is tangent to the horizon, and the proof that Eq. (6.4) implies Eq. (6.2).

The second of Eq. (6.7) can be expressed in the form

$$\frac{\partial}{\partial v}(e_A^\alpha \xi_\alpha) = -\frac{\partial}{\partial \theta^A}(k^\alpha \xi_\alpha) + 2\omega_A(k^\alpha \xi_\alpha), \quad (6.9)$$

where ω_A was introduced in Eq. (2.14). With $k^\alpha \xi_\alpha = 0$ and $\xi^\alpha = 0$ initially, this equation states that $e_A^\alpha \xi_\alpha = 0$ at all times. We therefore see that ξ^α must be directed along k^α , so that a point (v, θ^A) is necessarily mapped to a point (v', θ^A) on the same generator. This simply corresponds to a reparameterization $v \rightarrow v'(v, \theta^A)$ of the generators, and we conclude that ξ^α can be set equal to zero whenever Eq. (6.4) is enforced.

B. Metric and Lie derivatives

In the preferred gauge defined by Eqs. (6.2)–(6.4), the perturbed horizon is in the same coordinate position as the Kerr horizon, and the perturbed horizon metric is $\hat{\gamma}_{AB} = \hat{g}_{\alpha\beta} \hat{e}_A^\alpha \hat{e}_B^\beta = \gamma_{AB} + \lambda h_{\alpha\beta} e_A^\alpha e_B^\beta$, having used Eq. (6.3). This means that the metric perturbation is given by

$$\gamma_{AB}^1 = h_{\alpha\beta} e_A^\alpha e_B^\beta. \quad (6.10)$$

The Lie derivative of γ_{AB}^1 in the direction of k^α is calculated as $\mathcal{L}_k \gamma_{AB}^1 = (h_{\alpha\beta} e_A^\alpha e_B^\beta)_{;\gamma} k^\gamma$. The covariant derivative of e_A^α in the direction of k^α is evaluated in Eq. (2.14), and we have $\mathcal{L}_k \gamma_{AB}^1 = h_{\alpha\beta;\gamma} e_A^\alpha e_B^\beta k^\gamma + (\omega_A e_B^\alpha + e_A^\alpha \omega_B) h_{\alpha\beta} k^\beta$. Using Eq. (6.4), we arrive at

$$\mathcal{L}_k \gamma_{AB}^1 = 2\sigma_{AB} = h_{\alpha\beta;\gamma} e_A^\alpha e_B^\beta k^\gamma. \quad (6.11)$$

The Lie derivative of γ_{AB}^1 in the direction of ϕ^α is calculated in a similar way. Here we need the covariant derivative of e_A^α in the direction of ϕ^α , which is evaluated as

$$e_{A;\gamma}^\alpha \phi^\gamma = e_{A;\gamma}^\alpha e_C^\gamma \phi^C = (p_{AC} k^\alpha + \Gamma_{AC}^B e_B^\alpha) \phi^C,$$

in which we expressed ϕ^γ as $\phi^C e_C^\gamma$ — refer back to Sec. II C — and substituted Eq. (2.15). Gathering the results and using Eq. (6.10) as well as the gauge conditions of Eq. (6.4), we obtain $\mathcal{L}_\phi \gamma_{AB}^1 = h_{\alpha\beta;\gamma} e_A^\alpha e_B^\beta \phi^\gamma + 2\gamma_{D(A}^1 \Gamma_{B)C}^D \phi^C$. We now use the identity $c_A^D \equiv \phi^D|_A = \Gamma_{AC}^D \phi^C$ (which follows from the definition of intrinsic covariant differentiation and the fact that ϕ^C does not depend on the horizon coordinates) to manipulate the terms involving the horizon connection. We arrive at

$$\mathcal{L}_\phi \gamma_{AB}^1 = h_{\alpha\beta;\gamma} e_A^\alpha e_B^\beta \phi^\gamma + c_A^C \gamma_{CB}^1 + c_B^C \gamma_{CA}^1. \quad (6.12)$$

Recall that the antisymmetric two-tensor $c_{AB} \equiv -\phi_{A|B}$ was defined and evaluated in Sec. II C; see Eq. (2.17).

We now define a four-dimensional version of this tensor with the relation

$$c^{\alpha\beta} = c^{AB} e_A^\alpha e_B^\beta, \quad (6.13)$$

where the indices on c_{AB} are raised with γ^{AB} , the inverse of the Kerr horizon metric; this relation is inverted by $c_{AB} = c_{\alpha\beta} e_A^\alpha e_B^\beta$. With Eqs. (6.10) and (6.13) we have

$$c_A^C \gamma_{CB}^1 = c_{\alpha\gamma} h_{\delta\beta} e_A^\alpha e_B^\beta (\gamma^{CD} e_C^\gamma e_D^\delta).$$

Using now the completeness relations of Eq. (2.12) and the properties $c_{\alpha\gamma} k^\gamma = c_{\alpha\gamma} N^\gamma = 0$, we obtain $c_A^C \gamma_{CB}^1 = c_{\alpha\gamma} h_{\gamma\beta} e_A^\alpha e_B^\beta$ and Eq. (6.12) becomes

$$\mathcal{L}_\phi \gamma_{AB}^1 = h_{\alpha\beta;\gamma} e_A^\alpha e_B^\beta \phi^\gamma + (c_{\alpha\gamma} h_{\gamma\beta} + c_{\beta\gamma} h_{\gamma\alpha}) e_A^\alpha e_B^\beta. \quad (6.14)$$

Finally, the Lie derivative of γ_{AB}^1 in the direction of t^α is calculated as $\mathcal{L}_t \gamma_{AB}^1 = \mathcal{L}_k \gamma_{AB}^1 - \Omega_H \mathcal{L}_\phi \gamma_{AB}^1$. From Eqs. (6.11) and (6.14) we obtain

$$\mathcal{L}_t \gamma_{AB}^1 = h_{\alpha\beta;\gamma} e_A^\alpha e_B^\beta t^\gamma - \Omega_H (c_{\alpha\gamma} h_{\gamma\beta} + c_{\beta\gamma} h_{\gamma\alpha}) e_A^\alpha e_B^\beta. \quad (6.15)$$

C. Fluxes

It is a straightforward matter to substitute Eqs. (6.11), (6.14), and (6.15) into the flux formulae of Eqs. (4.22)–(4.24). We first obtain

$$\begin{aligned} \sigma^{AB} \mathcal{L}_k \gamma_{AB}^1 &= \frac{1}{2} F[k], \\ \sigma^{AB} \mathcal{L}_\phi \gamma_{AB}^1 &= \frac{1}{2} (F[\phi] + G), \\ \sigma^{AB} \mathcal{L}_t \gamma_{AB}^1 &= \frac{1}{2} (F[t] - \Omega_H G), \end{aligned}$$

where

$$F[\xi] \equiv h_{\alpha\beta;\gamma} k^\gamma h_{\mu\nu;\lambda} \xi^\lambda (e^{A\alpha} e_A^\mu) (e^{B\beta} e_B^\nu)$$

and

$$G \equiv h_{\alpha\beta;\gamma} k^\gamma (c_\mu^\lambda h_{\lambda\nu} + c_\nu^\lambda h_{\lambda\mu}) (e^{A\alpha} e_A^\mu) (e^{B\beta} e_B^\nu).$$

By virtue of Eq. (2.12), $e^{A\alpha}e_A^\mu = g^{\alpha\mu} + k^\alpha N^\mu + N^\alpha k^\mu$, and this identity can be used to simplify our expressions for $F[\xi]$ and G . To simplify things further we also write

$$k^\beta h_{\alpha\beta;\gamma}\xi^\gamma = (h_{\alpha\beta}k^\beta)_{;\gamma}\xi^\gamma - h_{\alpha\beta}k^\beta_{;\gamma}\xi^\gamma,$$

and we note that according to Eqs. (2.13) and (2.14), any tangential derivative of the form $k^\beta_{;\gamma}\xi^\gamma$ is necessarily proportional to k^β . The preceding equation therefore becomes

$$k^\beta h_{\alpha\beta;\gamma}\xi^\gamma = (h_{\alpha\beta}k^\beta)_{;\gamma}\xi^\gamma - p[\xi](h_{\alpha\beta}k^\beta),$$

where $p[\xi]$ is a proportionality factor that depends on the choice of vector ξ^α ; for example, $p[k] = \kappa$ and $p[\phi] = \omega_A\phi^A$. At this stage it is convenient to supplement the three gauge conditions of Eqs. (6.4) with the fourth condition implied by Eq. (6.5); adopting the radiation gauge allows us to set $h_{\alpha\beta}k^\beta = 0$ on the horizon, and therefore to discard all terms of the form $k^\beta h_{\alpha\beta;\gamma}\xi^\gamma$. This greatly simplifies our expressions for $F[\xi]$ and G . After carrying out these manipulations we obtain

$$F[\xi] = h_{\alpha\beta;\gamma}k^\gamma h^{\alpha\beta}_{;\delta}\xi^\delta$$

and

$$G = 2h_{\alpha\beta;\gamma}k^\gamma c^{\alpha\delta}h_\delta^\beta.$$

Substituting these results into Eqs. (4.22)–(4.24) we arrive at

$$\langle \dot{M} \rangle = \oint T_{\alpha\beta}k^\alpha t^\beta dS - \Omega_H \oint q dS, \quad (6.16)$$

$$\langle \dot{J} \rangle = - \oint T_{\alpha\beta}k^\alpha \phi^\beta dS - \oint q dS, \quad (6.17)$$

$$\frac{\kappa}{8\pi} \langle \dot{A} \rangle = \oint T_{\alpha\beta}k^\alpha k^\beta dS, \quad (6.18)$$

where

$$T_{\alpha\beta} \equiv \frac{1}{32\pi} \langle h_{\mu\nu;\alpha} h^{\mu\nu}_{;\beta} \rangle \quad (6.19)$$

and

$$q \equiv \frac{1}{16\pi} \langle h_{\mu\nu;\alpha} k^\alpha c^{\mu\lambda} h_\lambda^\nu \rangle. \quad (6.20)$$

We recall that these results are formulated in the radiation gauge of Eq. (6.5). And we mention that an alternative expression for q is

$$q = -\frac{1}{16\pi} \langle \phi^\alpha_{;\beta} h^\beta_\gamma h^\gamma_{\alpha;\mu} k^\mu \rangle; \quad (6.21)$$

this follows from substituting the (easily-derived) identity $c_{\alpha\beta} = -\phi_{\alpha;\beta} - \phi_{\alpha;\mu}N^\mu k_\beta + \phi_{\beta;\mu}N^\mu k_\alpha$ into Eq. (6.20) and simplifying the result.

The integrals involving $T_{\alpha\beta}$ in Eqs. (6.16)–(6.18) are formally identical to the flux formulae that would be obtained for a horizon perturbed by a matter field with

stress-energy tensor $T_{\alpha\beta}$ (see, for example, Sec. 6.4.2 of Ref. [37], or Sec. 4.3.6 of Ref. [30]). It is therefore tempting to view Eq. (6.19) as a definition of an effective stress-energy tensor for gravitational radiation crossing the event horizon. While in general the integrals involving q spoil this interpretation, we see that there exists an approximate regime in which the interpretation is sound: this is the high-frequency regime first investigated by Isaacson [31, 32]. Schematically, $T \sim (\nabla h)^2$ while $q \sim h\nabla h$, and the additional derivative ensures that $T_{\alpha\beta}$ dominates over q in the high-frequency limit. And indeed, our expression for $T_{\alpha\beta}$, as given by Eq. (6.19), does agree with Isaacson's effective stress-energy tensor. It should be noted that the time averaging involved in Eq. (6.19) is different from the spacetime (Brill-Hartle [39]) averaging used in Isaacson's construction; but it is plausible that the two averaging procedures are reconciled after $T_{\alpha\beta}$ is integrated over dS . It should also be noted that while Eq. (6.19) is formulated in the radiation gauge of Eq. (6.5), Isaacson has shown that the expression is actually gauge invariant in the high-frequency limit.

The flux formulae of Eqs. (6.16)–(6.18) are not limited to the high-frequency regime; they can be applied in general situations, provided that the metric perturbation $h_{\alpha\beta}$ satisfies the gauge conditions $h_{\alpha\beta}k^\beta = 0$ on the horizon. These formulae could in principle be used in tandem with Chrzanowski's metric reconstruction [38, 40, 41, 42] to calculate the absorption of mass and angular momentum by a Kerr black hole. But to proceed like this would be much more involved than to proceed directly with the curvature formalism of Sec. V. The flux formulae could also be used in the context of a Schwarzschild black hole, but the formulation given here is not optimal and I shall refine it in the following section. My main purpose in this section was to introduce the preferred gauge (which will be used also in Sec. VII) and to establish the preceding connection with Isaacson's effective stress-energy tensor [31, 32].

VII. METRIC FORMALISM FOR SCHWARZSCHILD BLACK HOLES

In this section I fulfill the promise made in Sec. VI, to translate the flux formulae of Eqs. (4.22)–(4.24) into a more practical language that involves the metric perturbations of a Schwarzschild black hole. The key aspects of the theory of first-order perturbations of this spacetime are summarized the Appendix.

A. Background spacetime

The Schwarzschild metric in Eddington-Finkelstein coordinates (v, r, θ, ϕ) is given by

$$ds^2 = -f dv^2 + 2 dv dr + r^2 d\Omega^2, \quad (7.1)$$

where $f = 1 - 2M/r$ and $d\Omega^2 = \Omega_{AB}d\theta^A d\theta^B = d\theta^2 + \sin^2\theta d\phi^2$. The subset of coordinates (v, θ, ϕ) is used on the horizon; v is a parameter on the null generators, and $\theta^A = (\theta, \phi)$ are comoving coordinates. In the spacetime coordinates (v, r, θ, ϕ) the basis vectors are $k^\alpha = (1, 0, 0, 0)$, $N^\alpha = (0, -1, 0, 0)$, $e_\theta^\alpha = (0, 0, 1, 0)$, and $e_\phi^\alpha = (0, 0, 0, 1)$. The metric of the unperturbed horizon is $\gamma_{AB} = r_+^2 \Omega_{AB}$, where $r_+ = 2M$.

B. Metric perturbation

The metric perturbation $\lambda h_{\alpha\beta}$ (denoted $\delta g_{\alpha\beta}$ in the Appendix) is cast in the radiation gauge of Eq. (6.5). We therefore impose $h_{\alpha\beta}k^\beta = 0$ on the perturbed horizon, which is still located at $r = r_+$ — see the discussion of Sec. VI A. The gauge conditions imply that the components $h_{v\alpha}$ of the metric perturbations all vanish. According to Eq. (6.10) the perturbation of the horizon metric is

$$\gamma_{AB}^1 = h_{\alpha\beta} e_A^\alpha e_B^\beta, \quad (7.2)$$

where e_A^α are the background basis vectors.

