# Microscopic derivation of rate equations for quantum transport 

S. A. Gurvitz and Ya. S. Prager<br>Department of Particle Physics, Weizmann Institute of Science, Rehovot 76100, Israel

(October 1, 2018)


#### Abstract

It is shown that under certain conditions the resonant transport in mesoscopic systems can be described by modified (quantum) rate equations, which resemble the optical Bloch equations with some additional terms. Detailed microscopic derivation from the many-body Schrödinger equation is presented. Special attention is paid to the Coulomb blockade and quantum coherence effects in coupled quantum dot systems. The distinction between classical and quantum descriptions of resonant transport is clearly manifested in the modified rate equations.


PACS numbers: $72.10 . \mathrm{Bg}, 73.40 . \mathrm{Gk}, 73.20 . \mathrm{Dx}, 05.60 .+\mathrm{w}$

## I. INTRODUCTION

Over the last decade a great interest has been paid to artificially fabricated nanostructures containing discrete number of quantum states. The discreteness of quantum states manifests itself in peculiar transport properties of these systems as, for instance, in the Coulomb blockade oscillations [1]. Actually, the study has been mostly concentrated on the quantum transport through single devices (quantum dots). In fact, more interesting quantum mechanical effects can be found in coupled nanostructures devices, where the quantum interference may strongly influence the resonance current. The impressive progress in microfabrication technology now allows to extend the experimental investigation to these systems too. For instance, the transport properties of coupled dots are presently under intensive study [2, 3 ].

For description of quantum transport through single quantum dot (quantum well) the "classical" rate equations are usually used (4) 6]. They can be derived either by using nonequilibrium Green's functions technique (7) or directly from the Schrödinger equation [8]. The situation is different for coupled wells with aligned levels. The quantum transport through these devices goes on via quantum superposition between the states in adjacent wells. It is thus quite obvious that non-diagonal density matrix elements would appear in the equations of motion. These terms have no classical counterparts, and therefore the classical rate equations have to be modified. A plausible modification of master equations for some particular cases of the resonance tunneling through double-dot structures has been proposed by Nazarov [9]. A more general case, though without account of Coulomb interaction, has been considered in [10], where modified rate equations have been proposed by using an analogy to the optical Bloch equations [11]. However, no microscopic derivation of the modified rate equations has been presented yet.

In this paper we derive the rate equations for a general case of resonant transport through mesoscopic systems, starting with the many-body Schrödinger equation, with special attention being paid to the Coulomb blockade and coherent quantum mechanical effects. Our
main goals are, first, to substantiate and generalize the previously suggested rate equations and second, to determine the region of validity of the rate equations for description of quantum transport in general. Also, we believe that the microscopic derivation of the rate equations will provide a better understanding of the correspondence between quantum and classical description of carrier transport in mesoscopic systems.

The plan of the paper is the following. In Sect. 2 we give a detailed derivation of the transport rate equations through a single quantum well (dot). In order to present our method most lucidly, we neglect in this section the Coulomb interaction and spin effects. These effects are considered in Sect. 3. In Sect. 4 we derive the modified rate equations for coupled well structures, taking into account the Coulomb and spin effects. An example of coherent resonant transport with inelastic transitions is studied in Sect. 5. The derivation of rate equations performed in this case allows to establish their correct form valid in general case of quantum transport. The general case and an example of coherent resonant transport with inelastic transitions in the presence of strong Coulomb blockade are presented in Sect. 6. The last section is a summary.

## II. SINGLE-WELL STRUCTURE

Let us consider a mesoscopic "device" consisting of a quantum well (dot), coupled to two separate electron reservoirs. The density of states in the reservoirs is very high (continuum). The dot, however, contains only isolated levels. We first demonstrate how to achieve the reduction of many-body Schroedinger equation to the rate equation in the simplest example, Fig. 1, with only one level, $E_{1}$, inside the dot. We also ignore the Coulomb electron-electron interaction inside the well and the spin degrees of freedom. Hence, only one electron may occupy the well. With the stand simplifications, the tunneling Hamiltonian of the entire system in the occupation number representation is

$$
\begin{equation*}
\mathcal{H}=\sum_{l} E_{l} a_{l}^{\dagger} a_{l}+E_{1} a_{1}^{\dagger} a_{1}+\sum_{r} E_{r} a_{r}^{\dagger} a_{r}+\sum_{l} \Omega_{l}\left(a_{l}^{\dagger} a_{1}+a_{1}^{\dagger} a_{l}\right)+\sum_{r} \Omega_{r}\left(a_{r}^{\dagger} a_{1}+a_{1}^{\dagger} a_{r}\right) . \tag{2.1}
\end{equation*}
$$

Here the subscripts $l$ and $r$ enumerate correspondingly the (very dense) levels in the left (emitter) and right (collector) reservoirs. For simplicity, we restrict ourselves to the zero temperature case. All the levels in the emitter and the collector are initially filled with electrons up to the Fermi energy $E_{F}^{L}$ and $E_{F}^{R}$, respectively. This situation will be treated as the "vacuum" state $|0\rangle$.

This vacuum state is unstable; the Hamiltonian Eq. (2.1) requires it to decay exponentially to a continuum state having the form $a_{1}^{\dagger} a_{l}|0\rangle$ with an electron in the level $E_{1}$ and a hole in the emitter continuum. These continuum states are also unstable and decay to states $a_{r}^{\dagger} a_{l}|0\rangle$ having a particle in the collector continuum as well as a hole in the emitter continuum, and no electron in the level $E_{1}$. The latter, in turn, are decaying into the states $a_{1}^{\dagger} a_{r}^{\dagger} a_{l} a_{l^{\prime}}|0\rangle$ and so on. The evolution of the whole system is described by the many-particle wave function, which is represented as

$$
\begin{equation*}
|\Psi(t)\rangle=\left[b_{0}(t)+\sum_{l} b_{1 l}(t) a_{1}^{\dagger} a_{l}+\sum_{l, r} b_{l r}(t) a_{r}^{\dagger} a_{l}+\sum_{l<l^{\prime}, r} b_{1 l l^{\prime} r}(t) a_{1}^{\dagger} a_{r}^{\dagger} a_{l} a_{l^{\prime}}+\ldots\right]|0\rangle, \tag{2.2}
\end{equation*}
$$

where $b(t)$ are the time-dependent probability amplitudes to find the system in the corresponding states described above with the initial condition $b_{0}(0)=1$, and all the other $b(0)$ 's being zeros. Substituting Eq. (2.2) into in the Schrödinger equation $i|\dot{\Psi}(t)\rangle=\mathcal{H}|\Psi(t)\rangle$, results in an infinite set of coupled linear differential equations for the amplitudes $b(t)$. Applying the Laplace transform

$$
\begin{equation*}
\tilde{b}(E)=\int_{0}^{\infty} e^{i E t} b(t) d t \tag{2.3}
\end{equation*}
$$

and taking account of the initial conditions, we transform the linear differential equations for $b(t)$ into an infinite set of algebraic equations for the amplitudes $\tilde{b}(E)$,

$$
\begin{align*}
& E \tilde{b}_{0}(E)-\sum_{l} \Omega_{l} \tilde{b}_{1 l}(E)=i  \tag{2.4a}\\
& \left(E+E_{l}-E_{1}\right) \tilde{b}_{1 l}(E)-\Omega_{l} \tilde{b}_{0}(E)-\sum_{r} \Omega_{r} \tilde{b}_{l r}(E)=0  \tag{2.4b}\\
& \left(E+E_{l}-E_{r}\right) \tilde{b}_{l r}(E)-\Omega_{r} \tilde{b}_{1 l}(E)-\sum_{l^{\prime}} \Omega_{l^{\prime}} \tilde{b}_{1 l^{\prime} r}(E)=0  \tag{2.4c}\\
& \left(E+E_{l}+E_{l^{\prime}}-E_{1}-E_{r}\right) \tilde{b}_{1 l l^{\prime} r}(E)-\Omega_{l^{\prime}} \tilde{b}_{l r}(E)+\Omega_{l} \tilde{b}_{l^{\prime} r}(E)-\sum_{r^{\prime}} \Omega_{r^{\prime}} \tilde{b}_{l l^{\prime} r r^{\prime}}(E)=0 \tag{2.4d}
\end{align*}
$$

Eqs. (2.4) can be substantially simplified. Let us replace the amplitude $\tilde{b}$ in the term $\sum \Omega \tilde{b}$ of each of the equations by its expression obtained from the subsequent equation. For example, substitute $\tilde{b}_{1 l}(E)$ from Eq. (2.4b) into Eq. (2.4a). We obtain

$$
\begin{equation*}
\left[E-\sum_{l} \frac{\Omega_{l}^{2}}{E+E_{l}-E_{1}}\right] \tilde{b}_{0}(E)-\sum_{l, r} \frac{\Omega_{l} \Omega_{r}}{E+E_{l}-E_{1}} \tilde{b}_{l r}(E)=i . \tag{2.5}
\end{equation*}
$$

Since the states in the reservoirs are very dense (continuum), one can replace the sums over $l$ and $r$ by integrals, for instance $\sum_{l} \rightarrow \int \rho_{L}\left(E_{l}\right) d E_{l}$, where $\rho_{L}\left(E_{l}\right)$ is the density of states in the emitter. Then the first sum in Eq. (2.5) becomes an integral which can be split into a sum of the singular and principal value parts. The singular part yields $-i \Theta\left(E_{F}^{L}+E-E_{1}\right) \Gamma_{L} / 2$, where $\Gamma_{L}=2 \pi \rho_{L}\left(E_{1}\right)\left|\Omega_{L}\left(E_{1}\right)\right|^{2}$ is the level $E_{1}$ partial width due to coupling to the emitter. Let us assume that $E_{F}^{L} \gg E_{1} \gg E_{F}^{R}$, i.e. the bias is large and the energy level is deeply inside the band. In this case the integration over $E_{l(r)}$-variables can be extended to $\pm \infty$. As a result, the theta-function can be replaced by one, and the principal part is merely included into redefinition of the energy $E_{1}$. Also, the second sum (integral) in Eq. (2.5) proves to be negligibly small. Indeed, let us replace $\tilde{b}_{l r} \rightarrow \tilde{b}\left(E_{l}, E_{r}, E\right)$, and assume weak energy dependence of $\Omega$ on $E_{l(r)}$. Then one finds from Eqs. (2.4) that the poles of the integrand in the $E_{l}\left(E_{r}\right)$-variable are on one side of the integration contour, and therefore this term vanishes.

