## A third-order exceptional point effect on the dynamics of a single particle in a time-dependent harmonic trap

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The time evolution of a single particle in a harmonic trap with time dependent frequency  $\omega(t)$  is well studied. Nevertheless here we show that, when the harmonic trap is opened (or closed) as function of time while keeping the adiabatic parameter  $\mu = [d\omega(t)/dt]/\omega^2(t)$  fixed, a sharp transition from an oscillatory to a monotonic exponential dynamics occurs at  $\mu = 2$ . At this transition point the time evolution has a third-order exceptional point (EP) at all instants. This situation, where an EP of a time-dependent Hermitian Hamiltonian is obtained at any given time, is very different from other known cases. Our finding is relevant to the dynamics of a single ion in a magnetic, optical, or rf trap, and of diluted gases of ultracold atoms in optical traps.

Exceptional points (EP) are degeneracies of non-Hermitian Hamiltonians[1, 2], associated with the coalescence of two or more eigenstates. The studies of EPs have substantially grown since the pioneering works of Carl Bender and his co-workers on  $\mathcal{PT}$ -symmetric Hamiltonians[3]. These Hamiltonians have a real spectrum, which becomes complex at the EP. However,  $\mathcal{PT}$ symmetry is not required to obtain an EP point, as in the case of a coalescence between two resonant states, leading to self-orthogonal states[4–6].

The physical effects of second-order EPs have already been demonstrated in different types of experiments. See for example the effect of EPs on cold atoms experiments[7], on the cross sections of electron scattering from hydrogen molecules[8], and on the linewidth of unstable lasers[9]. More direct realizations of EPs in microwave experiments are given in Ref.[10, 11] and in optical experiments in Ref.[12]. For theoretical studies that are relevant to these experiments see for example[8, 13– 18]. In addition, theoretical studies predict significant effects of second-order EPs on the photoionization of atoms[19–21] and the photodissociation of molecules[22– 24].

The possibility of higher-order EPs (where more than two eigenstates of the non-Hermitian Hamiltonian coalesce at the EP) has been discussed in the literature for time independent  $\mathcal{PT}$  symmetric Hamiltonians (see Ref.[25, 26] and references therein). The main effect of EPs (of any order) on the dynamics of  $\mathcal{PT}$ -symmetric systems is the sudden transition from a real spectrum to a complex energy spectrum associated with gain and loss processes.[14].

All above mentioned studies on the effects of EPs are related to non-Hermitian time-independent Hamiltonians. Note that non-Hermitian Hamiltonians can be obtained from Hermitian Hamiltonians by imposing outgoing boundary conditions on the eigenfunctions or including complex absorbing potentials[5]. This approach allows the description of resonance phenomena in systems with finite-lifetime metastable states.

Other studies considered time-periodic Hamiltonians where the EPs are associated with the quasi-energies of the Floquet operator which can be represented by a time-independent non-Hermitian matrix (see for example one of the first studies of EP in atomic physics in Ref.[19]). EPs were also studied in non-periodic systems in the context of Landau-Zener-Majorana transitions, where the EP was obtained only after analytic continuation of the actual Hamiltonian[27]. Finally, timedependent EPs have been used to control the quantum evolution of non-Hermitian systems[28].

In this paper we show that an EP for the time evolution of a Hermitian time dependent Hamiltonian can be obtained. Our method relies on the re-scaling of the time-axis, allowing to map the time-dependent problem to an effective time-independent one, with non-unitary evolution.

The harmonic oscillator system with changing frequency in the Heisenberg picture – The 1D harmonic oscillator system with changing frequency is defined as

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2(t)\hat{x}^2,$$
(1)

where m is the mass of the particle, and  $\hat{p}$  and  $\hat{x}$  are respectively the momentum and position operators. Despite its simplicity, this time-dependent Hermitian Hamiltonian can associated with a third-order exceptional point.