The odd-parity sector of the perturbations is described by Eqs. (A.14), (A.15) and it involves the functions h_r^{lm} and h_2^{lm} of the coordinates (v, r) . The combinations of Eq. (A.16) are gauge invariant, and they are used in Eq. (A.18) to form the Regge-Wheeler function $\Psi_{\text{RW}}^{lm}(v, r)$ [3], which is also gauge invariant. A simple calculation shows that near the horizon,

$$\Psi_{\text{RW}}^{lm} = \frac{1}{2r_+} \frac{\partial h_2^{lm}}{\partial v} + O(f),$$

so that $h_2(v, r_+) = 2r_+ \int^v \Psi_{\text{RW}}^{lm}(v', r_+) dv'$. The odd-parity sector of Eq. (7.2) is therefore

$$\gamma_{AB}^{1,\text{odd}}(v, \theta^A) = 2r_+ \sum_{lm} X_{AB}^{lm}(\theta^A) \int^v \Psi_{\text{RW}}^{lm}(v', r_+) dv', \quad (7.3)$$

where $X_{AB}^{lm}(\theta^A)$ are the odd-parity tensorial harmonics introduced in Eq. (A.8). Here and below, the sum over l is restricted to $l \geq 2$, and the sum over m extends from $-l$ to l .

The even-parity sector of the metric perturbations is described by Eqs. (A.20)–(A.22) and it involves the functions h_{rr}^{lm} , j_r^{lm} , K^{lm} , and G^{lm} of the coordinates (v, r) . The combinations of Eq. (A.23) are gauge invariant, and they are used in Eq. (A.24) to form the Zerilli-Moncrief function $\Psi_{\text{ZM}}^{lm}(v, r)$ [4, 13, 43], which is also gauge invariant. A simple calculation shows that near the horizon,

$$\Psi_{\text{ZM}}^{lm} = -\frac{4r_+^2}{l(l+1)(l^2+l+1)} \frac{\partial}{\partial v} \left[K^{lm} - \frac{1}{2}l(l+1)G^{lm} \right] + \frac{2r_+}{l(l+1)} K^{lm} + O(f).$$

On the other hand, an analysis of the linearized field equations near the horizon shows that in the absence of

sources, $K^{lm} = \frac{1}{2}l(l+1)G^{lm} + O(f)$, so that $K^{lm}(v, r_+) = \frac{1}{2}l(l+1)G^{lm}(v, r_+) = \frac{1}{2}l(l+1)r_+^{-1}\Psi_{\text{ZM}}^{lm}(v, r_+)$. The even-parity sector of Eq. (7.2) is therefore

$$\gamma_{AB}^{1,\text{even}}(v, \theta^A) = r_+ \sum_{lm} Z_{AB}^{lm}(\theta^A) \Psi_{\text{ZM}}^{lm}(v, r_+), \quad (7.4)$$

where $Z_{AB}^{lm}(\theta^A)$ are the even-parity tensorial harmonics introduced in Eq. (A.7).

The complete perturbation of the horizon metric is given by the sum of Eqs. (7.3) and (7.4),

$$\gamma_{AB}^1 = r_+ \sum_{lm} \left[2X_{AB}^{lm} \int^v \Psi_{\text{RW}}^{lm}(v') dv' + Z_{AB}^{lm} \Psi_{\text{ZM}}^{lm}(v) \right], \quad (7.5)$$

where we have set $\Psi_{\text{RW}}^{lm}(v') \equiv \Psi_{\text{RW}}^{lm}(v', r_+)$ and $\Psi_{\text{ZM}}^{lm}(v) \equiv \Psi_{\text{ZM}}^{lm}(v, r_+)$. Notice that this tensor is tracefree: $\Omega_{AB} \gamma_{AB}^1 = 0$. The fact that γ_{AB}^1 is related to the *integral* of the Regge-Wheeler function means that this variable is rather ill-suited to describe ingoing gravitational radiation crossing the event horizon; a similar statement is made in subsection 5 of the Appendix, about outgoing radiation at future null infinity.

The shear tensor is obtained by differentiating $\frac{1}{2}\gamma_{AB}^1$ with respect to v , and

$$\sigma_{AB} = \frac{r_+}{2} \sum_{lm} \left[2X_{AB}^{lm} \Psi_{\text{RW}}^{lm}(v) + Z_{AB}^{lm} \dot{\Psi}_{\text{ZM}}^{lm}(v) \right]; \quad (7.6)$$

this is also equal to $\frac{1}{2}\mathcal{L}_k \gamma_{AB}^1$ and $\frac{1}{2}\mathcal{L}_t \gamma_{AB}^1$. Because the spherical harmonics are all proportional to $e^{im\phi}$, we also have

$$\mathcal{L}_\phi \gamma_{AB}^1 = r_+ \sum_{lm} (im) \left[2X_{AB}^{lm} \int^v \Psi_{\text{RW}}^{lm}(v') dv' + Z_{AB}^{lm} \Psi_{\text{ZM}}^{lm}(v) \right]. \quad (7.7)$$

C. Fluxes

It is a straightforward task to substitute the preceding results for σ_{AB} , $\mathcal{L}_k \gamma_{AB}$, $\mathcal{L}_t \gamma_{AB}$, and $\mathcal{L}_\phi \gamma_{AB}$ into the flux formulae of Eqs. (4.22)–(4.24), and then to integrate over $dS = r_+^2 \sin\theta d\theta d\phi$. The integrations can be carried out explicitly with the help of the orthogonality relations of Eqs. (A.11), (A.12), and we arrive at

$$\langle \dot{M} \rangle = \frac{1}{64\pi} \sum_{lm} (l-1)l(l+1)(l+2) \times \left\langle 4|\Psi_{\text{RW}}^{lm}(v)|^2 + |\dot{\Psi}_{\text{ZM}}^{lm}(v)|^2 \right\rangle \quad (7.8)$$

and

$$\langle \dot{J} \rangle = \frac{1}{64\pi} \sum_{lm} (l-1)l(l+1)(l+2)(im) \times \left\langle 4\Psi_{\text{RW}}^{lm}(v) \int^v \bar{\Psi}_{\text{RW}}^{lm}(v') dv' + \dot{\Psi}_{\text{ZM}}^{lm}(v) \bar{\Psi}_{\text{ZM}}^{lm}(v) \right\rangle. \quad (7.9)$$

Notice that except for the substitution $u \rightarrow v$, these formulae are identical to Eqs. (A.26) and (A.27), which give the rates at which energy and angular momentum are transported to future null infinity. Note also that for a nonrotating black hole, the first law of black-hole mechanics reduces to $(\kappa/8\pi)\langle\dot{A}\rangle = \langle\dot{M}\rangle$. Finally, note that although it involves complex quantities, the expression for $\langle\dot{J}\rangle$ is real; this property follows from the identity $\Psi^{l,-m} = (-1)^m \bar{\Psi}^{lm}$, which is inherited from the spherical harmonics, and which is satisfied by both the Regge-Wheeler and Zerilli-Moncrief functions.

Equations (7.8) and (7.9) give the final form of the flux formulae for the case of a Schwarzschild black hole. The steps required to compute $\langle\dot{M}\rangle$, $\langle\dot{J}\rangle$, and $\langle\dot{A}\rangle$ are therefore these (see also Sec. I D): First, solve the Regge-Wheeler [3] and Zerilli [4] equations for the functions $\Psi_{\text{RW}}^{lm}(v, r)$ and $\Psi_{\text{ZM}}^{lm}(v, r)$ defined in the Appendix, for all relevant value of l and m . Second, evaluate the functions at $r = r_+$ and compute the integral of $\Psi_{\text{RW}}^{lm}(v, r_+)$ and the derivative of $\Psi_{\text{ZM}}^{lm}(v, r_+)$. Third, and finally, substitute these functions into the flux formulae of Eqs. (7.8)–(7.9) and sum over l and m .

These flux formulae were first presented and used by Martel [14] in his exploration of gravitational-wave processes associated with the motion of a small-mass body around a Schwarzschild black hole. Although he arrived at the correct results, the derivation of Eqs. (7.8) and (7.9) given by Martel is flawed — it incorporates both a conceptual and a computational error. The conceptual error is that Martel based his derivation on Isaacson’s effective stress-energy tensor for gravitational waves [31, 32], incorrectly assuming that the high-frequency description is always applicable near the event horizon of a black hole (as it always is near future null infinity). This assumption was motivated by the observation that for a stationary observer just above the event horizon, any incoming gravitational wave would appear highly blueshifted. While the observation is of course valid, the observer-dependent blueshift does not by itself produce a perturbation that satisfies the assumptions underlying Isaacson’s construction — the static Schwarzschild coordinates do not form a “steady” coordinate system near the horizon. Martel’s starting point was therefore Eqs. (6.16) and (6.17) without the integrations over q , and this should have led him to the wrong formula for the flux of angular momentum (the q integral does not contribute to $\langle\dot{M}\rangle$, because $\Omega_{\text{H}} = 0$ for a nonrotating black hole). That he nevertheless obtained Eq. (7.9) is due to a computational error that accidentally compensated for the absence of the q integral.

VIII. SMALL-HOLE/SLOW-MOTION APPROXIMATION FOR A SCHWARZSCHILD BLACK HOLE

In this and the following section I describe an application of the flux formulae obtained in Sec. V and VII. I

shall evaluate $\langle\dot{M}\rangle$ and $\langle\dot{J}\rangle$ in a small-hole/slow-motion (SH/SM) approximation in which the ratio M/\mathcal{R} , where M is the black-hole mass and \mathcal{R} the radius of curvature of the spacetime in which the black hole moves, is assumed to be small. I begin in this section with a nonrotating black hole, and I will consider the case of a rotating black hole in Sec. IX.

A. The SH/SM approximation

We imagine a situation in which a black hole of mass M is not at rest and isolated, but moves in a spacetime that may contain a number of additional bodies. The radius of curvature of this external spacetime is denoted \mathcal{R} , and although this may depend on M (if the geometry of the external spacetime is significantly influenced by the black hole), we assume that

$$M/\mathcal{R} \ll 1 \quad (\text{small-hole/slow-motion approximation}). \quad (8.1)$$

More precisely, we assume that M is much smaller than all of \mathcal{R} , \mathcal{L} , and \mathcal{T} , where \mathcal{L} is the scale of inhomogeneity in the external universe, and \mathcal{T} is the time scale over which changes occur in the external universe. To simplify the notation we take \mathcal{R} , \mathcal{L} , and \mathcal{T} to be of the same order of magnitude. (These quantities, and many of the concepts used throughout Secs. VIII and IX, are defined precisely in Thorne and Hartle [33]; the reader is referred to this paper for details.)

Near the black hole the spacetime resembles closely the spacetime of an isolated black hole: the gravitational field is strongly dominated by the hole’s contribution, and the influence of the external universe is weak. But the hole is not truly isolated, and it is slightly distorted by the tidal gravitational field supplied by the external universe. As a result of this interaction, the hole’s mass and angular momentum change with time, and we wish here to calculate these changes.

When viewed on the large scale \mathcal{R} , the black hole occupies a very small region of the actual spacetime, and this region can be idealized as a world line γ in the external spacetime. Let u^α be the (normalized) tangent vector to this world line, and call this the four-velocity of the black hole in the external spacetime. It can be shown that to a very good degree of accuracy, the motion of the black hole is geodesic in this spacetime [19, 44, 45, 46, 47, 48]. Let e_a^α (with the index a running from 1 to 3) be a set of orthonormal vectors attached to γ ; let these vectors be orthogonal to u^α and choose them to be parallel transported on the world line. The tetrad (u^α, e_a^α) defines a reference frame in a neighborhood of γ , and we shall call this frame the *local asymptotic rest frame* of the black hole in the external spacetime.

We assume that the Ricci tensor of the external spacetime vanishes in a neighborhood of γ , so that no matter will appear in the vicinity of the black hole. The curvature of the external spacetime in this neighborhood is therefore described entirely by the Weyl tensor

$C_{\alpha\gamma\beta\delta}$. The Weyl tensor evaluated on the world line can be decomposed in the tetrad (u^α, e_a^α) ; we write, for example, $C_{0c0d}(v) \equiv C_{\alpha\gamma\beta\delta}(\gamma)u^\alpha e_c^\gamma u^\beta e_d^\delta$ and $C_{0cbd}(v) \equiv C_{\alpha\gamma\beta\delta}(\gamma)u^\alpha e_c^\gamma e_b^\beta e_d^\delta$, with v denoting proper time on γ — it will later be identified with an advanced-time coordinate on the black-hole horizon. It is easy to show that the frame tensors [33]

$$\mathcal{E}_{ab}(v) = C_{0a0b}(v), \quad \mathcal{B}_{ab}(v) = \frac{1}{2}\varepsilon_a^{cd}C_{cdb0}(v), \quad (8.2)$$

where ε_{abc} is the permutation symbol (all frame indices are lowered and raised with δ_{ab} and its inverse, respectively), are symmetric and tracefree, and that their components comprise all ten independent components of the Weyl tensor. These frame tensors are the tidal gravitational fields supplied by the external universe, and these are responsible for the tidal distortion of the black hole.

It will be convenient to promote the tidal fields \mathcal{E}_{ab} and \mathcal{B}_{ab} to four-dimensional spacetime tensors. We therefore define

$$\mathcal{E}_{\alpha\beta} = \mathcal{E}_{ab}e_\alpha^a e_\beta^b, \quad \mathcal{B}_{\alpha\beta} = \mathcal{B}_{ab}e_\alpha^a e_\beta^b, \quad (8.3)$$

where $e_\alpha^a \equiv \delta^{ab}g_{\alpha\beta}e_b^\beta$. It is not difficult to show that these tensors are also given by

$$\mathcal{E}_{\alpha\beta} = C_{\mu\alpha\nu\beta}u^\mu u^\nu \quad (8.4)$$

and

$$\mathcal{B}_{\alpha\beta} = \frac{1}{2}u^\mu \varepsilon_{\mu\alpha}{}^{\gamma\delta} C_{\gamma\delta\beta\nu} u^\nu, \quad (8.5)$$

where the Levi-Civita tensor $\varepsilon_{\mu\alpha\nu\beta}$ and the Weyl tensor $C_{\mu\alpha\nu\beta}$ are evaluated on the world line γ .

As an example of a SH/SM situation, consider a black hole of mass M on a circular orbit of radius b in the gravitational field of an external body of mass M_{ext} . The radius of curvature of the external spacetime at the position of the black hole is such that $\mathcal{R}^{-2} \sim (M + M_{\text{ext}})/b^3$, and we have

$$\frac{M}{\mathcal{R}} \sim \frac{M}{M + M_{\text{ext}}} V^3, \quad V = \sqrt{\frac{M + M_{\text{ext}}}{b}}, \quad (8.6)$$

where V is a measure of the hole's orbital velocity. There are many ways by which M/\mathcal{R} can be made small. One way is to let $M/M_{\text{ext}} \ll 1$; then M/\mathcal{R} will be small irrespective of the magnitude of V . This is the *small-hole approximation*, which allows the small black hole to move at relativistic speeds in the strong gravitational field of the external body. Another way is to let $V \ll 1$; then M/\mathcal{R} will be small for all mass ratios. This is the *slow-motion approximation*, which allows the slowly-moving black hole to have a mass comparable to (or even much larger than) M_{ext} . These two limiting approximations are special cases of the fundamental requirement that M/\mathcal{R} be small; we therefore call the approximation $M/\mathcal{R} \ll 1$ the SH/SM approximation.

B. Metric of a tidally distorted black hole

My considerations thus far have been general, and they apply to rotating as well as nonrotating black holes. I now specialize to nonrotating black holes.