Applying analogous considerations to the other equations of the system (2.4), we finally arrive to the following set of equations:

$$
\begin{align*}
& \left(E+i \Gamma_{L} / 2\right) \tilde{b}_{0}(E)=i  \tag{2.6a}\\
& \left(E+E_{l}-E_{1}+i \Gamma_{R} / 2\right) \tilde{b}_{1 l}(E)-\Omega_{l} \tilde{b}_{0}(E)=0  \tag{2.6b}\\
& \left(E+E_{l}-E_{r}+i \Gamma_{L} / 2\right) \tilde{b}_{l r}(E)-\Omega_{r} \tilde{b}_{1 l}(E)=0  \tag{2.6c}\\
& \left(E+E_{l}+E_{l^{\prime}}-E_{1}-E_{r}+i \Gamma_{R} / 2\right) \tilde{b}_{1 l l^{\prime} r}(E)-\Omega_{l^{l^{\prime}}} \tilde{b}_{l r}(E)+\Omega_{l} \tilde{b}_{l^{\prime} r}(E)=0 \tag{2.6d}
\end{align*}
$$

where $\Gamma_{R}=2 \pi \rho_{R}\left(E_{1}\right)\left|\Omega_{R}\left(E_{1}\right)\right|^{2}$ is the level $E_{1}$ partial width due to coupling to the collector.

Now we introduce the density matrix of the "device". The Fock space of the quantum well consists of only two possible states, namely: $|a\rangle$ - the level $E_{1}$ is empty, and $|b\rangle$ the level $E_{1}$ is occupied. In this basis, the diagonal elements of the density matrix of the "device", $\sigma_{a a}$ and $\sigma_{b b}$, give the probabilities of the resonant level being empty or occupied, respectively. In our notation, these probabilities are represented as follows:

$$
\begin{align*}
\sigma_{a a} & =\left|b_{0}(t)\right|^{2}+\sum_{l, r}\left|b_{l r}(t)\right|^{2}+\sum_{l<l^{\prime}, r<r^{\prime}}\left|b_{l l^{\prime} r r^{\prime}}(t)\right|^{2}+\ldots \\
& \equiv \sigma_{a a}^{(0)}+\sigma_{a a}^{(1)}+\sigma_{a a}^{(2)}+\ldots,  \tag{2.7a}\\
\sigma_{b b} & =\sum_{l}\left|b_{1 l}(t)\right|^{2}+\sum_{l<l^{\prime}, r}\left|b_{1 l l^{\prime} r}(t)\right|^{2}+\sum_{l<l^{\prime}<l^{\prime \prime}, r<r^{\prime}}\left|b_{1 l l^{\prime} l^{\prime \prime} r r^{\prime}}(t)\right|^{2}+\ldots \\
& \equiv \sigma_{b b}^{(0)}+\sigma_{b b}^{(1)}+\sigma_{b b}^{(2)}+\ldots, \tag{2.7b}
\end{align*}
$$

where the index $n$ in $\sigma^{(n)}$ denotes the number of electrons in the collector. The current $I(t)$ flowing through the system is $I(t)=e \dot{N}_{R}(t)$, where $N_{R}(t)$ is the number of electrons accumulated in the collector, i.e.

$$
\begin{equation*}
N_{R}(t)=\sum_{n} n\left[\sigma_{a a}^{(n)}(t)+\sigma_{b b}^{(n)}(t)\right] \tag{2.8}
\end{equation*}
$$

The density submatrix elements are directly related to the amplitudes $\tilde{b}(E)$ through the inverse Laplace transform

$$
\begin{equation*}
\sigma^{(n)}(t)=\sum_{l \ldots, r \ldots} \int \frac{d E d E^{\prime}}{4 \pi^{2}} \tilde{b}_{l \ldots r \ldots \ldots}(E) \tilde{b}_{l \ldots r \ldots \ldots}^{*}\left(E^{\prime}\right) e^{i\left(E^{\prime}-E\right) t} \tag{2.9}
\end{equation*}
$$

By means of this equation one can transform Eqs. (2.6) for the amplitudes $b(E)$ into differential equations directly for the probabilities $\sigma^{(n)}(t)$. Consider, for instance, the term $\sigma_{b b}^{(0)}(t)=\sum_{l}\left|b_{1 l}(t)\right|^{2}$, Eq. (2.7b). Multiplying Eq. (2.6b) by $\tilde{b}_{1 l}^{*}\left(E^{\prime}\right)$ and then subtracting the complex conjugated equation with the interchange $E \leftrightarrow E^{\prime}$ we obtain

$$
\begin{align*}
\int \frac{d E d E^{\prime}}{4 \pi^{2}}\left(E^{\prime}-E-i \Gamma_{R}\right) \sum_{l} & \tilde{b}_{1 l}(E) \tilde{b}_{1 l}^{*}\left(E^{\prime}\right) e^{i\left(E^{\prime}-E\right) t} \\
& -\int \frac{d E d E^{\prime}}{4 \pi^{2}} 2 \operatorname{Im} \sum_{l} \Omega_{l} \tilde{b}_{1 l}(E) \tilde{b}_{0}^{*}\left(E^{\prime}\right) e^{i\left(E^{\prime}-E\right) t}=0 \tag{2.10}
\end{align*}
$$

One can easily deduce from Eq. (2.9) that the first integral in Eq. (2.10) equals to $-i\left[\dot{\sigma}_{b b}^{(0)}(t)+\right.$ $\left.\Gamma_{R} \sigma_{b b}^{(0)}(t)\right]$. Next, substituting

$$
\begin{equation*}
\tilde{b}_{1 l}(E)=\frac{\Omega_{l} \tilde{b}_{0}(E)}{E+E_{l}-E_{1}+i \Gamma_{R} / 2} \tag{2.11}
\end{equation*}
$$

from Eq. (2.6B) into the second integral of Eq. (2.10), and replacing $\sum_{l} \rightarrow \int \rho_{L}\left(E_{l}\right) d E_{l}$ one can can perform the $E_{l}$-integration in the integral, thus obtaining $i \Gamma_{L} \sigma_{a a}^{(0)}(t)$. Ultimately, Eq. (2.10) reads $\dot{\sigma}_{b b}^{(0)}(t)=\Gamma_{L} \sigma_{a a}^{(0)}(t)-\Gamma_{R} \sigma_{b b}^{(0)}(t)$. We can go on with this algebra for all the other amplitudes $\tilde{b}(t)$. For instance, the above procedure applied to Eq. (2.6d) converts it into a differential equation for the density-matrix element $\sigma_{b b}^{(1)}$, Eq. (2.7b). The only difference with the previous example is an appearance of the "cross terms", like $\sum \Omega_{l} \tilde{b}_{l^{\prime} r}(E) \Omega_{l^{\prime}} \tilde{b}_{l r}^{*}\left(E^{\prime}\right)$. Yet, these terms vanish after the integration over $E_{l(r)}$, just as the second term in Eq. (2.5)). The rest of the algebra remains the same, so one obtains $\dot{\sigma}_{b b}^{(1)}(t)=\Gamma_{L} \sigma_{a a}^{(1)}(t)-\Gamma_{R} \sigma_{b b}^{(1)}(t)$. Finally we arrive to the following infinite system of the chain equations for the diagonal elements, $\sigma_{a a}^{(n)}$ and $\sigma_{b b}^{(n)}$, of the density matrix,

$$
\begin{align*}
& \dot{\sigma}_{a a}^{(0)}(t)=-\Gamma_{L} \sigma_{a a}^{(0)}(t),  \tag{2.12a}\\
& \dot{\sigma}_{b b}^{(0)}(t)=\Gamma_{L} \sigma_{a a}^{(0)}(t)-\Gamma_{R} \sigma_{b b}^{(0)}(t),  \tag{2.12b}\\
& \dot{\sigma}_{a a}^{(1)}(t)=-\Gamma_{L} \sigma_{a a}^{(1)}(t)+\Gamma_{R} \sigma_{b b}^{(0)}(t),  \tag{2.12c}\\
& \dot{\sigma}_{b b}^{(1)}(t)=\Gamma_{L} \sigma_{a a}^{(1)}(t)-\Gamma_{R} \sigma_{b b}^{(1)}(t), \tag{2.12d}
\end{align*}
$$

Summing up these equations, one easily obtains differential equations for the total probabilities $\sigma_{a a}=\sum_{n} \sigma_{a a}^{(n)}$ and $\sigma_{b b}=\sum_{n} \sigma_{b b}^{(n)}$ :

$$
\begin{gather*}
\dot{\sigma}_{a a}=-\Gamma_{L} \sigma_{a a}+\Gamma_{R} \sigma_{b b},  \tag{2.13a}\\
\dot{\sigma}_{b b}=\Gamma_{L} \sigma_{a a}-\Gamma_{R} \sigma_{b b}, \tag{2.13b}
\end{gather*}
$$

which should be supplemented with the initial conditions

$$
\begin{equation*}
\sigma_{a a}(0)=1, \quad \sigma_{b b}(0)=0 \tag{2.14}
\end{equation*}
$$

Using Eqs. (2.8), (2.12) we obtain the total current

$$
\begin{equation*}
I(t)=e \dot{N}_{R}(t)=e \Gamma_{R}\left[\sigma_{b b}^{(0)}(t)+\sigma_{b b}^{(1)}(t)+\sigma_{b b}^{(2)}(t)+\ldots\right]=e \Gamma_{R} \sigma_{b b}(t) \tag{2.15}
\end{equation*}
$$

Thus the current $I(t)$ is directly proportional to the charge density in the well. Solving Eqs. (2.13) and substituting $\sigma_{b b}(t)$ into Eq. (2.15), we obtain (for $t \rightarrow \infty$ ) the standard formula for the dc resonant current,

$$
\begin{equation*}
I / e=\frac{\Gamma_{L} \Gamma_{R}}{\Gamma_{L}+\Gamma_{R}} . \tag{2.16}
\end{equation*}
$$

Notice that whereas the time-behavior of the current $I(t)$ depends on the initial condition, the stationary current $I=I(t \rightarrow \infty)$, Eq. (2.16), does not.

Equations (2.13), derived from the many-body Schrödinger equation, coincide with the classical rate equations in the sequential picture for the resonant tunneling, obtained using nonequilibrium quantum statistical mechanics technique [7]. In contrast, our approach starts directly from the many-body Schrödinger equation and will be straightforwardly extended to more complicated situations. Note, however, that the method can be applied only when the resonance energy is inside the band, and $\Gamma_{L, R} \ll E_{F}^{L}-E_{F}^{R}$. If the resonance is near the edge of the band, but the width of the resonance is much smaller than the band width, our method still can be applied, but only to the stationary case $(t \rightarrow \infty)$. Yet, the timedependent Scrödinger equation cannot be reduced to the rate equations (2.13), and therefore this case is not a subject of this paper.

