

Lattice Pseudospin Model for $\nu = 1$ Quantum Hall Bilayers

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We present a new theoretical approach to the study of $\nu = 1$ quantum Hall bilayer that is based on a systematic mapping of the microscopic Hamiltonian to an anisotropic $SU(4)$ spin model on a lattice. To study the properties of this model we generalize the Heisenberg model Schwinger boson mean field theory (SBMFT) of Arovas and Auerbach to spin models with anisotropy. We calculate the temperature dependence of experimentally observable quantities, including the spin magnetization, and the differential interlayer capacitance. Our theory represents a substantial improvement over the conventional Hartree-Fock picture which neglects quantum and thermal fluctuations, and has advantages over long-wavelength effective models that fail to capture important microscopic physics at all realistic layer separations. The formalism we develop can be generalized to treat quantum Hall bilayers at filling factor $\nu = 2$.

I. INTRODUCTION

The physics of high-mobility two-dimensional electron systems in strong perpendicular magnetic fields continues to produce surprises. Electronic states with strong correlations in this regime originate fundamentally from the macroscopic degeneracies, Landau levels, in the spectrum of the free-particle kinetic energy Hamiltonian. When spin, layer, or other degrees of freedom are important at low energies, correlations at integer filling factors can give rise to broken symmetry ground states¹ with interesting physical properties. Bilayer quantum Hall systems, whose ground states can have spontaneous interlayer phase coherence, provide the simplest and best characterized example of this tendency, and have been studied both theoretically and experimentally for more than ten years.^{2–23} The superfluid effects associated with spontaneous coherence are especially dramatic.²² Most work on these states has concentrated on the case of total lowest Landau level (LLL) filling factor ν equal to one. The quantum Hall effect, indicative of a gap for charged excitations, is observed in this system at sufficiently small interlayer separations d . The charge gap in this case must result entirely from interlayer correlations,²⁴ since individual layer states at filling factor $\nu = 1/2$ are compressible. As the layer separation or the electron density in each layer is decreased there is a phase transition, likely of first order in the absence of disorder, from a compressible state with no quantum Hall effect to a highly correlated incompressible state^{3,8,17} with spontaneous coherence. In this paper we present a new approach which can be used to study the properties of these interesting ordered states and how they vary with layer separation.

The layer degree of freedom in the quantum Hall bilayer can be conveniently described using a pseudospin language: pseudospin up corresponds to an electron in the top layer and pseudospin down to an electron in the bottom layer.⁹ The interlayer phase coherent quantum Hall state corresponds in this language to a pseudospin ferromagnet with an $\hat{x} - \hat{y}$ easy-plane. Our work is motivated by two limitations that apply to most theoretical work on bilayer quantum Hall ferromagnets. The usual assumption in most previous studies of $\nu = 1$ bilayers has been that the spin degree of freedom can be ignored, *i.e.* that spin dynamics are frozen out by the magnetic field. At finite temperatures, however, this assumption is usually not justified: the typical value of the Zeeman gap is only $\sim 1\text{K}$, which is often smaller than the pseudospin gap when the interlayer phase is fixed by finite tunneling. There is a broad range of nonzero temperatures for many samples in which there is a nontrivial interplay between spin and pseudospin fluctuations. A second set of limitations that applies to much theoretical work on quantum Hall bilayers follows from the use of a gradient expansion in deriving an effective continuum pseudospin model. It is sometimes bothersome in applying this approach that divergent coefficients, due to the long-range of the Coulomb interaction, appear in the energy functional when one goes beyond the gradient expansion's leading order. A more severe limitation, however, is the fact that this approach cannot address the physics that gives rise to the phase transition between ordered and disordered states which, as we explain in some detail, is due to competing interactions on microscopic length scales. The importance of correlations on microscopic length scales for the phase transition was already evident in early work where it was associated with soft collective excitations³ at a finite wavevector. The precise nature of the compressible state close to the transition is not completely understood at present and we believe that our approach can shed some light on this question.

In this paper we present a careful study of the spin-pseudospin physics of the $\nu = 1$ bilayer in the quantum Hall state which is free from these limitations. Our approach is based on a systematic mapping from the microscopic interacting electron problem to a model with $SU(4)$ generalized spins on a lattice and anisotropic interactions. $SU(4)$ group appears here since we take into account both spin and pseudospin degrees of freedom and thus single-electron states are four-component spinors. The *only* simplification necessary to derive this mapping is the assumption of the absence of charge fluctuations, reasonably well justified in the incompressible state. To take advantage of the existence of a charge gap in the quantum Hall regime, we use a complete orthonormal set of magnetic Wannier functions on a von Neumann lattice as the LLL orbital basis. Given this basis, and the neglect of charge fluctuations, we can derive a coupled spin-pseudospin Hamiltonian which turns out to be unexpectedly simple. Using the spin-Hamiltonian we derive, the full arsenal of analytic and numerical techniques that have been developed to solve spin problems on a lattice can be applied to quantum Hall systems. In this paper, we present a study of this Hamiltonian using the Schwinger boson mean field theory,²⁵ developed originally by Arovas and Auerbach for the $SU(2)$ invariant Heisenberg model, appropriately generalized for the anisotropic case. We obtain a variety of new results for the temperature dependence of some experimental observables, the spin magnetization and the interlayer differential capacitance in particular. We note that our approach is valid not only for the $\nu = 1$ case but can be easily generalized to a quantum Hall bilayer with any integer filling factor, and with any orbital character of nearby Landau levels. For bilayers, the case of $\nu = 2$ is also interesting and has been studied extensively.^{11,12,14–16} Since the Schwinger boson mean field theory has some limitations, we hope that the present results will motivate the application of more rigorous Monte Carlo or exact-diagonalization techniques to our model Hamiltonian.

The paper is organized as follows. In Sec. II we give a detailed derivation of our spin-pseudospin effective lattice Hamiltonian that we obtain by using a functional integral approach. In Sec. III we discuss the generalization of the

Schwinger boson mean field theory of Arovas and Auerbach to the case of spin models with anisotropy. In Sec. IV we apply these considerations to our spin-pseudospin model for bilayer quantum Hall systems, discussing the results of our calculations and commenting on their relevance to experiments. We conclude in Sec. V by briefly discussing the application of our approach to other broken symmetry states at integer filling factors in quantum Hall systems, including the case of $\nu = 2$ in bilayers.

II. GENERALIZED SPIN MODEL

A single-particle state of an electron in a QH bilayer is described by three quantum numbers: LLL orbital state quantum number i , spin σ and pseudospin τ . In the incompressible state at filling factor $\nu = 1$ one can assume at low enough temperatures that there are no charge fluctuations and that each LLL orbital state i is always occupied by exactly one electron. The model we derive starting from this assumption describes quantum fluctuations of the remaining spin and pseudospin degrees of freedom. We will allow electrons in each LLL orbital to be in the most general coherent superposition of spin and pseudospin eigenstates. To define our notation, we write the corresponding creation operator as

$$\psi_i^\dagger = \sum_{k=1}^4 z_{ik} c_{ik}^\dagger. \quad (1)$$

Here we have introduced 4-component spinor index $k = 1, \dots, 4$ to describe the mixed spin-pseudospin degree of freedom. $k = 1$ labels an up-spin electron in the top layer, $k = 2$ a down-spin top-layer electron, $k = 3$ a bottom-layer up-spin electron, and $k = 4$ a bottom-layer down-spin electron. c_{ik}^\dagger is the creation operator for an electron in the LLL orbital state i and with 4-spinor index k . z_{ik} are complex amplitudes that satisfy the normalization constraint

$$\sum_{k=1}^4 |z_{ik}|^2 = 1. \quad (2)$$

A single Slater determinant many-body wavefunction can be constructed by putting exactly one electron with an arbitrary spinor (1) (for $\nu = 1$) in every LLL orbital state i

$$|\Psi[z]\rangle = \prod_i \left(\sum_{k=1}^4 z_{ik} c_{ik}^\dagger \right) |0\rangle. \quad (3)$$

Our approach is based on the observation that this set of states is complete when charge fluctuations are neglected.

We assume that the bilayer is described by a Hamiltonian of the following form¹⁵

$$H = \sum_{k_1 k_2 i} c_{ik_1}^\dagger h_{k_1 k_2}^0 c_{ik_2} + \frac{1}{2} \sum_{k_1 k_2} \sum_{i_1 i_2 i_3 i_4} c_{i_1 k_1}^\dagger c_{i_2 k_2}^\dagger c_{i_4 k_2} c_{i_3 k_1} \left[\langle i_1 i_2 | V_+ | i_3 i_4 \rangle + \tau_{k_1 k_1}^z \tau_{k_2 k_2}^z \langle i_1 i_2 | V_- | i_3 i_4 \rangle \right]. \quad (4)$$

Here we have introduced 4×4 Pauli matrices τ^a corresponding to the usual Pauli operators acting on pseudospin degrees of freedom only. The analogous spin Pauli matrices will be denoted σ^a . $V_\pm = (V_S \pm V_D)/2$ and V_S and V_D are the 2D Coulomb interactions between electrons in the same and different layers respectively. The single particle part of the Hamiltonian, h^0 , consists in general of three terms, corresponding to an interlayer bias potential, an interlayer tunneling amplitude, and Zeeman coupling to the perpendicular magnetic field

$$h^0 = -\frac{\Delta_V}{2} \tau^z - \frac{\Delta_t}{2} \tau^x - \frac{\Delta_z}{2} \sigma^z. \quad (5)$$

Since the occupation of each orbital state is fixed at one for $\nu = 1$ in our approach, the orbital, or charge degree of freedom in (4) is irrelevant. We would therefore like to completely eliminate any reference to the charge degree of freedom present in the microscopic Hamiltonian, obtaining an effective Hamiltonian that refers only to spin and pseudospin variables. The most convenient and mathematically rigorous way to do this is provided by a functional integral approach. Assuming at each discrete imaginary time that the single Slater determinant states are complete,

and following standard lines for the derivation of path integral formulations of quantum statistical mechanics problems, we obtain the following formal expression for the partition function of a $\nu = 1$ quantum Hall ferromagnet:

$$Z = \int D[\bar{z}, z] e^{-S[\bar{z}, z]}, \quad (6)$$

where the imaginary time action is given by

$$S[\bar{z}, z] = \int_0^\beta d\tau \left[\sum_i (\langle z_i | \partial_\tau | z_i \rangle - \frac{\Delta V}{2} \langle z_i | \tau^z | z_i \rangle - \frac{\Delta t}{2} \langle z_i | \tau^x | z_i \rangle - \frac{\Delta z}{2} \langle z_i | \sigma^z | z_i \rangle) \right. \\ \left. + \frac{1}{2} \sum_{ij} (H_{ij} \langle z_i | \tau^z | z_i \rangle \langle z_j | \tau^z | z_j \rangle - F_{ij}^+ \langle z_i | z_j \rangle \langle z_j | z_i \rangle - F_{ij}^- \langle z_i | \tau^z | z_j \rangle \langle z_j | \tau^z | z_i \rangle) \right]. \quad (7)$$

Here $H_{ij} = \langle ij | V_- | ij \rangle$, and $F_{ij}^\pm = \langle ij | V_\pm | ji \rangle$ are the direct and exchange two-particle matrix elements of the Coulomb interaction.