The metric of a Schwarzschild black hole immersed in an external universe can be obtained by solving the Einstein field equations. Because the tidal potentials scale as $(r/\mathcal{R})^2 \ll 1$, where r is a measure of distance from the black hole, it is sufficient to linearize the equations with respect to the Schwarzschild solution; this is a standard application of black-hole perturbation theory. Explicit forms for the metric were obtained by Manasse [44], Alvi [49], Detweiler [50], and Poisson [51], and I summarize their results here. I follow the description of Ref. [51], but I switch from the retarded coordinates (u, r, θ^A) used there to a set of advanced coordinates (v, r, θ^A) which are well behaved on the event horizon; the expressions for the perturbed metric are identical, except for the correspondence $du \rightarrow -dv$.

The metric takes the form of an expansion in powers of r/\mathcal{R} , but it is correct to all orders in M/r . It is given by

$$g_{vv} = -f(1 + r^2 f \mathcal{E}^*) + O(r^3/\mathcal{R}^3), \quad (8.7)$$

$$g_{vr} = 1, \quad (8.8)$$

$$g_{vA} = -\frac{2}{3}r^3 f(\mathcal{E}_A^* + \mathcal{B}_A^*) + O(r^4/\mathcal{R}^3), \quad (8.9)$$

$$g_{AB} = r^2 \Omega_{AB} - \frac{1}{3}r^4 \left[\left(1 - \frac{2M^2}{r^2}\right) \mathcal{E}_{AB}^* + \mathcal{B}_{AB}^* \right] + O(r^5/\mathcal{R}^3), \quad (8.10)$$

where $f = 1 - 2M/r$. The irreducible tidal fields are defined by

$$\mathcal{E}^* = \sum_m \mathcal{E}_m Y^m, \quad (8.11)$$

$$\mathcal{E}_A^* = \frac{1}{2} \sum_m \mathcal{E}_m Y_{:A}^m, \quad (8.12)$$

$$\mathcal{E}_{AB}^* = \sum_m \mathcal{E}_m Z_{AB}^m, \quad (8.13)$$

$$\mathcal{B}_A^* = \frac{1}{2} \sum_m \mathcal{B}_m X_A^m, \quad (8.14)$$

$$\mathcal{B}_{AB}^* = - \sum_m \mathcal{B}_m X_{AB}^m, \quad (8.15)$$

where Y^m , $Y_{:A}^m$, Z_{AB}^m , X_A^m , and X_{AB}^m are the real spherical harmonics of degree $l = 2$ that are introduced in subsection 2 of the Appendix, and

$$\mathcal{E}_0 = \mathcal{E}_{33} = -(\mathcal{E}_{11} + \mathcal{E}_{22}), \quad (8.16)$$

$$\mathcal{E}_{1c} = 2\mathcal{E}_{13}, \quad (8.17)$$

$$\mathcal{E}_{1s} = 2\mathcal{E}_{23}, \quad (8.18)$$

$$\mathcal{E}_{2c} = \frac{1}{2}(\mathcal{E}_{11} - \mathcal{E}_{22}), \quad (8.19)$$

$$\mathcal{E}_{2s} = \mathcal{E}_{12}, \quad (8.20)$$

with corresponding relations defining \mathcal{B}_m . (Notice that a typographical error contained in Ref. [51] is hereby corrected.)

In the limit $M/r \rightarrow 0$ (keeping r/\mathcal{R} fixed), the metric of Eqs. (8.7)–(8.10) becomes the metric of the external spacetime expressed as an expansion in powers of r/\mathcal{R} about the timelike geodesic γ . In this limit the interpretation of v as proper time on the world line becomes precise, and $\mathcal{E}_{ab}(v)$, $\mathcal{B}_{ab}(v)$ are recognized as frame components of the Weyl tensor evaluated on γ . In the limit $r/\mathcal{R} \rightarrow 0$ (keeping M/r fixed), the metric of Eqs. (8.7)–(8.10) becomes the metric of an isolated Schwarzschild black hole expressed in ingoing Eddington-Finkelstein coordinates; there is no notion of a world line γ in this limit. For small values of r/\mathcal{R} and arbitrary values of M/r , the metric of Eqs. (8.7)–(8.10) describes a black hole distorted by the tidal gravitational fields supplied by the external universe.

The metric perturbation $h_{\alpha\beta}$ defined by Eqs. (8.7)–(8.10) satisfies the conditions $h_{\alpha\beta}k^\alpha k^\beta = h_{\alpha\beta}k^\alpha e_A^\beta = 0$ at $r = r_+ = 2M$, because h_{vv} and h_{vA} are all proportional to $f = 1 - 2M/r$. The metric perturbation therefore satisfies the preferred gauge conditions of Eq. (6.4). It can also be checked directly from the metric that the hypersurface $r = 2M$ is null, and that its generators move with constant values of θ^A . The parameter on the generators is v , and a short calculation reveals that the surface gravity κ is equal to its Schwarzschild value $(4M)^{-1}$ up to a fractional correction of order $(M/\mathcal{R})^3$.

C. Odd-parity contribution to shear

Although the metric of the tidally distorted black hole is already expressed in the preferred gauge, it is safer (and as it turns out, necessary) to calculate the shear tensor σ_{AB} by first obtaining the gauge-invariant Regge-Wheeler [3] and Zerilli-Moncrief [4, 13, 43] functions; the relation between these quantities is given by Eq. (7.6). We begin here with the odd-parity piece of the shear tensor. A description of this sector of the metric perturbations is provided in subsection 3 of the Appendix.

When $l = 2$ the odd-parity perturbations can be expanded as

$$h_{iA} = \sum_m h_i^m(v, r) X_A^m(\theta^A) \quad (8.21)$$

and

$$h_{AB} = \sum_m h_2^m(v, r) X_{AB}^m(\theta^A). \quad (8.22)$$

The combinations

$$\tilde{h}_i^m = h_i^m + \frac{1}{2}h_{2,i}^m - \frac{1}{r}r_{,i}h_2^m \quad (8.23)$$

are gauge invariant, and the Regge-Wheeler function is defined by

$$\Psi_{\text{RW}}^m = \frac{1}{r}r_{,i}\tilde{h}_i^m. \quad (8.24)$$

Comparison of Eqs. (8.21), (8.22) with Eqs. (8.9), (8.10) using Eqs. (8.14), (8.15) reveals that $h_v^m = -\frac{1}{3}r^3 f \mathcal{B}_m$, $h_r^m = 0$, and $h_2^m = \frac{1}{3}r^4 \mathcal{B}_m$. Equation (8.23) then gives $\tilde{h}_v^m = -\frac{1}{3}r^3 f \mathcal{B}_m$ and $\tilde{h}_r^m = \frac{1}{3}r^3 \mathcal{B}_m$, up to smaller terms proportional to $d\mathcal{B}_m/dv \sim \mathcal{R}^{-3}$. From Eq. (8.24) we obtain $\Psi_{\text{RW}}^m = 0$. This curious result leads to the conclusion that the metric of Eqs. (8.7)–(8.10) is not sufficiently accurate to calculate the Regge-Wheeler function, and therefore the shear tensor.

Fortunately, the Regge-Wheeler equation (A.19) is sufficiently simple that it can be solved directly. Assuming (as we shall verify below) that derivatives of Ψ_{RW} with respect to v can be neglected compared with spatial derivatives, the Regge-Wheeler equation for $l = 2$ reduces to

$$\left[r(r-2M)\frac{d^2}{dr^2} + 2M\frac{d}{dr} - 6\left(1 - \frac{M}{r}\right) \right] \Psi_{\text{RW}} = 0. \quad (8.25)$$

The solution that is well behaved at the horizon is $\Psi_{\text{RW}} \propto r^3$, and to produce the correct metric perturbation we write

$$\Psi_{\text{RW}}^m(v, r) = -\frac{1}{12}r^3 \dot{\mathcal{B}}_m(v), \quad (8.26)$$

where the overdot indicates differentiation with respect to v . While $\mathcal{B}_m(v)$ scales as \mathcal{R}^{-2} , its time derivative scales as \mathcal{R}^{-3} and v -derivatives of the Regge-Wheeler function are indeed much smaller than its spatial derivatives.

To see that Eq. (8.26) is indeed the correct solution to the Regge-Wheeler equation, we reconstruct the metric perturbation in the preferred gauge and show that it agrees with the odd-parity sector of Eqs. (8.7)–(8.10). We note first that according to Eq. (8.24), $\Psi_{\text{RW}} = r^{-1}(\tilde{h}_v + f\tilde{h}_r)$, where we have removed the label m for simplicity. On the other hand, the field equation (A.17) implies $\partial_v \tilde{h}_r + \partial_r(\tilde{h}_v + f\tilde{h}_r) = 0$. Solving these equations yields $\tilde{h}_v = -\frac{1}{3}r^3 f \mathcal{B}$ and $\tilde{h}_r = \frac{1}{3}r^3 \mathcal{B}$, up to smaller terms involving $\dot{\mathcal{B}}$. The actual metric perturbations are then recovered by using Eq. (8.23) along with the gauge condition $h_r = 0$. The equation for \tilde{h}_r gives $(\frac{1}{2}\partial_r - r^{-1})h_2 = \tilde{h}_r$, and this differential equation has $h_2 = \frac{1}{3}r^4 \mathcal{B}$ as solution; this agrees with our previous expression. Finally, the equation for \tilde{h}_v gives $h_v + \frac{1}{2}\partial_v h_2 = \tilde{h}_v$, and we obtain $h_v = -\frac{1}{3}r^3 f \mathcal{B}$ up to a smaller term involving $\dot{\mathcal{B}}$; this also agrees with our previous expression. We conclude that the Regge-Wheeler function of Eq. (8.26) is indeed compatible with the metric of Eqs. (8.7)–(8.10).

Substituting Eq. (8.26) into Eq. (7.6) we obtain

$$\sigma_{AB}^{\text{odd}}(v, \theta^A) = -\frac{1}{12}r_+^4 \sum_m \dot{\mathcal{B}}_m(v) X_{AB}^m(\theta^A). \quad (8.27)$$

According to Eq. (8.15), this can also be written as $\sigma_{AB}^{\text{odd}} = \frac{1}{12}r_+^4 \dot{\mathcal{B}}_{AB}^*$.

D. Even-parity contribution to shear

We turn next to the even-parity sector of the metric perturbations; the reader is referred to the description given in subsection 4 of the Appendix.

When $l = 2$ the even-parity perturbations can be expanded as

$$h_{ij} = \sum_m h_{ij}^m(v, r) Y^m(\theta^A), \quad (8.28)$$

$$h_{iA} = \sum_m j_i^m(v, r) Y_{:A}^m(\theta^A), \quad (8.29)$$

$$h_{AB} = r^2 \sum_m G^m(v, r) Z_{AB}^m(\theta^A), \quad (8.30)$$

where we have incorporated our knowledge that $K^m = 3G^m$; this follows because the metric perturbation of Eq. (8.10) is tracefree [refer also to the discussion that precedes Eq. (7.4)]. The combinations

$$\tilde{h}_{ij}^m = h_{ij}^m - 2\varepsilon_{(i,j)}^m, \quad \tilde{K}^m = 3G^m - \frac{2}{r} r^{,i} \varepsilon_i^m, \quad (8.31)$$

where $\varepsilon_i^m = j_i^m - \frac{1}{2} r^2 G_{,i}^m$ are gauge invariant, and the Zerilli-Moncrief function [4, 13, 43] is defined by

$$\Psi_{\text{ZM}}^m \equiv \frac{r}{3} \left[\tilde{K}^m + \frac{2}{\Lambda} (r^{,i} r^{,j} \tilde{h}_{ij}^m - r r^{,i} \tilde{K}_{,i}^m) \right], \quad (8.32)$$

where $\Lambda = 4 + 6M/r$.

Comparison of Eqs. (8.28)–(8.30) with Eqs. (8.7)–(8.10) using Eqs. (8.11)–(8.13) reveals that $h_{vv}^m = -r^2 f^2 \mathcal{E}_m$, $j_v^m = -\frac{1}{3} r^3 f \mathcal{E}_m$, $G^m = -\frac{1}{3} r^2 (1 - 2M^2/r^2) \mathcal{E}_m$, and $K^m = 3G^m$, with all other components vanishing. From this information the Zerilli-Moncrief function can be computed straightforwardly (no need to solve the Zerilli equation directly), and its value on the horizon is found to be

$$\Psi_{\text{ZM}}^m(v, r_+) = -\frac{1}{6} r_+^3 \mathcal{E}_m(v). \quad (8.33)$$

Substituting this into Eq. (7.6) we obtain

$$\sigma_{AB}^{\text{even}}(v, \theta^A) = -\frac{1}{12} r_+^4 \sum_m \dot{\mathcal{E}}_m(v) Z_{AB}^m(\theta^A). \quad (8.34)$$

According to Eq. (8.13), this can also be written as $\sigma_{AB}^{\text{even}} = -\frac{1}{12} r_+^4 \dot{\mathcal{E}}_{AB}^*$.

E. Shear and Weyl tensor

The sum of Eqs. (8.27) and (8.34) gives the complete shear tensor,

$$\sigma_{AB}(v, \theta^A) = -\frac{1}{12} r_+^4 \sum_m \left[\dot{\mathcal{E}}_m(v) Z_{AB}^m(\theta^A) + \dot{\mathcal{B}}_m(v) X_{AB}^m(\theta^A) \right], \quad (8.35)$$

and we observe that this 2-tensor is properly tracefree.

We can use Eqs. (8.35) and Eq. (4.7) to calculate the Weyl tensor C_{AB} on the horizon. We may neglect time derivatives and write $C_{AB} = \kappa \sigma_{AB}$, which gives

$$C_{AB} = -\frac{1}{24} r_+^3 \sum_m \left[\dot{\mathcal{E}}_m Z_{AB}^m + \dot{\mathcal{B}}_m X_{AB}^m \right]. \quad (8.36)$$

This expression shows that the (dimensionless) Weyl curvature on the horizon is of order $(M/\mathcal{R})^3$, which is a factor $M/\mathcal{R} \ll 1$ smaller than the asymptotic value of the Weyl curvature (for $r \gg M$).

For future reference we calculate the Weyl scalar Ψ from C_{AB} ; this is defined by Eq. (5.6) and related to the Weyl tensor in Eq. (5.5). From this equation and the properties of the vectors e^A we infer that $\Psi = C_{AB} e^A e^B$. With Eqs. (A.9) and (A.10) we can relate the tensorial harmonics Z_{AB}^m and X_{AB}^m to the spin-weighted spherical harmonics ${}_{\pm 2} Y^m$. (Please note that the vectors e^A used in the Appendix are rescaled versions of the vectors used here: $e^A = \epsilon^A/r_+$; in the notation of the Appendix we have $\Psi = C_{AB} \epsilon^A \epsilon^B / r_+^2$.) Simple algebra then gives

$$\Psi(v, \theta^A) = -\frac{\sqrt{6}}{24} r_+ \sum_m \left[\dot{\mathcal{E}}_m(v) - i \dot{\mathcal{B}}_m(v) \right] {}_2 Y^m(\theta^A). \quad (8.37)$$

This reveals that $\Psi = O(M/\mathcal{R}^3)$ on the horizon, while $\Psi = O(1/\mathcal{R}^2)$ asymptotically (for $r \gg M$). This shows once more that the Weyl curvature on the horizon is suppressed by a factor $M/\mathcal{R} \ll 1$ with respect to its asymptotic value.