## III. COULOMB BLOCKADE

Now we extend the approach of Sect. 2 to include the effects of Coulomb interaction. Consider again the quantum well in Fig. 1, taking into account the spin degrees of freedom (s). In this case the tunneling Hamiltonian (2.1) becomes

$$
\begin{align*}
\mathcal{H}= & \sum_{l, s} E_{l} a_{l s}^{\dagger} a_{l s}+\sum_{s} E_{1} a_{1 s}^{\dagger} a_{1 s}+\sum_{r, s} E_{r} a_{r s}^{\dagger} a_{r s} \\
& +\sum_{l, s} \Omega_{l}\left(a_{l s}^{\dagger} a_{1 s}+a_{1 s}^{\dagger} a_{l s}\right)+\sum_{r, s} \Omega_{r}\left(a_{r s}^{\dagger} a_{1 s}+a_{1 s}^{\dagger} a_{r s}\right)+U a_{1 s}^{\dagger} a_{1 s} a_{1,-s}^{\dagger} a_{1,-s} \tag{3.1}
\end{align*}
$$

where $s= \pm 1 / 2$, and $U$ is the Coulomb repulsion energy.

Writing down the many-body wave function, $|\Psi(t)\rangle$, in the occupation number representation, just as in Eq. (2.2), and then substituting it into the Schrödinger equation $i|\dot{\Psi}(t)\rangle=\mathcal{H}|\Psi(t)\rangle$, we find a system of coupled equations for the amplitudes $b(t)$

$$
\begin{align*}
& E \tilde{b}_{0}(E)-\sum_{l} \Omega_{l}\left[\tilde{b}_{\uparrow l}(E)+\tilde{b}_{\downarrow l}(E)\right]=i  \tag{3.2a}\\
& \left(E+E_{l}-E_{1}\right) \tilde{b}_{\uparrow l}(E)-\Omega_{l} \tilde{b}_{0}(E)-\sum_{l^{\prime}} \Omega_{l^{l^{\prime}}} \tilde{b}_{\uparrow \downarrow l l^{\prime}}(E)-\sum_{r} \Omega_{r} \tilde{b}_{l r}(E)=0  \tag{3.2b}\\
& \begin{aligned}
&\left(E+E_{l}-E_{r}\right) \tilde{b}_{l r}(E)-\Omega_{r} \tilde{b}_{\uparrow l}(E)-\sum_{l^{\prime}} \Omega_{l^{\prime}} {\left[\tilde{b}_{\uparrow l^{\prime}}(E)+\tilde{b}_{\downarrow l^{\prime}}(E)\right]=0 } \\
&\left(E+E_{l}+E_{l^{\prime}}-2 E_{1}-U\right) \tilde{b}_{\uparrow \downarrow l l^{\prime}}(E)-\Omega_{l^{\prime}} \tilde{b}_{\uparrow l}(E)-\Omega_{l} \tilde{b}_{\downarrow l^{\prime}}(E) \\
& \quad-\sum_{r} \Omega_{r}\left[\tilde{b}_{\uparrow l^{\prime} r}(E)+\tilde{b}_{\downarrow l l^{\prime} r}(E)\right]=0
\end{aligned} \tag{3.2c}
\end{align*}
$$

In order to shorten notations we eliminated the index (1) of the level $E_{1}$ in the amplitudes $b$, so that $\tilde{b}_{\uparrow(\downarrow) . . .}(t)$ denotes the probability amplitude to find one electron inside the well with spin up (down), and the amplitude $\tilde{b}_{\uparrow \downarrow \ldots}(t)$ is the probability amplitude to find two electrons inside the well.

Eqs. (3.2) can be simplified by using the same procedure as described in the previous section. For instance, by substituting $\tilde{b}_{l r}$ from Eq. (3.2d) and $\tilde{b}_{\uparrow l l^{\prime}}$ from Eq. (3.2d) into Eq. (3.2b), and neglecting the "cross terms" on the grounds of the same arguments as in the analysis of Eq. (2.5)), we obtain

$$
\begin{equation*}
\left[E+E_{l}-E-\int_{-\infty}^{E_{F}^{L}} \frac{\rho_{L}\left(E_{l^{\prime}}\right) \Omega^{2}\left(E_{l^{\prime}}\right) d E_{l^{\prime}}}{E+E_{l}+E_{l^{\prime}}-2 E_{1}-U}-\int_{E_{F}^{R}}^{\infty} \frac{\rho_{R}\left(E_{r}\right) \Omega^{2}\left(E_{r}\right) d E_{r}}{E+E_{l}-E_{r}}\right] \tilde{b}_{\uparrow l}(E)=0 \tag{3.3}
\end{equation*}
$$

Since $E_{l} \sim E_{1}$, the singular parts of the integrals in (3.3) are respectively $-i \Theta\left(E_{F}^{L}+E-\right.$ $\left.E_{1}+U\right) \Gamma_{L}^{\prime} / 2$ and $-i \Theta\left(E+E_{1}-E_{F}^{R}\right) \Gamma_{R} / 2$, where

$$
\begin{equation*}
\Gamma_{L(R)}=2 \pi \rho_{L(R)}\left(E_{1}\right)\left|\Omega_{L(R)}\left(E_{1}\right)\right|^{2}, \quad \Gamma_{L(R)}^{\prime}=2 \pi \rho_{L(R)}\left(E_{1}+U\right)\left|\Omega_{L(R)}\left(E_{1}+U\right)\right|^{2} \tag{3.4}
\end{equation*}
$$

Here $\rho_{L(R)}$ is the spin up or spin down density of states in the emitter (collector), $\rho_{L(R)} \equiv$ $\rho_{L(R) \uparrow}=\rho_{L(R) \downarrow}$. As in the previous section, we assume the resonance level being deeply inside the band, $E_{F}^{R} \ll E_{1} \ll E_{F}^{L}$. If, in addition, $E_{1}+U \ll E_{F}^{L}$, the theta-function in
the singular parts of the integrals in (3.3) can be replaced by one. In the opposite case, $E_{1}+U \gg E_{F}^{L}$, the corresponding singular part is zero.

Proceeding this way with the other equations of the system (3.2), we finally obtain

$$
\begin{align*}
& \left(E+i \Gamma_{L}\right) \tilde{b}_{0}(E)=i  \tag{3.5a}\\
& \left(E+E_{l}-E_{1}+i \Gamma_{L}^{\prime} / 2+i \Gamma_{R} / 2\right) \tilde{b}_{\uparrow l}(E)-\Omega_{l} \tilde{b}_{0}(E)=0  \tag{3.5b}\\
& \left(E+E_{l}-E_{r}+i \Gamma_{L}\right) \tilde{b}_{l r}(E)-\Omega_{r} \tilde{b}_{\uparrow l}(E)=0  \tag{3.5c}\\
& \left(E+E_{l}+E_{l^{\prime}}-2 E_{1}-U+i \Gamma_{R}^{\prime}\right) \tilde{b}_{\uparrow \downarrow l l^{\prime}}(E)-\Omega_{l} \tilde{b}_{l l^{\prime}}(E)+\Omega_{l^{\prime}} \tilde{b}_{\uparrow l}(E)=0 \tag{3.5d}
\end{align*}
$$

Eqs. (3.5) can be transformed into equations for the density matrix of the "device" by using the method of the previous section. Since the algebra remains essentially the same, we give only the final equations for the diagonal density matrix elements $\sigma_{a a}^{(n)}(t), \sigma_{b \downarrow \uparrow}^{(n)}(t)$, $\sigma_{b b \downarrow}^{(n)}(t)$ and $\sigma_{c c}^{(n)}(t)$. These are the probabilities to find: a) no electrons inside the well; b) one electron with spin up (down) inside the well, and c) two electrons inside the well, respectively. The index $n$ denotes the number of electrons accumulated in the collector. We obtain

$$
\begin{align*}
& \dot{\sigma}_{a a}^{(n)}=-2 \Gamma_{L} \sigma_{a a}^{(n)}+\Gamma_{R} \sigma_{b b \uparrow}^{(n-1)}+\Gamma_{R} \sigma_{b b \downarrow}^{(n-1)}  \tag{3.6a}\\
& \dot{\sigma}_{b b \uparrow}^{(n)}=-\left(\Gamma_{L}^{\prime}+\Gamma_{R}\right) \sigma_{b b \uparrow}^{(n)}+\Gamma_{L} \sigma_{a a}^{(n)}+\Gamma_{R}^{\prime} \sigma_{c c}^{(n-1)}  \tag{3.6b}\\
& \dot{\sigma}_{b b \downarrow}^{(n)}=-\left(\Gamma_{L}^{\prime}+\Gamma_{R}\right) \sigma_{b b \downarrow}^{(n)}+\Gamma_{L} \sigma_{a a}^{(n)}+\Gamma_{R}^{\prime} \sigma_{c c}^{(n-1)}  \tag{3.6c}\\
& \dot{\sigma}_{c c}^{(n)}=-2 \Gamma_{R}^{\prime} \sigma_{c c}^{(n)}+\Gamma_{L}^{\prime} \sigma_{b b \uparrow}^{(n)}+\Gamma_{L}^{\prime} \sigma_{b b \downarrow}^{(n)} \tag{3.6d}
\end{align*}
$$

These rate equations look as a generalization of the rate equations (2.12), if one allows the well to be occupied by two electrons. The Coulomb repulsion leads merely to a modification of the corresponding rates $\Gamma \rightarrow \Gamma^{\prime}$, due to increase of the two electron energy.

Summing up the partial probabilities we obtain for the total probabilities, $\sigma(t)=$ $\sum_{n} \sigma^{(n)}(t)$, the following equations:

$$
\begin{equation*}
\dot{\sigma}_{a a}=-2 \Gamma_{L} \sigma_{a a}+\Gamma_{R} \sigma_{b b \uparrow}+\Gamma_{R} \sigma_{b b \downarrow} \tag{3.7a}
\end{equation*}
$$

$$
\begin{gather*}
\dot{\sigma}_{b b \uparrow}=-\left(\Gamma_{L}^{\prime}+\Gamma_{R}\right) \sigma_{b b \uparrow}+\Gamma_{L} \sigma_{a a}+\Gamma_{R}^{\prime} \sigma_{c c}  \tag{3.7b}\\
\dot{\sigma}_{b b \downarrow}=-\left(\Gamma_{L}^{\prime}+\Gamma_{R}\right) \sigma_{b b \downarrow}+\Gamma_{L} \sigma_{a a}+\Gamma_{R}^{\prime} \sigma_{c c}  \tag{3.7c}\\
\dot{\sigma}_{c c}=-2 \Gamma_{R}^{\prime} \sigma_{c c}+\Gamma_{L}^{\prime} \sigma_{b b \uparrow}+\Gamma_{L}^{\prime} \sigma_{b b \downarrow}, \tag{3.7d}
\end{gather*}
$$

and for the current

$$
\begin{equation*}
I(t)=\sum_{n} n\left[\dot{\sigma}^{(n)}(t)\right]=e \Gamma_{R}\left[\sigma_{b b \uparrow}(t)+\sigma_{b b \downarrow}(t)\right]+2 e \Gamma_{R}^{\prime} \sigma_{c c}(t) \tag{3.8}
\end{equation*}
$$