Using this path integral representation as an intermediary, we can now write the Hamiltonian in terms of spin and pseudospin operators. The complex fields in this path integral can be thought of as bosonic coherent state variables and we can introduce bosonic creation operators a_{ik}^\dagger corresponding to complex variables z_{ik} . The normalization condition (2) translates into the single-occupancy constraint for the a bosons

$$\sum_{k=1}^4 a_{ik}^\dagger a_{ik} = 1. \quad (8)$$

The action (7) is identical to that which arises from a coherent state path integral representation of the partition function for a system of bosons described by the following Hamiltonian

$$H = \sum_{ik_1 k_2} a_{ik_1}^\dagger h_{k_1 k_2}^0 a_{ik_2} \\ + \frac{1}{2} \sum_{ij} \sum_{k_1 k_2} \left[H_{ij} a_{ik_1}^\dagger a_{jk_2}^\dagger a_{ik_1} a_{jk_2} \tau_{k_1 k_1}^z \tau_{k_2 k_2}^z - F_{ij}^+ a_{ik_1}^\dagger a_{jk_2}^\dagger a_{ik_2} a_{jk_1} - F_{ij}^- a_{ik_1}^\dagger a_{jk_2}^\dagger a_{ik_2} a_{jk_1} \tau_{k_1 k_1}^z \tau_{k_2 k_2}^z \right]. \quad (9)$$

One can think of the bosons we introduce as Schwinger bosons representing “spins” which are generators of the SU(4) group (the usual physical spins are SU(2) generators). The SU(4) generators are written in terms of the Schwinger boson creation and annihilation operators as follows

$$S_i^{k_1 k_2} = a_{ik_1}^\dagger a_{ik_2}. \quad (10)$$

The Hamiltonian (9) can now be rewritten in terms of the SU(4) “spin” operators $S_i^{k_1 k_2}$

$$H = \sum_{ik_1 k_2} S_i^{k_1 k_2} h_{k_1 k_2}^0 \\ + \frac{1}{2} \sum_{ij} \sum_{k_1 k_2} \left[H_{ij} S_i^{k_1 k_1} S_j^{k_2 k_2} \tau_{k_1 k_1}^z \tau_{k_2 k_2}^z - F_{ij}^+ S_i^{k_1 k_2} S_j^{k_2 k_1} - F_{ij}^- S_i^{k_1 k_2} S_j^{k_2 k_1} \tau_{k_1 k_1}^z \tau_{k_2 k_2}^z \right]. \quad (11)$$

We have thus mapped the original problem of interacting electrons in a quantum Hall bilayer onto a generalized “spin” problem. Each LLL orbital i (“lattice site” in the spin model language) is occupied by an SU(4) “spin” and “spins” on different sites are coupled via a long-range anisotropic interaction (we will rewrite these spin operators below in terms of more physically transparent SU(2) spin and pseudospin operators).

So far we have not specified the LLL orbital basis we are using. In fact, the accuracy of the model that follows from neglecting charge fluctuations does depend on the single-particle representation we employ. It is obvious that the usually used orbit-center quantum numbers in Landau or symmetric gauges are not good choices for the complete set of orbital labels since they are not localized and the energy penalty for double occupation will vanish in the thermodynamic limit. Our approach is best used in combination with a Wannier-like von Neumann lattice basis set for a Landau level,^{30,31} in which orbitals are centered on lattice sites and the unit cell area equals the area per flux quantum, $2\pi\ell^2$, to accommodate exactly one electron with a corresponding SU(4) “spin” per site at $\nu = 1$.

We choose a square lattice with lattice constant $\sqrt{2\pi\ell^2}$. It is not immediately obvious that such a basis exists. A strong magnetic field imposes certain restrictions on the localization properties of magnetic orbitals,^{27–29} and it is well established, for example, that a set of linearly independent *and* exponentially localized single-particle orbitals in the LLL does not exist. However, it turns out to be possible^{30,31} to construct a complete orthonormal set of Wannier-like eigenfunctions, which, although not exponentially localized, have a well defined Gaussian core and a power law fall off at large distances. Following^{30,31} we refer to these orthonormal basis states as magnetic Wannier functions. The procedure one uses to construct such a basis set is very much like the one used to construct the usual Wannier functions in a crystal. One starts from the set of minimum uncertainty wavepackets for electrons in the LLL, centered at the sites of the square lattice described above. The difference from the case of a crystal here is that this set is *overcomplete*, as shown by Perelomov.²⁷ One then constructs Bloch functions from linear combinations of the minimum uncertainty wavepackets and Fourier transforms them to obtain the Wannier functions. There are subtleties in this procedure and we refer the reader to the appendix of our paper and to the original papers^{30,31} for further details. For the following discussion it is important to note that the procedure used to construct the basis states depends on the size of the system, so that the spin and pseudospin interactions we discuss below have values that depend on the overall system size, converging to well defined values in the thermodynamic limit.

The “spin model” (11) is not very useful as it is because we do not have much intuition about SU(4) “spins”. However, it turns out to be possible to further rewrite (11) in a much more useful form. To proceed one notes that the set of SU(4) generators (10) is not unique.³² We define a different set of generators, more appropriate for our purpose, as follows. Define total spin and pseudospin operators on each lattice site

$$\begin{aligned}\mathbf{S}_i &= \frac{1}{2} \sum_{k_1 k_2} a_{ik_1}^\dagger \vec{\sigma}_{k_1 k_2} a_{ik_2} \\ \mathbf{T}_i &= \frac{1}{2} \sum_{k_1 k_2} a_{ik_1}^\dagger \vec{\tau}_{k_1 k_2} a_{ik_2}.\end{aligned}\tag{12}$$

It is then possible to show that the original SU(4) generators given by (10) can be expressed in terms of a new set of generators $\{S^a, T^b, S^a T^b\}$ as follows

$$\begin{aligned}S_i^{11} &= \frac{1}{4} + \frac{1}{2}(S_i^z + T_i^z) + S_i^z T_i^z \\ S_i^{22} &= \frac{1}{4} - \frac{1}{2}(S_i^z - T_i^z) - S_i^z T_i^z \\ S_i^{33} &= \frac{1}{4} + \frac{1}{2}(S_i^z - T_i^z) - S_i^z T_i^z \\ S_i^{44} &= \frac{1}{4} - \frac{1}{2}(S_i^z + T_i^z) + S_i^z T_i^z \\ S_i^{12} &= \frac{1}{2}S_i^+ + S_i^+ T_i^z \\ S_i^{13} &= \frac{1}{2}T_i^+ + S_i^z T_i^+ \\ S_i^{14} &= S_i^+ T_i^+ \\ S_i^{23} &= S_i^- T_i^+ \\ S_i^{24} &= \frac{1}{2}T_i^+ - S_i^z T_i^+ \\ S_i^{34} &= \frac{1}{2}S_i^+ - S_i^+ T_i^z.\end{aligned}\tag{13}$$

Now the SU(4) “spin” Hamiltonian (11) can be rewritten, up to a constant, as the following coupled spin-pseudospin Hamiltonian

$$\begin{aligned}H &= - \sum_i [\Delta_V T_i^z + \Delta_t T_i^x + \Delta_z S_i^z] \\ &+ \sum_{ij} \left[(2H_{ij} - \frac{1}{2}F_{ij}^S) T_i^z T_j^z - \frac{1}{2}F_{ij}^D \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp - \frac{1}{2}F_{ij}^S \mathbf{S}_i \cdot \mathbf{S}_j \right. \\ &\left. - 2F_{ij}^S (\mathbf{S}_i \cdot \mathbf{S}_j) T_i^z T_j^z - 2F_{ij}^D (\mathbf{S}_i \cdot \mathbf{S}_j) (\mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp) \right].\end{aligned}\tag{14}$$

Here $F^{S,D} = F^+ \pm F^-$.

This Hamiltonian (14) is one of the main results of our paper. Let us emphasize that this Hamiltonian is *exact*, given the assumed absence of charge fluctuations. It is written in terms of the usual physical spin- $\frac{1}{2}$ operators and therefore all the great variety of methods available for spin models can be applied to it. Note also the quite unexpected simplicity of (14). All the effects of spin-pseudospin interaction are contained in the two terms with only four spin- $\frac{1}{2}$ operators.

The *classical* ground state of (14) is one with uniform spin-polarization in the direction of the applied Zeeman field and uniform pseudospin that, for $\Delta_t = 0$ and $\Delta_V = 0$, is in an arbitrary direction in the $\hat{x} - \hat{y}$ plane. It is *identical* to the Hartree-Fock ground state of the original microscopic Hamiltonian. Due to the anisotropic character of (14) there will be corrections to the Hartree-Fock picture even at zero temperature. These corrections can be expected to be quite strong, since we are dealing with spin- $\frac{1}{2}$ operators, i.e. the system is far away from the semiclassical regime. In fact, as we comment later, corrections to Hartree-Fock theory grow in importance as the layer separation is increased. Although these quantum fluctuations can, and have,³³ been addressed in a fully microscopic model at the generalized-random-phase-approximation (GRPA) level, the explicit removal of charge degrees of freedom in the approximate Hamiltonian we have derived allows a wider variety of more powerful theoretical techniques to be applied. In particular, the GRPA corresponds to a linearized spin-wave theory approximation to the above Hamiltonian. Later we will go beyond this level approximation, by applying Schwinger boson mean field theory to our model Hamiltonian.

We now discuss some simple properties that can be used to simplify (14) before embarking on a more detailed study. We note that $H_{ii} = F_{ii}^-$, i.e. that on-site direct and exchange Coulomb matrix elements are identical. It follows that the on-site contributions to the Hamiltonian add to a constant term that has no effect on spin and pseudospin dynamics, a property we use below. We also note that in the absence of the single-particle terms, the Hamiltonian has the correct $SU(2)_{spin} \times U(1)_{pseudospin}$ symmetry. Correspondingly there are two Goldstone modes associated with the spontaneous breaking of these symmetries. Their dispersion relations are readily evaluated in the linear spin wave approximation. For $\Delta_V = 0$ we find that

$$\begin{aligned} E_k^{spin} &= \Delta_z + F_0^+ - F_k^+ \\ E_k^{pseudospin} &= \sqrt{(\Delta_t + F_0^D + H_k - F_k^+)^2 - (H_k - F_k^-)^2}. \end{aligned} \quad (15)$$

In Fig. 1 we show dispersions (15) evaluated for a 20×20 square lattice in the $(1, 1)$ direction using $\Delta_t = \Delta_z = 0.01$ and $d/\ell = 1.4$. All energies in this paper will be in units of $e^2/\epsilon\ell$, the characteristic energy scale for all fractional quantum Hall systems. Note that in this example, which corresponds to reasonably typical values, the pseudospin gap is appreciably larger than the spin gap, even though the bare values of tunneling and Zeeman coupling are the same. This difference is due to pseudospin fluctuations present in the ground state, even in a linearized spin-wave theory, and originates from the fact that the Hamiltonian is invariant under all spin rotations, but only under pseudospin rotations around the \hat{z} axis, the $U(1)$ symmetry referred to above. The dip in the pseudospin wave dispersion at the boundary of the Brillouin zone signals the emerging development of antiferromagnetic instability, which eventually destroys the long-range pseudospin ferromagnetic order. It is this feature of the microscopic physics that is missing in long-wavelength effective models.

Let us now briefly consider the simplified Hamiltonian in which spins are frozen out ($\mathbf{S}_i \cdot \mathbf{S}_j = S_i^z S_j^z = 1/4$), valid at $T = 0$ in the presence of a Zeeman field:

$$H = - \sum_i [\Delta_V T_i^z + \Delta_t T_i^x] + \sum_{ij} [(2H_{ij} - F_{ij}^S) T_i^z T_j^z - F_{ij}^D \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp]. \quad (16)$$