F. Fluxes

The shear tensor of Eq. (8.35) is equal to $\frac{1}{2} \mathcal{L}_k \gamma_{AB}^1 = \frac{1}{2} \mathcal{L}_t \gamma_{AB}^1$, and the metric perturbation $\gamma_{AB}^1(v, \theta^A)$ can be obtained by direct integration with respect to v . Differentiation with respect to ϕ then gives $\mathcal{L}_\phi \gamma_{AB}^1$; this can be worked out by using the explicit forms for Z_{AB}^m and X_{AB}^m gathered from subsection 2 of the Appendix. Finally, these results can be substituted into the flux formulae of Eqs. (4.22)–(4.24), and integration over $dS = r_+^2 \sin \theta d\theta d\phi$ is readily carried out using the known angular dependence contained in the spherical harmonics. A straightforward computation yields

$$\begin{aligned} \langle \dot{M} \rangle &= \frac{8M^6}{45} \langle 3\dot{\mathcal{E}}_0^2 + \dot{\mathcal{E}}_{1c}^2 + \dot{\mathcal{E}}_{1s}^2 + 4\dot{\mathcal{E}}_{2c}^2 + 4\dot{\mathcal{E}}_{2s}^2 \\ &\quad + 3\dot{\mathcal{B}}_0^2 + \dot{\mathcal{B}}_{1c}^2 + \dot{\mathcal{B}}_{1s}^2 + 4\dot{\mathcal{B}}_{2c}^2 + 4\dot{\mathcal{B}}_{2s}^2 \rangle \end{aligned}$$

and

$$\begin{aligned} \langle \dot{J} \rangle &= -\frac{8M^6}{45} \langle \dot{\mathcal{E}}_{1c} \mathcal{E}_{1s} - \dot{\mathcal{E}}_{1s} \mathcal{E}_{1c} + 8\dot{\mathcal{E}}_{2c} \mathcal{E}_{2s} - 8\dot{\mathcal{E}}_{2s} \mathcal{E}_{2c} \\ &\quad + \dot{\mathcal{B}}_{1c} \mathcal{B}_{1s} - \dot{\mathcal{B}}_{1s} \mathcal{B}_{1c} + 8\dot{\mathcal{B}}_{2c} \mathcal{B}_{2s} - 8\dot{\mathcal{B}}_{2s} \mathcal{B}_{2c} \rangle, \end{aligned}$$

where \mathcal{E}_m and \mathcal{B}_m are the harmonic components of the tidal gravitational fields introduced in Eqs. (8.16)–(8.20).

These results can be expressed in terms of invariants formed from \mathcal{E}_{ab} and \mathcal{B}_{ab} , the components of the tidal fields in the local asymptotic rest frame of the moving black hole. We also need the derivatives of these fields with respect to v (denoted with an overdot), and the unit vector $s^a \equiv (0, 0, 1)$ that points in the direction of the third coordinate axis. (This direction is preferred because the angles θ and ϕ refer to it.) In terms of these quantities, the previous expressions become

$$\langle \dot{M} \rangle = \frac{16M^6}{45} \langle \dot{\mathcal{E}}_{ab} \dot{\mathcal{E}}^{ab} + \dot{\mathcal{B}}_{ab} \dot{\mathcal{B}}^{ab} \rangle \quad (8.38)$$

and

$$\langle \dot{J} \rangle = -\frac{32M^6}{45} \varepsilon_{acd} \langle \dot{\mathcal{E}}^a_b \mathcal{E}^{bc} + \dot{\mathcal{B}}^a_b \mathcal{B}^{bc} \rangle s^d, \quad (8.39)$$

where ε_{acd} is the three-dimensional permutation symbol. We also have $(\kappa/8\pi) \langle \dot{A} \rangle = \langle \dot{M} \rangle$. We note that when $\langle \dot{J} \rangle$ is expressed in the covariant form of Eq. (8.39), what is actually meant by $\langle \dot{J} \rangle$ is the rate of change of the component of the angular-momentum vector in the direction of s^a ; in three-dimensional vectorial language appropriate in the local asymptotic rest frame, $\langle \dot{J} \rangle \equiv \langle \dot{J}^a \rangle s_a$, where J^a is the vectorial angular momentum.

Alternatively, the flux formulae can be expressed in terms of the spacetime tensors of Eq. (8.4) and (8.5). The translation is effected by Eq. (8.3) and the identity $\varepsilon_{abc} = u^\mu \varepsilon_{\mu\alpha\beta\gamma} e_a^\alpha e_b^\beta e_c^\gamma$, where $\varepsilon_{\mu\alpha\beta\gamma}$ is the Levi-Civita tensor. We find

$$\langle \dot{M} \rangle = \frac{16M^6}{45} \langle \dot{\mathcal{E}}_{\alpha\beta} \dot{\mathcal{E}}^{\alpha\beta} + \dot{\mathcal{B}}_{\alpha\beta} \dot{\mathcal{B}}^{\alpha\beta} \rangle \quad (8.40)$$

and

$$\langle \dot{J} \rangle = -\frac{32M^6}{45} u^\mu \varepsilon_{\mu\alpha\gamma\delta} \langle \dot{\mathcal{E}}^\alpha_\beta \mathcal{E}^{\beta\gamma} + \dot{\mathcal{B}}^\alpha_\beta \mathcal{B}^{\beta\gamma} \rangle s^\delta, \quad (8.41)$$

where $s^\alpha = s^a e_a^\alpha$ is a unit spatial vector, and $\dot{\mathcal{E}}_{\alpha\beta} \equiv \mathcal{E}_{\alpha\beta;\mu} u^\mu$, $\dot{\mathcal{B}}_{\alpha\beta} \equiv \mathcal{B}_{\alpha\beta;\mu} u^\mu$ are the proper-time derivative of the tidal gravitational fields. We recall that in this SH/SM description, all vectors and tensors refer to the spacetime of the external universe in which the black hole moves. From Eqs. (8.38)–(8.41) we gather that $\langle \dot{M} \rangle$ scales as M^6/\mathcal{R}^6 , while $\langle \dot{J} \rangle$ scales as M^6/\mathcal{R}^5 . To the best of my knowledge, Eqs. (8.38)–(8.41) have never appeared before in the literature.

G. Comparison with Thorne, Hartle, and Zhang

The rate of change of angular momentum for a general body interacting with a tidal gravitational field was calculated, in the regime $M/\mathcal{R} \ll 1$, by Thorne and Hartle [33]; they obtained the expression

$$\langle \dot{J}^a \rangle = -\varepsilon^a_{bc} \left\langle M^b_d \mathcal{E}^{dc} + \frac{4}{3} J^b_d \mathcal{B}^{dc} \right\rangle, \quad (8.42)$$

where M_{ab} is the body's mass quadrupole moment, while J_{ab} is its current quadrupole moment (both defined in terms of the structure of the gravitational field outside the arbitrary body). Zhang [52], on the other hand, calculated the rate at which the body changes its mass; he obtained an expression equivalent to

$$\langle \dot{M} \rangle = \frac{1}{2} \left\langle M_{ab} \dot{\mathcal{E}}^{ab} + \frac{4}{3} J_{ab} \dot{\mathcal{B}}^{ab} \right\rangle. \quad (8.43)$$

We wish to show that our previous results are compatible with these expressions.

An isolated Schwarzschild black hole is spherically symmetric, and its intrinsic quadrupole moments vanish: $M_{ab} = J_{ab} = 0$. But a black hole immersed in an external universe is tidally distorted and therefore acquires nonvanishing moments. It is easy to see that Eqs. (8.38), (8.39) are compatible with the general results of Eqs. (8.42), (8.43) if the tidally-induced quadrupole moments of a nonrotating black hole are given by

$$M_{ab} = \frac{32M^6}{45} \dot{\mathcal{E}}_{ab} \quad (8.44)$$

and

$$J_{ab} = \frac{8M^6}{15} \dot{\mathcal{B}}_{ab}. \quad (8.45)$$

These scale as $M^3(M/\mathcal{R})^3$, and they both involve the *rates of change* of the tidal gravitational fields. This is a rather surprising result, as one would expect the quadrupole deformation of a tidally-distorted body to be proportional to the tidal gravitational field itself, instead of its time derivative. But the time derivative is present, and its origin can be traced back to Eq. (8.37): the Weyl curvature at the horizon is proportional to the time derivative of the asymptotic curvature. Since it is the horizon curvature that produces the black-hole distortion, this explains why a time derivative enters Eqs. (8.44) and (8.45).

H. Black hole in a circular binary: Slow-motion approximation

If we specialize to a slow-motion situation, the tidal gravitational fields of the external universe can be approximated by

$$\mathcal{E}_{ab} \simeq \Phi_{,ab}, \quad \mathcal{B}_{ab} \simeq 0, \quad (8.46)$$

where Φ is a Newtonian potential. For concreteness we take the Newtonian field to be produced by an external body of mass M_{ext} located at $\mathbf{r}_{\text{ext}}(t)$ relative to the system's center of mass. Then $\Phi(\mathbf{x}) = -M_{\text{ext}}/|\mathbf{x} - \mathbf{r}_{\text{ext}}|$, and we exclude the contribution $-M/|\mathbf{x} - \mathbf{r}|$ from the black hole because this does not produce a tidal field at the position $\mathbf{r}(t)$ of the black hole. Also for concreteness we take the orbit to be circular, and we let $b \equiv |\mathbf{r} - \mathbf{r}_{\text{ext}}|$ be

the constant relative separation between the two bodies. The orbital angular velocity is

$$\Omega = \sqrt{\frac{M + M_{\text{ext}}}{b^3}}, \quad (8.47)$$

and the relative position vector is $\mathbf{r} - \mathbf{r}_{\text{ext}} \equiv \boldsymbol{\rho} = b(\cos \Omega t, \sin \Omega t, 0) \equiv b\hat{\boldsymbol{\rho}}(t)$. The relative velocity vector is $\mathbf{V} = b\Omega(-\sin \Omega t, \cos \Omega t, 0) \equiv b\Omega\hat{\boldsymbol{\phi}}(t)$. For simplicity we align the spin vector in the direction of the orbital angular momentum: $\mathbf{s} = (0, 0, 1)$.

Using this information we calculate $\mathcal{E}_{ab}(t) = (M_{\text{ext}}/b^3)(\delta_{ab} - 3\hat{\rho}_a\hat{\rho}_b)$ and

$$\dot{\mathcal{E}}_{ab} = -\frac{3M_{\text{ext}}\Omega}{b^3}(\hat{\rho}_a\hat{\phi}_b + \hat{\phi}_a\hat{\rho}_b).$$

Substituting this into Eq. (8.38) gives

$$\langle \dot{M} \rangle = \frac{32}{5}\eta^2 \left(\frac{M}{M + M_{\text{ext}}} \right)^4 V^{18}, \quad (8.48)$$

where $\eta = MM_{\text{ext}}/(M + M_{\text{ext}})^2$ is a dimensionless reduced-mass parameter and

$$V = \sqrt{\frac{M + M_{\text{ext}}}{b}} \ll 1 \quad (8.49)$$

is the relative orbital velocity. The rate of change of the hole's angular momentum can be obtained directly from this and the rigid-rotation relation $\langle \dot{J} \rangle = \Omega^{-1}\langle \dot{M} \rangle$; this gives

$$\langle \dot{J} \rangle = \frac{32}{5}\eta^2 \left(\frac{M}{M + M_{\text{ext}}} \right)^4 (M + M_{\text{ext}})V^{15}. \quad (8.50)$$

These results agree (in a limit of no black-hole rotation) with earlier expressions obtained by Alvi [18]. In the regime $M_{\text{ext}} \ll M$ they also agree with earlier results derived by Poisson and Sasaki [16].

I. Black hole in a circular binary: Small-hole approximation

We now allow the black hole to move rapidly in the strong gravitational field of another Schwarzschild hole of mass M_{ext} ; to comply with the SH/SM condition $M/\mathcal{R} \ll 1$ we now impose $M/M_{\text{ext}} \ll 1$, as was discussed in Sec. VIII A. Once more we choose the orbit to be circular. In the standard Schwarzschild coordinates (t, r, θ, ϕ) used in the background spacetime of the large black hole, the orbital radius is b and the four-velocity of the small hole is $u^\alpha = \gamma(1, 0, 0, \Omega)$, where $\gamma = (1 - 3M_{\text{ext}}/b)^{-1/2}$ is a normalization factor and

$$\Omega = \sqrt{\frac{M_{\text{ext}}}{b^3}} \quad (8.51)$$

is the angular velocity. We again align the spin vector in the direction of the orbital angular momentum, so

that $s^\alpha = (0, 0, -1/b, 0)$. Calculation of $\mathcal{E}_{\alpha\beta}$, $\mathcal{B}_{\alpha\beta}$ using Eqs. (8.4), (8.5), and substitution into Eqs. (8.40), (8.41) gives

$$\langle \dot{M} \rangle = \frac{32}{5} \left(\frac{M}{M_{\text{ext}}} \right)^6 V^{18} \frac{(1 - V^2)(1 - 2V^2)}{(1 - 3V^2)^2} \quad (8.52)$$

and

$$\langle \dot{J} \rangle = \frac{32}{5} \left(\frac{M}{M_{\text{ext}}} \right)^6 M_{\text{ext}} V^{15} \frac{(1 - V^2)(1 - 2V^2)}{(1 - 3V^2)^2}, \quad (8.53)$$

where $V = \sqrt{M_{\text{ext}}/b} \leq 6^{-1/2}$ is a measure of the hole's orbital velocity. These results agree with those of the preceding subsection in a common domain of validity defined by $M \ll M_{\text{ext}}$ and $V \ll 1$. To the best of my knowledge, the results of Eqs. (8.52) and (8.53), complete with all-order relativistic corrections, have never appeared before in the literature.

IX. SMALL-HOLE/SLOW-MOTION APPROXIMATION FOR A KERR BLACK HOLE

In this section I apply the SH/SM approximation introduced in Sec. V III A to the flux formulae derived in Sec. V D, Eqs. (5.23)–(5.29). I will proceed much as in Sec. VIII, except that I will deal with curvature perturbations — and the Teukolsky equation [2] — instead of metric perturbations.

A. Flux formulae in the SH/SM approximation

We begin by isolating, in Eqs. (5.27) and (5.28), the terms for which $m = 0$:

$$\begin{aligned} \langle \dot{M} \rangle &= \frac{1}{2}(r_+^2 + a^2) \int \langle |\Phi_+^0|^2 \rangle \sin \theta d\theta \\ &+ \frac{r_+^2 + a^2}{4\kappa} \sum_{m \neq 0} \left[2\kappa \int \langle |\Phi_+^m|^2 \rangle \sin \theta d\theta \right. \\ &\left. - im\Omega_H \int \langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \rangle \sin \theta d\theta \right], \quad (9.1) \end{aligned}$$

$$\begin{aligned} \langle \dot{J} \rangle &= -\frac{r_+^2 + a^2}{4\kappa} \sum_{m \neq 0} (im) \\ &\times \int \langle \bar{\Phi}_+^m \Phi_-^m - \Phi_+^m \bar{\Phi}_-^m \rangle \sin \theta d\theta. \quad (9.2) \end{aligned}$$

We recall the definitions

$$\Phi_+^m(v, \theta) = e^{\kappa v} \int_v^\infty e^{-(\kappa - im\Omega_H)v'} \Psi^m(v', \theta) dv', \quad (9.3)$$

$$\Phi_-^m(v, \theta) = \int_{-\infty}^v e^{im\Omega_H v'} \Psi^m(v', \theta) dv', \quad (9.4)$$

where

$$\Psi(v, r_+, \theta, \psi) = \sum_m \Psi^m(v, \theta) e^{im\psi} \quad (9.5)$$

is the horizon Weyl scalar introduced in Eq. (5.6).