Eqs．（3．7），（3．8）can be solved most easily for dc current，$I=I(t \rightarrow \infty)$ ．In this case $\dot{\sigma}=0$ ，and Eqs．（3．7）turn into the system of linear algebraic equations．One also finds from Eq．（3．7）that $\sigma_{a a}+\sigma_{b b \uparrow}+\sigma_{b b \downarrow}+\sigma_{c c}=1$ ．The latter implies that dc current does not depend on the initial conditions．Finally we obtain

$$
\begin{equation*}
I / e=\frac{2 \Gamma_{L} \Gamma_{R}^{\prime}\left(\Gamma_{L}^{\prime}+\Gamma_{R}\right)}{\Gamma_{L} \Gamma_{L}^{\prime}+2 \Gamma_{L} \Gamma_{R}^{\prime}+\Gamma_{R} \Gamma_{R}^{\prime}} \tag{3.9}
\end{equation*}
$$

If $E_{1} \ll E_{F}^{L} \ll E_{1}+U$ ，one finds from Eq．（3．3）that $\Gamma_{L}^{\prime}=0$ ，so that the state with two electrons inside the well is not available．In this case one obtains from Eq．（3．9）for the dc current

$$
\begin{equation*}
I / e=\frac{2 \Gamma_{L} \Gamma_{R}}{2 \Gamma_{L}+\Gamma_{R}} \tag{3.10}
\end{equation*}
$$

It is interesting to note that this result is different from Eq．（2．16），although in both cases only one electron can occupy the well．However，if the Coulomb repulsion effect is small， i．e．$\Gamma_{L, R}^{\prime}=\Gamma_{L, R}$ ，Eq．（3．9）does produce the same result as Eq．（2．16），provided the density of states is doubled due to the spin degrees of freedom．

One can also consider the case when the Fermi level in the right reservoir $E_{F}^{R}$ lies above the resonance level $E_{1}$ ，but below $E_{1}+U$ ，so that $\Gamma_{R}=0$ ，Eq．（3．3）．Then the resonant transitions of electrons from the left to the right reservoirs can go only through the state with two electrons inside the well．Using Eq．（3．9）one finds for the dc current

$$
\begin{equation*}
I / e=\frac{2 \Gamma_{L}^{\prime} \Gamma_{R}^{\prime}}{\Gamma_{L}^{\prime}+2 \Gamma_{R}^{\prime}} \tag{3.11}
\end{equation*}
$$

which coincides with the result found by Glazmann and Matveev［⿴囗十

## IV. DOUBLE-WELL STRUCTURE

## A. Non-interacting electrons.

Now we turn to the coherent case of resonant tunneling. Let us consider the coupledwell structure, shown in Fig. 2. We assume that both levels $E_{1,2}$ are inside the band, i.e. $E_{F}^{R} \ll E_{1}, E_{2} \ll E_{F}^{L}$. In order to make our derivation as clear as possible, we begin with the case of no spin degrees of freedom and no Coulomb interaction. The tunneling Hamiltonian for this system is

$$
\begin{align*}
\mathcal{H}= & \sum_{l} E_{i} a_{l}^{\dagger} a_{l}+E_{1} a_{1}^{\dagger} a_{1}+E_{2} a_{2}^{\dagger} a_{2}+\sum_{r} E_{r} a_{r}^{\dagger} a_{r} \\
& +\Omega_{0}\left(a_{1}^{\dagger} a_{2}+a_{2}^{\dagger} a_{1}\right)+\sum_{l} \Omega_{l}\left(a_{l}^{\dagger} a_{1}+a_{1}^{\dagger} a_{l}\right)+\sum_{r} \Omega_{r}\left(a_{r}^{\dagger} a_{2}+a_{2}^{\dagger} a_{r}\right) \tag{4.1}
\end{align*}
$$

where $a_{1,2}^{\dagger}, a_{1,2}$ are creation and annihilation operators for an electron in the first or the second well, respectively. All the other notations are taken from Sect. 2. The many-body wave function describing this system can be written in the occupation number representation as

$$
\begin{align*}
|\Psi(t)\rangle & =\left[b_{0}(t)+\sum_{l} b_{1 l}(t) a_{1}^{\dagger} a_{l}+\sum_{l, r} b_{l r}(t) a_{r}^{\dagger} a_{l}\right. \\
& \left.+\sum_{l} b_{2 l}(t) a_{2}^{\dagger} a_{l}+\sum_{l l^{\prime}} b_{12 l l^{\prime}}(t) a_{1}^{\dagger} a_{2}^{\dagger} a_{l} a_{l^{\prime}}+\sum_{l<l^{\prime}, r} b_{1 l l^{\prime} r}(t) a_{1}^{\dagger} a_{r}^{\dagger} a_{l} a_{l^{\prime}}+\ldots\right]|0\rangle \tag{4.2}
\end{align*}
$$

Substituting Eq. (4.2) into the Shrödinger equation with the Hamiltonian (4.1) and performing the Laplace transform, we obtain an infinite set of the coupled equations for the amplitudes $\tilde{b}(t)$ :

$$
\begin{align*}
& E \tilde{b}_{0}(E)-\sum_{l} \Omega_{l} \tilde{b}_{1 l}(E)=i  \tag{4.3a}\\
& \left(E+E_{l}-E_{1}\right) \tilde{b}_{1 l}(E)-\Omega_{l} \tilde{b}_{0}(E)-\Omega_{0} \tilde{b}_{2 l}(E)=0  \tag{4.3b}\\
& \left(E+E_{l}-E_{2}\right) \tilde{b}_{2 l}(E)-\Omega_{0} \tilde{b}_{1 l}(E)-\sum_{l^{\prime}} \Omega_{l^{\prime}} \tilde{b}_{12 l l^{\prime}}(E)-\sum_{r} \Omega_{r} \tilde{b}_{r l}(E)=0  \tag{4.3c}\\
& \left(E+E_{l}+E_{l^{\prime}}-E_{1}-E_{2}\right) \tilde{b}_{12 l l^{\prime}}(E)-\Omega_{l^{\prime}} \tilde{b}_{2 l}(E)+\Omega_{l} \tilde{b}_{2 l^{\prime}}(E)-\sum_{r} \Omega_{r} \tilde{b}_{1 l l^{\prime} r}(E)=0 \tag{4.3d}
\end{align*}
$$

Using exactly the same procedure as in the previous sections, Eq. (2.5), we transform Eqs. (4.3) into the following set of equations:

$$
\begin{align*}
& \left(E+i \Gamma_{L} / 2\right) \tilde{b}_{0}(E)=i  \tag{4.4a}\\
& \left(E+E_{l}-E_{1}\right) \tilde{b}_{1 l}(E)-\Omega_{l} \tilde{b}_{0}(E)-\Omega_{0} \tilde{b}_{2 l}(E)=0  \tag{4.4b}\\
& \left(E+E_{l}-E_{2}+i \Gamma_{L} / 2+i \Gamma_{R} / 2\right) \tilde{b}_{2 l}(E)-\Omega_{0} \tilde{b}_{1 l}(E)=0  \tag{4.4c}\\
& \left(E+E_{l}+E_{l^{\prime}}-E_{1}-E_{2}+i \Gamma_{R} / 2\right) \tilde{b}_{12 l l^{\prime}}(E)-\Omega_{l^{\prime}} \tilde{b}_{2 l}(E)+\Omega_{l} \tilde{b}_{l^{\prime} r}(E)=0 \tag{4.4d}
\end{align*}
$$

The amplitudes $b(t)$ determine the density submatrix of the system, $\sigma_{i j}^{(n)}$, in the corresponding Fock space: $|a\rangle$ - the levels $E_{1,2}$ are empty, $|b\rangle$ - the level $E_{1}$ is occupied, $|c\rangle$ - the level $E_{2}$ is occupied, $|d\rangle$ - the both level $E_{1,2}$ are occupied; the index $n$ denotes the number of electrons in the collector. The matrix elements of the density matrix of the "device" can be written as

$$
\begin{align*}
\sigma_{a a} & =\sum_{n} \sigma_{a a}^{(n)} \equiv\left|b_{0}(t)\right|^{2}+\sum_{l, r}\left|b_{l r}(t)\right|^{2}+\sum_{l<l^{\prime}, r<r^{\prime}}\left|b_{l l^{\prime} r r^{\prime}}(t)\right|^{2}+\ldots  \tag{4.5a}\\
\sigma_{b b} & =\sum_{n} \sigma_{b b}^{(n)} \equiv \sum_{l}\left|b_{1 l}(t)\right|^{2}+\sum_{l<l^{\prime}, r}\left|b_{1 l l^{\prime} r}(t)\right|^{2}+\sum_{l<l^{\prime}<l^{\prime \prime}, r<r^{\prime}}\left|b_{1 l l^{\prime} l^{\prime \prime} r r^{\prime}}(t)\right|^{2}+\ldots  \tag{4.5b}\\
\sigma_{c c} & =\sum_{n} \sigma_{c c}^{(n)} \equiv \sum_{l}\left|b_{2 l}(t)\right|^{2}+\sum_{l<l^{\prime}, r}\left|b_{2 l l^{\prime} r}(t)\right|^{2}+\sum_{l<l^{\prime}<l^{\prime \prime}, r<r^{\prime}}\left|b_{2 l l^{\prime} l^{\prime \prime} r r^{\prime}}(t)\right|^{2}+\ldots  \tag{4.5c}\\
\sigma_{d d} & =\sum_{n} \sigma_{d d}^{(n)} \equiv \sum_{l<l^{\prime}}\left|b_{12 l l^{\prime}}(t)\right|^{2}+\sum_{l<l^{\prime}<l^{\prime \prime}<l^{\prime \prime \prime}, r<r^{\prime}}\left|b_{12 l l^{\prime} l^{\prime \prime} l^{\prime \prime \prime} r r^{\prime}}(t)\right|^{2}+\ldots  \tag{4.5d}\\
\sigma_{b c} & =\sum_{n} \sigma_{b c}^{(n)} \equiv \sum_{l} b_{1 l}(t) b_{2 l}^{*}(t)+\sum_{l<l^{\prime}, r} b_{1 l l^{\prime} r}(t) b_{2 l l^{\prime} r}^{*}(t)+\ldots \tag{4.5e}
\end{align*}
$$

Now we transform Eqs. (4.4) into differential equations for $\sigma^{(n)}(t)$. Consider for instance the term $\sigma_{b b}^{(0)}=\sum_{l}\left|b_{1 l}(t)\right|^{2}$, Eq. (4.5b), where the amplitudes $b_{1 l}$ are determined by Eq. (4.4b). Multiplying Eq. (4.4b) by $\tilde{b}_{1 l}^{*}\left(E^{\prime}\right)$ and subtracting the complex conjugate equation with $E \leftrightarrow E^{\prime}$, we find

$$
\begin{align*}
\sum_{l}\left(E^{\prime}-E\right) \tilde{b}_{1 l}(E) \tilde{b}_{1 l}^{*}\left(E^{\prime}\right)-\sum_{l} \Omega_{l} & {\left[\tilde{b}_{0}^{*}\left(E^{\prime}\right) \tilde{b}_{1 l}(E)-\tilde{b}_{0}(E) \tilde{b}_{1 l}^{*}\left(E^{\prime}\right)\right] } \\
& -\Omega_{0} \sum_{l}\left[\tilde{b}_{2 l}^{*}\left(E^{\prime}\right) \tilde{b}_{1 l}(E)-\tilde{b}_{2 l}(E) \tilde{b}_{1 l}^{*}\left(E^{\prime}\right)\right]=0 \tag{4.6}
\end{align*}
$$