We study the model (1) in the framework of Ref.[29,

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30], where it is shown that, due to the closed commutation relations between the operators  $\hat{p}^2$ ,  $\hat{x}^2$ ,  $\hat{p}\hat{x} + \hat{x}\hat{p}$ , the model forms a SU(1,1) algebra. As a basis set for this algebra we choose the Hamiltonian, the Lagrangian and the x - p anti-commutator:

$$\hat{O}_1 = \hat{H} \tag{2}$$

$$\hat{O}_2 = \hat{L} = \hat{H} - m\omega^2(t)\hat{x}^2 \tag{3}$$

$$\hat{O}_3 = \hat{D} = \omega(t)(\hat{x}\hat{p} + \hat{p}\hat{x})/2$$
 (4)

Any commutator between operators in the algebra can be expressed as a linear combination of these operators:

$$[\hat{O}_k, \hat{O}_j]_{j=1,2,3} = \sum_{k=1}^3 C_{jk}^l \hat{O}_l \tag{5}$$

where the  $C_{jk}^l$  are the structure factors of the SU(1,1) algebra[31]. The Heisenberg picture for the dynamics which is associated with the operators  $\hat{O}_j$  is described as

$$\frac{d\hat{O}_j}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{O}_j] + \frac{\partial\hat{O}_j}{\partial t},\tag{6}$$

where j = 1, 2, 3.

These equations are explicitly given by

$$\frac{d}{dt}\hat{H} = \frac{\partial\hat{H}}{\partial t} = m\dot{\omega}\omega\hat{x}^2 = \omega\mu(\hat{H} - \hat{L})$$

$$\frac{d}{dt}\hat{L} = \frac{i}{\hbar}[\hat{H},\hat{L}] + \frac{\partial\hat{L}}{\partial t} = -2\omega\hat{D} - \omega\mu(\hat{H} - \hat{L}) \qquad (7)$$

$$\frac{d}{dt}\hat{D} = \frac{i}{\hbar}[\hat{H},\hat{D}] + \frac{\partial\hat{D}}{\partial t} = 2\omega L + \omega\mu\hat{D}$$

Here we defined the dimensionless "adiabatic parameter"

$$\mu = \left[\frac{1}{\omega^2(t)}\right] \frac{d\omega}{dt}.$$
(8)

The equations of motion (7) conserve the "Casimir" operator[32]  $\hat{C}(t) = [\hat{H}^2(t) - \hat{L}^2(t) - \hat{D}^2(t)]/\omega^2(t)$  by satisfying  $d\hat{C}/dt = 0$ .

In what follows we will focus on the specific case of  $\mu = \text{const}$ , corresponding to the frequency profile

$$\omega(t) = \frac{\omega(0)}{1 - \mu\omega(0)t},\tag{9}$$

In experiments the harmonic trap is varied between two extreme values,  $\omega_{open}$  and  $\omega_{closed}$ . The compression factor is given by  $\omega_{closed}/\omega_{open}$ . For positive values of the adiabatic parameter  $\omega(0) = \omega_{open}$  and  $\omega(t_f) = \omega_{closed}$ . For negative values  $\omega(0) = \omega_{closed}$  and  $\omega(t_f) = \omega_{open}$ . In both cases  $t_f = |\mu|(\omega_{open}^{-1} - \omega_{closed}^{-1})$ .

The parameter  $\mu$  sets the degree of adiabaticity of the process. For  $\mu \to 0$ , the dynamics is perfectly adiabatic and the system follows the eigenvalues of the instantaneous Hamiltonian. In contrast, for  $\mu \to \pm \infty$ , the change of the Hamiltonian is so fast that the system does not have time to change at all. As we will show, these two limits are separated by an exceptional point. A similar

effect is known to occur in the vicinity of quantum critical points (See for example Ref.[33]) and is here shown in time-dependent non-periodic harmonic traps.

It is convenient to introduce the dimensionless time variable  $\tau = (1/\mu) \log(\omega(t)/\omega(0))$ , satisfying  $d\tau = \omega(t)dt$ , and rewrite (7) as

$$i\frac{d\mathbf{O}(\tau)}{d\tau} = \mathcal{H}_{\text{Heis}}\mathbf{\vec{O}}(\tau) \equiv \left(i\mu\mathbf{I} + \mathcal{H}_{\text{Heis}}^{\text{Tr}=0}\right)\mathbf{\vec{O}}(\tau) \qquad (10)$$

where  $\vec{O} = \{H, L, D\}$ -components, **I** is the unit matrix and the traceless operator  $\mathcal{H}_{\text{Heis}}^{\text{Tr}=0}$  is defined by

$$\mathcal{H}_{\text{Heis}}^{\text{Tr}=0} = i \begin{pmatrix} 0 & -\mu & 0 \\ -\mu & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}.$$
 (11)

