Quantization of electromagnetic field in inhomogeneous dispersive dielectrics

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Abstract

Canonical quantization of electromagnetic field inside the time–spatially dispersive inhomogeneous dielectrics is presented. Interacting electromagnetic and matter excitation fields create the closed system, Hamiltonian of which may be diagonalized by generalized polariton transformation. Resulting dispersion relations coincide with the classical ones obtained by the solution of wave equation, the corresponding mode decomposition is, however, orthogonal and complete in the enlarged Hilbert space.

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I. INTRODUCTION

The investigation of electromagnetic field in dielectrics has attracted growing attention recently [1–5]. Particularly, quantum aspects of this problem are of current interest due to the potential applications in technology of nanostructures. This research includes investigation of quantum wells embedded in microcavities [6] and generation and propagation of nonclassical states of light. In this connection, the quantization of electromagnetic field in inhomogenous time-spatially dispersive linear medium represents a nontrivial problem [7].

Our considerations are motivated by standard electromagnetic theory [8]. Suppose for concreteness the geometry of closed cavity (R=1) with dispersive inhomogeneity (refractive index $n(\Omega)$) along the z–axis as sketched in the Fig. 1. For the sake of simplicity the s–polarization of electric field only will be assumed in the folloving. Using the Maxwell equations, the eigenmodes of the cavity with the time dependance $e^{i\Omega t}$ may be specified as solution of Helmholtz (time independent) wave equation

$$\left[\Delta + \frac{\Omega^2}{c^2}(1+\theta(z)\chi(\Omega))\right]E = 0, \tag{1}$$

susceptibility being $\chi(\Omega) = \chi' - i\chi'' = n^2(\Omega) - 1$. The inhomogeneity is included in the characteristic function

$$\theta(z) = \begin{cases} 1 & \text{for } |z| \leq l/2, \\ 0 & \text{for } |z| > l/2. \end{cases}$$

This equation may be simply solved in the regions where the coefficients are continuous function of z. The electric field E must be continuous together with its first derivation $\frac{\partial E}{\partial z}$ for each $|z| \leq L/2$. The boundary conditions at $\pm L/2$ are given as $E(\pm L/2) = 0$ and they yield the dispersion relation

$$\begin{aligned} Q'_z \tan\left[Q_z \frac{L-l}{2}\right] &= Q_z \cot\left[Q'_z \frac{l}{2}\right], \\ Q'_z \tan\left[Q_z \frac{L-l}{2}\right] &= -Q_z \tan\left[Q'_z \frac{l}{2}\right], \end{aligned} \tag{2}$$

where

$$Q_z^2 = \frac{\Omega^2}{c^2} - q^2 \quad \text{and} \quad Q_z'^2 = n^2(\Omega)\frac{\Omega^2}{c^2} - q^2.$$

Here $\mathbf{q}$ represents the 2D component of wave vector paralel to the boundaries. This transcendent equation may be solved yielding the discrete set of eigenvalues. Nevertheless, a simple analysis shows that the corresponding eigenfunctions $E_{q,m}(z)$ are not orthogonal since the "potential" $\chi(\Omega)$ depends on the frequency. This is the source of theoretical troubles, since the decomposition of electric field is becoming questionable. The quasinormal modes in leaky macrocavity were used in Refs. [9–11].

The purpose of this contribution is to clarify the quantum meaning of this classical electromagnetic problem. Adopting the canonical quantization scheme formulated by Huttner and Barnett [5], we will formulate and solve the problem of quantization of electromagnetic field in closed cavity with dispersive inhomogeneity. Normal modes are associated with (generalized) polariton transformation. Two extreme cases of this formulation may be distinguished as problem of exciton confined in quantum well embedded in microcavity [6] and the above mentioned classical problem of leaky macrocavity. Polariton solution exactly yields the orthogonal decomposition. Additional degrees of freedom are associated with matter excitations. For the sake of simplicity all the quantum considerations will be performed for special form of singular refractive index. In Appendix A this result will be extended to the general form of an arbitrary refractive index fulfilling the Kramers–Kronig relations.