After applying the inverse Laplace transform, Eq. (2.9), the first term in this equation becomes $-i \dot{\sigma}_{b b}^{(0)}(t)$. Next, substituting

$$
\begin{equation*}
\tilde{b}_{1 l}(E)=\frac{\Omega_{l} \tilde{b}_{0}(E)+\Omega_{0} \tilde{b}_{2 l}(E)}{E+E_{l}-E_{1}} \tag{4.7}
\end{equation*}
$$

from Eq. (4.4b) into the second term of Eq. (4.6), and replacing the sum by an integral over $E_{l}$, we reduce this term to $i \Gamma_{L} \tilde{b}_{0}(E) \tilde{b}_{0}^{*}\left(E^{\prime}\right)$. After the inverse Laplace transform it becomes $i \Gamma_{L} \sigma_{a a}^{(0)}(t)$. Notice that the "cross term", $\propto \Omega_{0} \Omega_{l} \tilde{b}_{0} \tilde{b}_{2 l}$, does not contribute to the integral over $E_{l}$, since the poles of the integrand in the $E_{l}$-variable lie on one side of the integration contour (cf. the second term of Eq. (2.5)). The third term of Eq. (4.6) turns to be $\Omega_{0}\left[\sigma_{b c}^{(0)}(t)-\sigma_{c b}^{(0)}(t)\right]$, after the inverse Laplace transform. Finally we obtain a differential equation for the density submatrix element $\sigma_{b b}^{(0)}$,

$$
\begin{equation*}
\dot{\sigma}_{b b}^{(0)}(t)=\Gamma_{L} \sigma_{a a}^{(0)}+i \Omega_{0}\left(\sigma_{b c}^{(0)}-\sigma_{c b}^{(0)}\right) \tag{4.8}
\end{equation*}
$$

In contrast to the rate equations of the previous sections, the diagonal matrix element $\sigma_{b b}$ is coupled with the off-diagonal density matrix element $\sigma_{b c}$.

The corresponding differential equation for $\sigma_{b c}$ can be easily obtained by multiplying Eq. (4.4b) by $\tilde{b}_{2 l}^{*}\left(E^{\prime}\right)$ with subsequent subtracting the complex conjugated Eq. (4.4d), multiplied by $\tilde{b}_{1 l}$. Afterwords by integrating over $E_{l}$ we obtain

$$
\begin{equation*}
\dot{\sigma}_{b c}^{(0)}=i\left(E_{2}-E_{1}\right) \sigma_{b c}^{(0)}+i \Omega_{0}\left(\sigma_{b b}^{(0)}-\sigma_{c c}^{(0)}\right)-\frac{1}{2}\left(\Gamma_{L}+\Gamma_{R}\right) \sigma_{b c}^{(0)} . \tag{4.9}
\end{equation*}
$$

Eventually we arrive to the following set of equations for $\sigma^{(n)}$

$$
\begin{align*}
& \dot{\sigma}_{a a}^{(n)}=-\Gamma_{L} \sigma_{a a}^{(n)}+\Gamma_{R} \sigma_{c c}^{(n-1)}  \tag{4.10a}\\
& \dot{\sigma}_{b b}^{(n)}=\Gamma_{L} \sigma_{a a}^{(n)}+\Gamma_{R} \sigma_{d d}^{(n-1)}+i \Omega_{0}\left(\sigma_{b c}^{(n)}-\sigma_{c b}^{(n)}\right),  \tag{4.10b}\\
& \dot{\sigma}_{c c}^{(n)}=-\Gamma_{R} \sigma_{c c}^{(n)}-\Gamma_{L} \sigma_{c c}^{(n)}-i \Omega_{0}\left(\sigma_{b c}^{(n)}-\sigma_{c b}^{(n)}\right),  \tag{4.10c}\\
& \dot{\sigma}_{d d}^{(n)}=-\Gamma_{R} \sigma_{d d}^{(n)}+\Gamma_{L} \sigma_{c c}^{(n)}  \tag{4.10d}\\
& \dot{\sigma}_{b c}^{(n)}=i\left(E_{2}-E_{1}\right) \sigma_{b c}^{(n)}+i \Omega_{0}\left(\sigma_{b b}^{(n)}-\sigma_{c c}^{(n)}\right)-\frac{1}{2}\left(\Gamma_{L}+\Gamma_{R}\right) \sigma_{b c}^{(n)} . \tag{4.10e}
\end{align*}
$$

Using Eqs. (4.10) we can find the charge accumulated in the collector, $N_{R}(t)$, and subsequently, the total current, $e \dot{N}(t)$, as given by

$$
\begin{equation*}
I(t) / e=\dot{N}(t)=\sum_{n} n\left[\dot{\sigma}_{a a}^{(n)}(t)+\dot{\sigma}_{b b}^{(n)}(t)+\dot{\sigma}_{c c}^{(n)}(t)+\dot{\sigma}_{d d}^{(n)}(t)\right]=\Gamma_{R}\left[\sigma_{c c}(t)+\sigma_{d d}(t)\right] \tag{4.11}
\end{equation*}
$$

As in the previous examples, the current is proportional to the total probability of finding an electron in the well adjacent to the right reservoir. The off-diagonal elements of the density matrix do not appear in Eq. (4.11).

Summing up over $n$ in Eqs. (4.10), we obtain the system of differential equations for the density matrix elements of the device

$$
\begin{align*}
& \dot{\sigma}_{a a}=-\Gamma_{L} \sigma_{a a}+\Gamma_{R} \sigma_{c c}  \tag{4.12a}\\
& \dot{\sigma}_{b b}=\Gamma_{L} \sigma_{a a}+\Gamma_{R} \sigma_{d d}+i \Omega_{0}\left(\sigma_{b c}-\sigma_{c b}\right)  \tag{4.12b}\\
& \dot{\sigma}_{c c}=-\Gamma_{R} \sigma_{c c}-\Gamma_{L} \sigma_{c c}-i \Omega_{0}\left(\sigma_{b c}-\sigma_{c b}\right)  \tag{4.12c}\\
& \dot{\sigma}_{d d}=-\Gamma_{R} \sigma_{d d}+\Gamma_{L} \sigma_{c c}  \tag{4.12d}\\
& \dot{\sigma}_{b c}=i\left(E_{2}-E_{1}\right) \sigma_{b c}+i \Omega_{0}\left(\sigma_{b b}-\sigma_{c c}\right)-\frac{1}{2}\left(\Gamma_{L}+\Gamma_{R}\right) \sigma_{b c} \tag{4.12e}
\end{align*}
$$

Eqs. (4.12)) resemble the optical Bloch equations [11]. Note that the coupling with the reservoirs produces purely negative contribution into the non-diagonal matrix element's dynamic equation, Eq. (4.12®), thus causing damping of this matrix element.

Eqs. (4.12) are solved most easily for the stationary current, $I=I(t \rightarrow \infty)$. Using $\sigma_{a a}+\sigma_{b b}+\sigma_{c c}+\sigma_{d d}=1$, we obtain

$$
\begin{equation*}
I / e=\left(\frac{\Gamma_{L} \Gamma_{R}}{\Gamma_{L}+\Gamma_{R}}\right) \frac{\Omega_{0}^{2}}{\Omega_{0}^{2}+\Gamma_{L} \Gamma_{R} / 4+\epsilon^{2} \Gamma_{L} \Gamma_{R} /\left(\Gamma_{L}+\Gamma_{R}\right)^{2}}, \tag{4.13}
\end{equation*}
$$

where $\epsilon=E_{2}-E_{1}$. This result coincides with the one found in the framework of one-electron approach [12, 13].

## B. Coulomb blockade.

The extension of the rate equations (4.12) for the case of spin and Coulomb interaction is done exactly in the same way as in Sect 3 . Here also the rate equations for the device
density matrix are obtained only for $E_{1,2}+U$ being inside or outside the band, but not close to the band edges $\left(E_{F}^{R} \ll E_{1,2}+U \ll E_{F}^{L}\right.$ or $\left.E_{1,2}+U \gg E_{F}^{L}\right)$. Eventually we arrive to the rate equations of type Eqs. (4.12), but with the number of the available states of the device changed due to additional (spin) degrees of freedom and Coulomb blockade restrictions. The Coulomb repulsion manifests itself also in a modification of the transition amplitude $\Omega$ and the rates $\Gamma$ 's, Eq. (3.4).

In the case of large Coulomb repulsion, some of electron states of the device are outside the band (the Coulomb blockade). As a result, the number of the equations is reduced. Consider, for instance, the situation where the Coulomb interaction $U$ of two electrons in the same well so large that $E_{1,2}+U \gg E_{F}^{L}$, but the Coulomb repulsion of two electrons in different wells, $\bar{U}$, is much smaller, so that $E_{1,2}+\bar{U} \ll E_{F}^{L}$. Then the state of two electrons in the same well is not available, but two electrons can occupy different wells. In this case the rate equations for the corresponding density matrix elements of the device are

$$
\begin{align*}
& \dot{\sigma}_{a a}=-2 \Gamma_{L} \sigma_{a a}+\Gamma_{R}\left(\sigma_{c c \uparrow}+\sigma_{c c \downarrow}\right)  \tag{4.14a}\\
& \dot{\sigma}_{b b \uparrow}=\Gamma_{L} \sigma_{a a}-\Gamma_{R}^{\prime}\left(\sigma_{d d \uparrow \uparrow}+\sigma_{d d \uparrow \downarrow}\right)+i \Omega_{0}\left(\sigma_{b c \uparrow}-\sigma_{c b \uparrow}\right)  \tag{4.14b}\\
& \dot{\sigma}_{c c \uparrow}=-\Gamma_{R} \sigma_{c c \uparrow}-2 \Gamma_{L}^{\prime} \sigma_{c c \uparrow}-i \Omega_{0}\left(\sigma_{b c \uparrow}-\sigma_{c b \uparrow}\right)  \tag{4.14c}\\
& \dot{\sigma}_{d d \uparrow \uparrow}=-\Gamma_{R}^{\prime} \sigma_{d d \uparrow \uparrow}+\Gamma_{L}^{\prime} \sigma_{c c \uparrow}  \tag{4.14d}\\
& \dot{\sigma}_{b c \uparrow}=i\left(E_{2}-E_{1}\right) \sigma_{b c \uparrow}+i \Omega_{0}\left(\sigma_{b b \uparrow}-\sigma_{c c \uparrow}\right)-\frac{1}{2}\left(2 \Gamma_{L}^{\prime}+\Gamma_{R}\right) \sigma_{b c \uparrow} \tag{4.14e}
\end{align*}
$$

where $\Gamma_{L(R)}^{\prime}=2 \pi \rho_{L(R)}\left(E_{1}+\bar{U}\right)\left|\Omega_{L(R)}\left(E_{1}+\bar{U}\right)\right|^{2}$. Here for the shortness we wrote only the equations for the "spin up" component of the density matrix. The same equations are obtained for the "spin down" components of the density matrix. The total current is

$$
\begin{equation*}
I / e=\Gamma_{R}\left(\sigma_{c c \uparrow}+\sigma_{c c \downarrow}\right)+\Gamma_{R}^{\prime}\left(\sigma_{d d \uparrow \uparrow}+\sigma_{d d \uparrow \downarrow}+\sigma_{d d \downarrow \uparrow}+\sigma_{d d \downarrow \downarrow}\right) . \tag{4.15}
\end{equation*}
$$