We can further simplify Eq.(10) by performing the transformation

$$\vec{\mathbf{O}}(\tau(t)) \to \frac{1}{\omega(t)} \vec{\mathbf{O}}(\tau(t)) \quad \Rightarrow \quad \frac{d\vec{\mathbf{O}}(\tau)}{d\tau} \to \frac{d\vec{\mathbf{O}}(\tau)}{d\tau} - \mu \mathbf{I}.$$
(12)

The resulting equation of motion  $id\vec{\mathbf{O}}/d\tau = \mathcal{H}_{\text{Heis}}^{\text{Tr}=0}\vec{\mathbf{O}}$ is equivalent to a time dependent Schrödinger equation with a non-Hermitian time-independent Hamiltonian. The matrix  $\mathcal{H}_{\text{Heis}}^{\text{Tr}=0}$  is  $\mathcal{PT}$ -symmetric[34] and its three eigenvalues

$$E_0 = 0; \quad E_{\pm} = \pm \sqrt{4 - \mu^2}$$
 (13)

are real for  $|\mu| \leq 2$ . The corresponding eigenvectors are given by,  $v_0 = (1, 0, -\mu/2)$ ,  $v_{\pm} = (\mu, \pm i\sqrt{4-\mu^2}, -2)/\mu$ .

In contrast to the Schrödinger equation, the population of the eigenvectors in a physical state is not arbitrary, but must satisfy several constraints. For example, for  $|\mu| < 2$ , the eigenvectors  $v_+$  and  $v_-$  are complex and any physical state must populate them with an equal weight, in order to keep the expectation values H, D, and L real. In addition,  $v_+$  and  $v_-$  have a zero Casimir constant  $\langle \hat{C} \rangle = H^2 - L^2 - D^2 = 0$ . Due to the uncertainty relation  $\langle \hat{C} \rangle \geq \hbar^2/4$  any physical state must necessarily populate the eigenstate  $v_0$  with non-zero weight as well. Thus, for a generic initial state, we expect more than one eigenvector to be occupied, leading to an oscillatory behavior that we describe below.

The third-order EP for the time evolution operator of the Hermitian time-dependent harmonic oscillator – The matrix  $\mathcal{H}_{\text{hies}}^{\text{Tr}=0}$  has a third order EP at  $|\mu| = 2$ . At this point all three eigenvalues and the corresponding eigenvectors (in the *HLD* space) coalesce. As a consequence,  $\left[\mathcal{H}_{\text{Heis}}^{\text{Tr}=0}(\mu = \pm 2)\right]^3 = 0$  while  $\left[\mathcal{H}_{\text{Heis}}^{\text{Tr}=0}(\mu = \pm 2)\right]^2 \neq 0$ , demonstrating that the present EP is of third order.

At the EP the matrix  $\mathcal{H}_{\text{Heis}}^{\text{Tr}=0}$  has one single eigenvector, which is "self-orthogonal". To show this property,

it is necessary to multiply the right and left eigenvectors of the non-symmetric  $\mathcal{H}_{\text{Heis}}^{\text{Tr}=0}$ . At the EP the right eigenvector is (1,0,-1) while the left eigenvector is (1,0,1). Their product is equal to zero showing that the eigenvector is "self-orthogonal" (see for example Chapter 9 in

Let us first discuss the situation where the initial state is Gaussian (e.g., a ground state of the harmonic oscillator trap at t = 0 or a thermal state). The evolution in time is described by a Gaussian Wigner distribution with elliptic contour plots. Using the normalized coordinates,  $x(m\omega)^{1/2}$  and  $p/(m\omega)^{1/2}$ , the standard deviation of the narrow axis (N) and the wide axis (W), are given by:

Ref.[5]).

$$\sigma_{W,N}^2 = \frac{H \pm \sqrt{L^2 + D^2}}{\omega} \,. \tag{14}$$

Note that by virtue of the Casimir constant a most general state satisfies  $\sigma_N \sigma_W \geq 1/2$ . For  $\omega = \text{const} (\mu = 0)$ the evolution simply mixes L and D leaving  $\sqrt{L^2 + D^2}$ constant. Hence even though for  $\omega = \text{const}$  the distribution rotates in phase space and changes the variance of position and momentum, the width of the narrow and wide axis of the distribution remains fixed.