Consider the evolution of (16) as one increases the interlayer separation from 0. At $d = 0$, $2H_{ij} - F_{ij}^S = -F_{ij}^D$ and the system in the absence of tunneling and bias is an $SU(2)$ invariant pseudospin ferromagnet. For $0 < d < d^*$, the F_{ij}^D term dominates and the bilayer ground state is an easy-plane pseudospin ferromagnet. At a certain critical value of the interlayer separation d^* a quantum phase transition occurs to a state with no long-range easy-plane ferromagnetic order. To understand more deeply why this occurs we plot in Fig.2 and 3 the pseudospin couplings $J_{ij}^{Pz} = 2H_{ij} - F_{ij}^S$ and $J_{ij}^{P\perp} = F_{ij}^D$ for two different values of the interlayer separation. As can be seen from the figures, increasing the interlayer separation primarily affects J_{ij}^{Pz} coupling, changing its character from short-range ferromagnetic at $d/\ell = 0.5$ to long-range ($\propto 1/r^3$) antiferromagnetic at $d/\ell = 1.4$ (we use the word *antiferromagnetic* here just to emphasize the sign change of the interaction, it is not immediately clear that the ground state is actually antiferromagnetic, although it very likely is). The change in J_{ij}^{Pz} is accompanied by a weakening of the in-plane ferromagnetic interaction $J_{ij}^{P\perp}$. At present, experimental samples have a d/ℓ value that is not very far below the critical value. We therefore expect that the effective interactions illustrated for the $d/\ell = 1.4$ case are representative of the current typical experimental situation. This change in character reflects a change in the relative importance of

exchange interactions within the layers (F_{ij}^S), which are not d dependent and the electrostatic interactions between the dipoles (H_{ij}) that are created by \hat{z} direction pseudospin polarization and whose moments are proportional to d . For $d/\ell \sim 1.5$, electrostatics dominates, $J_{ij}^P z$ is antiferromagnetic, and competition develops between two qualitatively different potential ground states. *Our analysis makes it clear that it is this developing competition that is responsible for the phase transition that occurs in bilayer quantum Hall systems for $d/\ell \sim 1.5$.* In our model which neglects charge fluctuations, the ground state at large d is likely an antiferromagnet with Ising anisotropy, as it appears from the analysis of the classical energetics of Eq.(16). In experiment, the loss of the quantum Hall effect at this transition indicates that it is accompanied by the loss of a charge gap, making charge fluctuations at least somewhat important. However, even if we neglect the charge fluctuations, i.e. still assume that the system can be described by the pseudospin model (16) in the compressible region, the precise nature of the pseudospin state above the transition is not immediately obvious, although as we have pointed out above, it is likely to be antiferromagnetic. The $T_i^z T_j^z$ term generally favors an easy axis antiferromagnetic ordering of pseudospins. However, the long-range character of the $T_i^z T_j^z$ coupling (since it originates from Coulomb interactions between dipoles associated with \hat{z} -direction pseudospin polarization, one can expect it to fall off as $1/r^3$) could introduce frustration and a nontrivial spin liquid state. A recent Quantum Monte Carlo study³⁴ of a quantum XXZ model with nearest- and next-nearest-neighbor interactions, which is similar to (16), has shown that both Neel and striped phases are possible, depending on the relative strength of the interactions. Our model could in principle produce more exotic phases due to the truly long-range character of spin-spin interactions in our case. The phase transition between the XY-ferromagnetic and Neel or striped phases in³⁴ was shown to be first order, which gives additional support to the validity of our model, since according to recent exact diagonalization studies,¹⁷ the compressible-incompressible transition in $\nu = 1$ bilayers is very likely of first order as well. We believe that the pseudospin Hamiltonian (16) captures important parts of the physics of this still poorly understood quantum phase transition and deserves further study.

Although we concentrate in this paper on the case of bilayer quantum Hall systems at $\nu = 1$ in the lowest Landau level, the same formalism can be used to derive spin-pseudospin models for other cases of interest. For example, pseudospin stripe states³⁵ are expected at $\nu = 1$ in the case of Landau levels with orbital index $N > 2$, and Ising ferromagnets are expected¹ in bilayers when the orbital indices in the two layers differ. In our formalism these different phases will appear naturally because of changes in the effective pseudospin-pseudospin interactions. The effective interactions we calculate depend slightly on the finite size of the system in which the magnetic Wannier states are orthogonalized, but approach a definite value in the thermodynamic limit. Effective interactions for either the thermodynamic limit, or for any specified finite size system, are available from the authors on request for any of the above cases.

To investigate finite-temperature properties of our spin-pseudospin model for lowest Landau level bilayers, one needs to go beyond linear spin wave theory (15) which is not adequate in two space dimensions (2D) at any finite temperature due to the absence of long-range order. A simple but powerful method, the Schwinger boson mean field theory (SBMFT), was proposed by Arovas and Auerbach.²⁵ It does not break spin rotational invariance at finite temperatures, yet correctly reproduces linearized spin wave theory in the semiclassical limit. For the SU(2) Heisenberg model SBMFT qualitatively reproduces the low temperature continuum field theory results.^{36,37} For 1D integer antiferromagnetic spin chains, SBMFT gives the correct value of the Haldane gap in the ground state. However, it fails in the case of the half-integer chains, where Berry phase effects missed by SBMFT result in a gapless excitation spectrum at zero temperature.

In the next section we briefly recount some features of the original theory, developed for the SU(2) invariant Heisenberg model, that will be important for our discussion. We then show how to achieve a proper generalization of SBMFT for spin models with anisotropy, a procedure that requires some care. We then apply our generalized SBMFT to the model Hamiltonian derived in the present section.

III. SCHWINGER BOSON MEAN FIELD THEORY FOR SPIN MODELS WITH ANISOTROPY

We first briefly review the original SBMFT proposed by Arovas and Auerbach²⁵ for the SU(2) invariant spin model. Our motivation here is to lay the ground work for the generalization to anisotropic interaction models that is required for the application to our model Hamiltonian. Consider the 2D Heisenberg model on a square lattice with only nearest-neighbor interactions

$$H = -\frac{1}{2}J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j. \quad (17)$$

The sum is over neighboring *sites* with the $1/2$ factor correcting for double counting. For $J > 0$ the ground state of this model is ferromagnetic. In 2D long-range order is destroyed at any finite temperature, strong short-range

ferromagnetic (or antiferromagnetic for $J < 0$) correlations are present for temperatures up to $\sim |J|$, however. A way to build a simple theory of this *spin liquid* state uses the Schwinger boson representation of spin operators

$$\begin{aligned} S_i^z &= \frac{1}{2}(a_{i1}^\dagger a_{i1} - a_{i2}^\dagger a_{i2}) \\ S_i^+ &= a_{i1}^\dagger a_{i2} \\ S_i^- &= a_{i2}^\dagger a_{i1} \\ a_{i1}^\dagger a_{i1} + a_{i2}^\dagger a_{i2} &= 2S. \end{aligned} \tag{18}$$

Correlations in the spin liquid are then described by *bond operators*, which are bilinear forms in Schwinger boson operators from nearest-neighbor sites. Nonzero expectation values for these bond operators represent *short-range order*. There are of course many possible bond operators one can define, however *only the $SU(2)$ invariant ones are relevant*, since rotational invariance is preserved in the spin liquid state. There are only two $SU(2)$ invariant bond operators:

$$\begin{aligned} F_{ij}^\dagger &= a_{i1}^\dagger a_{j1} + a_{i2}^\dagger a_{j2} \\ A_{ij}^\dagger &= a_{i1}^\dagger a_{j2} - a_{i2}^\dagger a_{j1}. \end{aligned} \tag{19}$$

The operators F^\dagger and A^\dagger are not independent, but are connected by the following operator identity

$$: F_{ij}^\dagger F_{ij} : + A_{ij}^\dagger A_{ij} = 4S^2, \tag{20}$$

where colons represent normal ordering. One can write the spin Hamiltonian (17) in terms of these bond operators as follows

$$H = -\frac{J}{8} \sum_{\langle ij \rangle} \left(: F_{ij}^\dagger F_{ij} : - A_{ij}^\dagger A_{ij} \right) \tag{21}$$

From (21) the physical meaning of the two bond operators is clear. Nonzero expectation value of F^\dagger represents short-range ferromagnetic order, of A^\dagger —short-range antiferromagnetic order. Using (20) one can eliminate one of the bond operator products from (21), depending on the kind of correlations one expects to have (determined by the sign of J in the simple Heisenberg model). Assuming $J > 0$ one has

$$H = -\frac{J}{4} \sum_{\langle ij \rangle} \left(: F_{ij}^\dagger F_{ij} : - 2S^2 \right). \tag{22}$$

In the original approach of Arovas and Auerbach this representation is generalized to N Schwinger boson flavors, making it possible to set up a systematic expansion of the functional integral representation of the partition function in powers of $1/N$, starting from the saddle point approximation. We are, however, only interested in the saddle point, or mean field, solution itself and therefore will use a more simple-minded approach. We assume that the bond operator has a nonzero expectation value

$$Q = \langle F_{ij}^\dagger \rangle = \langle F_{ij} \rangle, \tag{23}$$

and perform a Hartree-Fock decoupling of (22). The mean field Hamiltonian, neglecting constants, is

$$\begin{aligned} H &= -\frac{JQ}{4} \sum_{\langle ij \rangle} \left(F_{ij}^\dagger + F_{ij} \right) \\ &= -\frac{JQ}{4} \sum_{\langle ij \rangle} \left(a_{i1}^\dagger a_{j1} + a_{i2}^\dagger a_{j2} + h.c. \right). \end{aligned} \tag{24}$$

The constraint on Schwinger boson occupation numbers in (18) is imposed on average by introducing a chemical potential term in the mean field Hamiltonian to yield

$$H = \lambda \sum_i \left(a_{i1}^\dagger a_{i1} + a_{i2}^\dagger a_{i2} \right) - \frac{JQ}{4} \sum_{\langle ij \rangle} \left(a_{i1}^\dagger a_{j1} + a_{i2}^\dagger a_{j2} + h.c. \right), \tag{25}$$

or in Fourier space

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} (a_{\mathbf{k}1}^\dagger a_{\mathbf{k}1} + a_{\mathbf{k}2}^\dagger a_{\mathbf{k}2}), \quad (26)$$

where

$$\epsilon_{\mathbf{k}} = \lambda - 2JQ\gamma_{\mathbf{k}}, \quad (27)$$

and $\gamma_{\mathbf{k}} = \frac{1}{2}(\cos(k_x a) + \cos(k_y a))$ for a square lattice with lattice constant a . It is convenient to redefine λ to make the notation more physically meaningful, and also to facilitate the solution of SBMFT equations by constraining the form of $\epsilon_{\mathbf{k}}$

$$\lambda \rightarrow \lambda + 2JQ. \quad (28)$$

Then $\epsilon_{\mathbf{k}}$ becomes

$$\epsilon_{\mathbf{k}} = \lambda + 2JQ(1 - \gamma_{\mathbf{k}}). \quad (29)$$

One completes the model by writing down the selfconsistency equations

$$\begin{aligned} \frac{2}{N} \sum_{\mathbf{k}} n_B(\epsilon_{\mathbf{k}}) &= 2S \\ Q &= 2S - \frac{2}{N} \sum_{\mathbf{k}} (1 - \gamma_{\mathbf{k}}) n_B(\epsilon_{\mathbf{k}}), \end{aligned} \quad (30)$$

where N is the total number of lattice sites.

The first of equations (30) is the constraint on the *average* number of Schwinger bosons per site, while the second is the self-consistent expression for the bond operator expectation value. The physical meaning of $\epsilon_{\mathbf{k}}$ is now clear. The second term in (29) is just the gapless magnon dispersion with a self-consistently renormalized stiffness constant. The gap λ reflects the finite correlation length in the spin liquid at finite temperatures. As the temperature goes to zero and the system size N goes to infinity, $\lambda \rightarrow 0$, $Q \rightarrow 2S$ and the gapless quadratic spin wave dispersion is recovered. Note however, that the mean field quasiparticles in SBMFT are *not* magnons. Due to the fact that the constraint on the total number of Schwinger bosons on each site is enforced in (30) only on average, unphysical excitations which change the local number of Schwinger bosons are allowed in the mean field theory.

It is straightforward to show that at finite temperatures SBMFT reproduces the *renormalized classical*³⁷ correlation length for 2D quantum ferromagnets and antiferromagnets. For 1D integer antiferromagnetic chains at $T = 0$, the Haldane gap is also correctly described.

Generally, SBMFT has proven to be a very successful and useful tool. One can think of it as an analog of the Weiss molecular field theory for quantum spin liquids. It is accurate enough to describe subtle quantum effects, yet simple enough to treat complicated spin models, such as (14). We refer the reader to the original paper of Arovas and Auerbach and Auerbach's textbook³⁸ for further details on the application of SBMFT to the SU(2) Heisenberg model. For a discussion of the application of the SBMFT approach to frustrated antiferromagnets see.³⁹

We now generalize the theory outlined above to anisotropic models. It is clear how to do this if one recalls the crucial principle of the SBMFT: one needs to write the Hamiltonian in terms of products of bond operators invariant under the transformations of the symmetry group of the Hamiltonian. To be concrete, let us consider the following model

$$H = \frac{1}{2} \sum_{\langle ij \rangle} (J^z S_i^z S_j^z - J^\perp \mathbf{S}_i^\perp \cdot \mathbf{S}_j^\perp). \quad (31)$$

This Hamiltonian has the same $Z_2 \times U(1)$ symmetry as the pseudospin part of our spin-pseudospin model (14) and shares the frustration that can occur between antiferromagnetic $S^z - S^z$ interactions and ferromagnetic $S^\perp - S^\perp$ interactions. To illustrate our strategy we assume a square lattice with only nearest neighbor spins interacting. We will also discuss only the case $J^\perp > J^z$ when the classical ground state is ferromagnetically ordered in the $\hat{x} - \hat{y}$ plane, which is the case relevant case for bilayer quantum Hall ferromagnets. Other cases can be analyzed in a similar way.