To see how we may specialize Eqs. (9.3) and (9.4) to the SH/SM approximation, consider first an integral of the form $F_-(v) = \int_{-\infty}^v e^{i\omega v'} f(v') dv'$, and suppose that $f(v')$ varies on a time scale τ that is large compared with ω^{-1} . (We also suppose that f switches off sufficiently rapidly in the infinite past so that the integral converges.) Then $F_-(v)$ can be evaluated by successive integration by parts, each integration generating a relative correction of order $\epsilon \equiv (\omega\tau)^{-1} \ll 1$. To leading order, $F_-(v) = -i\omega^{-1} f(v) e^{i\omega v} [1 + O(\epsilon)]$. Consider next an integral of the form $F_+(v) = \int_v^\infty e^{-\lambda v'} f(v') dv'$, where the real part of λ is assumed to be positive; here we suppose that $\epsilon' \equiv (\lambda\tau)^{-1} \ll 1$. Integration by parts in this case leads to $F_+(v) = \lambda^{-1} f(v) e^{-\lambda v} [1 + O(\epsilon')]$. These simple manipulations allow us, within the stated conditions, to approximate the integrals by local expressions. This is the technique we shall employ to evaluate Eqs. (9.3) and (9.4).

In this way we obtain

$$\Phi_+^m = \frac{\Psi^m(v, \theta) e^{im\Omega_H v}}{\kappa - im\Omega_H} \left[1 + O\left(\frac{1}{(\kappa - im\Omega_H)\tau}\right) \right]$$

and

$$\Phi_-^m = \frac{\Psi^m(v, \theta) e^{im\Omega_H v}}{im\Omega_H} \left[1 + O\left(\frac{1}{im\Omega_H\tau}\right) \right],$$

where τ is the time scale associated with changes in $\Psi^m(v, \theta)$. To see how the conditions $\kappa\tau \gg 1$ and $\Omega_H\tau \gg 1$ relate to the SH/SM approximation, we first recall that changes in $\Psi(v, r_+, \theta, \psi)$ are governed by processes taking place in the external universe, so that $\tau \sim \mathcal{R}$. We also express κ and Ω_H in terms of the black-hole mass M and its dimensionless rotational parameter $\chi \equiv a/M \equiv J/M^2$:

$$\kappa = \frac{\sqrt{1 - \chi^2}}{2M(1 + \sqrt{1 - \chi^2})}, \quad \Omega_H = \frac{\chi}{2M(1 + \sqrt{1 - \chi^2})}; \quad (9.6)$$

we recall that χ is limited to the interval $0 \leq \chi \leq 1$ and that $r_+ = M(1 + \sqrt{1 - \chi^2})$. In orders of magnitude we have $\kappa \sim 1/M$ and $\Omega_H \sim \chi/M$, and to achieve $\kappa\tau \gg 1$ and $\Omega_H\tau \gg 1$ we need $M/\mathcal{R} \ll 1$ and $M/\mathcal{R} \ll \chi$, respectively. The stronger condition is

$$M/\mathcal{R} \ll \chi, \quad (9.7)$$

and we take this to be the precise statement of the small-hole/slow-motion condition when we deal with rotating black holes. Notice that by virtue of Eq. (9.7), the no-rotation limit $\chi \rightarrow 0$ will be inaccessible in our analysis; this case was treated separately in Sec. VIII. We shall write our previous results as

$$\Phi_+^m = \frac{\Psi^m(v, \theta) e^{im\Omega_H v}}{\kappa - im\Omega_H} \left[1 + O(M/\mathcal{R}) \right], \quad (9.8)$$

$$\Phi_-^m = \frac{\Psi^m(v, \theta) e^{im\Omega_H v}}{im\Omega_H} \left[1 + O(M/\mathcal{R}) \right], \quad (9.9)$$

with the understanding that the error terms are really of order $M/(\chi\mathcal{R})$, and therefore small by virtue of Eq. (9.7). For the remainder of this section we assume that χ is of order unity, and we allow ourselves to lose sight of this distinction.

Substituting Eqs. (9.8) and (9.9) into Eq. (9.1) reveals that each $m \neq 0$ term vanishes to leading order in M/\mathcal{R} ; what remains is

$$\langle \dot{M} \rangle = \frac{r_+^2 + a^2}{2\kappa^2} \int \langle |\Psi^0(v, \theta)|^2 \rangle \sin\theta d\theta + O(M^5/\mathcal{R}^5). \quad (9.10)$$

The scaling of the error term follows from the facts that each contribution to a $m \neq 0$ term is of order $(M/\mathcal{R})^4$, but that the cancellation suppresses this by a factor of (at least) M/\mathcal{R} . An *a priori* estimate of the surviving term in Eq. (9.10) indicates that it is of order $(M/\mathcal{R})^4$, but we shall see that it is in fact of order $(M/\mathcal{R})^6$. Inserting Eqs. (9.8) and (9.9) into Eq. (9.2) gives

$$\langle J \rangle = -\frac{r_+^2 + a^2}{2\Omega_H} \sum_{m \neq 0} (\kappa^2 + m^2\Omega_H^2)^{-1} \times \int \langle |\Psi^m(v, \theta)|^2 \rangle \sin\theta d\theta, \quad (9.11)$$

and this is of order M^5/\mathcal{R}^4 .

B. Weyl scalar: asymptotic values

To proceed further we must compute the functions $\Psi^m(v, \theta)$ that enter into the simplified flux formulae of Eqs. (9.10) and (9.11). This requires solving the Teukolsky equation for the Weyl scalar $\psi_0(v, r, \theta, \psi)$, with appropriate boundary data provided by the conditions in the external universe. In this subsection I specify these boundary conditions; in the next subsection (Sec. IX C) I tackle the integration of the Teukolsky equation and construct $\Psi^m(v, \theta)$ from the solution. My presentation in these two subsections will not stray very far from what is contained in Sec. 3.5 of the book by D'Eath [19]; and my end results will be equivalent to his.

The function $\psi_0(v, r, \theta, \psi)$ we shall work with is

$$\psi_0 = -C_{\alpha\gamma\beta\delta}^1 k^\alpha m^\gamma k^\beta m^\delta, \quad (9.12)$$

where $C_{\alpha\gamma\beta\delta}^1$ is the perturbation of the Weyl tensor, while $k^\alpha \equiv k^\alpha(\mathbf{K})$ and $m^\alpha \equiv m^\alpha(\mathbf{K})$ are members of *Kinnersley's* null tetrad [2, 36]. The relation between $\psi_0 \equiv \psi_0(\mathbf{K})$ and $\Psi(v, r_+, \theta, \psi)$ is given by Eq. (5.6).

We wish to calculate ψ_0 in the asymptotic regime $r \gg r_+$, assuming that r is still much smaller than \mathcal{R} , the radius of curvature of the external spacetime. The asymptotic values will be constructed from $\mathcal{E}_{ab}(v)$ and $\mathcal{B}_{ab}(v)$, the tidal gravitational fields introduced in Sec. VIII A — Eq. (8.2); recall that lower-case Latin indices refer to the hole's local asymptotic rest frame, and that the hole's angular-momentum vector is directed along the third coordinate axis.

In the asymptotic regime $r \gg r_+$ the coordinates (v, r, θ, ψ) are easily related to a set of Lorentzian coordinates (t, x, y, z) that are adapted to the frame (u^α, e_a^α) ; the relations are $t = v - r$, $x = r \sin \theta \cos \psi$, $y = r \sin \theta \sin \psi$, and $z = r \cos \theta$. In this regime the null vector k^α can be decomposed as $k^\alpha \sim u^\alpha + r^\alpha$, where u^α is the hole's velocity vector in the external spacetime, and r^α is a spacelike vector that points radially outward. In the asymptotic Lorentzian coordinates (t, x, y, z) we have $u^\alpha = (1, 0, 0, 0)$ and $r^\alpha = (0, \sin \theta \cos \psi, \sin \theta \sin \psi, \cos \theta)$. In the limit we also have $m^\alpha \sim 2^{-1/2}(0, \cos \theta \cos \psi - i \sin \psi, \cos \theta \sin \psi + i \cos \psi, -\sin \theta)$.

We have seen in Sec. VIII A that in the vicinity of the black hole (the region $r \ll \mathcal{R}$, which includes the asymptotic region $r \gg r_+$), the Weyl tensor of the external spacetime can be decomposed into the symmetric, trace-free fields \mathcal{E}_{ab} and \mathcal{B}_{ab} . In the asymptotic coordinates (t, x, y, z) the decomposition is given by $C_{tatb} = \mathcal{E}_{ab}$, $C_{tabc} = -\varepsilon_{bcd}\mathcal{B}_a^d$, and $C_{acbd} = \delta_{ab}\mathcal{E}_{cd} + \delta_{cd}\mathcal{E}_{ab} - \delta_{ad}\mathcal{E}_{bc} - \delta_{bc}\mathcal{E}_{ad}$. In these relations the Weyl tensor, and the tidal gravitational fields \mathcal{E}_{ab} and \mathcal{B}_{ab} , are evaluated on the black hole's world line in the external spacetime; they are functions of t (or v) only.

According to Eq. (9.12), the asymptotic value of ψ_0 is $-C_{\alpha\gamma\beta\delta}(u^\alpha + r^\alpha)m^\gamma(u^\beta + r^\beta)m^\delta$, with $C_{\alpha\gamma\beta\delta}$ denoting the Weyl tensor of the external spacetime evaluated on the hole's world line. Using the information provided in the preceding two paragraphs, we obtain the explicit expression

$$\psi_0 \sim -2\mathcal{E}_{ab}m^am^b + 2r^am^b\varepsilon_{abc}\mathcal{B}_d^cm^d, \quad (9.13)$$

where r^a and m^a are the spatial components of the vectors r^α and m^α , respectively. The angular dependence contained in these vectors is encoded in spin-weighted spherical harmonics of degree $l = 2$ (see subsection 2 of the Appendix for a definition). It is convenient to introduce a set given by

$${}_2Y_2^0(\theta, \psi) = -\frac{3}{2}\sin^2\theta, \quad (9.14)$$

$${}_2Y_2^{\pm 1}(\theta, \psi) = -\sin\theta(\cos\theta \mp 1)e^{\pm i\psi}, \quad (9.15)$$

$${}_2Y_2^{\pm 2}(\theta, \psi) = \frac{1}{4}(1 \mp 2\cos\theta + \cos^2\theta)e^{\pm 2i\psi}. \quad (9.16)$$

This set is not normalized; we have instead $\int |{}_2Y_2^0|^2 d\Omega = 24\pi/5$, $\int |{}_2Y_2^{\pm 1}|^2 d\Omega = 16\pi/5$, and $\int |{}_2Y_2^{\pm 2}|^2 d\Omega = 4\pi/5$. If we also introduce

$$\alpha_0 = \mathcal{E}_{11} + \mathcal{E}_{22}, \quad (9.17)$$

$$\alpha_{\pm 1} = \mathcal{E}_{13} \mp i\mathcal{E}_{23}, \quad (9.18)$$

$$\alpha_{\pm 2} = \mathcal{E}_{11} - \mathcal{E}_{22} \mp 2i\mathcal{E}_{12} \quad (9.19)$$

and

$$\beta_0 = \mathcal{B}_{11} + \mathcal{B}_{22}, \quad (9.20)$$

$$\beta_{\pm 1} = \mathcal{B}_{13} \mp i\mathcal{B}_{23}, \quad (9.21)$$

$$\beta_{\pm 2} = \mathcal{B}_{11} - \mathcal{B}_{22} \mp 2i\mathcal{B}_{12}, \quad (9.22)$$

then it is straightforward to show that Eq. (9.13) is equivalent to

$$\psi_0 \sim -\sum_m [\alpha_m(v) + i\beta_m(v)] {}_2Y_2^m(\theta, \psi). \quad (9.23)$$

This is the asymptotic value of the Weyl scalar $\psi_0(v, r, \theta, \psi)$ in the regime $r_+ \ll r \ll \mathcal{R}$, expressed in terms of the tidal gravitational fields $\mathcal{E}_{ab}(v)$ and $\mathcal{B}_{ab}(v)$.

C. Teukolsky equation

To relate $\Psi^m(v, \theta)$ to the asymptotic value of ψ_0 obtained in Eq. (9.23) it is necessary to solve the Teukolsky equation [2] for $s = 2$ and $l = 2$. Because the v -dependence of the solution enters through the tidal gravitational fields $\mathcal{E}_{ab}(v)$ and $\mathcal{B}_{ab}(v)$, and because this dependence is slow (time scale of order \mathcal{R}), it is actually sufficient to integrate the *time-independent* Teukolsky equation. We therefore write

$$\psi_0(v, r, \theta, \psi) = -\sum_m [\alpha_m(v) + i\beta_m(v)] R_m(r) {}_2Y_2^m(\theta, \psi), \quad (9.24)$$

where $R_m(r)$ is a radial function normalized so that $R_m(r \gg r_+) \sim 1$; this function must be a solution to Eq. (2.10) of Teukolsky and Press [1], in which we set $\omega = 0$.

The explicit form of the radial equation is

$$\left\{ x(1+x) \frac{d^2}{dx^2} + [3(2x+1) + 2im\gamma] \frac{d}{dx} + 4im\gamma \frac{2x+1}{x(1+x)} \right\} R_m(x) = 0, \quad (9.25)$$

where

$$x = \frac{r - r_+}{r_+ - r_-} \quad (9.26)$$

is a new independent variable, and

$$\gamma = \frac{a}{r_+ - r_-}; \quad (9.27)$$

we recall that $r_\pm = M \pm \sqrt{M^2 - a^2}$. The relevant solution to Eq. (9.25) is

$$R_m(r) = A_m x^{-2} (1+x)^{-2} F(-4, 1; -1 + 2im\gamma; -x), \quad (9.28)$$

in which the hypergeometric function is actually an ordinary polynomial of degree 4 in the variable $-x$. Equation (9.28) is essentially Eq. (5) from Ref. [18], and the superficial difference is attributed to the fact that Alvi works in Boyer-Lindquist coordinates instead of our Kerr coordinates. This is also Eq. (3.7) in Chapter VI of Teukolsky's Ph.D. dissertation [53], and Eq. (9.28) is equivalent to Eq. (3.5.7) of Ref. [19]. The constant A_m must be chosen so that the radial function approaches unity when

$x \rightarrow \infty$; a simple calculation shows that it must be given by

$$A_m = -\frac{i}{6}m\gamma(1 + im\gamma)(1 + 4m^2\gamma^2) \quad (9.29)$$

when $m \neq 0$.