For  $|\mu| < 2$  time oscillations of measurable quantities are obtained. A convenient measure to capture this oscillatory dynamics is given by the ratio:

$$\rho = \frac{\sigma_W}{\sigma_N} \tag{15}$$

where by definition  $\rho \geq 1$ . The time evolution of  $\rho$  is shown in Fig. 1a as function of the time-dependent compression factor  $\omega(t)/\omega(0)$  in a log scale. Since the logarithm of the compression factor is equal to the new time variable  $\tau$  (multiplied by  $\mu$ ), in this scale  $\rho(t)$  shows periodic oscillations. In the original time variable t, the system displays non-periodic oscillations with the same amplitude. The periodicity in the  $\tau$  time coordinate is determined by the eigenvalue difference:

$$T_{\tau} = \frac{2\pi}{\frac{1}{2}(E_{+} - E_{-})} = \frac{2\pi}{\sqrt{4 - \mu^{2}}}$$
(16)

The visibility of the fringes pattern that appear in Fig. 1a is given by the simple expression:

$$V = \frac{\rho_{max} - \rho_{min}}{\rho_{max} + \rho_{min}} = \frac{|\mu|}{2}.$$
 (17)

At  $|\mu| = 2$  the visibility becomes one, and the oscillations disappear for  $|\mu| \ge 2$ .

We now comment on the time evolution starting from a generic non-Gaussian initial state. Of course, in this case H,L and D are not sufficient to completely determine the state. Nevertheless for  $\mu = \text{const}$  the expectation values of H,L and D follow exactly the dynamics described above. It is possible to show that the expression (17) is valid for any initial state satisfying  $\mu D(0) + L(0)^2/(4H(0)) > 0$ . In particular, this condition is fulfilled by any initial state that is stationary with respect to the initial Hamiltonian  $\hat{H}(0)$ .

A similar transition between oscillatory and monotonous behavior appears in other properly scaled quantities (see Eq.12), like  $\langle \hat{H} - \hat{L} \rangle / \omega(t) = m\omega(t) \langle x^2 \rangle$ and  $\langle \hat{H} + \hat{L} \rangle / \omega(t) = \langle p^2 \rangle / (m\omega(t))$ , as shown in Fig. 1b. However note that for initial states which are not eigenstates of the harmonic trap these quantities display additional trivial oscillations that show up even for  $\omega = \text{const.}$ 

The EP corresponds to a transition from an underdamped to an over-damped harmonic oscillator– We now present a different approach which will clarify the relation between the present problem and energy-dissipative systems. Our approach is based on the equivalence between quantum and classical evolution of quadratic Hamiltonians. To reproduce the quantum mechanical results one simply needs to complement the classical equations of motion by stochastic initial distributions, given by the Wigner transform of the initial state.

In our case, the relevant equation of motion is Newton's law

$$\left[\frac{d^2}{dt^2} + \omega^2(t)\right]x(t) = 0 \tag{18}$$

By applying the transformation  $d\tau = \omega(t)dt$ , or  $\frac{d}{dt} = \omega(t)\frac{d}{d\tau}$ , we obtain

$$\frac{d^2}{dt^2}x = \frac{d}{dt}(\omega(t)\frac{d}{d\tau}x) = \omega'(t)\frac{d}{d\tau}f + \omega^2(t)\frac{d}{d\tau}x \qquad (19)$$

In the specific case  $\mu = \text{const}$ , the equation of motion then becomes

$$\left[\frac{d^2}{d\tau^2} + \mu \frac{d}{d\tau} + 1\right] x(\tau) = 0 \tag{20}$$

Here we obtain the time-independent non-Hermitian equation of motion of a damped harmonic oscillator. The EP  $|\mu| = 2$  corresponds to the transition between an under-damped and over-damped oscillator, as can be seen by the Fourier transform of (20), leading to:

$$\lambda_{\pm} = \frac{i\mu \pm \sqrt{4-\mu^2}}{2} \tag{21}$$

For  $|\mu| > 2$  both eigenfrequencies are pure imaginary, leading to the disappearance of the oscillatory behavior.