There are four bond operators that are invariant under symmetry operations of the Hamiltonian:

$$\begin{aligned}
F_{ij}^\dagger &= a_{i1}^\dagger a_{j1} + a_{i2}^\dagger a_{j2} \\
A_{ij}^\dagger &= a_{i1}^\dagger a_{j2}^\dagger - a_{i2}^\dagger a_{j1}^\dagger \\
X_{ij}^\dagger &= a_{i1}^\dagger a_{j2}^\dagger + a_{i2}^\dagger a_{j1}^\dagger \\
Z_{ij}^\dagger &= a_{i1}^\dagger a_{j1} - a_{i2}^\dagger a_{j2}.
\end{aligned} \tag{32}$$

The first two are the usual SU(2) ferromagnetic and antiferromagnetic bond operators. The two new bond operators, X_{ij}^\dagger and Z_{ij}^\dagger represent XY and easy axis ferromagnetic correlations. They are connected by the familiar identity

$$: Z_{ij}^\dagger Z_{ij} : + X_{ij}^\dagger X_{ij} = 4S^2. \tag{33}$$

As in the SU(2) Heisenberg model case, one can write the Hamiltonian in terms of the products of invariant bond operators. There are again several ways to do this, and we need to choose the one that is appropriate to describe the correlations in the ground state. In the case where the ground state is an XY-ferromagnet, the desired mean field theory follows from the following form for the Hamiltonian

$$H = \frac{1}{8} \sum_{\langle ij \rangle} \left[J^z \left(: F_{ij}^\dagger F_{ij} : - X_{ij}^\dagger X_{ij} \right) - J^\perp \left(X_{ij}^\dagger X_{ij} + : F_{ij}^\dagger F_{ij} : - 4S^2 \right) \right]. \tag{34}$$

We will see later that the resulting Schwinger boson mean field theory correctly reproduces quasiclassical dynamics in the $S \rightarrow \infty, T \rightarrow 0$ limit.

Regrouping terms and neglecting a constant contribution, the Hamiltonian can be rewritten as

$$H = -\frac{1}{8} \sum_{\langle ij \rangle} \left[(J^\perp - J^z) : F_{ij}^\dagger F_{ij} : + (J^\perp + J^z) X_{ij}^\dagger X_{ij} \right]. \tag{35}$$

To proceed it is convenient to first rotate coordinates by $\pi/2$ around the y-axis. The X^\dagger bond operator then changes to

$$X_{ij}^\dagger = a_{i1}^\dagger a_{j1}^\dagger - a_{i2}^\dagger a_{j2}^\dagger. \tag{36}$$

F_{ij}^\dagger does not change of course, because it is an SU(2) invariant. As before, we introduce expectation values of the bond operators

$$\begin{aligned}
Q &= \langle F_{ij}^\dagger \rangle = \langle F_{ij} \rangle \\
P &= \langle X_{ij}^\dagger \rangle = \langle X_{ij} \rangle,
\end{aligned} \tag{37}$$

and perform a Hartree-Fock decoupling. The resulting mean field Hamiltonian is

$$\begin{aligned}
H &= \sum_{\mathbf{k}} \left[\lambda - (J^\perp - J^z) Q \gamma_{\mathbf{k}} \right] \left(a_{\mathbf{k}1}^\dagger a_{\mathbf{k}1} + a_{\mathbf{k}2}^\dagger a_{\mathbf{k}2} \right) \\
&\quad - \frac{1}{2} (J^\perp + J^z) P \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \left(a_{\mathbf{k}1}^\dagger a_{-\mathbf{k}1}^\dagger + a_{\mathbf{k}1} a_{-\mathbf{k}1} - a_{\mathbf{k}2}^\dagger a_{-\mathbf{k}2}^\dagger - a_{\mathbf{k}2} a_{-\mathbf{k}2} \right).
\end{aligned} \tag{38}$$

We introduce the following notation to make the subsequent equations readable

$$\begin{aligned}
A_{\mathbf{k}1} &= \lambda + (J^\perp + J^z) P + (J^\perp - J^z) Q (1 - \gamma_{\mathbf{k}}) \\
A_{\mathbf{k}2} &= \lambda + h + (J^\perp + J^z) P + (J^\perp - J^z) Q (1 - \gamma_{\mathbf{k}}) \\
B_{\mathbf{k}1,2} &= \pm (J^\perp + J^z) P \gamma_{\mathbf{k}} \\
\epsilon_{\mathbf{k}1,2} &= \sqrt{A_{\mathbf{k}1,2}^2 - B_{\mathbf{k}1,2}^2}.
\end{aligned} \tag{39}$$

Here we have added magnetic field h in the \hat{x} -direction and redefined λ as before.

Eq.(38) in this notation becomes

$$H = \sum_{\mathbf{k}, m=1,2} \left[A_{\mathbf{k}m} a_{\mathbf{k}m}^\dagger a_{\mathbf{k}m} - \frac{1}{2} B_{\mathbf{k}m} \left(a_{\mathbf{k}m}^\dagger a_{-\mathbf{k}m}^\dagger + a_{\mathbf{k}m} a_{-\mathbf{k}m} \right) \right]. \tag{40}$$

Hamiltonian (40) is diagonalized by a standard Bogoliubov transformation

$$\begin{aligned}
a_{\mathbf{k}m} &= \cosh \theta_{\mathbf{k}m} \alpha_{\mathbf{k}m} + \sinh \theta_{\mathbf{k}m} \alpha_{-\mathbf{k}m}^\dagger \\
\cosh 2\theta_{\mathbf{k}m} &= \frac{A_{\mathbf{k}m}}{\epsilon_{\mathbf{k}m}} \\
\sinh 2\theta_{\mathbf{k}m} &= \frac{B_{\mathbf{k}m}}{\epsilon_{\mathbf{k}m}},
\end{aligned} \tag{41}$$

and

$$H = \sum_{\mathbf{k}m} \epsilon_{\mathbf{k}m} \alpha_{\mathbf{k}m}^\dagger \alpha_{\mathbf{k}m}. \tag{42}$$

The parameters in the mean field Hamiltonian are fixed by solving the self-consistency equations

$$\begin{aligned}
\frac{1}{N} \sum_{\mathbf{k}} \left\{ \frac{A_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] + \frac{A_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right] - 1 \right\} &= 2S \\
Q = 2S - \frac{1}{N} \sum_{\mathbf{k}} (1 - \gamma_{\mathbf{k}}) \left\{ \frac{A_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] + \frac{A_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right] - 1 \right\} \\
P = \frac{1}{N} \sum_{\mathbf{k}} \left\{ \frac{B_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] - \frac{B_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right] \right\}.
\end{aligned} \tag{43}$$

As can be seen from (43), there are corrections to linearized spin-wave theory even in the $T = 0$ limit in the anisotropic spin model case. This difference is not unexpected since the XY-ferromagnetic order parameter does not commute with the Hamiltonian and has quantum fluctuations in the ground state. If both quasiclassical ($S \rightarrow \infty$) and $T \rightarrow 0$ limits are taken, it is easy to see from Eq.(39) and (43) that $\lambda \rightarrow 0$, Q and $P \rightarrow 2S$, and one recovers the magnon dispersion of linear spin wave theory.

At low temperatures a system described by (31) must have *quasi long range* order (*i.e.* power law correlations) and undergo a Kosterlitz-Thouless phase transition at a finite temperature. Unfortunately, our theory fails to reproduce these results. Instead our SBMFT always implies a nonzero gap in the Schwinger boson spectrum, and exponential correlation functions, at all finite temperatures. This deficiency can be verified by a simple inspection of the first of equations (43), which represents the constraint on the number of Schwinger bosons, enforced on average. Assuming $\lambda = 0$, the Schwinger bosons dispersion (39) is linear for small k . Taking into account only the leading contribution to the integral over wavevector, we can take $n_B(\epsilon_{\mathbf{k}}) \sim T/\epsilon_{\mathbf{k}}$. Then it is clear that the integral over \mathbf{k} in (43) diverges logarithmically at the origin in 2D. Therefore at any finite temperature the gap in the Schwinger boson spectrum must be nonzero, which translates to exponentially decaying spin correlations as the gap introduces a spatial scale into the problem. The reason for this failure is the presence of unphysical fluctuations in the local number of Schwinger bosons allowed by the soft (enforced only on average) constraint in (43). The existence of quasi long range order depends critically on the fact that the physical low energy excitations in (31) (spin waves) are mostly in the $\hat{x} - \hat{y}$ plane. Unphysical fluctuations of the Schwinger boson occupation numbers in the mean field theory violate this constraint and therefore destroy the quasi long range order. In the SU(2) case, correlations are not so delicate and therefore the SBMFT works out qualitatively correctly.

This deficiency is not a very serious one, however, since in most cases of interest a small pseudospin magnetic field is present due to tunneling. We are concerned mainly with calculating quantities like the pseudospin magnetization at finite tunneling and the transverse susceptibility, which should not be extremely sensitive to the precise character of long-range correlations. We have checked the mean field values of the gap in the XY-model case are indeed much smaller at all temperatures than the ones in the isotropic Heisenberg model. As is true for any mean field theory, we do not expect our theory to be quantitatively correct, but to capture the general trends in the temperature and magnetic field dependence of experimentally observable quantities. Finally, it seems likely that this problem can be corrected by taking into account gaussian fluctuations around the saddle point solution (43). Gaussian fluctuations of the field λ will reintroduce the local constraint and therefore eliminate the unphysical excitations. This issue will be examined in the future work.

We have solved the mean field equations (43) for finite lattices with up to 100×100 sites and periodic boundary conditions which make our grid of wavevectors discrete. Results for the pure XY-model case ($J^z = 0$) are summarized in Figs. 4 and 5. Fig. 4 shows the in-plane magnetization for different values of the magnetic field in the \hat{x} -direction. Typical temperature dependences of the short-range order parameters Q and P are shown in the inset. At a temperature of the order of J^\perp , there is an unphysical transition to the state with no short range correlations. This is a well known problem of SBMFT which occurs in the SU(2) case as well.

The most interesting quantity for us is the transverse susceptibility

$$\chi^{zz} = \frac{d\langle S^z \rangle}{dh^z} \Big|_{h^z=0}. \quad (44)$$

as it is the most relevant quantity for experiments in the case of quantum Hall bilayers. It can be written in terms of imaginary time spin-spin correlation function as

$$\chi^{zz} = \frac{1}{N} \sum_{ij} \int_0^{1/T} d\tau \langle S_i^z(\tau) S_j^z(0) \rangle. \quad (45)$$

The SBMFT result for Eq. (45) can be evaluated using the imaginary time path integral technique. The result is

$$\begin{aligned} \chi^{zz} = \frac{1}{4N} \sum_{\mathbf{k}} \left[\left(\frac{A_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \frac{A_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} + \frac{B_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \frac{B_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} + 1 \right) \frac{n_B(\epsilon_{\mathbf{k}2}) - n_B(\epsilon_{\mathbf{k}1})}{\epsilon_{\mathbf{k}1} - \epsilon_{\mathbf{k}2}} \right. \\ \left. + \left(\frac{A_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \frac{A_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} + \frac{B_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \frac{B_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} - 1 \right) \frac{n_B(\epsilon_{\mathbf{k}1}) + n_B(\epsilon_{\mathbf{k}2}) + 1}{\epsilon_{\mathbf{k}1} + \epsilon_{\mathbf{k}2}} \right]. \quad (46) \end{aligned}$$

The temperature dependence of the transverse susceptibility for the case of XY-model with no in-plane field is shown in Fig.5. The monotonic increase of the susceptibility with temperature reflects the softening of the in-plane order due to thermal fluctuations. At high temperatures χ^{zz} must follow the usual $1/T$ paramagnetic susceptibility temperature dependence. Therefore χ^{zz} must have maximum at an intermediate temperatures beyond the range over which SBMFT is reliable. Recent quantum Monte Carlo simulation results⁴⁰ support this picture. Our result for the zero temperature susceptibility $\chi^{zz} = 0.1955$ is reasonably close to the value quantum Monte Carlo result 0.2096 from.⁴⁰ It is interesting to note that in this case SBMFT apparently gives the correct result for the susceptibility, unlike in the SU(2) Heisenberg model case, where it is overestimated by a factor of 3/2. The temperature dependence we obtain is, however, stronger than in⁴⁰ presumably due to the fact that SBMFT allows unphysical low energy out-of-plane spin fluctuations.