II. CANONICAL QUANTIZATION

Let us formulate canonical description of interaction of transversal electromagnetic field with matter. Neglecting other losses the Lagrangian reads

$$L = \int d^3\mathbf{r} \;\; \mathcal{L}(\mathbf{r}), \tag{3}$$

where Lagrangian density is $\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{mat} + \mathcal{L}_{int}$,

$$\mathcal{L}_{em} = \frac{\epsilon_0}{2}[\dot{\mathbf{A}}^2 - c^2(\nabla \times \mathbf{A})^2], \tag{4}$$

$$\mathcal{L}_{mat} = \frac{\rho}{2}\,[\dot{\mathbf{X}}^2 - \omega_0^2\mathbf{X}^2], \tag{5}$$

$$\mathcal{L}_{int} = -\alpha \mathbf{A} \cdot \dot{\mathbf{X}}, \tag{6}$$

boldface characters denote vectors and dot means time derivation $\frac{\partial}{\partial t}$. Electromagnetic part is represented by vector field $\mathbf{A}$ defined in the whole cavity. Polarisation part is modeled by harmonic oscillator field with amplitude vector $\mathbf{X}$, which is non–zero only in the interval of inhomogeneity $|z| \leq l/2$, ω_0 being the frequency and ρ density. Interaction of both fields is characterized by interaction constant α. For the simplicity, linearly polarized fields with polarization paralel to discontinuity planes $z = \pm l/2$ will be assumed (s–polarization) in the following. The vector field $\mathbf{A}$ may be interpreted as electric intensity and both fields may be represented as (real) scalar fields. The Lagrange-Euler equations then read

$$\epsilon_0 \Big[\frac{\partial^2}{\partial t^2} - c^2 \Delta\Big] A + \alpha \frac{\partial}{\partial t} X = 0, \tag{7}$$

$$\rho \Big[\frac{\partial^2}{\partial t^2} + \omega_0^2\Big] X - \alpha \frac{\partial}{\partial t} A = 0 \tag{8}$$

yielding exactly the wave equation (1) for susceptibility

$$\chi = \frac{\alpha^2}{\epsilon_0 \rho (\omega_0^2 - \Omega^2)}. \tag{9}$$

A. Free field decompositions

The interacting fields may be quantized using expansion in orthogonal basis relevant to respective free fields. Standard approach [5] may be used in the plane xy perpendicular to the direction of inhomogeneity, since $\mathbf{q}$ is conserved due to the translation symmetry. In the z–direction, the eigenfunction for light and matter excitation parts should be distinguished. Assuming $e^{i\mathbf{q}\tau}/(2\pi)$ dependence, τ being the projection of 3D $\mathbf{r}$ vector into the xy–plane, $\{\varphi_m(z)\}$ are the solutions of time–independent wave equation

$$\Big[\frac{d^2}{dz^2} + Q_m^2\Big] \varphi_m(z) = 0; \quad Q_m^2 = \frac{\Omega_m^2}{c^2} - q^2, \tag{10}$$

fulfilling the given boundary conditions on the interval $|z| \leq L/2$. Assuming for concreteness perfect reflection on the end mirrors, we have $Q_m = \pi m/L; \quad m = 1, 2, 3, ...$ Consequently frequencies are quantized as $\Omega_m(q) = c\sqrt{(\pi m/L)^2 + q^2},\ q = |\mathbf{q}|$. Corresponding eigenfunctions are given as $\varphi_m(z) = \sqrt{2}\sin[m\pi(z/L + 1/2)]$. The inhomogeneity of matter excitations is included in the definition of eigenfunctions $\chi_{\mathbf{q},\xi}(\mathbf{r})$, since they are non–zero only on the interval $|z| \leq l/2$. Two sets of functions $\varphi_{\mathbf{q},m}(\mathbf{r})$ and $\chi_{\mathbf{q},\xi}(\mathbf{r})$ defined in the 3D space are orthogonal and complete in the volumes of quantization

$$\int d^3\mathbf{r}\ \varphi_{\mathbf{q},m}(\mathbf{r})\varphi^*_{\mathbf{q}',n}(\mathbf{r}) = \delta(\mathbf{q}-\mathbf{q}')\delta_{m,n}, \quad \sum_m \int d^2\mathbf{q}\ \varphi_{\mathbf{q},m}(\mathbf{r})\varphi^*_{\mathbf{q},m}(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),$$
$$\int d^3\mathbf{r}\ \chi_{\mathbf{q},\xi}(\mathbf{r})\chi^*_{\mathbf{q},\eta}(\mathbf{r}) = \delta(\mathbf{q}-\mathbf{q}')\delta_{\xi,\eta}, \quad \sum_\xi \int d^2\mathbf{q}\ \chi_{\mathbf{q},\xi}(\mathbf{r})\chi^*_{\mathbf{q},\xi}(\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')\theta(z).$$