Physical realization of the third-order EP in experiments – Although our Hamiltonian can be realized in any controllable Harmonics trap (optics, plasma, ...), we will consider here the case of either a single particle (ion[35]), or a dilute atomic cloud[36–38], in time-dependent confining traps. The realization with atomic clouds allows the measurement of expectation values in a single-shot experiment. Complications of the dynamics due to the

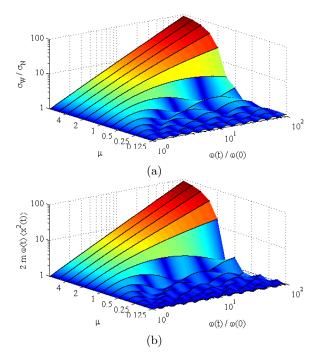


FIG. 1: (a) The ratio,  $\rho$ , between the narrow and wide axis in the Gaussian Wigner distribution as a function of the time dependent compression factor  $\omega(t)/\omega(0)$  for different values of the adiabatic parameter  $\mu \geq 0$ . For  $\mu < 0$  the same plot is obtained where now the compression factor is taken as  $\omega(0)/\omega(t)$ . At the exceptional point (EP)  $\mu = 2$  the dynamics changes from oscillatory to monotonous. (b) Same plot for the variance of the position operator normalized by the instantaneous frequency,  $2\langle x^2(t)\rangle\omega(t)$ . The initial state is the ground state of the Hamiltonian (1) at t = 0 where  $\langle x^2 \rangle = 1/(2m\omega(0))$ .

atom-atom interactions can be avoided (minimized) by setting the atomic scattering length to zero in the vicinity of a Feshbach resonance[39].

The procedure to observe the EP effect on the dynamics of time-dependent Hermitian Hamiltonian is as follows:

(1) Equilibrate the matter in a harmonic trap characterized by the frequency  $\omega(0)$ .

(2) Vary the frequency of the trap as function of time from  $\omega(0)$  to  $\omega(t_f)$ , while imposing a constant adiabatic parameter  $\mu$ , as shown in Eq.9.

(3) At time  $\{0 < t_n < t_f\}_{n=1,2...,N}$  measure either the spatial distribution of matter inside the trap[38, 40–44], or the momentum distribution in a time of flight experiment, by suddenly turning off the trap.

(4) Scale the variance of the measured data (position or momentum) by the instantaneous frequency:  $\langle x^2(t) \rangle \rightarrow \omega(t) \langle x^2(t) \rangle$  and  $\langle p^2(t) \rangle \rightarrow \langle x^2(t) \rangle / \omega(t)$ . These quantities are then plotted as function of the compression factor  $\omega(t)/\omega(0)$  for different values of  $\mu$  as in Fig.1b, showing oscillatory behavior for any  $|\mu| < 2$ , and exponential behavior for  $|\mu| \geq 2$ .

To avoid the need to use an eigenstate of the Hamiltonian at t = 0, one should plot the ratio between the narrow and wide axis of the Wigner distribution of the propagated wavepacket  $\sigma_N$  and  $\sigma_W$ , defined above. This ratio can be measured as follows. At time  $t_n$ , rather than performing a direct measurement, we propose to keep the frequency of the trap unchanged at  $\omega = \omega(t_n)$ and to measure the variance of the position (or momentum) as function of time. The minimum and the maximum of  $m\omega(t_n)\langle x^2(t)\rangle$  are respectively  $\sigma_N^2$  and  $\sigma_W^2$ . This method is perhaps more time-consuming, but guarantees the independence of the result on the initial preparation provided that  $\mu D(0) + L(0)^2/(4H(0)) > 0$ . Note that an initial stationary state (either an eigenstate of the initial Hamiltonian or a thermal state) satisfies this condition.

Concluding remarks – The dramatic effect of EPs of non-Hermitian time-independent Hamiltonian systems on the dynamics is in the focus of recent theoretical and experimental studies in various fields of physics (as for example in optical or microwave experiments where the material has a complex index of refraction). Here we show that the dynamics of a system described by a time-dependent Hermitian Hamiltonian can be strongly affected by a *third-order EP* of an effective timeindependent Hamiltonian. The fact that the dynamics of the Hermitian time-dependent harmonic oscillator can be explained by the existence of an EP at all instants shows the richness of the dynamics of one of the most basic model Hamiltonians, which constitutes a milestone in a large variety of fields in physics. Our finding is both interesting for fundamental theoretical reasons and useful to control the dynamics of a single-ion and of a diluted BEC in a time-dependent traps.

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- T. Kato, Perturbation Theory of Linear Operators (Springer, Berlin, 1966).
- [2] W. D. Heiss and H. L. Harney, Eur. Phys. J. D 17, 149 (2001).
- [3] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007) and references therein.
- [4] N. Moiseyev and S. Friedland, N. Moiseyev and S. Friedland, Phys. Rev. A 22, 618 (1980).
- [5] N. Moiseyev, Non-Hermitian Quantum Mechanics, Cambridge University Press, Cambridge 2011.
- [6] E. Narevicius, P. Serra and N. Moiseyev, Europhys. Lett., 62, 789-794 (2003).
- [7] M.V. Berry and D.H. O'Dell, J. Phys. A 32, 2093 (1998).