In closing this section, we remark that Timm and Jensen⁴¹ have recently suggested a different strategy for generalizing SBMFT to anisotropic spin models. These authors employed a more formal $1/N$ expansion approach. We believe that our approach is better for practical calculations since, unlike the method of Timm and Jensen, it reproduces the correct semiclassical dynamics in the limit $S \rightarrow \infty$ at the mean field level.

IV. SCHWINGER BOSON MEAN FIELD THEORY FOR THE $\nu = 1$ BILAYER EFFECTIVE SPIN-PSEUDOSPIN MODEL.

In this section we apply the formalism developed above to the effective spin-pseudospin model of the $\nu = 1$ quantum Hall bilayer given by (14). The Hamiltonian (14) has an SU(2) invariant spin system coupled to a $Z_2 \times U(1)$ invariant pseudospin system. Therefore our strategy will be to decouple the spin and pseudospin parts of (14) in a mean field approximation and use SBMFT to study the decoupled spin and pseudospin Hamiltonians. The coupling is restored by solving the resulting equations self-consistently. We expect this to be a good approximation due to the strong short-range correlations in our model and the long-range character of the couplings in (14).

The decoupled spin and pseudospin Hamiltonians are given by

$$H^S = -\Delta_z \sum_i S_i^z - \sum_{ij} \left[\frac{1}{2} F_{ij}^S + 2F_{ij}^S \langle T_i^z T_j^z \rangle + 2F_{ij}^D \langle \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp \rangle \right] \mathbf{S}_i \cdot \mathbf{S}_j, \quad (47)$$

and

$$\begin{aligned} H^P = - \sum_i [\Delta_v T_i^z + \Delta_t T_i^x] \\ + \sum_{ij} \left[\left(2H_{ij} - \frac{1}{2} F_{ij}^S - 2F_{ij}^S \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \right) T_i^z T_j^z - \left(\frac{1}{2} F_{ij}^D + 2F_{ij}^D \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \right) \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp \right]. \quad (48) \end{aligned}$$

To simplify the following equations we define effective couplings for the spin and pseudospin Hamiltonians:

$$\begin{aligned}
J_{ij}^S &= \frac{1}{2}F_{ij}^S + 2F_{ij}^S \langle T_i^z T_j^z \rangle + 2F_{ij}^D \langle \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp \rangle \\
J_{ij}^{Pz} &= 2H_{ij} - \frac{1}{2}F_{ij}^S - 2F_{ij}^S \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle \\
J_{ij}^{P\perp} &= \frac{1}{2}F_{ij}^D + 2F_{ij}^D \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle.
\end{aligned} \tag{49}$$

In terms of these couplings (47) and (48) are written as

$$H^S = -\Delta_z \sum_i S_i^z - \sum_{ij} J_{ij}^S \mathbf{S}_i \cdot \mathbf{S}_j, \tag{50}$$

$$H^P = -\sum_i [\Delta_V T_i^z + \Delta_t T_i^x] + \sum_{ij} [J_{ij}^{Pz} T_i^z T_j^z - J_{ij}^{P\perp} \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp]. \tag{51}$$

We now analyze the effective spin and pseudospin Hamiltonians using the methods developed in section III. Let us consider H^S first. As before, we represent spin operators by two Schwinger bosons, define ferromagnetic bond operators F_{ij}^\dagger and rewrite the Hamiltonian in terms of these operators

$$H^S = -\frac{\Delta_z}{2} \sum_i (a_{i1}^\dagger a_{i1} - a_{i2}^\dagger a_{i2}) - \frac{1}{2} \sum_{ij} J_{ij}^S : F_{ij}^\dagger F_{ij} :. \tag{52}$$

The main difference from the simple Heisenberg model we considered before is that here we have long-range spin-spin couplings. Correspondingly, bond operators are also defined for all pairs of sites, not just the nearest-neighbor ones.

We define expectation value of the bond operator for each pair of sites

$$Q_{ij}^S = \langle F_{ij}^\dagger \rangle = \langle F_{ij} \rangle, \tag{53}$$

and perform a Hartree-Fock decoupling, adding a chemical potential term to account for the constraint

$$H^S = \sum_i \left[\left(\lambda - \frac{\Delta_z}{2} \right) a_{i1}^\dagger a_{i1} + \left(\lambda + \frac{\Delta_z}{2} \right) a_{i2}^\dagger a_{i2} \right] - \frac{1}{2} \sum_{ij} \tilde{J}_{ij}^S (a_{i1}^\dagger a_{j1} + a_{i2}^\dagger a_{j2} + h.c.), \tag{54}$$

where we have introduced notation $\tilde{J}_{ij}^S = J_{ij}^S Q_{ij}^S$.

Taking advantage of translational invariance we rewrite (54) in Fourier space, simultaneously diagonalizing it and reducing the number of variables

$$H^S = \sum_{\mathbf{k}} \left[\left(\lambda - \frac{\Delta_z}{2} - \tilde{J}_{\mathbf{k}}^S \right) a_{\mathbf{k}1}^\dagger a_{\mathbf{k}1} + \left(\lambda + \frac{\Delta_z}{2} - \tilde{J}_{\mathbf{k}}^S \right) a_{\mathbf{k}2}^\dagger a_{\mathbf{k}2} \right]. \tag{55}$$

As before, it is convenient to redefine λ to make the notation more physical and also to facilitate numerical solution of the self-consistent equations,

$$\lambda \rightarrow \lambda + \frac{\Delta_z}{2} + \tilde{J}_0^S. \tag{56}$$

Then H^S becomes

$$H^S = \sum_{\mathbf{k}} \left(\epsilon_{\mathbf{k}1} a_{\mathbf{k}1}^\dagger a_{\mathbf{k}1} + \epsilon_{\mathbf{k}2} a_{\mathbf{k}2}^\dagger a_{\mathbf{k}2} \right), \tag{57}$$

where

$$\begin{aligned}
\epsilon_{\mathbf{k}1} &= \lambda + \tilde{J}_0^S - \tilde{J}_{\mathbf{k}}^S \\
\epsilon_{\mathbf{k}2} &= \lambda + \Delta_z + \tilde{J}_0^S - \tilde{J}_{\mathbf{k}}^S.
\end{aligned} \tag{58}$$

The two kinds of Schwinger bosons now have different dispersions due to the asymmetry introduced by the magnetic field. The a_2^\dagger Schwinger bosons become the usual magnons in the limit $T \rightarrow 0$ and $N \rightarrow \infty$. On the other hand, the

bosons created by a_1^\dagger Bose condense in the same limit which signals the appearance of long-range order at $T = 0$ in the thermodynamic limit. The connection between the condensation of Schwinger bosons and long-range magnetic order is discussed in more detail in.⁴² We will always consider finite temperatures and finite system sizes and therefore will not be concerned with this.

The system of self-consistent equations one has to solve now is as follows

$$\begin{aligned} \frac{1}{N} \sum_{\mathbf{k}} [n_B(\epsilon_{\mathbf{k}1}) + n_B(\epsilon_{\mathbf{k}2})] &= 1 \\ Q_{\mathbf{k}}^S &= n_B(\epsilon_{\mathbf{k}1}) + n_B(\epsilon_{\mathbf{k}2}). \end{aligned} \quad (59)$$

One therefore has $N + 1$ coupled equations to solve instead of 2 in the case of the Heisenberg model with nearest neighbor interactions. It is useful to define another expectation value which will be used later,

$$\tilde{Q}_{\mathbf{k}}^S = n_B(\epsilon_{\mathbf{k}1}) - n_B(\epsilon_{\mathbf{k}2}). \quad (60)$$

These equations have to be solved self-consistently with the corresponding equations for the effective pseudospin Hamiltonian that differ from our detailed discussion of the near-neighbor interaction model in a similar manner. The effective pseudospin Hamiltonian is

$$H^P = - \sum_i \Delta t T_i^x + \sum_{ij} [J_{ij}^{Pz} T_i^z T_j^z - J_{ij}^{P\perp} \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp], \quad (61)$$

or, in terms of Schwinger bosons and bond operators

$$H^P = - \frac{\Delta t}{2} \sum_i (b_{i1}^\dagger b_{i2} + b_{i2}^\dagger b_{i1}) - \frac{1}{4} \sum_{ij} [(J_{ij}^{P\perp} - J_{ij}^{Pz}) : F_{ij}^\dagger F_{ij} : + (J_{ij}^{P\perp} + J_{ij}^{Pz}) X_{ij}^\dagger X_{ij}]. \quad (62)$$

Here we use a different notation for Schwinger boson creation and annihilation operators to distinguish them from the analogous operators representing real spins.

It is convenient to first rotate coordinates to diagonalize the tunneling term

$$H^P = - \frac{\Delta t}{2} \sum_i (b_{i1}^\dagger b_{i1} - b_{i2}^\dagger b_{i2}) - \frac{1}{4} \sum_{ij} [(J_{ij}^{P\perp} - J_{ij}^{Pz}) : F_{ij}^\dagger F_{ij} : + (J_{ij}^{P\perp} + J_{ij}^{Pz}) X_{ij}^\dagger X_{ij}], \quad (63)$$

so that the bond operators are

$$\begin{aligned} F_{ij}^\dagger &= b_{i1}^\dagger b_{j1} + b_{i2}^\dagger b_{j2} \\ X_{ij}^\dagger &= b_{i1}^\dagger b_{j1}^\dagger - b_{i2}^\dagger b_{j2}^\dagger. \end{aligned} \quad (64)$$

Introducing expectation values for the bond operators,

$$\begin{aligned} Q_{ij}^P &= \langle F_{ij}^\dagger \rangle = \langle F_{ij} \rangle \\ P_{ij}^P &= \langle X_{ij}^\dagger \rangle = \langle X_{ij} \rangle, \end{aligned} \quad (65)$$

then performing Hartree-Fock decoupling and introducing renormalized couplings,

$$\begin{aligned} \tilde{J}_{ij}^- &= \frac{1}{2} (J_{ij}^{P\perp} - J_{ij}^{Pz}) Q_{ij}^P, \\ \tilde{J}_{ij}^+ &= \frac{1}{2} (J_{ij}^{P\perp} + J_{ij}^{Pz}) P_{ij}^P, \end{aligned} \quad (66)$$

we obtain the following mean field Schwinger boson Hamiltonian

$$\begin{aligned} H^P &= \sum_i \left[\left(\lambda - \frac{\Delta t}{2} \right) b_{i1}^\dagger b_{i1} + \left(\lambda + \frac{\Delta t}{2} \right) b_{i2}^\dagger b_{i2} \right] \\ &\quad - \frac{1}{2} \sum_{ij} \left[\tilde{J}_{ij}^- (b_{i1}^\dagger b_{j1} + b_{i2}^\dagger b_{j2} + h.c.) + \tilde{J}_{ij}^+ (b_{i1}^\dagger b_{j1}^\dagger - b_{i2}^\dagger b_{j2}^\dagger + h.c.) \right]. \end{aligned} \quad (67)$$

Fourier transforming and redefining λ as before

$$H^P = \sum_{\mathbf{k}} \left[\left(\lambda + \tilde{J}_0^+ + \tilde{J}_0^- - \tilde{J}_{\mathbf{k}}^- \right) b_{\mathbf{k}1}^\dagger b_{\mathbf{k}1} + \left(\lambda + \Delta_t + \tilde{J}_0^+ + \tilde{J}_0^- - \tilde{J}_{\mathbf{k}}^- \right) b_{\mathbf{k}2}^\dagger b_{\mathbf{k}2} - \frac{1}{2} \tilde{J}_{\mathbf{k}}^+ \left(b_{\mathbf{k}1}^\dagger b_{-\mathbf{k}1}^\dagger - b_{\mathbf{k}2}^\dagger b_{-\mathbf{k}2}^\dagger + h.c. \right) \right]. \quad (68)$$

As before, we introduce the notation

$$\begin{aligned} A_{\mathbf{k}1} &= \lambda + \tilde{J}_0^+ + \tilde{J}_0^- - \tilde{J}_{\mathbf{k}}^- \\ A_{\mathbf{k}2} &= \lambda + \Delta_t + \tilde{J}_0^+ + \tilde{J}_0^- - \tilde{J}_{\mathbf{k}}^- \\ B_{\mathbf{k}1,2} &= \pm \tilde{J}_{\mathbf{k}}^+ \\ \epsilon_{\mathbf{k}1,2} &= \sqrt{A_{\mathbf{k}1,2}^2 - B_{\mathbf{k}1,2}^2}. \end{aligned} \quad (69)$$