It is quite clear that the "spin up" and "spin down" components of the density matrix are equal, i.e. $\sigma_{b b \uparrow}=\sigma_{b b \downarrow}=\sigma_{b b}$, the same holding for $\sigma_{c c}, \sigma_{d d}$ components. Therefore Eqs. (4.14), (4.15) can be rewritten as

$$
\begin{align*}
& \dot{\sigma}_{a a}=-2 \Gamma_{L} \sigma_{a a}+2 \Gamma_{R} \sigma_{c c}  \tag{4.16a}\\
& \dot{\sigma}_{b b}=\Gamma_{L} \sigma_{a a}+2 \Gamma_{R}^{\prime} \sigma_{d d}+i \Omega_{0}\left(\sigma_{b c}-\sigma_{c b}\right)  \tag{4.16b}\\
& \dot{\sigma}_{c c}=-\Gamma_{R} \sigma_{c c}-2 \Gamma_{L}^{\prime} \sigma_{c c}-i \Omega_{0}\left(\sigma_{b c}-\sigma_{c b}\right)  \tag{4.16c}\\
& \dot{\sigma}_{d d}=-\Gamma_{R}^{\prime} \sigma_{d d}+\Gamma_{L}^{\prime} \sigma_{c c}  \tag{4.16d}\\
& \dot{\sigma}_{b c}=i\left(E_{2}-E_{1}\right) \sigma_{b c}+i \Omega_{0}\left(\sigma_{b b}-\sigma_{c c}\right)-\frac{1}{2}\left(2 \Gamma_{L}^{\prime}+\Gamma_{R}\right) \sigma_{b c} \tag{4.16e}
\end{align*}
$$

and

$$
\begin{equation*}
I / e=2 \Gamma_{R} \sigma_{c c}+4 \Gamma_{R}^{\prime} \sigma_{d d} \tag{4.17}
\end{equation*}
$$

Using $\sigma_{a a}+2 \sigma_{b b}+2 \sigma_{c c}+4 \sigma_{d d}=1$ we obtain for the dc current

$$
\begin{equation*}
I / e=\left(\frac{2 \Gamma_{L} \Gamma_{R}^{\prime}}{2 \Gamma_{L}^{\prime}+\Gamma_{R}}\right) \frac{\Omega_{0}^{2}}{4 \Omega_{0}^{2} \frac{\Gamma_{L} \Gamma_{L}^{\prime}+\Gamma_{L} \Gamma_{R}^{\prime}+\Gamma_{R} \Gamma_{R}^{\prime} / 4}{\left(2 \Gamma_{L}^{\prime}+\Gamma_{R}\right)^{2}}+\frac{\Gamma_{L} \Gamma_{R}^{\prime}}{2}+\epsilon^{2} \frac{2 \Gamma_{L} \Gamma_{R}^{\prime}}{\left(2 \Gamma_{L}^{\prime}+\Gamma_{R}\right)^{2}}}, \tag{4.18}
\end{equation*}
$$

where $\epsilon=E_{2}-E_{1}$. Notice that the current (4.18) differs from that given by Eq. (4.13) even for $\Gamma_{L}^{\prime}=\Gamma_{L}$ and $\Gamma_{R}^{\prime}=\Gamma_{R}$, despite the fact that in the both cases only one electron can occupy each of the wells.

It is interesting to compare our result with that of Stoof and Nazarov [14] for the case of strong Coulomb repulsion between two electrons in different wells ( $E_{1,2}+\bar{U} \gg E_{F}^{L}$ ), where only one electron can be found inside the system. It corresponds to $\Gamma_{L}^{\prime}=0$. In this case the dc current given by Eq. (4.18) is

$$
\begin{equation*}
I / e=\frac{\Gamma_{R} \Omega_{0}^{2}}{\Omega_{0}^{2}\left(2+\Gamma_{R} / 2 \Gamma_{L}\right)+\Gamma_{R}^{2} / 4+\epsilon^{2}} \tag{4.19}
\end{equation*}
$$

This result is slightly different from that obtained by Stoof and Nazarov (by the factor two in front of $\Gamma_{L}$ ). The difference stems from the account of spin components in the rate equations, which has not been done in (14].

## V. INELASTIC PROCESSES

As an example of a system with coherent tunneling accompanied by inelastic scattering, let us consider the coupled-dot structure shown in Fig. 3. In this system a resonant current
flows due to inelastic transition from the upper to the lower level in the left well. For simplicity, we restrict ourselves to non-interacting spin-less electrons. The Coulomb interaction and the spin effects can be accounted for precisely in the same way as we did in the previous sections, namely by allowing for states with doubly occupied levels (excluding states violating Coulomb restrictions) and modifying transition amplitudes and inelastic rates.

The tunneling Hamiltonian of the system has the following structure

$$
\begin{align*}
\mathcal{H} & =\sum_{l} E_{l} a_{l}^{\dagger} a_{l}+E_{1} a_{1}^{\dagger} a_{1}+E_{2} a_{2}^{\dagger} a_{2}+E_{3} a_{3}^{\dagger} a_{3}+\sum_{\alpha} E_{\alpha}^{p h} c_{\alpha}^{\dagger} c_{\alpha}+\sum_{r} E_{r} a_{r}^{\dagger} a_{r}+\Omega_{0}\left(a_{2}^{\dagger} a_{3}+a_{3}^{\dagger} a_{2}\right) \\
& +\sum_{l} \Omega_{l}\left(a_{l}^{\dagger} a_{1}+a_{1}^{\dagger} a_{l}\right)+\sum_{\alpha} \Omega_{\alpha}^{p h}\left(a_{2}^{\dagger} a_{1} c_{\alpha}^{\dagger}+a_{1}^{\dagger} a_{2} c_{\alpha}\right)+\sum_{r} \Omega_{r}\left(a_{r}^{\dagger} a_{3}+a_{3}^{\dagger} a_{r}\right) \tag{5.1}
\end{align*}
$$

Here the subscript $\alpha$ enumerates the states in the phonon bath and $\Omega_{\alpha}^{p h}$ is the corresponding coupling. The many particle time-dependent wave function of the system is

$$
\begin{align*}
|\Psi(t)\rangle & =\left[b_{0}(t)+\sum_{l} b_{1 l}(t) a_{1}^{\dagger} a_{l}+\sum_{l, \alpha} b_{2 l \alpha}(t) a_{2}^{\dagger} a_{l} c_{\alpha}^{\dagger}+\sum_{l, \alpha} b_{3 l \alpha}(t) a_{3}^{\dagger} a_{l} c_{\alpha}^{\dagger}+\right. \\
& \left.+\sum_{l<l^{\prime}, \alpha} b_{12 l l^{\prime} \alpha}(t) a_{1}^{\dagger} a_{2}^{\dagger} a_{l} a_{l^{\prime}} c_{\alpha}^{\dagger}+\sum_{l<l^{\prime}, \alpha} b_{13 l l^{\prime} \alpha}(t) a_{1}^{\dagger} a_{3}^{\dagger} a_{l} a_{l^{\prime}} c_{\alpha}^{\dagger}+\ldots\right]|0\rangle . \tag{5.2}
\end{align*}
$$

Repeating the procedure of the previous sections we find the following set of equations for the Laplace transformed amplitudes, $\tilde{b}(E)$ :

$$
\begin{align*}
& \left(E+i \Gamma_{L} / 2\right) \tilde{b}_{0}=i  \tag{5.3a}\\
& \left(E+E_{l}-E_{1}+i \Gamma_{i n} / 2\right) \tilde{b}_{1 l}-\Omega_{l} \tilde{b}_{0}=0  \tag{5.3b}\\
& \left(E+E_{l}-E_{\alpha}-E_{2}+i \Gamma_{L} / 2\right) \tilde{b}_{2 l \alpha}-\Omega_{\alpha}^{p h} \tilde{b}_{1 l}-\Omega_{0} \tilde{b}_{3 l \alpha}=0  \tag{5.3c}\\
& \left(E+E_{l}-E_{\alpha}-E_{3}+i \Gamma_{L} / 2+i \Gamma_{R} / 2\right) \tilde{b}_{3 l \alpha}-\Omega_{0} \tilde{b}_{2 l \alpha}=0  \tag{5.3d}\\
& \left(E+E_{l}+E_{l^{\prime}}-E_{1}-E_{2}-E_{\alpha}\right) \tilde{b}_{12 l l^{\prime} \alpha}-\Omega_{l^{\prime}} \tilde{b}_{2 l \alpha}+\Omega_{l} \tilde{b}_{2 l^{\prime} \alpha}-\Omega_{0} \tilde{b}_{13 l l^{\prime} \alpha}=0  \tag{5.3e}\\
& \left(E+E_{l}+E_{l^{\prime}}-E_{1}-E_{3}-E_{\alpha}+i \Gamma_{i n} / 2+i \Gamma_{R} / 2\right) \tilde{b}_{13 l l^{\prime} \alpha} \\
& \quad-\Omega_{0} \tilde{b}_{12 l l^{\prime} \alpha}-\Omega_{l^{\prime}} \tilde{b}_{3 l \alpha}+\Omega_{l} \tilde{b}_{3 l^{\prime} \alpha}=0 \tag{5.3f}
\end{align*}
$$

where $\Gamma_{i n}=2 \pi \rho_{p h}\left|\Omega^{p h}\right|^{2}$ is the partial width of the level $E_{1}$ due to phonon emission and $\rho_{p h}$ is the density of phonon states.