- [8] E. Narevicius and N. Moiseyev, Phys. Rev. Lett. 84, 1681 (2000).
- [9] M. V. Berry, J. Mod. Opt. **50**, 63 (2003).
- [10] C. Dembowski et-al Phys. Rev. Lett. 86, 787 (2001).
- [11] B. Dietz  $et\mathchar`-al$  Phys. Rev. Lett.  $106,\,150403$  (2011) and references therein.
- [12] C. E. Rüter *et-al* Nature Physics **6**, 192 (2010).
- [13] S. Rotter et al. Phys. Rev. E 69, 046208 (2004).
- [14] S. Klaiman, U. Günther, and N. Moiseyev, Phys. Rev. Lett. 101, 080402 (2008).
- [15] K. G. Makris *et-al* Int. J. Theor. Phys. **50**, 1019, (2011);
   Phys. Rev. A **81**, 063807, (2010).
- [16] R. Uzdin, A. A. Mailybaev, and N. Moiseyev, J. Phys. A bf 44, 435302 (2011).
- [17] M. V. Berry and R. Uzdin, J. Phys. A 44, 435303 (2011).
- [18] R. Uzdin and N. Moiseyev, Phys. Rev. A 85, 031804(R) (2012).
- [19] O. Latinne *et al* Phys. Rev. Lett. **74**, 46 (1995).
- [20] H. Cartarius, J. Main, and G. Wunner Phys. Rev. Lett. 99, 173003 (2007).
- [21] H. Cartarius and N. Moiseyev, Phys.Rev. A 84, 013419 (2011).
- [22] R. Lefebvre *et al.*, Phys. Rev. Lett. **103**, 123003 (2009).
- [23] O. Atabek *et al.* Phys. Rev. Lett. **106**, 173002 (2011).
- [24] I. Gilary and N. Moiseyev, J. Phys. B 45, 051002 (2012).
- [25] U. Günther and F. Stefani, Czech. J. Phys. 55, 1099 (2005)
- [26] E. M. Graefe et al, J. Phys. A 41, 255206 (2008).

- [27] R. Uzdin and N. Moiseyev, J. Phys. A 45, 444033, (2012).
- [28] R. Uzdin et al. J. Phys. A 45, 415304 (2012).
- [29] Y. Rezek and R. Kosloff, New Journal of Physics 8, 83 (2006).
- [30] Y. Rezek, P. Salomon, K. H. Hoffman and R. Kosloff, Euro. Phys. Lett. 85, 30008 (2009).
- [31] R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications, Dover Books on Mathematics, 2006.
- [32] The Casimir constant is simply the "Interval" in special relativity. See U. Günther and B. Samsonov, Phys. Rev. A 78, 042115, (2008).
- [33] A. Polkovnikov et al., Rev. Mod. Phys. 83, 863 (2011)
- [34] The  $\mathcal{P}$  operator for this Hamiltonian matrix is simply  $\mathcal{P} = diagonalmatrix([1, -1, 1])$  where  $\mathcal{P}^2 = \mathbf{I}$  and  $\mathcal{P}[\mathcal{H}_{Heis}^{Tr=0}]^* \mathcal{H}_{Heis}^{Tr=0}\mathcal{P} = 0.$
- [35] O. Abah *et al.* Phys. Rev. Lett. **109**, 203006 (2012).
- [36] D.S. Jin, Phys. Rev. Lett. **77**, 420 (1996)
- [37] M.-O. Mewes, Phys. Rev. Lett. 77, 988 (1996)
- [38] Y. Castin, R. Dum, Phys. Rev. Lett. 77, 5315 (1996)
- [39] C. Chin et al., Rev. Mod. Phys. 82, 1225 (2010).
- [40] M.R. Andrews, et al., Science 273, 84 (1996)
- [41] C. Bradley, C.A. Sackett, and R.G. Hulet, Phys. Rev. Lett. 78, 985 (1997)
- [42] S. Levy, E. Lahoud, I. Shomroni, and J. Steinhuser, Nature, 449, 579 (2007).
- [43] O. Lahav et al., Phys. Rev. Lett 105, 240401 (2010).
- [44] I. Shammas et al., Phys. Rev. Lett 109, 195301 (2012).