Diagonalizing (68) by Bogoliubov transformation we finally obtain

$$H^P = \sum_{\mathbf{k}} \left(\epsilon_{\mathbf{k}1} \beta_{\mathbf{k}1}^\dagger \beta_{\mathbf{k}1} + \epsilon_{\mathbf{k}2} \beta_{\mathbf{k}2}^\dagger \beta_{\mathbf{k}2} \right). \quad (70)$$

As usual the parameters in the mean field Hamiltonian are fixed by solving the self-consistency equations

$$\begin{aligned} \frac{1}{N} \sum_{\mathbf{k}} \left\{ \frac{A_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] + \frac{A_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right] - 1 \right\} &= 1 \\ Q_{\mathbf{k}}^P &= \frac{A_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] + \frac{A_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right] - 1 \\ P_{\mathbf{k}}^P &= \frac{B_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] - \frac{B_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right]. \end{aligned} \quad (71)$$

The systems of equations (59) and (71) have to be supplemented by the equations for the spin-spin and pseudospin-pseudospin correlation functions, which couple the spin and pseudospin systems. They can be evaluated straightforwardly using Wick's theorem. For unbiased bilayers

$$\begin{aligned} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle &= \frac{3}{8} |Q_{ij}^S|^2 - \frac{1}{8} |\tilde{Q}_{ij}^S|^2 + \langle S_i^z \rangle^2 \\ \langle T_i^z T_j^z \rangle &= \frac{1}{8} \left(|Q_{ij}^P|^2 - |P_{ij}^P|^2 - |\tilde{Q}_{ij}^P|^2 + |\tilde{P}_{ij}^P|^2 \right) \\ \langle \mathbf{T}_i^\perp \cdot \mathbf{T}_j^\perp \rangle &= \frac{1}{4} \left(|Q_{ij}^P|^2 + |P_{ij}^P|^2 \right) + \langle T_i^x \rangle^2, \end{aligned} \quad (72)$$

where

$$\begin{aligned} \tilde{Q}_{\mathbf{k}}^P &= \langle b_{\mathbf{k}1}^\dagger b_{\mathbf{k}2} + b_{\mathbf{k}2}^\dagger b_{\mathbf{k}1} \rangle = \frac{A_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] - \frac{A_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right] \\ \tilde{P}_{\mathbf{k}}^P &= \langle b_{\mathbf{k}1} b_{-\mathbf{k}1} + b_{\mathbf{k}2} b_{-\mathbf{k}2} \rangle = \frac{B_{\mathbf{k}1}}{\epsilon_{\mathbf{k}1}} \left[n_B(\epsilon_{\mathbf{k}1}) + \frac{1}{2} \right] + \frac{B_{\mathbf{k}2}}{\epsilon_{\mathbf{k}2}} \left[n_B(\epsilon_{\mathbf{k}2}) + \frac{1}{2} \right]. \end{aligned} \quad (73)$$

Equations (59), (71), (72) constitute the full set of equations of the SBMFT for the $\nu = 1$ quantum Hall bilayer. Fig.'s 6–10 summarize some of the results we have obtained by solving these equations numerically. The calculations were performed for lattices with up to 20×20 sites.

The polarization response of the pseudospin to a field that is transverse to the ordering direction corresponds physically to the charge transferred between layers in response to a bias voltage. In the Hartree-Fock approximation this quantity has no temperature dependence for temperatures below the charge gap. Our results for the temperature dependence of the differential capacitance at $d/\ell = 0.5, 1.0$ and 1.4 at several different values of the Zeeman coupling strength Δ_z are summarized in Fig.'s 6–8. The limit $\Delta_z \rightarrow \infty$ corresponds to the usual assumption that the spin degree of freedom is frozen out. The large increase of susceptibility compared to the Hartree-Fock value at zero temperature in Fig.6 is an artifact of SBMFT, which overestimates the value of the spin-spin correlation function in

Eq.(49) leading to an overestimate of the transverse susceptibility. We did not try to fix this problem by normalizing the value of the correlation function by hand since we are mainly concerned with the qualitative features of the temperature dependence of susceptibility rather than exact values for its magnitude.

One can see that the temperature dependences in Fig.6 are qualitatively different from those in the XY-model case in Fig.5. The source of this difference is the dependence of the effective pseudospin interactions in Eq.(49) on the spin polarization, as is evident from the strong dependence of the differential capacitance on Zeeman coupling. It is clear from thermodynamic considerations that the transverse pseudospin coupling is approximately inversely proportional to the difference between the in-plane and transverse effective couplings. From Eq.(49) we see that both become less ferromagnetic (or more antiferromagnetic) as correlations in the spin system are suppressed and the exchange contributions to the pseudospin effective interactions are reduced. The coupling between spin and pseudospin fluctuations therefore tends to make the easy-plane anisotropy strengthen and the transverse susceptibility weaken with increasing temperature. This effect competes with the increase in susceptibility with temperature that holds for the XY-model with temperature independent coupling constants. Our calculations indicate that the former effect dominates as small d/ℓ while the latter effect dominates at larger d/ℓ . This qualitative discussion does not account for subtle effects associated with the longer range of the interaction between pseudospin \hat{z} components, or for role played by the relative sign of the XY and Ising pseudospin interaction constants. The transverse pseudospin susceptibility can be measured directly by measuring the capacitance of the bilayer system.⁴³ We emphasize that the temperature dependences shown in these figures are all due to correlated quantum and thermal fluctuations and would be absent in a Hartree-Fock theory.

In Fig.9 and 10 the temperature dependences of the spin and in-plane pseudospin magnetization for different values of interlayer separation are plotted. The spin density in bilayer systems can be measured by measuring the Knight shift in an optically pumped NMR experiment⁴⁴ while the pseudospin polarization can be extracted from optical absorption experiments.⁴⁵ The strong dependence of spin magnetization on the interlayer separation that we find is another signature of spin-pseudospin coupling.

V. CONCLUDING REMARKS

In this paper we have presented a theory of the finite temperature properties of $\nu = 1$ quantum Hall bilayers that accounts for both spin and layer degrees of freedom. The theory is based on an approximate mapping of the microscopic interacting electron problem onto a problem of interacting spins and pseudospins on a lattice with a prescribed Hamiltonian specified in Eq.(14). Our mapping is exact when fluctuations in the charge density, summed over the two layers, are neglected. To investigate the finite temperature properties of this model, we first generalize the Schwinger boson mean field theory of Arovas and Auerbach to anisotropic spin Hamiltonians. Using this approach, we are able to predict qualitatively the character of the temperature dependence of several experimentally observable quantities like the spin magnetization and transverse pseudospin susceptibility, or interlayer differential capacitance. The temperature dependences we find reflect a subtle interplay between spin and pseudospin fluctuations. Spin fluctuations in bilayer quantum Hall systems can be neglected only at temperatures significantly smaller than the unenhanced spin-splitting gap, which is $\sim 1K$ in typical circumstances. When spin fluctuations are neglected our calculation is closely related to the microscopic generalized random-phase-approximation (RPA) calculation of Joglekar *et al.*³³ The linear spin-wave theory approximation to our spin-pseudospin model is equivalent to the microscopic generalized RPA; by neglecting charge fluctuations we are able to go beyond this level, approximating the spin-wave interactions of bilayer quantum Hall systems using Schwinger boson mean field theory. Our spin-pseudospin model is not however coupled in any way to any particular technique and the full array of numerical and analytic methods that have been developed for quantum spin models could be employed.

Our formalism could in principle be generalized to integer filling factors greater than one. To illustrate the complications involved we discuss briefly the case of $\nu = 2$. Quantum Hall bilayers at $\nu = 2$ have been studied actively in recent years and a number of interesting theoretical results concerning broken symmetry ground states have also been obtained^{11,12,14-16} for this case. In particular, a *canted antiferromagnetic* phase is predicted in which total spins in the two layers have opposite tilts away from the magnetic field and spin-dependent spontaneous coherence is established between the opposite layers. Thus spin-pseudospin interplay is important at $\nu = 2$ even for ground state properties. At $\nu = 2$ the single Slater determinant wavefunctions that span the Hilbert space of the system have the following form

$$|\Psi[z]\rangle = \prod_{i,\alpha=1,2} \left(\sum_{k=1}^4 z_{ik}^{\alpha} c_{ik}^{\dagger} \right) |0\rangle, \quad (74)$$

i.e. there are two occupied orbitals corresponding to each magnetic Wannier state i . In this case, orthonormality conditions for these four orbitals provide four constraints on what will become coherent state labels in the boson path integral. Following this line, the $\nu = 2$ spin-pseudospin Hamiltonian is

$$\begin{aligned}
H = & - \sum_{i,\alpha} [\Delta_V T_{i\alpha}^z + \Delta_t T_{i\alpha}^x + \Delta_z S_{i\alpha}^z] \\
& + \sum_{ij,\alpha\beta} \left[(2H_{ij} - \frac{1}{2} F_{ij}^S) T_{i\alpha}^z T_{j\beta}^z - \frac{1}{2} F_{ij}^D \mathbf{T}_{i\alpha}^\perp \cdot \mathbf{T}_{j\beta}^\perp - \frac{1}{2} F_{ij}^S \mathbf{S}_{i\alpha} \cdot \mathbf{S}_{j\beta} \right. \\
& \left. - 2F_{ij}^S (\mathbf{S}_{i\alpha} \cdot \mathbf{S}_{j\beta}) T_{i\alpha}^z T_{j\beta}^z - 2F_{ij}^D (\mathbf{S}_{i\alpha} \cdot \mathbf{S}_{j\beta}) (\mathbf{T}_{i\alpha}^\perp \cdot \mathbf{T}_{j\beta}^\perp) \right]. \tag{75}
\end{aligned}$$

Each lattice site now has two spins and two pseudospins corresponding to two occupied orbitals in (74). It would be interesting to study the correspondence between this Hamiltonian and two similar effective models introduced recently.^{46,47} Eq.(75) is substantially more complicated than (14) and even the classical ground state, again identical to the Hartree-Fock ground state, is known only numerically for general external fields. The presence of implicit orthonormality constraints in Eq.(75) is a disadvantage of this approach, since it reduces the transparency of the Hamiltonian. We believe, nevertheless, that Eq.(75) could be a good starting point for a theory of correlations and thermal fluctuations in $\nu = 2$ quantum Hall ferromagnets.

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APPENDIX

In this section we explain in more detail how our effective interactions are evaluated. A key step is the construction of the magnetic Wannier basis, and we start by briefly summarizing the results of Rashba *et al.*³¹ We want to construct a complete set of Wannier functions localized at the sites of a square lattice with lattice constant $a = \sqrt{2\pi\ell^2}$. We start from an overcomplete set of minimum uncertainty wave packets centered on square lattice sites. The minimum uncertainty wavepacket localized at the origin is

$$c_{00}(\mathbf{r}) = \frac{1}{\sqrt{2\pi\ell^2}} \exp\left(-\frac{r^2}{4\ell^2}\right). \tag{76}$$

c_{00} is just the zero angular momentum eigenfunction in the symmetric gauge. To construct wavepackets localized at other sites of the square lattice we need to translate c_{00} wavepacket from the origin to each of the lattice sites. Let the lattice sites be

$$\mathbf{r}_{mn} = ma\hat{x} + na\hat{y}. \tag{77}$$

Define

$$c_{mn}(\mathbf{r}) = T_{ma\hat{x}} T_{na\hat{y}} c_{00}(\mathbf{r}). \tag{78}$$

Here

$$T_{\mathbf{R}} = e^{-\frac{i}{\hbar} \mathbf{R} \cdot (\mathbf{p} - \frac{e}{c} \mathbf{A})} \tag{79}$$

is the magnetic translation operator (this expression is valid in the symmetric gauge only). In the symmetric gauge

$$\mathbf{A} = \frac{B}{2}(-y, x, 0). \tag{80}$$

Substituting (80) into (79) one obtains

$$T_{ma\hat{x}}T_{na\hat{y}} = (-1)^{mn} \exp\left[\frac{i}{2\ell^2}\hat{z}\cdot(\mathbf{r}\times\mathbf{r}_{mn})\right] \exp\left(-\frac{i}{\hbar}\mathbf{r}_{mn}\cdot\mathbf{p}\right). \quad (81)$$

Therefore c_{mn} is given by

$$c_{mn}(\mathbf{r}) = \frac{(-1)^{mn}}{\sqrt{2\pi\ell^2}} \exp\left[-\frac{(\mathbf{r}-\mathbf{r}_{mn})^2}{4\ell^2} + \frac{i}{2\ell^2}\hat{z}\cdot(\mathbf{r}\times\mathbf{r}_{mn})\right]. \quad (82)$$

The functions c_{mn} are not orthogonal and form an overcomplete set due to the following identity established by Perelomov

$$\sum_{mn=-\infty}^{\infty} (-1)^{m+n} c_{mn}(\mathbf{r}) = 0. \quad (83)$$

First we will construct a complete set of orthonormal Bloch functions from the set $\{c_{mn}\}$. Assume a normalization plaquette of area $L^2 = 2\pi\ell^2 N_\phi$, where N_ϕ is the number of magnetic flux quanta. Define a Bloch function at quasimomentum \mathbf{k} as follows

$$\Psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{N_\phi\nu(\mathbf{k})}} \sum_{mn=-\infty}^{\infty} c_{mn}(\mathbf{r}) \exp(i\mathbf{k}\mathbf{r}_{mn}). \quad (84)$$

Here $\nu(\mathbf{k})$ is a momentum-dependent normalization factor. The allowed values of quasimomentum are determined from the boundary conditions. If we assume periodic boundary conditions with respect to magnetic translations $T_{ma\hat{x}}, T_{na\hat{y}}$ the allowed values are $k_{x,y} = 2\pi n_{x,y}/L$.