The case $m = 0$ must be considered separately. It is formally obtained by setting $\gamma = 0$ in Eq. (9.25), and Eq. (9.27) shows that this amounts to letting $a = 0$. For $m = 0$, therefore, Eq. (9.25) reduces to Teukolsky's radial equation in Schwarzschild spacetime; the independent variable is now given by $x = r/r_+ - 1$. The relation between $\Psi^m(v, \theta)$ and the asymptotic value of ψ_0 was already worked out, for Schwarzschild spacetime, in Sec. VIII E — Eq. (8.37). There it was revealed that it is of the schematic form $\Psi^m \sim r_+ \psi_0(v, r \gg r_+, \theta, \psi)$, and that it involves a derivative of the asymptotic field with respect to v . This relation is very different from what was anticipated in Eq. (9.24), and therefore different from what is known to be true for $m \neq 0$. These considerations imply that for $m = 0$ and $a \neq 0$, the relation between $\Psi^m(v, \theta)$ and the asymptotic value of ψ_0 comes with an additional factor of M/\mathcal{R} relative to terms with $m \neq 0$. We conclude that it is appropriate to neglect the $m = 0$ term in Eq. (9.24), which becomes

$$\begin{aligned} \psi_0 = & - \sum_{m \neq 0} A_m [\alpha_m(v) + i\beta_m(v)] x^{-2} (1+x)^{-2} \\ & \times F(-4, 1; -1 + 2im\gamma; -x) {}_2Y_2^m(\theta, \psi), \end{aligned} \quad (9.30)$$

where $x = (r - r_+)/r_+$, $\gamma = a/r_+$, A_m is given by Eq. (9.29), and $\alpha_m(v)$, $\beta_m(v)$ are listed in Eqs. (9.17)–(9.22).

The functions $\Psi^m(v, \theta)$ are obtained by substituting Eq. (9.30) into Eq. (5.6) and taking the limit $r \rightarrow r_+$, or $x \rightarrow 0$; we recall that $\psi_0(v, r, \theta, \psi)$ is the Weyl scalar constructed with the Kinnersley tetrad, and that $\Psi(v, r_+, \theta, \psi)$ is decomposed as in Eq. (9.5). Simple algebra, using $a = \chi M$, $r_{\pm} = M(1 \pm \sqrt{1 - \chi^2})$, and $\gamma = \frac{1}{2}\chi(1 - \chi^2)^{-1/2}$, yields

$$\begin{aligned} \Psi^m(v, \theta) = & -\frac{im\chi(1 - \chi^2)^{3/2}}{12(1 + \sqrt{1 - \chi^2})^2} (1 + im\gamma)(1 + 4m^2\gamma^2) \\ & \times [\alpha_m(v) + i\beta_m(v)] {}_2Y_2^m(\theta, 0). \end{aligned} \quad (9.31)$$

This result holds when $m \neq 0$, and it reveals that $\Psi^{m \neq 0}$ is of order \mathcal{R}^{-2} ; as we have seen, when $m = 0$ we have instead the Schwarzschild result $\Psi^0 = O(M/\mathcal{R}^3)$.

D. Fluxes

We now insert Eq. (9.31) into the approximate flux formulae of Eqs. (9.10) and (9.11). The fact that $\Psi^0 = O(M/\mathcal{R}^3)$ implies that the error term of Eq. (9.10) is in fact dominant, and we obtain

$$\langle \dot{M} \rangle = O(M^5/\mathcal{R}^5). \quad (9.32)$$

This result indicates that to calculate $\langle \dot{M} \rangle$ requires information that is not accessible to the leading-order analysis carried out here. To go beyond this leading-order calculation should be feasible, but this lies beyond the scope of this work.

A more definite result can be obtained for $\langle \dot{J} \rangle$. Substitution of Eq. (9.31) into Eq. (9.11) and integration over θ — recall the explicit forms of the spin-weighted spherical harmonics specified by Eqs. (9.14)–(9.16) — returns

$$\begin{aligned} \langle \dot{J} \rangle = & -\frac{M^5\chi}{45} \sum_{m \neq 0} [1 + (m^2 - 1)\chi^2] [4 + (m^2 - 4)\chi^2] \\ & \times \langle |\alpha_m(v) + i\beta_m(v)|^2 \rangle \end{aligned}$$

after simplification; notice that a factor of m^2 is canceled by the integral $\int |{}_2Y_2^{m \neq 0}(\theta, 0)|^2 \sin \theta d\theta = 8/(5m^2)$. This becomes

$$\begin{aligned} \langle \dot{J} \rangle = & -\frac{2}{45}M^5\chi \left[(4 - 3\chi^2) \langle \mathcal{E}_{13}^2 + \mathcal{E}_{23}^2 + \mathcal{B}_{13}^2 + \mathcal{B}_{23}^2 \rangle \right. \\ & + 4(1 + 3\chi^2) \langle (\mathcal{E}_{11} - \mathcal{E}_{22})^2 + 4\mathcal{E}_{12}^2 \\ & \left. + (\mathcal{B}_{11} - \mathcal{B}_{22})^2 + 4\mathcal{B}_{12}^2 \rangle \right] \end{aligned}$$

after using Eqs. (9.17)–(9.22).

At this stage we introduce the invariants

$$E_1 = \mathcal{E}_{ab}\mathcal{E}^{ab} = \mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta}, \quad (9.33)$$

$$E_2 = \mathcal{E}_{ab}s^b\mathcal{E}_c^a s^c = \mathcal{E}_{\alpha\beta}s^\beta\mathcal{E}_\gamma^\alpha s^\gamma, \quad (9.34)$$

$$E_3 = (\mathcal{E}_{ab}s^a s^b)^2 = (\mathcal{E}_{\alpha\beta}s^\alpha s^\beta)^2, \quad (9.35)$$

and

$$B_1 = \mathcal{B}_{ab}\mathcal{B}^{ab} = \mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta}, \quad (9.36)$$

$$B_2 = \mathcal{B}_{ab}s^b\mathcal{B}_c^a s^c = \mathcal{B}_{\alpha\beta}s^\beta\mathcal{B}_\gamma^\alpha s^\gamma, \quad (9.37)$$

$$B_3 = (\mathcal{B}_{ab}s^a s^b)^2 = (\mathcal{B}_{\alpha\beta}s^\alpha s^\beta)^2, \quad (9.38)$$

where the unit vector s^a gives the direction of the black hole's spin in the local asymptotic rest frame, and $s^\alpha = s^a e_a^\alpha$ is the corresponding spacetime vector. We therefore have $J^a = J s^a$, $J = \chi M^2$, and $\langle \dot{J} \rangle = \langle \dot{J}^a \rangle s_a$. In terms of these invariants we have, for example, $\mathcal{E}_{13}^2 + \mathcal{E}_{23}^2 = E_2 - E_3$ and $(\mathcal{E}_{11} - \mathcal{E}_{22})^2 + 4\mathcal{E}_{12}^2 = 2E_1 - 4E_2 + E_3$. Our final expression for the rate of change of angular momentum is

$$\begin{aligned} \langle \dot{J} \rangle = & -\frac{2}{45}M^5\chi \left[8(1 + 3\chi^2) \langle E_1 + B_1 \rangle \right. \\ & - 3(4 + 17\chi^2) \langle E_2 + B_2 \rangle \\ & \left. + 15\chi^2 \langle E_3 + B_3 \rangle \right]. \end{aligned} \quad (9.39)$$

This result reveals that $\langle \dot{J} \rangle = O(M^5/\mathcal{R}^4)$.

The first law of black-hole mechanics implies that the rate of change of the horizon area is given by $(\kappa/8\pi)\langle \dot{A} \rangle = -\Omega_H \langle \dot{J} \rangle + O(M^5/\mathcal{R}^5)$, with a leading term scaling as

M^4/\mathcal{R}^4 . The ratio Ω_{H}/κ can be expressed in terms of M and $\chi \equiv a/M$, and we obtain

$$\begin{aligned} \langle \dot{A} \rangle = & \frac{16\pi}{45} \frac{M^5 \chi^2}{\sqrt{1-\chi^2}} \left[8(1+3\chi^2) \langle E_1 + B_1 \rangle \right. \\ & - 3(4+17\chi^2) \langle E_2 + B_2 \rangle \\ & \left. + 15\chi^2 \langle E_3 + B_3 \rangle \right]; \end{aligned} \quad (9.40)$$

this scales as M^5/\mathcal{R}^4 . This result is equivalent to Eq. (3.5.39) of the book by D'Eath [19].

E. Comparison with Thorne and Hartle

The rate of change of angular momentum for a general body interacting with a tidal gravitational field was calculated by Thorne and Hartle [33] and their result displayed in Eq. (8.42). We wish to compare this general expression with our result for $\langle \dot{J} \rangle$ displayed in Eq. (9.39); recall that $\langle \dot{J} \rangle = \langle \dot{J}^a \rangle s_a$, with s^a giving the direction of the angular-momentum vector.

The quadrupole moments of a Kerr black hole immersed in an external universe include an *intrinsic* component that would be present even if the black hole were isolated, and an *induced* component that comes from the hole's tidal distortion. We write

$$M_{ab} = M_{ab}^{\text{intrinsic}} + M_{ab}^{\text{induced}}, \quad (9.41)$$

$$J_{ab} = J_{ab}^{\text{intrinsic}} + J_{ab}^{\text{induced}}, \quad (9.42)$$

and we know that [33]

$$M_{ab}^{\text{intrinsic}} = \frac{1}{3} M^3 \chi^2 (\delta_{ab} - 3s_a s_b), \quad J_{ab}^{\text{intrinsic}} = 0. \quad (9.43)$$

We recall that M_{ab} is the hole's mass quadrupole moment, while J_{ab} is its current quadrupole moment; both tensors are symmetric and tracefree. We wish to see if we can determine M_{ab}^{induced} , J_{ab}^{induced} and establish compatibility between Eq. (8.42) and (9.39).

It is easy to show, by substituting Eq. (9.43) into Eq. (8.42), that the coupling between the *intrinsic* moments and the tidal gravitational fields does not affect the magnitude of the angular-momentum vector; the only effect is to produce a precession of J^a described by $\dot{J}^a = \varepsilon^a_{bc} \Omega_{\text{P}}^b J^c$, where $\Omega_{\text{P}}^a \equiv -M\chi \mathcal{E}^a_b s^b$ is the precessional angular velocity. We conclude that only the *induced* moments will contribute to $\langle \dot{J} \rangle$, and we now seek to determine them.

To ease the comparison between Eq. (8.42) and (9.39) we set $s^a = (0, 0, 1)$ and compute $\langle \dot{J} \rangle \equiv \langle \dot{J}^3 \rangle$, which we compare with the result displayed immediately before Eq. (9.33). This reveals that M_{ab}^{induced} is partially determined by the relations $M_{11} - M_{22} \propto 16(1+3\chi^2)\mathcal{E}_{12}$, $M_{12} \propto -4(1+3\chi^2)(\mathcal{E}_{11} - \mathcal{E}_{22})$, $M_{13} \propto (4-3\chi^2)\mathcal{E}_{23}$, and $M_{23} \propto -(4-3\chi^2)\mathcal{E}_{23}$, where the (unique) constant of proportionality is equal to $\frac{2}{45}M^5\chi$. Analogous relations

link $\frac{4}{3}J_{ab}^{\text{induced}}$ to \mathcal{B}_{ab} . These relations determine M_{ab}^{induced} (and J_{ab}^{induced}) up to a term proportional to $\delta_{ab} - 3s_a s_b$; the coefficient must be a scalar formed from \mathcal{E}_{ab} (or \mathcal{B}_{ab}), δ_{ab} , s_a , and ε_{abc} , and the only possible candidate is an arbitrary function of χ multiplying $\mathcal{E}_{ab} s^a s^b$ (or $\mathcal{B}_{ab} s^a s^b$).

We therefore arrive at

$$\begin{aligned} M_{ab}^{\text{induced}} = & \frac{2}{45} M^5 \chi \left[\lambda(\chi) (\delta_{ab} - 3s_a s_b) \mathcal{E}_{cd} s^c s^d \right. \\ & + 8(1+3\chi^2) \mathcal{E}^c_{(a} \varepsilon_{b)cd} s^d \\ & \left. + 30\chi^2 s_{(a} \varepsilon_{b)cd} s^c \mathcal{E}^d_e s^e \right] \end{aligned} \quad (9.44)$$

for the mass quadrupole moment, and

$$\begin{aligned} J_{ab}^{\text{induced}} = & \frac{1}{30} M^5 \chi \left[\mu(\chi) (\delta_{ab} - 3s_a s_b) \mathcal{B}_{cd} s^c s^d \right. \\ & + 8(1+3\chi^2) \mathcal{B}^c_{(a} \varepsilon_{b)cd} s^d \\ & \left. + 30\chi^2 s_{(a} \varepsilon_{b)cd} s^c \mathcal{B}^d_e s^e \right] \end{aligned} \quad (9.45)$$

for the current quadrupole moment, where $\lambda(\chi)$ and $\mu(\chi)$ are unknown functions of the hole's rotational parameter. We conclude that our results are indeed compatible with the general results of Thorne and Hartle [33]. We observe that the relationships between the induced moments and the tidal gravitational fields have the schematic form $M^{\text{induced}} \sim M^5 \mathcal{E}$, $J^{\text{induced}} \sim M^5 \mathcal{B}$ (but with a complicated tensorial structure), and that the moments scale as $M^3(M/\mathcal{R})^2$. These results follow expectation, and they are markedly different from those of Sec. VIII G; recall that for a nonrotating black hole the relationships involve an additional factor of M and a time derivative.

We emphasize that the induced moments are only partially determined: the functions $\lambda(\chi)$ and $\mu(\chi)$ cannot be determined by the comparison with Thorne and Hartle, because the terms to which they belong in M_{ab} and J_{ab} do not affect the magnitude of the angular-momentum vector. They produce instead a small fractional correction of order $(M/\mathcal{R})^2$ to Ω_{P}^a , the precessional angular velocity.

F. Comparison between Kerr and Schwarzschild results

The main results of Sec. VIII, Eqs. (8.38) and (8.39), or equivalently Eqs. (8.40) and (8.41), hold to leading order in $M/\mathcal{R} \ll 1$, and they reveal that for a Schwarzschild black hole, $\langle \dot{M} \rangle = O(M^6/\mathcal{R}^6)$ and $\langle \dot{J} \rangle = O(M^6/\mathcal{R}^5)$. On the other hand, the main results of this section, Eqs. (9.32) and (9.39), hold to leading order in $M/\mathcal{R} \ll \chi$, and they reveal that for a Kerr black hole, $\langle \dot{M} \rangle = O(M^5/\mathcal{R}^5)$ and $\langle \dot{J} \rangle = O(M^5/\mathcal{R}^4)$. The scalings are very different, and the condition $M/\mathcal{R} \ll \chi$ implies that the Schwarzschild results cannot straightforwardly be obtained from the Kerr results in a limit $\chi \rightarrow 0$.

The origin of the difference in scalings can easily be understood in the special case of rigid rotation, for which

$\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ are given by Eqs. (4.29)–(4.31),

$$\langle \dot{M} \rangle = \Omega(\Omega - \Omega_{\text{H}})\mathcal{K}, \quad \langle \dot{J} \rangle = (\Omega - \Omega_{\text{H}})\mathcal{K},$$

where \mathcal{K} is defined by Eq. (4.32) and $\Omega = O(\mathcal{R}^{-1})$ is the hole’s angular velocity in the external spacetime. This argument was first presented to me by Kip Thorne (personal communication), and it was then elaborated on by Alvi [18].