The cross–products are given by matrix elements

$$\int \varphi_{\mathbf{q},m}(\mathbf{r})\chi^*_{\mathbf{q}',\xi}(\mathbf{r})d^3\mathbf{r} = \delta(\mathbf{q}-\mathbf{q}')\ K_{m,\xi},$$
$$K_{m,\xi} = \int_{-l/2}^{l/2} \varphi_m(z)\chi_\xi(z)dz, \quad K_{m,\xi} = K^*_{m,\xi}. \tag{11}$$

Here the full set of functions $\{\chi_\xi(z)\}$ create an orthogonal and complete system of functions on the interval $|z| \leq l/2$. The explicit form of correct exciton wave functions $\chi_{\mathbf{q},\xi}(z)$ represents very complex problem, solution of which is beyond the scope of this contribution. The basis used in decomposition of matter excitations will be considered as given and index ξ will be intuitively interpreted as energy levels of free matter excitations. In the following the wave functions of electromagnetic field will be consistently enumerated by Latin indices, whereas Greek ones will be used for the matter excitations.

Even if we started with the Lagrangian of harmonic oscillator field (5), we will incorporate into our scheme also a little more general models corresponding to the finite number of energy levels ξ in the decomposition of harmonic oscillator. This technique corresponds to quantization with confinements tending to spatial dispersion. Particularly, the matter excitations in the ground state only correspond to the case of single exciton confined in quantum well [6]. For this purpose, the range of sumation in respective decompositions of matter excitations will not be specified explicitly and will be mentioned in the discussion of the final results only. The respective expansions used in the following then read

$$A(\mathbf{r},t) = \frac{1}{2\pi}\sum_m \int d^2\mathbf{q}\; A_{\mathbf{q},m}(t)\; \varphi_{\mathbf{q},m}(\mathbf{r}), \tag{12}$$

$$X(\mathbf{r},t) = \frac{1}{2\pi}\sum_\xi \int d^2\mathbf{q}\; X_{\mathbf{q},\xi}(t)\; \chi_{\mathbf{q},\xi}(\mathbf{r}). \tag{13}$$

B. Hamiltonian formalism

The total Lagrangian $L = L_{em} + L_{mat} + L_{int}$ may be quantized as

$$L_{em} = \epsilon_0 \sum_m \int' d^2\mathbf{q}\Big[|\dot{A}_{\mathbf{q},m}|^2 - \Omega_m^2(q)|A_{\mathbf{q},m}|^2\Big], \tag{14}$$

$$L_{mat} = \rho \sum_\xi \int' d^2\mathbf{q}\Big[|\dot{X}_{\mathbf{q},\xi}|^2 - \omega_0^2|X_{\mathbf{q},\xi}|^2\Big], \tag{15}$$

$$L_{int} = -\alpha \sum_{m,\xi} \int' d^2\mathbf{q}\; K_{m,\xi}\Big[A_{\mathbf{q},m}\cdot \dot{X}^*_{\mathbf{q},\xi} + c.c.\Big]. \tag{16}$$

Here the prime means the integration over the half of the reciprocal space. Canonically conjugated variables are given as

$$A^*_{\mathbf{q},m} \to P_{\mathbf{q},\mathbf{m}} = \frac{\partial L}{\partial \dot{A}^*_{\mathbf{q},m}} = \epsilon_0 \dot{A}_{\mathbf{q},m}, \tag{17}$$

$$X^*_{\mathbf{q},\xi} \to Y_{\mathbf{q},\xi} = \frac{\partial L}{\partial \dot{X}^*_{\mathbf{q},\xi}} = \rho \dot{X}_{\mathbf{q},\xi} - \alpha \sum_n K_{n,\xi} A_{\mathbf{q},n}, \tag{18}$$

and as the complex conjugated relations. Hamiltonian reads

$$H = \epsilon_0 \sum_m \int' d^2\mathbf{q}\; \Big[|\dot{A}_{\mathbf{q},m}|^2 + \Omega_m^2(q)|A_{\mathbf{q},m}|^2\Big] + \rho \sum_\xi \int' d^2\mathbf{q}\; \Big[|\dot{X}_{\mathbf{q},\xi}|^2 + \omega_0^2|X_{\mathbf{q},\xi}|^2\Big]. \tag{19}$$