The Bloch functions are assumed to be normalized to unity over the normalization plaquette.

$$\int_A d\mathbf{r} |\Psi_{\mathbf{k}}(\mathbf{r})|^2 = 1. \quad (85)$$

The normalization factor is given by

$$\nu(\mathbf{k}) = a \sum_{mn=-\infty}^{\infty} c_{mn}(0) \cos(\mathbf{k}\mathbf{r}_{mn}). \quad (86)$$

It turns out that $\nu(\mathbf{k})$ goes to zero at the corners of the Brillouin zone $k_{x,y} = \pm\pi/a$. At these points the Bloch function has to be calculated by a careful limiting procedure (for details see³¹). The result is

$$\Psi_{\mathbf{k}_0}(\mathbf{r}) = \frac{i}{a\sqrt{2N_\phi\gamma}} \sum_{mn=-\infty}^{\infty} (-1)^{m+n} \bar{z}_{mn} c_{mn}(\mathbf{r}), \quad (87)$$

where $\gamma = -\frac{1}{a} \sum_{mn=-\infty}^{\infty} (-1)^{m+n} c_{mn}(0) x_m^2$. Now we can construct magnetic Wannier functions by inverse Fourier transform

$$W_{mn}(\mathbf{r}) = \frac{1}{\sqrt{N_\phi}} \sum_{\mathbf{k}} \Psi_{\mathbf{k}}(\mathbf{r}) \exp(-i\mathbf{k}\mathbf{r}_{mn}). \quad (88)$$

The functions W_{nm} are orthonormal and form a complete set by construction. We evaluated our effective spin and pseudospin interactions numerically by inserting these functions in direct and exchange Coulomb matrix elements $\langle ij|V_{S,D}|ij\rangle$ and $\langle ij|V_{S,D}|ji\rangle$, where $i = (nm)$. The calculation is best done by Fourier transforming the Coulomb interactions $V_{S,D}$

$$\begin{aligned} V_S(\mathbf{q}) &= \frac{2\pi e^2}{\epsilon q} \\ V_D(\mathbf{q}) &= \frac{2\pi e^2}{\epsilon q} e^{-qd}, \end{aligned} \quad (89)$$

and evaluating the plane-wave matrix elements $\langle i|e^{i\mathbf{q}\mathbf{r}}|j\rangle$. The Coulomb interaction matrix elements are then given by

$$\langle ij|V_{S,D}|ij\rangle = \frac{1}{L^2} \sum_{\mathbf{q}} \langle i|e^{i\mathbf{q}\mathbf{r}}|i\rangle V_{S,D}(\mathbf{q}) \langle j|e^{-i\mathbf{q}\mathbf{r}}|j\rangle, \quad (90)$$

and

$$\langle ij|V_{S,D}|ji\rangle = \frac{1}{L^2} \sum_{\mathbf{q}} \langle i|e^{i\mathbf{q}\mathbf{r}}|j\rangle V_{S,D}(\mathbf{q}) \langle j|e^{-i\mathbf{q}\mathbf{r}}|i\rangle. \quad (91)$$

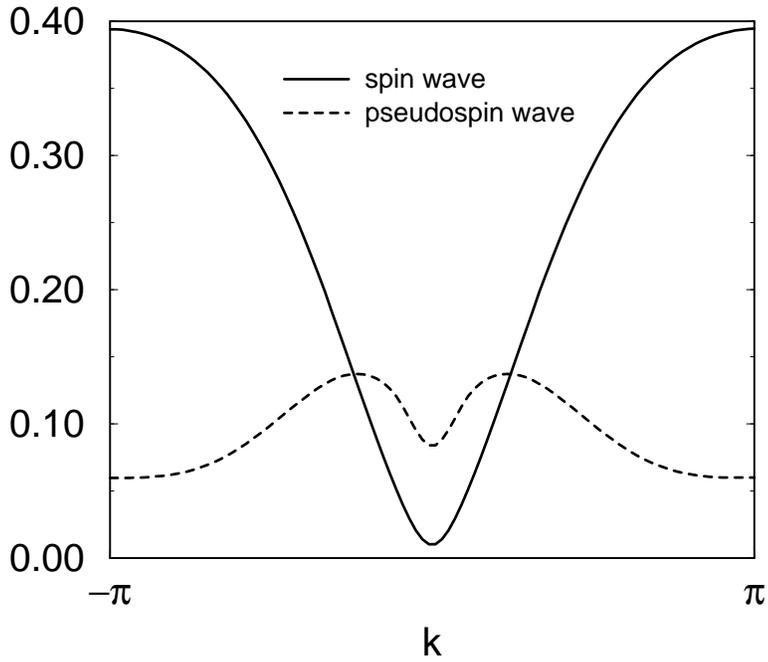


FIG. 1. Spin and pseudospin wave dispersions for a 20×20 lattice and $d/\ell = 1.4$ in the (1,1) direction. The dip in the pseudospin wave dispersion at the Brillouin zone boundary signals the development of antiferromagnetic instability which destroys the long-range XY-ferromagnetic order.

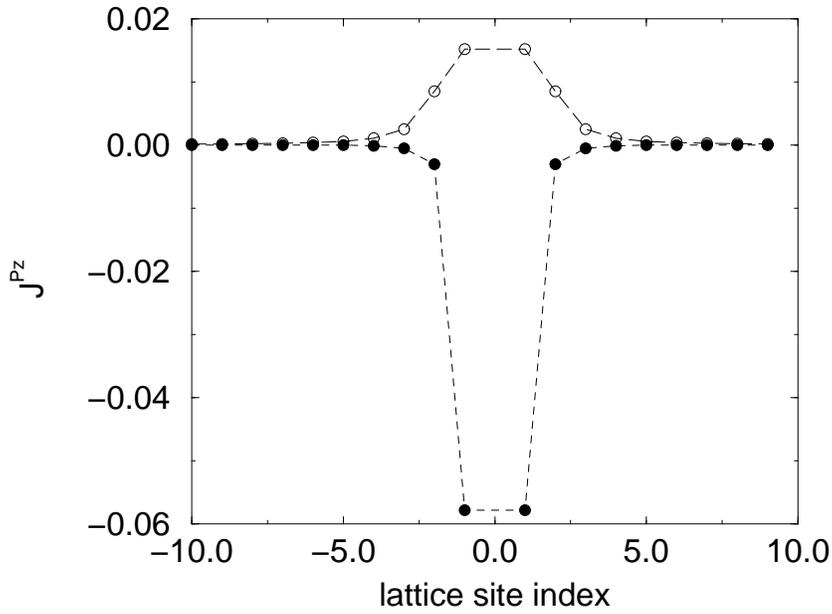


FIG. 2. Effective pseudospin-pseudospin interaction J_{ij}^{Pz} in Eq.(16) for $d/\ell = 0.5$ (filled circles) and $d/\ell = 1.4$ (open circles). J_{ij}^{Pz} changes its character from short-range ferromagnetic at $d/\ell = 0.5$ to long-range antiferromagnetic at $d/\ell = 1.4$, eventually making XY-ferromagnetic pseudospin state unstable.

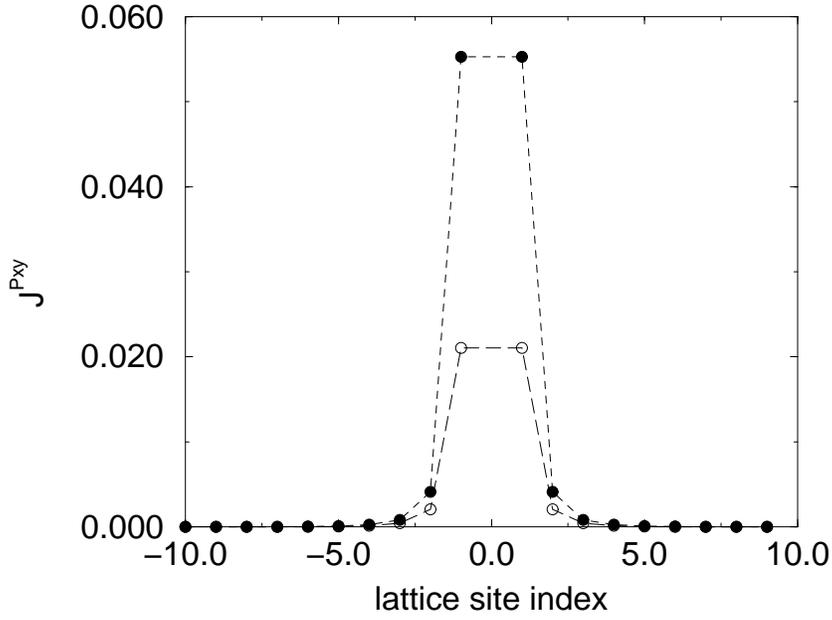


FIG. 3. Effective pseudospin-pseudospin interaction in Eq.(16) $J_{ij}^{P\perp}$ for $d/\ell = 0.5$ (filled circles) and $d/\ell = 1.4$ (open circles). Pseudospin XY-ferromagnetic interactions are weakening as the interlayer separation is increased due to the weakening of interlayer exchange interactions.

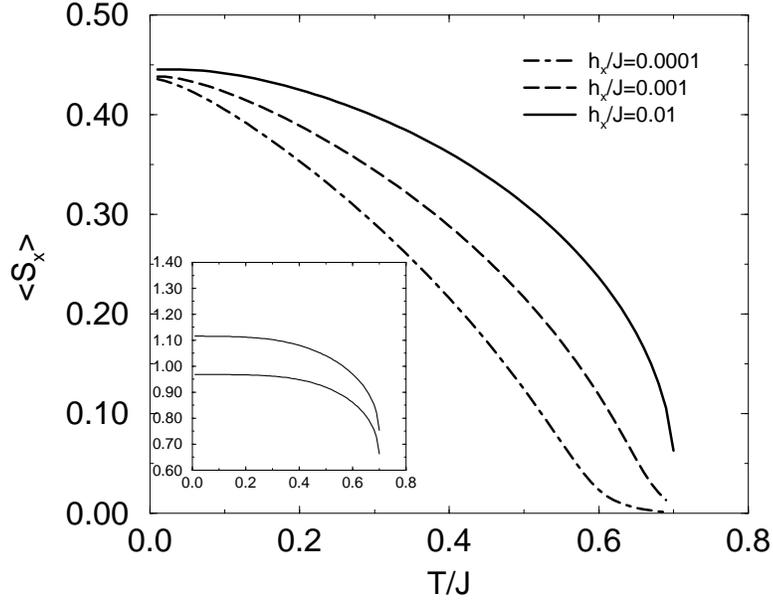


FIG. 4. In-plane magnetization for the pure XY-model case ($J^z = 0$) of Eq.(31). SBMFT short range order parameters Q and P are shown in the inset. Magnetization is reduced from its maximum value even at zero temperature due to the quantum fluctuations induced by anisotropy.