Suppose, as we shall show below, that $\mathcal{K} = O(M^6/\mathcal{R}^4)$. In the case of nonrotating black hole we have $\Omega_{\text{H}} = 0$, and it follows that $\langle \dot{M} \rangle = \Omega^2\mathcal{K} = O(M^6/\mathcal{R}^6)$ and $\langle \dot{J} \rangle = \Omega\mathcal{K} = O(M^6/\mathcal{R}^5)$; those are precisely the scalings obtained previously for a Schwarzschild black hole. The situation is different for a rotating black hole. In this case the condition $M/\mathcal{R} \ll \chi$ implies that $\Omega \ll \Omega_{\text{H}}$, and we have instead $\langle \dot{M} \rangle = -\Omega\Omega_{\text{H}}\mathcal{K} = O(M^5/\mathcal{R}^5)$ and $\langle \dot{J} \rangle = -\Omega_{\text{H}}\mathcal{K} = O(M^5/\mathcal{R}^4)$; those are precisely the scalings obtained previously for a Kerr black hole. Notice that in the case of Kerr, $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ are both proportional to Ω_{H} and therefore to χ ; this observation is confirmed by Eq. (9.39).

The different scalings reflect the different technical meanings assigned to the phrase “small-hole/slow-motion approximation.” For a Schwarzschild black hole we impose $M/\mathcal{R} \ll 1$ and we naturally have $\Omega \gg \Omega_{\text{H}}$; for a Kerr black hole we impose instead $M/\mathcal{R} \ll \chi$ and we consequently have $\Omega \ll \Omega_{\text{H}}$. In generic situations (that is, in the absence of rigid rotation) the scaling argument given previously continues to apply, but $\Omega \sim \mathcal{R}^{-1}$ is now interpreted as an inverse time scale associated with changes in the Weyl tensor of the external spacetime.

An expression for \mathcal{K} can be obtained from the approximate relation $\langle \dot{J} \rangle = -\Omega_{\text{H}}\mathcal{K}$ which holds for a rotating black hole. We obtain

$$\begin{aligned} \mathcal{K} = & \frac{4}{45}M^6(1 + \sqrt{1 - \chi^2}) \left[8(1 + 3\chi^2)\langle E_1 + B_1 \rangle \right. \\ & - 3(4 + 17\chi^2)\langle E_2 + B_2 \rangle \\ & \left. + 15\chi^2\langle E_3 + B_3 \rangle \right], \end{aligned} \quad (9.46)$$

and we confirm that indeed, $\mathcal{K} = O(M^6/\mathcal{R}^4)$. Notice that there is no obstacle to taking the limit $\chi \rightarrow 0$ of this expression.

G. Black hole in a circular binary: Slow-motion approximation

We now specialize the results of Sec. IX D to a Kerr black hole placed on a circular orbit in the weak gravitational field of an external body of mass M_{ext} . This is the slow-motion approximation, and we shall repeat here most of the steps described in Sec. VIII H.

As before the tidal gravitational fields of the external universe are approximated by $\mathcal{E}_{ab} \simeq \Phi_{,ab}$ and $\mathcal{B}_{ab} \simeq 0$, where $\Phi = -M_{\text{ext}}/|\mathbf{x} - \mathbf{r}_{\text{ext}}|$ is the Newtonian potential associated with the external body. As before the black

hole is moving on a circular orbit, and we assume that the orbital angular momentum vector is either aligned or anti-aligned with the hole’s spin vector: $\hat{\mathbf{L}} \cdot \mathbf{s} \equiv \epsilon = \pm 1$. The hole’s orbital angular velocity is then

$$\Omega = \epsilon \sqrt{\frac{M + M_{\text{ext}}}{b^3}}, \quad (9.47)$$

where b is the orbital radius; the angular velocity is positive when the orbital and spin angular momenta are aligned, and it is negative when they are anti-aligned.

A simple calculation, along the lines of what was presented in Sec. VIII H, yields

$$\langle \dot{M} \rangle = -\epsilon \frac{8}{5} \eta^2 \left(\frac{M}{M + M_{\text{ext}}} \right)^3 \chi(1 + 3\chi^2)V^{15} \quad (9.48)$$

and

$$\langle \dot{J} \rangle = -\frac{8}{5} \eta^2 \left(\frac{M}{M + M_{\text{ext}}} \right)^3 (M + M_{\text{ext}})\chi(1 + 3\chi^2)V^{12}, \quad (9.49)$$

where $\eta = MM_{\text{ext}}/(M + M_{\text{ext}})^2$ is a dimensionless reduced-mass parameter and

$$V = \sqrt{\frac{M + M_{\text{ext}}}{b}} \ll 1 \quad (9.50)$$

is the relative orbital velocity. Notice that while we could not calculate $\langle \dot{M} \rangle$ in the general case described in Sec. IX D, here it is simply given by $\Omega\langle \dot{J} \rangle$ because the black hole is in rigid rotation around M_{ext} . The results of Eq. (9.48) and (9.49) agree with earlier expressions obtained by Alvi [18]. In the regime $M_{\text{ext}} \gg M$ they also agree with earlier results derived by Tagoshi, Mano, and Takasugi [17].

H. Black hole in a circular binary: Small-hole approximation

We now allow the Kerr black hole to move rapidly in the strong gravitational field of a Schwarzschild hole of mass M_{ext} . We no longer restrict the size of V but we now impose $M \ll M_{\text{ext}}$; this is the small-hole approximation, and we shall repeat here most of the steps described in Sec. VIII I.

Once more we take the orbit to be circular. In the standard Schwarzschild coordinates (t, r, θ, ϕ) used in the background spacetime of the large black hole, the orbital radius is b and the four-velocity of the small hole is $u^\alpha = \gamma(1, 0, 0, \Omega)$, where $\gamma = (1 - 3M_{\text{ext}}/b)^{-1/2}$ is a normalization factor and

$$\Omega = \epsilon \sqrt{\frac{M_{\text{ext}}}{b^3}} \quad (9.51)$$

is the angular velocity; as in the preceding subsection $\epsilon = \pm 1$ gives the orientation of the orbital angular momentum vector relative to the hole’s spin vector, $s^\alpha =$

$(0, 0, -1/b, 0)$. Calculation of $\mathcal{E}_{\alpha\beta}$, $\mathcal{B}_{\alpha\beta}$ using Eqs. (8.4), (8.5), and substitution into Eqs. (9.33)–(9.39) gives

$$\langle \dot{M} \rangle = -\epsilon \frac{8}{5} \left(\frac{M}{M_{\text{ext}}} \right)^5 \chi (1 + 3\chi^2) V^{15} \times \frac{(1 - 2V^2) \left(1 - \frac{4+27\chi^2}{4+12\chi^2} V^2 \right)}{(1 - 3V^2)^2} \quad (9.52)$$

and

$$\langle \dot{J} \rangle = -\frac{8}{5} \left(\frac{M}{M_{\text{ext}}} \right)^5 M_{\text{ext}} \chi (1 + 3\chi^2) V^{12} \times \frac{(1 - 2V^2) \left(1 - \frac{4+27\chi^2}{4+12\chi^2} V^2 \right)}{(1 - 3V^2)^2}, \quad (9.53)$$

where $V = \sqrt{M_{\text{ext}}/b} \leq 6^{-1/2}$ is a measure of the hole's orbital velocity. These results agree with those of the preceding subsection in a common domain of validity defined by $M \ll M_{\text{ext}}$ and $V \ll 1$. To the best of my knowledge, the results of Eqs. (9.47) and (9.48), complete with all-order relativistic corrections, have never appeared before in the literature.

I. Black hole in a static tidal field

For completeness we explore another special case of Eq. (9.39), in which the rotating black hole is at rest in a static tidal gravitational field. We wish to calculate the rate at which this black hole loses its angular momentum. This calculation was presented many times before, most notably by Hartle [54, 55], Teukolsky [53], Chrzanowski [56], Thorne, Price, and Macdonald [28], and Alvi [18]. The point of this subsection is to illustrate how easily the classic spin-down result of Eq. (9.55) follows from Eq. (9.39).

We assume that the tidal gravitational field is purely electric in the local asymptotic rest frame of the black hole, and that it is axially symmetric in the arbitrary direction of the unit vector n^a . With these specifications we have

$$\mathcal{E}_{ab} = -\frac{1}{2} \mathcal{E} (\delta_{ab} - 3n_a n_b), \quad \mathcal{B}_{ab} = 0, \quad (9.54)$$

where $\mathcal{E} \equiv \mathcal{E}_{ab} n^a n^b$. If, for example, the tidal field is produced by a body of mass M_{ext} maintained at a fixed position $\mathbf{r} = b\mathbf{n}$ relative to the black hole, then $\mathcal{E} = -2M_{\text{ext}}/b^3$. We assume that the black hole's angular momentum makes an angle α with respect to the direction of n^a , so that $s_a n^a = \cos \alpha$. We then have $\mathcal{E}_{ab} s^b = \frac{1}{2} \mathcal{E} (3 \cos \alpha n_a - s_a)$, $\mathcal{E}_{ab} s^a s^b = \frac{1}{2} \mathcal{E} (3 \cos^2 \alpha - 1)$, and the invariants of Eqs. (9.33)–(9.35) are easily computed. After simplification we find that Eq. (9.39) reduces to

$$\dot{J} = -\frac{2}{5} \mathcal{E}^2 M^5 \chi \sin^2 \alpha \left[1 - \frac{3}{4} \left(1 - 5 \sin^2 \alpha \right) \chi^2 \right], \quad (9.55)$$

which is the classic spin-down formula.

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Note added

Conversations with Kip Thorne and John Friedman (whom I thank) made me understand that the discussion of induced quadrupole moments inserted in Secs. VIII G and IX E is incomplete. I should have realized that the tidal-heating formulae of Eqs. (8.42) and (8.43) allow the determination of M_{ab} only up to a term proportional to \mathcal{E}_{ab} , and the determination of J_{ab} up to a term proportional to \mathcal{B}_{ab} . Such terms do not participate in the tidal heating and leave $\langle \dot{M} \rangle$ and $\langle \dot{J}^a \rangle$ unchanged. It is therefore possible for a tidally distorted Schwarzschild black hole to have quadrupole moments given by $M_{ab} = aM^5 \mathcal{E}_{ab} + (32/45)M^6 \dot{\mathcal{E}}_{ab}$ and $J_{ab} = a'M^5 \mathcal{B}_{ab} + (8/15)M^6 \dot{\mathcal{B}}_{ab}$, where a and a' are undetermined dimensionless constants. Such moments would scale as M^5/\mathcal{R}^2 , which is the expected scaling.

APPENDIX: PERTURBATIONS OF A SCHWARZSCHILD BLACK HOLE

In this Appendix I collect a few key results from the theory of gravitational perturbations of a Schwarzschild black hole [3, 4, 35, 43]. I employ a covariant/gauge-invariant formalism that was inspired by the work of Gerlach & Sengupta [57, 58] and Sarbach & Tiglio [59]. These results are presented without derivation; details can be found in Martel's PhD dissertation [10].

1. Background metric

The Schwarzschild metric is expressed as

$$ds^2 = g_{ij} dx^i dx^j + r^2 \Omega_{AB} d\theta^A d\theta^B, \quad (\text{A.1})$$

in a form that is covariant under two-dimensional coordinate transformations $x^i \rightarrow x'^i$. The indices i, j, k, \dots run over the values 0 and 1, and the indices A, B, C, \dots run over the values 2 and 3. The traditional Schwarzschild coordinates are $x^i = (t, r)$, and in the text we use the ingoing Eddington-Finkelstein coordinates $x^i = (v, r)$, where $v = t + r + 2M \ln(r/2M - 1)$, with M denoting the black-hole mass. In the metric of Eq. (A.1), r is viewed as a scalar function of the arbitrary coordinates x^i , and $\Omega_{AB} = \text{diag}(1, \sin^2 \theta)$ is the metric on the unit two-sphere.

We use g_{ij} and its inverse to lower and raise all lower-case Latin indices. And in this Appendix, contrary to

previous usage in the body of the paper, we use Ω_{AB} and its inverse to lower and raise all upper-case Latin indices. We indicate covariant differentiation with respect to a connection compatible with g_{ij} with a dot: $g_{ij,k} = 0$. And we indicate covariant differentiation with respect to a connection compatible with Ω_{AB} with a colon: $\Omega_{AB:C} = 0$.

2. Spherical harmonics

The tensorial nature of the spherical harmonics refers to the unit two-sphere, whose metric is Ω_{AB} . The definitions adopted below agree with those of Regge and Wheeler [3]. The Levi-Civita tensor on the unit two-sphere is denoted ε_{AB} , with $\varepsilon_{\theta\phi} = \sin\theta$.

The scalar harmonics are the usual spherical-harmonic functions $Y^{lm}(\theta^A)$, which satisfy the eigenvalue equation $\Omega^{AB} Y_{:AB}^{lm} + l(l+1)Y^{lm} = 0$.

Vectorial spherical harmonics come in two types. The *even-parity* harmonics are

$$Y_{:A}^{lm} \quad (\text{even parity}), \quad (\text{A.2})$$

while the *odd-parity* harmonics are

$$X_A^{lm} = -\varepsilon_A^B Y_{:B}^{lm} \quad (\text{odd parity}). \quad (\text{A.3})$$

The vectorial harmonics satisfy the following orthogonality relations:

$$\int \bar{Y}_{lm}^{:A} Y_{:A}^{l'm'} d\Omega = \int \bar{X}_{lm}^A X_A^{l'm'} d\Omega = l(l+1) \delta_{ll'} \delta_{mm'} \quad (\text{A.4})$$

and

$$\int \bar{Y}_{lm}^{:A} X_A^{l'm'} d\Omega = 0, \quad (\text{A.5})$$

where an overbar indicates complex conjugation and $d\Omega = \sin\theta d\theta d\phi$.