Standard quantization is prescribed by commutation relations between operators

$$\Big[A_{\mathbf{q},m}, P^*_{\mathbf{q}',n}\Big] = i\hbar\delta_{m,n}\; \delta(\mathbf{q}-\mathbf{q}'), \tag{20}$$

$$\Big[X_{\mathbf{q},\xi}, Y^*_{\mathbf{q}',\eta}\Big] = i\hbar\delta_{\xi,\eta}\; \delta(\mathbf{q}-\mathbf{q}'). \tag{21}$$

Annihilation operators of electromagnetic field

$$a_{\mathbf{q},m} = \sqrt{\frac{\epsilon_0}{2\hbar\Omega_m(q)}}\Big[\Omega_m(q)A_{\mathbf{q},m} + \frac{i}{\epsilon_0}P_{\mathbf{q},m}\Big] \tag{22}$$

and matter excitations

$$b_{\mathbf{q},\xi} = \sqrt{\frac{\rho}{2\hbar\omega_0}}\Big[\omega_0 X_{\mathbf{q},\xi} + \frac{i}{\rho}Y_{\mathbf{q},\xi}\Big] \tag{23}$$

are fulfilling the ordinary boson commutation relations

$$\Big[a_{\mathbf{q},m}, a^{\dagger}_{\mathbf{q}',n}\Big] = \delta_{m,n}\delta(\mathbf{q}-\mathbf{q}'), \tag{24}$$

$$\Big[b_{\mathbf{q},\xi}, b^{\dagger}_{\mathbf{q}',\eta}\Big] = \delta_{\xi,\eta}\delta(\mathbf{q}-\mathbf{q}'). \tag{25}$$

The definition may be extended to full space of $\mathbf{q}$ vectors as

$$a_{-\mathbf{q},m} = \sqrt{\frac{\epsilon_0}{2\hbar\Omega_m(q)}}\Big[\Omega_m(q)A^{*}_{\mathbf{q},m} + \frac{i}{\epsilon_0}P^{*}_{\mathbf{q},m}\Big], \tag{26}$$

and

$$b_{-\mathbf{q},\xi} = \sqrt{\frac{\rho}{2\hbar\omega_0}}\Big[\omega_0 X^{*}_{\mathbf{q},\xi} + \frac{i}{\rho}Y^{*}_{\mathbf{q},\xi}\Big]. \tag{27}$$

Hamiltonian then reads

$$\begin{aligned} H = {} & \sum_m \int d^2\mathbf{q}\ \ \hbar\Omega_m(q)\ a^{\dagger}_{\mathbf{q},m}a_{\mathbf{q},m} + \sum_\xi \int d^2\mathbf{q}\ \ \hbar\omega_0\ \ b^{\dagger}_{\mathbf{q},\xi}b_{\mathbf{q},\xi} + \\ & + \frac{i\hbar}{2}G\sqrt{\omega_0}\sum_{n,\xi}\int d^2\mathbf{q}\ \ \frac{K_{n,\xi}}{\sqrt{\Omega_n(q)}}(a_{\mathbf{q},n} + a^{\dagger}_{-\mathbf{q},n})\cdot(b^{\dagger}_{\mathbf{q},\xi} - b_{-\mathbf{q},\xi}) \\ & + \frac{\hbar}{4}G^2\ \sum_{n,m}\int d^2\mathbf{q}\frac{D_{n,m}}{\sqrt{\Omega_m(q)\Omega_n(q)}}(a_{\mathbf{q},n} + a^{\dagger}_{-\mathbf{q},n})\cdot(a_{\mathbf{q},m} + a^{\dagger}_{-\mathbf{q},m}), \end{aligned} \tag{28}$$

where

$$D_{n,m} = \sum_\xi K_{n,\xi}K_{m,\xi} \tag{29}$$

and the effective interaction constant is abbreviated as

$$G = \alpha/\sqrt{\epsilon_0\rho}. \tag{30}$$

The Hamiltonian of this type has already been investigated recently in connection with polariton effects [6] and exactly corresponds to the many–exciton lines interacting with eletromagnetic field discussed in Ref. [12].