The density matrix elements of the device is $\sigma_{i j}(t)=\sum_{n} \sigma_{i j}^{(n)}(t)$, where $\sigma_{i j}^{(n)}(t)$, are related to the amplitudes $\tilde{b}(E)$ via Eq. (2.9). All possible states electron states of the device are shown in Fig. 4. Using the previous section procedure for diagonal matrix elements we obtain master equations analogous to Eq. (4.12), in which transitions between isolated levels $E_{2}$ and $E_{3}$ take place through the coupling with non-diagonal matrix elements. These equations have the appearance of the optical Bloch equation [11]. However, the master equation for the non-diagonal matrix element, $\sigma_{e f}$, contains an additional term. Therefore, we present the derivation of the master equations for "coherences" $\sigma_{e f}$ and $\sigma_{c d}$ in some detail. Consider for example the non-diagonal density submatrix elements $\sigma_{c d}^{(0)}=\sum_{l, \alpha} b_{2 l \alpha}(t) b_{3 l \alpha}^{*}(t)$ and $\sigma_{e f}^{(0)}=\sum_{l<l^{\prime}, \alpha} b_{12 l l^{\prime} \alpha}(t) b_{13 l l^{\prime} \alpha}^{*}(t)$. The differential equation for $\sigma_{c d}^{(0)}(t)$ can be obtained by multiplying Eq. (5.3d) by $\tilde{b}_{3 l \alpha}^{*}\left(E^{\prime}\right)$ with subsequent subtraction of the complex conjugated Eq. (5.3d), multiplied by $\tilde{b}_{2 l \alpha}(E)$. Then using Eq. (2.9), we obtain

$$
\begin{equation*}
\dot{\sigma}_{c d}^{(0)}=i\left(E_{3}-E_{2}\right) \sigma_{c d}^{(0)}+i \Omega_{0}\left(\sigma_{c c}^{(0)}-\sigma_{d d}^{(0)}\right)-\frac{1}{2}\left(2 \Gamma_{L}+\Gamma_{R}\right) \sigma_{c d}^{(0)} \tag{5.4}
\end{equation*}
$$

Similarly, multiplying Eq. (5.3e) by $\tilde{b}_{13 l l^{\prime} \alpha}^{*}\left(E^{\prime}\right)$ and Eq. (5.3f) by $\tilde{b}_{12 l l^{\prime} \alpha}(E)$, we find the differential equation for $\sigma_{e f}^{(0)}(t)$

$$
\begin{equation*}
\dot{\sigma}_{e f}^{(0)}=i\left(E_{3}-E_{2}\right) \sigma_{e f}^{(0)}+i \Omega_{0}\left(\sigma_{e e}^{(0)}-\sigma_{f f}^{(0)}\right)-\frac{1}{2}\left(\Gamma_{i n}+\Gamma_{R}\right) \sigma_{e f}^{(0)}-i \Delta . \tag{5.5}
\end{equation*}
$$

where

$$
\begin{align*}
\Delta=\sum_{l<l^{\prime}, \alpha} & \int \frac{d E d E^{\prime}}{4 \pi^{2}}\left[\tilde{b}_{13 l l^{\prime} \alpha}^{*}\left(E^{\prime}\right) \Omega_{l^{\prime}} \tilde{b}_{2 l \alpha}(E)-\tilde{b}_{13 l l^{\prime} \alpha}^{*}\left(E^{\prime}\right) \Omega_{l} \tilde{b}_{2 l^{\prime} \alpha}(E)\right. \\
& \left.-\tilde{b}_{12 l l^{\prime} \alpha}(E) \Omega_{l^{\prime}} \tilde{b}_{3 l \alpha}^{*}\left(E^{\prime}\right)+\tilde{b}_{12 l l^{\prime} \alpha}(E) \Omega_{l} \tilde{b}_{3 l^{\prime} \alpha}^{*}\left(E^{\prime}\right)\right] e^{i\left(E^{\prime}-E\right) t} \tag{5.6}
\end{align*}
$$

Substituting the amplitudes $\tilde{b}_{12 l l^{\prime} \alpha}$ from Eq. (5.3e) and $\tilde{b}_{13 l l^{\prime} \alpha}^{*}$ from Eq. (5.3f) into Eq. (5.6), and replacing the sum over $l\left(l^{\prime}\right)$ by the corresponding integral, we find $-i \Delta=\Gamma_{L} \sigma_{c d}^{(0)}$. It implies that the non-diagonal density matrix $\sigma_{e f}$ given by Eq. (5.5), is coupled with $\sigma_{c d}$ via a single electron transition from the emitter to the left well. Such a term does not appear in the Bloch equations, which deal with two-level systems.

Summing up over $n$ in the rate equations for the density submatrix $\sigma_{i j}^{(n)}(t)$ we obtain the set of rate equations for the density matrix of the device

$$
\begin{align*}
& \dot{\sigma}_{a a}=-\Gamma_{L} \sigma_{a a}+\Gamma_{R} \sigma_{d d}  \tag{5.7a}\\
& \dot{\sigma}_{b b}=\Gamma_{L} \sigma_{a a}-\Gamma_{i n} \sigma_{b b}+\Gamma_{R} \sigma_{f f}  \tag{5.7b}\\
& \dot{\sigma}_{c c}=\Gamma_{i n} \sigma_{b b}+i \Omega\left(\sigma_{c d}-\sigma_{d c}\right)+\Gamma_{R} \sigma_{g g}-\Gamma_{L} \sigma_{c c}  \tag{5.7c}\\
& \dot{\sigma}_{d d}=-\Gamma_{R} \sigma_{d d}+i \Omega\left(\sigma_{d c}-\sigma_{c d}\right)-\Gamma_{L} \sigma_{d d}  \tag{5.7d}\\
& \dot{\sigma}_{e e}=\Gamma_{L} \sigma_{c c}+i \Omega\left(\sigma_{e f}-\sigma_{f e}\right)+\Gamma_{R} \sigma_{h h}  \tag{5.7e}\\
& \dot{\sigma}_{f f}=\Gamma_{L} \sigma_{d d}-\Gamma_{R} \sigma_{f f}+i \Omega\left(\sigma_{f e}-\sigma_{e f}\right)-\Gamma_{i n} \sigma_{f f}  \tag{5.7f}\\
& \dot{\sigma}_{g g}=\Gamma_{i n} \sigma_{f f}-\Gamma_{R} \sigma_{g g}-\Gamma_{L} \sigma_{g g}  \tag{5.7~g}\\
& \dot{\sigma}_{h h}=\Gamma_{L} \sigma_{g g}-\Gamma_{R} \sigma_{h h}  \tag{5.7h}\\
& \dot{\sigma}_{c d}=i\left(E_{3}-E_{2}\right) \sigma_{c d}+i \Omega\left(\sigma_{c c}-\sigma_{d d}\right)-1 / 2\left(2 \Gamma_{L}+\Gamma_{R}\right) \sigma_{c d}  \tag{5.7i}\\
& \dot{\sigma}_{e f}=i\left(E_{3}-E_{2}\right) \sigma_{e f}+i \Omega\left(\sigma_{e e}-\sigma_{f f}\right)-1 / 2\left(\Gamma_{i n}+\Gamma_{R}\right) \sigma_{e f}+\Gamma_{L} \sigma_{c d} \tag{5.7j}
\end{align*}
$$

and the resonant current flowing through this system is $I / e=\Gamma_{R}\left[\sigma_{d d}+\sigma_{f f}+\sigma_{g g}+\sigma_{h h}\right]$.

## VI. GENERAL CASE

Now utilizing the results obtained in the previous sections we can write the rate equations for the general case. These equations describing the time evolution of the density matrix $\sigma_{a b}(t)$ of the device are as follows:

$$
\begin{align*}
\dot{\sigma}_{a a}= & i \sum_{b(\neq a)} \Omega_{a b}\left(\sigma_{a b}-\sigma_{b a}\right)-\sigma_{a a} \sum_{d(\neq a)} \Gamma_{a \rightarrow d}+\sum_{c(\neq a)} \sigma_{c c} \Gamma_{c \rightarrow a},  \tag{6.1a}\\
\dot{\sigma}_{a b}= & i\left(E_{b}-E_{a}\right) \sigma_{a b}+i\left(\sum_{b^{\prime}(\neq b)} \sigma_{a b^{\prime}} \Omega_{b^{\prime} b}-\sum_{a^{\prime}(\neq a)} \Omega_{a a^{\prime}} \sigma_{a^{\prime} b}\right) \\
& -\frac{1}{2} \sigma_{a b}\left(\sum_{d(\neq a)} \Gamma_{a \rightarrow d}+\sum_{d(\neq b)} \Gamma_{b \rightarrow d}\right)+\frac{1}{2} \sum_{a^{\prime} b^{\prime} \neq a b} \sigma_{a^{\prime} b^{\prime}}\left(\Gamma_{a^{\prime} \rightarrow a}+\Gamma_{b^{\prime} \rightarrow b}\right), \tag{6.1b}
\end{align*}
$$

where $\Omega_{a b}$ denote the couplings between non-orthogonal isolated states, as for instance between the levels in adjacent wells, and $\sigma_{b a}=\sigma_{a b}^{*}$. The width $\Gamma_{a \rightarrow b}$ is the probability per unit
time for the system to make a transition from the state $|a\rangle$ to the state $|b\rangle$ of the device due to the tunneling to (or from) the reservoirs, or due to interaction with the phonon bath, or any other interaction, generated by a continuum state medium. Notice that Eq. (6.1a) for diagonal elements has a classical rate equation form, except for the first term. This term describes transitions between isolated states through the coupling with non-diagonal terms. Therefore it is responsible for coherent (quantum) effects in the transport.

The non-diagonal matrix elements are described by Eq. (6.1b), which resembles the corresponding Bloch equation, supplemented with an additional term. The latter appears whenever a one-electron transition converts the state $\left|a^{\prime}\right\rangle$ into $|a\rangle$ and the state $\left|b^{\prime}\right\rangle$ into $|b\rangle$. The positive sign of the additional term calls forth a suspicion that Eq. (6.1b) might have unbounded solutions. This is not the case, however, since any positive contribution from the additional term in the equation for $\dot{\sigma}_{a b}$ has its negative counterpart in the equation for $\dot{\sigma}_{a^{\prime} b^{\prime}}$, which corresponds to the conversion $\left(a^{\prime} b^{\prime}\right) \rightarrow(a b)$ and originates from the third term in the rhs of Eq. (6.1b). Moreover, the negative contributions, corresponding to the conversions $(a b) \rightarrow\left(a^{\prime} b\right)$ and $(a b) \rightarrow\left(a b^{\prime}\right)$ have no positive counterparts. Therefore the coupling with continuum leads to negative total balance, and hence, to the damping of non-diagonal matrix elements.

The current through the mesoscopic device is the time derivative of the total charge accumulated in the collector. We find that the current is totally determined through the diagonal elements of the density matrix of the device by the following relation

$$
\begin{equation*}
I(t)=e \sum_{c} \sigma_{c c}(t) \Gamma_{R}^{(c)} \tag{6.2}
\end{equation*}
$$

where $|c\rangle$ are the occupied states in the well adjacent to the collector, and $\Gamma_{R}^{(c)}$ is the partial width of the state $|c\rangle$ due to tunneling to the collector.