Tensorial spherical harmonics come in the same two types. The *even-parity* harmonics are

$$Y^{lm} \Omega_{AB}, \quad Y_{:AB}^{lm} \quad (\text{even parity}). \quad (\text{A.6})$$

It is useful to define also

$$Z_{AB}^{lm} = Y_{:AB}^{lm} + \frac{1}{2} l(l+1) Y^{lm} \Omega_{AB}. \quad (\text{A.7})$$

By virtue of the eigenvalue equation for the scalar harmonics, $\Omega^{AB} Z_{AB}^{lm} = 0$; these harmonics are therefore tracefree. The *odd-parity* harmonics are

$$X_{AB}^{lm} = -X_{(A:B)}^{lm} \quad (\text{odd parity}); \quad (\text{A.8})$$

these are also tracefree: $\Omega^{AB} X_{AB}^{lm} = 0$. We record the following relations between the tensorial harmonics and the spherical harmonics of spin-weight $s = \pm 2$ [60]:

$$Z_{AB}^{lm} = \frac{1}{2} \sqrt{(l-1)l(l+1)(l+2)} \times (-2Y^{lm} \epsilon_A \epsilon_B + 2Y^{lm} \bar{\epsilon}_A \bar{\epsilon}_B) \quad (\text{A.9})$$

and

$$X_{AB}^{lm} = \frac{i}{2} \sqrt{(l-1)l(l+1)(l+2)} \times (-2Y^{lm} \epsilon_A \epsilon_B - 2Y^{lm} \bar{\epsilon}_A \bar{\epsilon}_B), \quad (\text{A.10})$$

where the vectors $\epsilon_A \equiv (1, i \sin\theta)/\sqrt{2}$ satisfy $\Omega^{AB} \epsilon_A \epsilon_B = \Omega^{AB} \bar{\epsilon}_A \bar{\epsilon}_B = 0$ and $\Omega^{AB} \epsilon_A \bar{\epsilon}_B = 1$. The tensorial harmonics satisfy the following orthogonality relations:

$$\begin{aligned} \int \bar{Z}_{lm}^{AB} Z_{AB}^{l'm'} d\Omega &= \int \bar{X}_{lm}^{AB} X_{AB}^{l'm'} d\Omega \\ &= \frac{1}{2} (l-1)l(l+1)(l+2) \delta_{ll'} \delta_{mm'} \end{aligned} \quad (\text{A.11})$$

and

$$\int \bar{Z}_{lm}^{AB} X_{AB}^{l'm'} d\Omega = 0. \quad (\text{A.12})$$

The spherical-harmonic functions constructed thus far are complex, and they are all proportional to $e^{im\phi}$. We shall also need a set of real spherical harmonics of degree $l = 2$, which we denote Y^m , $Y_{:A}^m$, X_A^m , $Y_{:AB}^m$, Z_{AB}^m , and X_{AB}^m . The label \mathbf{m} runs over the set $\{0, 1c, 1s, 2c, 2s\}$; the numerical part of this label refers to the azimuthal index m , and the letter indicates whether the corresponding scalar-harmonic function is proportional to $\cos(m\phi)$ or $\sin(m\phi)$. Explicitly,

$$\begin{aligned} Y^0 &= \frac{1}{2} (3 \cos^2 \theta - 1), \\ Y^{1c} &= \sin \theta \cos \theta \cos \phi, \\ Y^{1s} &= \sin \theta \cos \theta \sin \phi, \\ Y^{2c} &= \sin^2 \theta \cos 2\phi, \\ Y^{2s} &= \sin^2 \theta \sin 2\phi. \end{aligned} \quad (\text{A.13})$$

The vectorial and tensorial harmonics are generated by acting on Y^m with the same differential operators as those involved in Eqs. (A.2), (A.3), (A.6)–(A.8).

3. Odd-parity perturbations

The odd-parity perturbations of the Schwarzschild metric are those which are expanded in terms of odd-parity spherical harmonics. This sector of the metric perturbation is given by

$$\delta g_{iA}(x^i, \theta^A) = h_i(x^i) X_A^{lm}(\theta^A), \quad (\text{A.14})$$

$$\delta g_{AB}(x^i, \theta^A) = h_2(x^i) X_{AB}^{lm}(\theta^A). \quad (\text{A.15})$$

We suppress usage of the lm label on the fields h_i and h_2 , and it is understood that the right-hand sides are summed over l and m . It can be shown that the combinations

$$\tilde{h}_i = h_i + \frac{1}{2} h_{2,i} - \frac{1}{r} r_{,i} h_2 \quad (\text{A.16})$$

are invariant under odd-parity gauge transformations. The linearized Einstein field equations are then naturally expressed in terms of \tilde{h}_i and its covariant derivatives. One of these equations is required in the text: In the absence of sources it can be shown that

$$\tilde{h}^i{}_{;i} = 0. \quad (\text{A.17})$$

The remaining field equations can be manipulated to form a one-dimensional wave equation for the master variable

$$\Psi_{\text{RW}} \equiv \frac{1}{r} r^{;i} \tilde{h}_i, \quad (\text{A.18})$$

which is evidently gauge invariant. The function $\Psi_{\text{RW}}(x^i)$ is known as the Regge-Wheeler function [3], and in the absence of sources it satisfies the differential equation

$$\square \Psi_{\text{RW}} - \left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3} \right] \Psi_{\text{RW}} = 0, \quad (\text{A.19})$$

where $\square \Psi \equiv g^{ij} \Psi_{;ij}$ is the one-dimensional wave operator acting on the scalar function $\Psi(x^i)$. It is well understood that in a specified gauge, all components of the odd-parity metric perturbation can be reconstructed from the Regge-Wheeler function.

4. Even-parity perturbations

The even-parity perturbations are expanded in terms of even-parity spherical harmonics. This sector of the metric perturbation is given by

$$\delta g_{ij}(x^i, \theta^A) = h_{ij}(x^i) Y^{lm}(\theta^A), \quad (\text{A.20})$$

$$\delta g_{iA}(x^i, \theta^A) = j_i(x^i) Y_{;A}^{lm}(\theta^A), \quad (\text{A.21})$$

$$\delta g_{AB}(x^i, \theta^A) = r^2 [K(x^i) Y^{lm}(\theta^A) \Omega_{AB} + G(x^i) Y_{;AB}^{lm}(\theta^A)]. \quad (\text{A.22})$$

Once more we suppress usage of the lm label on the fields h_{ij} , j_i , K , and G , and it is understood that the right-hand sides are summed over l and m . The combinations

$$\tilde{h}_{ij} = h_{ij} - 2\varepsilon_{(i,j)}, \quad \tilde{K} = K - \frac{2}{r} r^{;i} \varepsilon_i, \quad (\text{A.23})$$

where $\varepsilon_i = j_i - \frac{1}{2} r^2 G_{;i}$, are invariant under even-parity gauge transformations. The linearized Einstein field equations are then naturally expressed in terms of these fields and their covariant derivatives. They can be manipulated to form a one-dimensional wave equation for the master variable

$$\Psi_{\text{ZM}} \equiv \frac{2r}{l(l+1)} \left[\tilde{K} + \frac{2}{\Lambda} (r^{;i} r^{;j} \tilde{h}_{ij} - r r^{;i} \tilde{K}_{;i}) \right], \quad (\text{A.24})$$

where $\Lambda \equiv (l-1)(l+2) + 6M/r$. The function $\Psi_{\text{ZM}}(x^i)$ is known as the Zerilli-Moncrief function [4, 43], and it is evidently gauge-invariant; it satisfies a differential equation similar to Eq. (A.19), but with a more complicated potential. The normalization of the Zerilli-Moncrief function is chosen so as to agree with the definition proposed by Lousto and Price [13].

5. Waveforms and energy radiated at infinity

When examined near future null infinity, the gravitational perturbations of Eqs. (A.14), (A.15), (A.20)–(A.22) can be presented in an outgoing-radiation gauge that permits an easy identification of the radiative field. It can be shown that the two fundamental polarizations of the gravitational waves are given by

$$h_+ - ih_\times = \frac{1}{2r} \sum_{lm} \sqrt{(l-1)l(l+1)(l+2)} \left[\Psi_{\text{ZM}}^{lm}(u) - 2i \int^u \Psi_{\text{RW}}^{lm}(u') du' \right] {}_{-2}Y^{lm}(\theta^A), \quad (\text{A.25})$$

where $u = t - r - 2M \ln(r/2M - 1)$ is retarded time, and ${}_{-2}Y^{lm}(\theta^A)$ are spherical harmonics of spin-weight $s = -2$. The fact that the waveforms are expressed in terms of an *integral* of the Regge-Wheeler function means that this master variable is rather ill-suited to describe the radiative aspects of the metric perturbation. An alternative choice of master variable, which is free of this blemish, was proposed by Cunningham, Price, and Moncrief [61, 62]; it was recently revived by Jhingan and Tanaka [63].

The energy and angular momentum radiated to infinity are given by

$$\begin{aligned} \langle \dot{E} \rangle &= \frac{1}{64\pi} \sum_{lm} (l-1)l(l+1)(l+2) \\ &\quad \times \left\langle 4|\Psi_{\text{RW}}^{lm}(u)|^2 + |\dot{\Psi}_{\text{ZM}}^{lm}(u)|^2 \right\rangle, \quad (\text{A.26}) \\ \langle \dot{J} \rangle &= \frac{1}{64\pi} \sum_{lm} (l-1)l(l+1)(l+2)(im) \\ &\quad \times \left\langle 4\Psi_{\text{RW}}^{lm}(u) \int^u \bar{\Psi}_{\text{RW}}^{lm}(u') du' \right. \\ &\quad \left. + \dot{\Psi}_{\text{ZM}}^{lm}(u) \bar{\Psi}_{\text{ZM}}^{lm}(u) \right\rangle. \quad (\text{A.27}) \end{aligned}$$

The expressions are very similar to the horizon-flux formulae of Eqs. (7.8) and (7.9).

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- [1] S. A. Teukolsky and W. H. Press, *Astrophys. J.* **193**, 443 (1974).
- [2] S. A. Teukolsky, *Astrophys. J.* **185**, 635 (1973).
- [3] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).
- [4] F. J. Zerilli, *Phys. Rev. D* **2**, 2141 (1970).
- [5] V. P. Frolov and I. D. Novikov, *Black hole physics: Basic concepts and new developments* (Kluwer, Dordrecht, The Netherlands, 1998).
- [6] W. Krivan, P. Laguna, and P. Papadopoulos, *Phys. Rev. D* **54**, 4728 (1997).
- [7] W. Krivan, P. Laguna, P. Papadopoulos, and N. Andersson, *Phys. Rev. D* **56**, 3395 (1997).
- [8] L. M. Burko and G. Khanna, *Phys. Rev. D* **67**, 081502(R) (2004).
- [9] M. A. Scheel, A. L. Erickcek, L. M. Burko, L. E. Kidder, H. P. Pfeiffer, and S. A. Teukolsky, *Phys. Rev. D* **69**, 104006 (2004).
- [10] K. Martel, Ph.d. thesis, University of Guelph (2003).
- [11] C. Gundlach, R. H. Price, and J. Pullin, *Phys. Rev. D* **49**, 883 (1994).
- [12] C. Gundlach, R. H. Price, and J. Pullin, *Phys. Rev. D* **49**, 890 (1994).
- [13] C. O. Lousto and R. H. Price, *Phys. Rev. D* **55**, 2124 (1997).
- [14] K. Martel, *Phys. Rev. D* **69**, 044025 (2004).
- [15] K. Martel and E. Poisson, *Phys. Rev. D* **66**, 084001 (2002).
- [16] E. Poisson and M. Sasaki, *Phys. Rev. D* **51**, 5753 (1995).
- [17] H. Tagoshi, S. Mano, and E. Takasugi, *Prog. Theor. Phys.* **98**, 829 (1997).
- [18] K. Alvi, *Phys. Rev. D* **64**, 104020 (2001).
- [19] P. D. D'Eath, *Black holes: Gravitational interactions* (Clarendon Press, Oxford, 1996).
- [20] R. H. Price and J. T. Whelan, *Phys. Rev. Lett.* **87**, 231101 (2001).
- [21] S. A. Hughes, *Phys. Rev. D* **64**, 064004 (2001).
- [22] A. Ashtekar and B. Krishnan, *Phys. Rev. Lett.* **89**, 261101 (2002).
- [23] A. Ashtekar and B. Krishnan, *Phys. Rev. D* **68**, 104030 (2003).
- [24] I. Booth and S. Fairhurst, *Phys. Rev. Lett.* **92**, 011102 (2004).
- [25] S. A. Hayward, *Energy conservation for dynamical black holes* (2004), <http://xxx.lanl.gov/abs/gr-qc/0404077>.
- [26] S. W. Hawking and J. B. Hartle, *Commun. Math. Phys.* **27**, 283 (1972).
- [27] R. H. Price and K. S. Thorne, *Phys. Rev. D* **33**, 915 (1986).
- [28] K. S. Thorne, R. H. Price, and D. A. Macdonald, *Black holes: The membrane paradigm* (Yale University Press, New Haven, 1986).
- [29] R. M. Wald, *General relativity* (University of Chicago Press, Chicago, 1984).
- [30] E. Poisson, *A relativist's toolkit: The mathematics of black-hole mechanics* (Cambridge University Press, Cambridge, England, 2004).
- [31] R. A. Isaacson, *Phys. Rev.* **166**, 1263 (1968).
- [32] R. A. Isaacson, *Phys. Rev.* **166**, 1272 (1968).
- [33] K. S. Thorne and J. B. Hartle, *Phys. Rev. D* **31**, 1815 (1985).
- [34] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
- [35] S. Chandrasekhar, *The mathematical theory of black holes* (Clarendon Press, Oxford, England, 1983).
- [36] W. Kinnersley, *J. Math. Phys.* **10**, 1195 (1969).
- [37] B. Carter, in *General relativity: An Einstein centenary survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
- [38] P. L. Chrzanowski, *Phys. Rev. D* **11**, 2042 (1975).
- [39] D. R. Brill and J. B. Hartle, *Phys. Rev. B* **135**, 271 (1964).
- [40] R. M. Wald, *Phys. Rev. Lett.* **41**, 203 (1978).
- [41] C. O. Lousto and B. F. Whiting, *Phys. Rev. D* **66**, 024026 (2002).
- [42] A. Ori, *Phys. Rev. D* **67**, 124010 (2003).
- [43] V. Moncrief, *Ann. Phys. (N.Y.)* **88**, 323 (1974).
- [44] F. K. Manasse, *J. Math. Phys.* **4**, 746 (1963).
- [45] R. E. Kates, *Phys. Rev. D* **22**, 1853 (1980).
- [46] Y. Mino, M. Sasaki, and T. Tanaka, *Phys. Rev. D* **55**, 3457 (1997).
- [47] T. C. Quinn and R. M. Wald, *Phys. Rev. D* **56**, 3381 (1997).
- [48] E. Poisson, *Living Rev. Relativity* **7** (2004), 6. [Online article]: cited on June 22, 2004, <http://www.livingreviews.org/lrr-2004-6>.
- [49] K. Alvi, *Phys. Rev. D* **61**, 124013 (2000).
- [50] S. Detweiler, *Phys. Rev. Lett.* **86**, 1931 (2001).
- [51] E. Poisson, *Phys. Rev. D* **69**, 084007 (2004).
- [52] X.-H. Zhang, *Phys. Rev. D* **31**, 3130 (1986).
- [53] S. A. Teukolsky, Ph.d. thesis, California Institute of Technology (1973).
- [54] J. B. Hartle, *Phys. Rev. D* **8**, 1010 (1973).
- [55] J. B. Hartle, *Phys. Rev. D* **9**, 2749 (1974).
- [56] P. L. Chrzanowski, *Phys. Rev. D* **13**, 806 (1975).
- [57] U. H. Gerlach and U. K. Sengupta, *Phys. Rev. D* **19**, 2268 (1979).
- [58] U. H. Gerlach and U. K. Sengupta, *Phys. Rev. D* **22**, 1300 (1980).
- [59] O. Sarbach and M. Tiglio, *Phys. Rev. D* **64**, 084016 (2001).
- [60] J. Goldberg, A. MacFarlane, E. Newman, F. Rohrlich, and E. Sudarshan, *J. Math. Phys.* **8**, 2155 (1967).
- [61] C. T. Cunningham, R. H. Price, and V. Moncrief, *Astrophys. J.* **224**, 643 (1978).
- [62] C. T. Cunningham, R. H. Price, and V. Moncrief, *Astrophys. J.* **230**, 870 (1979).
- [63] S. Jhingan and T. Tanaka, *Phys. Rev. D* **67**, 104018 (2003).