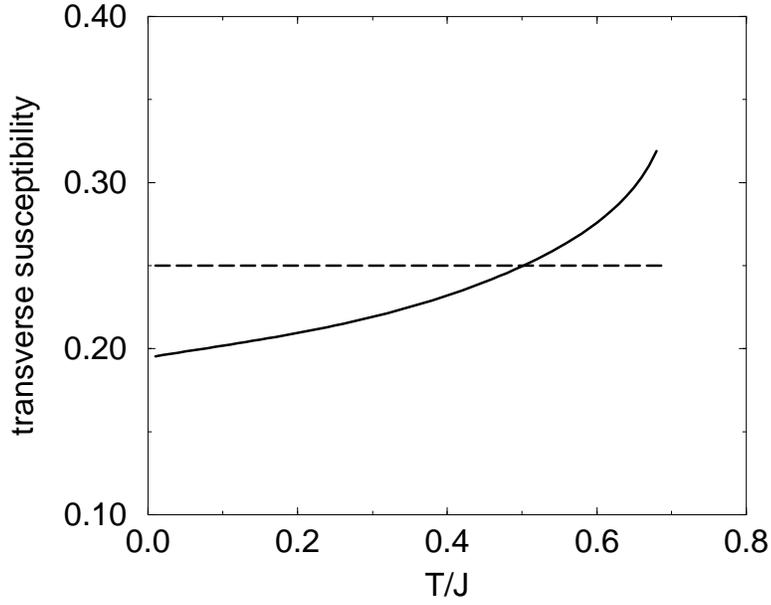


FIG. 5. Temperature dependence of the transverse susceptibility for the XY-model (solid line). The classical zero temperature susceptibility is shown by a dashed line. Susceptibility increases with temperature due to softening of the in-plane spin order by thermal fluctuations.

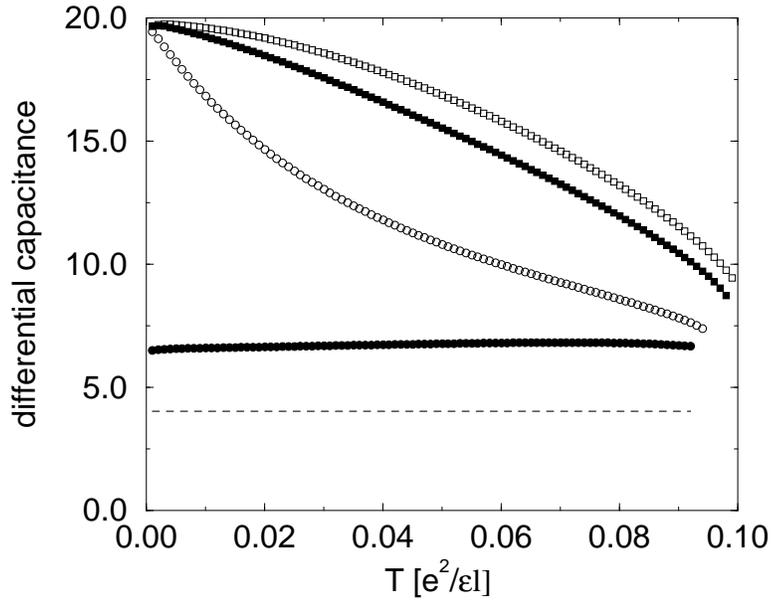


FIG. 6. Temperature dependence of the differential capacitance of the bilayer $d\langle T^z \rangle / d\Delta_V$ for $d/\ell = 0.5$ and $\Delta_z = 0$. (filled circles), $\Delta_z = 0.001$ (open circles), $\Delta_z = 0.005$ (filled squares) and $\Delta_z = 0.01$ (open squares). The constant Hartree-Fock susceptibility is shown by a dashed line. The large increase of the capacitance compared to the Hartree-Fock value is an artifact of SBMFT. The main effect determining the temperature dependence of the capacitance at $d/\ell = 0.5$ is the suppression of spin-polarization by thermal fluctuations which influences effective pseudospin-pseudospin interactions (see Eq.(49)).

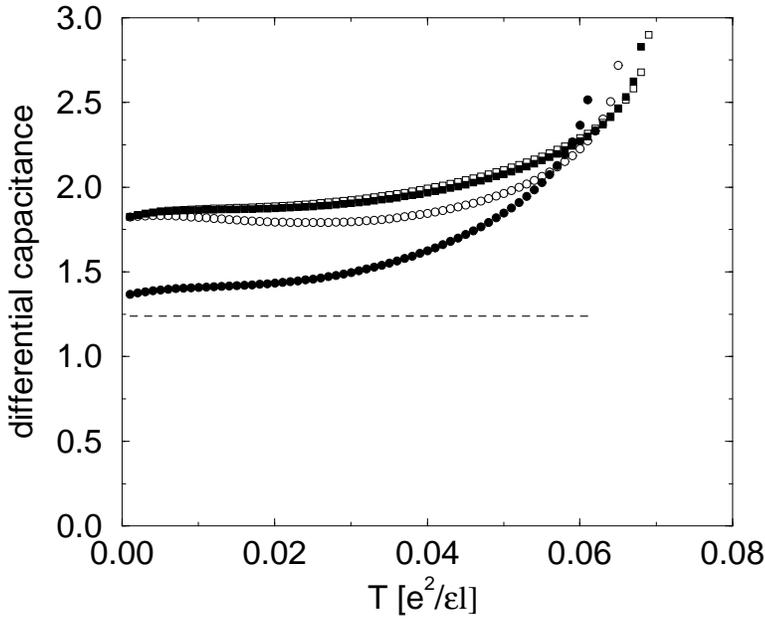


FIG. 7. Temperature dependence of the differential capacitance of the bilayer $d\langle T^z \rangle / d\Delta_V$ for $d/\ell = 1.0$ and $\Delta_z = 0$. (filled circles), $\Delta_z = 0.001$ (open circles), $\Delta_z = 0.005$ (filled squares) and $\Delta_z = 0.01$ (open squares). The constant Hartree-Fock susceptibility is shown by a dashed line. The influence of spin-polarization on the pseudospin system is weaker here compared to the case $d/\ell = 0.5$ and the temperature dependence of the differential capacitance is mainly determined by the softening of the XY-ferromagnetic pseudospin order by thermal fluctuations, although there still is a substantial dependence on the Zeeman-coupling strength.

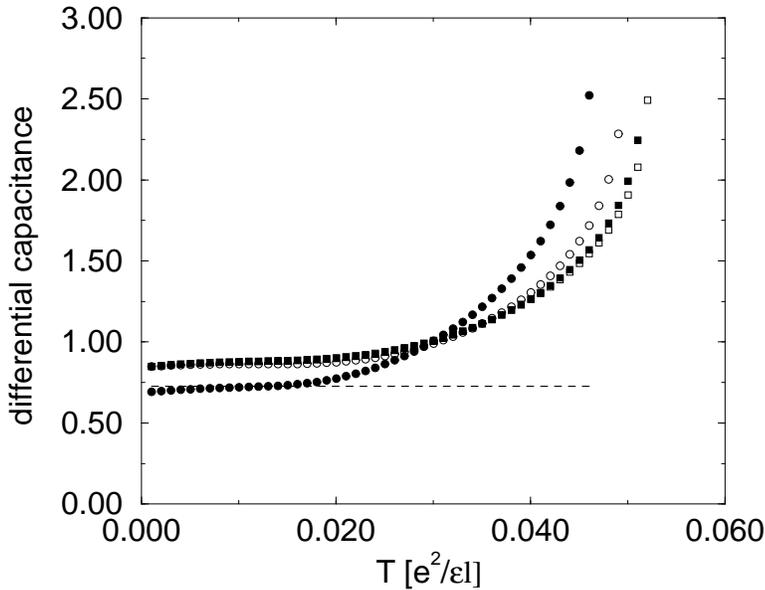


FIG. 8. Temperature dependence of the differential capacitance of the bilayer $d\langle T^z \rangle / d\Delta_V$ for $d/\ell = 1.4$ and $\Delta_z = 0$. (filled circles), $\Delta_z = 0.001$ (open circles), $\Delta_z = 0.005$ (filled squares) and $\Delta_z = 0.01$ (open squares). The constant Hartree-Fock susceptibility is shown by a dashed line. The spin-polarization influence has become very small both in the temperature and Zeeman-coupling dependence.

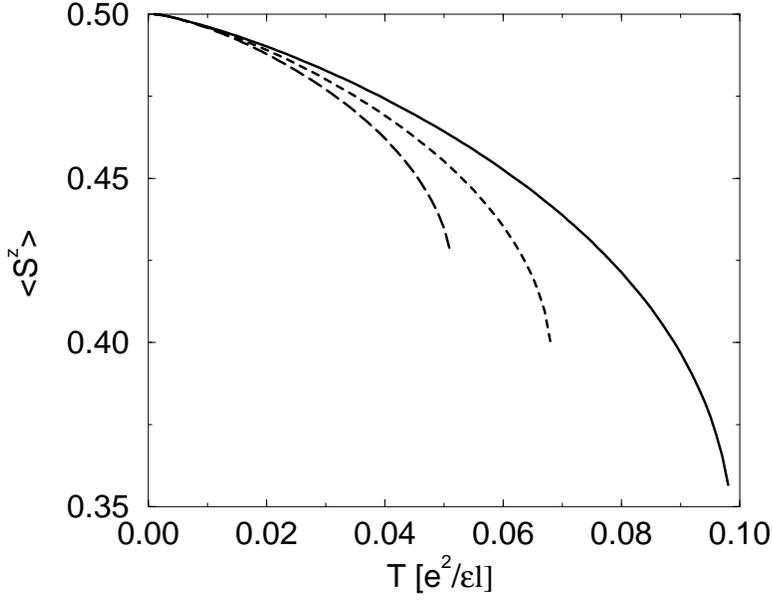


FIG. 9. Temperature dependence of the spin magnetization of the bilayer system for $\Delta_z = 0.005$, $\Delta_t = 0.001$ and $d/\ell = 0.5$ (solid line), $d/\ell = 1.0$ (dashed line) and $d/\ell = 1.4$ (long dashed line). The strong dependence of the spin magnetization on the interlayer separation is a signature of the spin-pseudospin coupling.

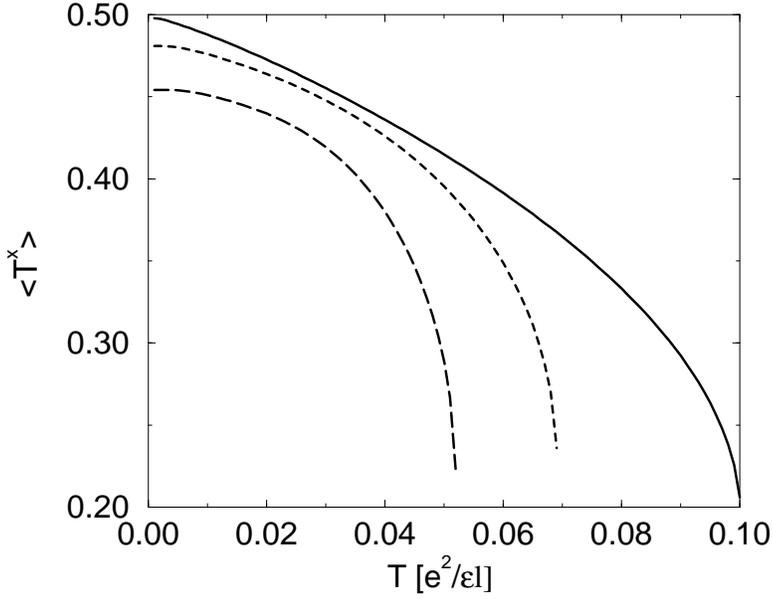


FIG. 10. Temperature dependence of the in-plane pseudospin magnetization of the bilayer for $\Delta_t = 0.001$, $\Delta_z = 0.005$ and $d/\ell = 0.5$ (solid line), $d/\ell = 1.0$ (dashed line) and $d/\ell = 1.4$ (long dashed line). Pseudospin XY-ferromagnetic order is weakened by quantum fluctuations as the interlayer separation is increased.

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