III. DISPERSION RELATIONS

General form of polariton transformation diagonalizing the Hamiltonian is given as

$$B_{\mathbf{q},\Omega} = \sum_m W_{\mathbf{q},m} a_{\mathbf{q},m} + \sum_\xi X_{\mathbf{q},\xi} b_{\mathbf{q},\xi} + \sum_m Y_{\mathbf{q},m} a^\dagger_{-\mathbf{q},m} + \sum_\xi Z_{\mathbf{q},\xi} b^\dagger_{-\mathbf{q},\xi}. \tag{31}$$

Index m exhausts all the cavity modes and ξ similarly does all the modes of decomposition of matter excitations. Standard diagonalization condition

$$\Big[B_{\mathbf{q},\Omega}, H\Big] = \hbar\Omega B_{\mathbf{q},\Omega} \tag{32}$$

yields the dispersion relation for eigenfrequency Ω and relations for coefficients in (31). The anticipated operator solution is normalized with respect to the boson commutation relation

$$[B_{\mathbf{q},\Omega}, B^\dagger_{\mathbf{q}',\Omega'}] = \delta(\mathbf{q} - \mathbf{q}')\delta_{\Omega,\Omega'}. \tag{33}$$

Straightforward but lengthy calculations lead to the following equations (dependence on $\mathbf{q}$ will be omitted for brevity)

$$(\Omega_m - \Omega)W_m + \frac{1}{2}G^2 \sum_k \frac{D_{m,k}}{\sqrt{\Omega_m \Omega_k}}(W_k - Y_k) + \frac{i}{2}G\frac{\sqrt{\omega_0}}{\sqrt{\Omega_m}} \sum_\xi K_{m,\xi}(X_\xi + Z_\xi) = 0, \tag{34}$$

$$(\Omega_m + \Omega)Y_m - \frac{1}{2}G^2 \sum_k \frac{D_{m,k}}{\sqrt{\Omega_m \Omega_k}}(W_k - Y_k) - \frac{i}{2}G\frac{\sqrt{\omega_0}}{\sqrt{\Omega_m}} \sum_\xi K_{m,\xi}(X_\xi + Z_\xi) = 0, \tag{35}$$

$$(\Omega - \omega_0)X_\xi + \frac{i}{2}G\sqrt{\omega_0} \sum_m \frac{K_{m,\xi}}{\sqrt{\Omega_m}}(W_m - Y_m) = 0, \tag{36}$$

$$(\Omega + \omega_0)Z_\xi - \frac{i}{2}G\sqrt{\omega_0} \sum_m \frac{K_{m,\xi}}{\sqrt{\Omega_m}}(W_m - Y_m) = 0. \tag{37}$$

Finally, the dispersion relation follows as the condition for existence of non–trivial solution of linear equations for T_m

$$T_m = \frac{G^2\Omega^2}{(\omega_0^2 - \Omega^2)(\Omega_m^2 - \Omega^2)} \sum_n D_{m,n} T_n, \tag{38}$$

where $T_m = (W_m - Y_m)/\sqrt{\Omega_m}$.