Although the non-diagonal density-matrix elements do not enter explicitly in Eq. (6.2), they are coupled with diagonal matrix elements in the rate equations (the first term in (6.1a)), and therefore influence the resonant current. The coupling with non-diagonal elements always appears in the rate equation, whenever a carrier jumps from one to another
isolated states inside the device. In the absence of such transition as, for instance, in resonant tunneling through a single well, the diagonal and non-diagonal matrix elements are decoupled and the evolution of diagonal density-matrix elements is described by the classical rate equation.

Hence, the distinction between isolated and continuum states becomes very essential in the description of quantum transport. At first sight, it may seem that in a real situation such a distinction can hardly be carried out, since there are no pure isolated states. For instance, a single electron state inside the device is always coupled with the continuum states of phonons. However, the corresponding density of states would display peaks in energy dependence, and they can be considered as isolated states. Indeed, if we have written equations like Eqs. (2.5), (3.3), etc. for such a system, the contribution from these peaks in the integrals over continuum states would generate a coupling with non-diagonal density matrix elements in the rate equations, just as in a transition between two isolated states, Eqs. (6.1).

As an example of application of Eq. (6.1) in the case of strong Coulomb blockade, we consider the system shown in Fig. 5. The wells may represent two coupled dots. An electron tunnels from the emitter to the first well, and then to the second well into the upper level $E_{2}$. After that it can either relax inelastically into the lower level $E_{3}$ due to interaction with the phonon bath, and then tunnel into the collector, or tunnel out directly into the collector. Here $\Gamma_{i n}$ and $\Gamma_{R}^{\prime}$ are the partial widths of the upper level, $E_{2}$, due to coupling to phonon reservoir and the collector, respectively, and $\Gamma_{R}$ is the width of the level $E_{3}$ due to coupling to the collector. Let's assume that the Coulomb blockade prevents the system from being occupied by two electrons, even in different wells. Then there are four possible states of the device $|a\rangle$ - all the levels $E_{1,2,3}$ are empty; $|b\rangle$ - the level $E_{1}$ is occupied; $|c\rangle$ - the level $E_{2}$ is occupied; $|d\rangle$ - the level $E_{3}$ is occupied. It is clear that the density matrix elements for an electron with spin up and spin down inside the system are equal, $\sigma_{b b \uparrow}=\sigma_{b b \downarrow}=\sigma_{b b}$, and the same holds for $\sigma_{c c}$ and $\sigma_{d d}$. Hence, Eqs. (6.1) can be written in this case as

$$
\begin{align*}
& \dot{\sigma}_{a a}=-2 \Gamma_{L} \sigma_{a a}+2 \Gamma_{R}^{\prime} \sigma_{c c}+2 \Gamma_{R} \sigma_{d d}  \tag{6.3a}\\
& \dot{\sigma}_{b b}=i \Omega_{0}\left(\sigma_{b c}-\sigma_{c b}\right)+\Gamma_{L} \sigma_{a a}  \tag{6.3b}\\
& \dot{\sigma}_{c c}=-i \Omega_{0}\left(\sigma_{b c}-\sigma_{c b}\right)-\left(\Gamma_{R}^{\prime}+\Gamma_{i n}\right) \sigma_{c c}  \tag{6.3c}\\
& \dot{\sigma}_{d d}=-\Gamma_{R} \sigma_{d d}+\Gamma_{i n} \sigma_{c c}  \tag{6.3d}\\
& \dot{\sigma}_{b c}=i\left(E_{2}-E_{1}\right) \sigma_{b c}+i \Omega_{0}\left(\sigma_{b b}-\sigma_{c c}\right)-\frac{1}{2}\left(\Gamma_{R}^{\prime}+\Gamma_{i n}\right) \sigma_{b c} \tag{6.3e}
\end{align*}
$$

and the dc current $I$, Eq. (6.2), is given by

$$
\begin{equation*}
I / e=2 \sigma_{c c} \Gamma_{R}^{\prime}+2 \sigma_{d d} \Gamma_{R} \tag{6.4}
\end{equation*}
$$

Using $\sigma_{a a}+2 \sigma_{b b}+2 \sigma_{c c}+2 \sigma_{d d}=1$, Eqs. (6.3) can be easily solved for $t \rightarrow \infty$, yielding for the dc current

$$
\begin{equation*}
I / e=\left(\frac{2 \Gamma_{L} \Gamma_{R}}{\Gamma_{i n}+\Gamma_{R}^{\prime}}\right) \frac{\Omega_{0}^{2}}{\Omega_{0}^{2} \frac{2 \Gamma_{i n} \Gamma_{L}+\Gamma_{i n} \Gamma_{R}+4 \Gamma_{L} \Gamma_{R}+\Gamma_{R} \Gamma_{R}^{\prime}}{\left(\Gamma_{i n}+\Gamma_{R}^{\prime}\right)^{2}}+\frac{\Gamma_{L} \Gamma_{R}}{2}+\epsilon^{2} \frac{2 \Gamma_{L} \Gamma_{R}}{\left(\Gamma_{i n}+\Gamma_{R}^{\prime}\right)^{2}}}, \tag{6.5}
\end{equation*}
$$

This result shows very peculiar dependence of the dc current of the inelastic width $\Gamma_{i n}$. One could expect, at least for $\Gamma_{R}^{\prime} \sim \Gamma_{R}$, that the current should increase when $\Gamma_{i n}$ grows. However, as follows from Eq. (6.3), the current $I \rightarrow 0$ for $\Gamma_{i n} \rightarrow \infty$ (cf. with another example in [10]). In fact, such an unexpected behavior of the dc current would always take place in the presence of coherent transitions between isolated states in carrier transport. For instance, it can be traced even in a more simple case of the resonant tunneling through a double well structure, Eq. (4.19). One finds that $I \rightarrow 0$ when $\Gamma_{R} \rightarrow \infty$. This phenomenon can be understood by analyzing Eq. (6.1b) for non-diagonal density matrix elements. In contrast with the rate equation for diagonal matrix elements, Eq. (6.1a), the coupling with continuum states always leads to damping of non-diagonal matrix elements. Since the transport through isolated states goes only via non-diagonal density-matrix elements, Eq. (6.1a), the total current would always decrease with the growth of the corresponding partial widths.

## VII. SUMMARY

In this paper we have studied quantum transport in mesoscopic systems (quantum dots) containing finite number of isolated quantum states. Starting with the many-particle wave function in the occupation number representation, and integrating out the continuum states, we have found the equations of motion for the density submatrix of the system. These equations have a form of the master (rate) equations for diagonal density matrix element. But in addition, non-diagonal density matrix elements, responsible for transitions between isolated quantum states, appear in these equations. If, however, these transitions are generated by a continuum states medium, the diagonal and non-diagonal density matrix elements become decoupled, and the quantum transport is described by classical rate equations.

It follows from our derivation that the reduction of many-body Schrödinger equation to the modified rate equations for density submatrix of the device can be performed only if two conditions are met: first, the energy states of the system which carry the resonant transport must be inside the bias, $E_{F}^{L}-E_{F}^{R}$; second, the width of these states is much smaller than the bias. If the second condition is satisfied, but the resonant levels of the device are close to band edges, our rate equations cannot be derived. Yet, the method still can be used for dc current. However, when the bias is less than the level width, the continuum states of the reservoir cannot be integrated out in the manner of Sect. 2, and our method cannot be applied.

We have compared some of our results with ones obtained earlier in the literature. For example, for the resonant tunneling through a single dot we obtained the same result as Glazman and Matveev [4]. In the case of resonant tunneling through double-dot structure we found simple analytical expression for dc current under condition of strong Coulomb repulsion inside the dots, when no more than one electron can occupy the dots. The obtained expression is very close to that found in [9, 14].

As an application of our equations we considered a more complicated case of the resonant tunneling in a coupled dot system, where the inelastic process takes place in the course of
transport. It was found that the resonant current decreases with the growth of the inelastic width. We found that this anomalous behavior always emerges whenever coherent transitions are accompanied by inelastic processes.

## REFERENCES

[1] D.V. Averin and K.K. Likharev, in em Mesoscopic Phenomena in Solids, edited by B. Altshuler, P.A. Lee, and R.A. Webb (Elsevier, Amsterdam, 1991).
[2] N.C. van der Vaart, S.F. Godijn, Y.V. Nazarov, C.J.P.M. Hartmans, J.E. Mooij, L.W. Molenkamp, and C.T. Foxon, Phys. Rev. Lett. 74, 4702 (1995).
[3] F.R. Waugh, M.J. Berry, D.J. Mar, R.M. Westervelt, K.L. Campman and A.C. Gossard, Phys. Rev. Lett. 75, 4702 (1995).
[4] L.I. Glazman and K.A. Matveev, Pis'ma Zh. Eksp. Theor. Fiz. 48, 403 (1988) [JETP Lett. 48, 445 (1988)].
[5] D.V. Averin and A.N. Korotkov, Zh. Eksp. Teor. Fiz. 97, 1661 (1990) [Sov. Phys.-JETP 70, 937 (1990)].
[6] C.W.J. Beenakker, Phys. Rev. B 44,1646 (1991)
[7] J. H. Davies, S. Hershfield, P. Hyldgaard, and J. W. Wilkins, Phys. Rev. B 47, 4603 (1993).
[8] S.A. Gurvitz, H.J. Lipkin, and Ya.S. Prager, Mod. Phys. Lett. B 8, 1377 (1994).
[9] Yu.V. Nazarov, Physica B 189, 57 (1993).
[10] S.A. Gurvitz, H.J. Lipkin, and Ya.S. Prager, WIS-93/69/July -PH, condmat@babbage.sissa.it/9307051.
[11] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, Atom-Photon Interactions: Basic Processes and Applications (Wiley, New York, 1992).
[12] S. A. Gurvitz, Phys. Rev. B 44, 11924 (1991).
[13] D. Sokolovski, Phys. Lett. A 132 (1988) 381.
[14] T.H. Stoof and Yu.V. Nazarov, cond-mat@babbage.sissa.it/9506036.

## FIGURES

FIG. 1. Resonant transport through a single quantum well structure.

FIG. 2. Resonant transport through a double-well structure.

FIG. 3. Resonant transport through a double-well structure in the presence of inelastic process.

FIG. 4. All possible electron states of the device, shown in Fig. 3: $|a\rangle$ - all the levels $E_{1,2,3}$ are empty; $|b\rangle$ - the upper level, $E_{1}$, is occupied; $|c\rangle$ - the lower level, $E_{2}$, is occupied; $|d\rangle$ - the level $E_{3}$ is occupied; $|e\rangle$ - the levels $E_{1}$ and $E_{2}$ are occupied; $|f\rangle$ - the levels $E_{1}$ and $E_{3}$ are occupied; $|g\rangle$ - the levels $E_{2}$ and $E_{3}$ are occupied; $|h\rangle$ - all the levels $E_{1,2,3}$ are occupied.

FIG. 5. Resonant transport through a double-well structure in the presence of inelastic process with strong Coulomb blockade effects.


Fig. 1


Fig. 2


Fig. 3


Fig. 4


Fig. 5