The algebraic treatment may be equivalently replaced by an analytical method. The recurent system of linear equations (38) may be rewritten to the form of integro–differential equation. Hence the quantum problem is related to the classical solution of wave equation with dispersive inhomogeneous medium. Let us define formally the function

$$A(\mathbf{r}) = \sum_m \int d^2\mathbf{q}\, T_m\, \varphi_{\mathbf{q},m}(\mathbf{r}),$$

which is continuous and has continuous derivation dA/dz on the interval $|z| \leq L/2$. After simple manipulations using relations (10), (11) and (29) we find that it fulfills the equation

$$\left[\frac{d^2}{dz^2} + Q_z^2\right] A(z) = -\theta(z)\frac{G^2\Omega^2}{c^2(\omega_0^2 - \Omega^2)} \int_{-l/2}^{l/2} \sum_\xi \chi_\xi(z)\chi_\xi(z')A(z')\, dz'\ , \tag{39}$$

where $Q_z^2 = (\Omega/c)^2 - q^2$. This alternative representation of (38) may be interpreted as scalar wave equation with the time–spatially dispersive inhomogeneity along the z–axis. Consequently, we are able to associate the quantum problem–diagonalization of the Hamiltonian (28), with the classical solution in electromagnetic theory. This analogy represents a powerful tool for specification of dispersion relations [13]. The wave equation (39) may be solved separately in the regions where all the functions are continuous. Dispersion relation is then given as necessary condition for continuity of the solution and its first derivation on the boundary $|z| = l/2$. The general solution of eq. (39) on the interval $|z| \leq l/2$ is given as superposition of particular and fundamental solutions

$$A(z) = \sum_\xi c_\xi \int dz' G(z,z')\chi_\xi(z') + A_2\, e^{iQ_z z} + B_2\, e^{-iQ_z z}, \tag{40}$$

A_2 and B_2 being general multiplicators in fundamental solution, $G(z,z')$ being the Green function of the operator $d^2/dz^2 + Q_z^2$. We may take the explicit form

$$G(z,z') = -\frac{1}{2Q_z}\sin(Q_z|z - z'|).$$

Coefficients c_ξ of particular solution are given in accordance with the relations

$$c_\xi \equiv -\frac{G^2\Omega^2}{c^2(\omega_0^2 - \Omega^2)} \int dz\chi_\xi(z)A(z) = -\frac{G^2\Omega^2}{c^2(\omega_0^2 - \Omega^2)}\Big\{\sum_\eta c_\eta \int dz\, dz'\chi_\xi(z)G(z,z')\chi_\eta(z')$$
$$+A_2 \int dz'\ \chi_\xi(z')e^{iQ_z z'} + B_2 \int dz'\ \chi_\xi(z')e^{-iQ_z z'}\Big\}. \tag{41}$$

Parametrizing the solutions on the intervals $L/2 \leq |z| \leq l/2$ as $A_j e^{iQz} + B_j e^{-iQz};\ \ j = 1, 3$ the dispersion relation may be found in closed form for limited number of terms ξ for any mode decomposition $\chi_\xi(z)$.

Algebraic method will be used in the following sections III A and III B, whereas the analytical method will be applied on the analogy of classical problem in the section III C.

A. One–exciton dispersion relation

Closed form of dispersion relation may be find easily in some special cases. Particularly, exciton confined in quantum well (QW) is characterized by the only energy level in the expansion of matter excitations $\xi = 0$ (ground state of excitations). The integro–differential equation (39) indicates the spatial dispersion. Nevertheless, the dispersion relation may be found without solving it, since the coefficients $D_{m,k}$ are factorized as product $D_{n,m} = K_{n,0} \cdot K_{m,0}$. Equations (38) then directly yield the necessary condition

$$\frac{G^2\Omega^2}{\omega_0^2 - \Omega^2} \sum_n \frac{K_{m,0}^2}{\Omega_m^2 - \Omega^2} = 1, \tag{42}$$

representing the dispersion relation of QW polaritons embedded in microcavity [6].

B. Many–exciton dispersion relations

Dispersion relation for many excitons interacting with electromagnetic field will be demonstrated on the explicit example of two excitons ($\xi = 0, 1$). Taking into account the form of the kernel $D_{m,n} = K_{m,0}K_{n,0} + K_{m,1}K_{n,1}$, the system of linear equations (38) may be rewritten as

$$\begin{aligned} x_0 &= x_0 \frac{G^2\Omega^2}{\omega_0^2 - \Omega^2} \sum_m \frac{K_{m,0}^2}{\Omega_m^2 - \Omega^2} + x_1 \frac{G^2\Omega^2}{\omega_0^2 - \Omega^2} \sum_m \frac{K_{m,0}K_{m,1}}{\Omega_m^2 - \Omega^2} \\ x_1 &= x_0 \frac{G^2\Omega^2}{\omega_0^2 - \Omega^2} \sum_m \frac{K_{m,0}K_{m,1}}{\Omega_m^2 - \Omega^2} + x_1 \frac{G^2\Omega^2}{\omega_0^2 - \Omega^2} \sum_m \frac{K_{m,1}^2}{\Omega_m^2 - \Omega^2}, \end{aligned} \tag{43}$$

where $x_\xi = \sum_m T_m K_{m,\xi}; \xi = 0, 1$. Dispersion relation for two exciton then reads

$$\left[1-\frac{G^2\Omega^2}{\omega_0^2-\Omega^2}\sum_m\frac{K_{m,0}^2}{\Omega_m^2-\Omega^2}\right]\left[1-\frac{G^2\Omega^2}{\omega_0^2-\Omega^2}\sum_m\frac{K_{m,1}^2}{\Omega_m^2-\Omega^2}\right]=\left[\frac{G^2\Omega^2}{\omega_0^2-\Omega^2}\sum_m\frac{K_{m,0}K_{m,1}}{\Omega_m^2-\Omega^2}\right]^2. \quad (44)$$

The sums involved in the dispersion relation may further be expressed in the analytical form as done in Ref. [6], but this is beyond the scope of this paper. Let us only note that the application of analytical method yields directly the closed form of dispersion relations.

C. Classical dispersion relations

As the last special example, the dispersion relations resulting from the classical electrodynamics in dispersive inhomogeneity (1) will be considered from the quantum viewpoint. This case is characterized by the decomposition of matter excitations, which is complete on the interval of inhomogenity (i.e. all the energy lines of excitations are included). Relation (29) then reads

$$D_{n,m}=\sum_\xi\int_{-l/2}^{l/2}\varphi_n(z)\chi_\xi(z)dz\cdot\int_{-l/2}^{l/2}\varphi_m(z')\chi_\xi(z')dz'=\int_{-l/2}^{l/2}\varphi_n(z)\varphi_m(z)dz. \quad (45)$$

Spatial dispersion in (39) disappears yielding scalar wave equation identical with Lagrange–Euler equation (1) for the susceptibility (9) and for the dispersion relation (2). Consequently, the classical solution yielding non–orthogonal eigenmode functions was completed by the fully quantum treatment characterized by the same dispersion relations but orthogonal decomposition.

IV. CONCLUSION

The problem of canonical quantization of electromagnetic field in linear dispersive inhomogeneous media was formulated in terms of overlaping of wave functions related to quantization of free electromagnetic and matter excitation fields. Diagonalization is given by generalized polariton (Hopfield) transformation. Within the classical electrodynamics, it may be also interpreted as the wave in spatially dispersive medium. The description of macroscopic inhomogeneity and quantum well polaritons may be unified in this way into the same framework.

Even if quantum and classical problems yield the same dispersion relations, there is a difference between both the treatments. Since the electromagnetic field itself is not conserved, the respective eigenfunctions are not orthogonal representing the system of quasinormal modes [9–11]. They may be used for description of electromagnetic field inside the cavity, however since completeness and orthogonality relations should be redefined, the description is more complicated than in the ordinary case of orthogonal modes. On the other hand the quantum solution is characterized by an ordinary orthogonal diagonalization in the form of generalized polariton transformation acting on the space of coupled electromagnetic and matter excitation fields. Standard description of the time evolution may be used, since the normal modes are orthogonal and complete.

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APPENDIX A: INHOMOGENEITY WITH AN ARBITRARY REFRACTIVE INDEX

The theory developed above describes the lossless dispersive inhomogeneity characterized by the real part of the susceptibility (9)

$$\chi'(\Omega) = \frac{G^2}{\omega_0^2 - \Omega^2}. \tag{A1}$$

Principle of superposition may be used now to get an arbitrary susceptibility as done in Ref. [14]. The electromagnetic field will interact with an ansamble of independent oscillators with Lagrangian densities as in (5,6), distinguished by frequencies $\omega_0, \omega_1, ...$ etc. and by different parameters α_0, α_1, etc. and ρ_0, ρ_1, etc.. In the limit of continuous frequency distribution ω, the parameters are considered as frequency dependent $\alpha(\omega), \rho(\omega)$ yielding the total contribution to the Lagrangian density

$$\mathcal{L}_{mat} = \int_0^\infty d\omega \frac{\rho(\omega)}{2} [\dot{\mathbf{X}}_\omega^2 - \omega^2 \mathbf{X}_\omega^2], \tag{A2}$$

$$\mathcal{L}_{int} = -\int_0^\infty d\omega \alpha(\omega) \mathbf{A} \cdot \dot{\mathbf{X}}_\omega. \tag{A3}$$

Further development of the theory runs similarly as in the single–frequency case: Matter operators are denoted by modal index ξ and by an additional continuous index ω. The effective interaction parameter (30) is freqency dependent $G(\omega)$. The final system of linear equations (38) reads

$$T_m = \frac{\Omega^2}{\Omega_m^2 - \Omega^2} \int_0^\infty \frac{G^2(\omega)}{\omega^2 - \Omega^2}\, d\omega\, \sum_n D_{m,n} T_n, \tag{A4}$$

what corresponds to the real part of the susceptibility

$$\chi'(\Omega) = \int_0^\infty d\omega \frac{G^2(\omega)}{\omega^2 - \Omega^2}. \tag{A5}$$

Due to the Kramers–Kronig relations [14], the real and imaginary part of susceptibility are mutually related by Hilbert transformation as

$$\chi'(\Omega) = \frac{2}{\pi} \int_0^\infty \frac{\omega \chi''(\omega)}{\omega^2 - \Omega^2} d\omega, \tag{A6}$$

$$\chi''(\Omega) = -\frac{2\Omega}{\pi} \int_0^\infty \frac{\chi'(\omega)}{\omega^2 - \Omega^2} d\omega. \tag{A7}$$

The imaginary part of lossless and general cases therefore read

$$\chi''(\Omega) = -\pi G^2 \delta(\Omega^2 - \omega_0^2) \tag{A8}$$

and

$$\chi''(\Omega) = -\pi \int_0^\infty d\omega G^2(\omega) \delta(\omega^2 - \Omega^2), \tag{A9}$$

respectively. The relations (A5) and (A9) represent the desired extension of the theory with single–frequency of matter excitations and singular susceptibility into an arbitrary case characterized by general refractive index (susceptibility).

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FIGURES

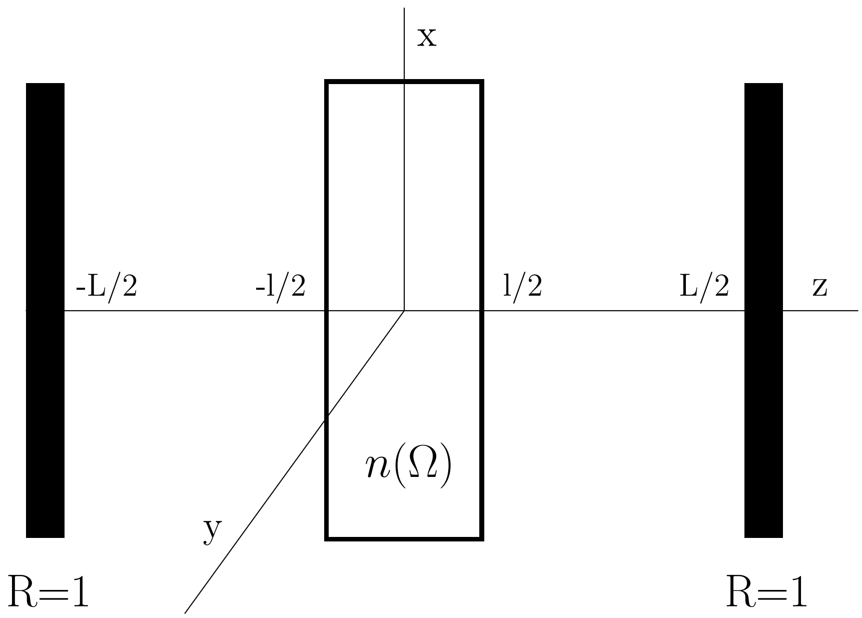

FIG. 1. Geometry of closed cavity with inhomogeneity in the z–direction.