

Can bilayer graphene become a fractional metal?

A.O. Sboychakov,¹ A.L. Rakhmanov,¹ A.V. Rozhkov,¹ and Franco Nori^{2,3}

¹*Institute for Theoretical and Applied Electrodynamics,
Russian Academy of Sciences, 125412 Moscow, Russia*

²*Advanced Science Institute, RIKEN, Wako-shi, Saitama, 351-0198, Japan*

³*Department of Physics, University of Michigan, Ann Arbor, MI 48109-1040, USA*

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It is known that electron interactions can cause a perfect spin polarization of the Fermi surface of a metal. In such a situation only half of the non-interacting Fermi surface is available, and thus this phase is commonly referred to as a ‘half-metal’. Here we argue that, in multi-band electronic systems with nesting, further ‘fractionalization’ of the Fermi surface is possible. Taking the AA bilayer graphene as a convenient test case, we demonstrate that, under suitable conditions imposed on the electron interactions, doped AA bilayer graphene can host a ‘quarter-metal’ state. In such a state, only one quarter of the non-interacting Fermi surface (Fermi contour) reaches the Fermi energy. At higher doping level, other ‘fractional’ metals can emerge. We briefly analyze the transport properties of these proposed phases.

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Introduction.— In usual metals, the total spin polarization of the charge carriers at the Fermi surface is zero. A strong electron-electron interaction can lift the spin degeneracy, and induce spin polarization of the states at the Fermi surface. In the extreme case of the so-called half-metals [1–3], this polarization is perfect: all states at the Fermi energy have identical spin projection. Indeed, various rather different systems with transition-metal atoms are found to be half-metals [4–7]. The existence of spin-polarized currents in these half-metals makes them promising materials for applications in spintronics [3, 8]. Several papers [9–13] predict half-metallicity in carbon-based systems. The half-metals free of heavy atoms could be of interest for bio-compatible applications and carbon-based electronics [14–19].

In previous works [20, 21] we have proposed a mechanism for half-metallicity in electronic systems with weak interactions. This requires the existence of two Fermi surface sheets with nesting between them. These sheets are referred below as having ‘electron/hole charge flavors’ [22]. When doped, the spin-density wave (SDW) or charge-density wave (CDW) insulator state in such a model is replaced by this type of half-metallic state.

In a multi-band system with nesting, besides spin, an additional discrete quantum number ξ emerges, enumerating pairs of nested Fermi surface sheets. In such a situation, one may wonder if a many-body state with an additional polarization with respect to ξ could be realized. The stability of this peculiar conducting state, which we call below “fractional metal” (FraM), is the main topic of this paper.

It follows from the FraM definition that only a material with sufficiently complex multi-sheet Fermi surface with nesting might host a FraM phase. This requirement makes AA bilayer graphene (AA-BLG) a promising candidate to be a FraM. The AA-BLG is less studied than

the Bernal stacked (AB) bilayer. Yet, AA-BLG samples have been manufactured [23–26]. Moreover, progress in van der Waals heterostructures fabrication [27] allows one to hope that more efforts will be undertaken in the direction of producing high-quality AA-BLG samples. As in other graphene structures, the low-energy states of the AA-BLG can be classified by their proximity to either the \mathbf{K}_1 or \mathbf{K}_2 Dirac point. A given Dirac point is encircled by an electron Fermi surface sheet and a hole sheet; altogether there are four Fermi surface sheets in the whole Brillouin zone. We argue that, for such a degenerate Fermi surface structure and under rather common assumptions about the electron-electron coupling, doped AA-BLG could enter the FraM phase. We investigate the stability of this phase and also briefly discuss its most immediate properties, such as transport of spin and valley quanta, and peculiar features of superconductivity.

Model.— The electronic properties of AA-BLG are described by the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$, where \hat{H}_0 is the single-electron part and \hat{H}_{int} corresponds to the interaction between quasiparticles. For AA-BLG [19]:

$$\hat{H}_0 = -t \sum_{\langle \mathbf{mn} \rangle l \sigma} d_{\mathbf{m}l0\sigma}^\dagger d_{\mathbf{n}l1\sigma} - t_0 \sum_{\mathbf{n}a\sigma} d_{\mathbf{n}0a\sigma}^\dagger d_{\mathbf{n}1a\sigma} + \text{H.c.} - \mu n. \quad (1)$$

Here $d_{\mathbf{m}l a \sigma}^\dagger$ ($d_{\mathbf{m}l a \sigma}$) is the creation (annihilation) operator of an electron with spin projection σ in layer l [$l = 0$ ($l = 1$) corresponds to upper (lower) layer] on sublattice a [$a = 0$ ($a = 1$) represents sublattice A (B)] at the position \mathbf{m} . Also, $n = \sum_{\mathbf{n}l a \sigma} d_{\mathbf{n}l a \sigma}^\dagger d_{\mathbf{n}l a \sigma}$ is the total charge density, μ is the chemical potential, and $\langle \dots \rangle$ denotes nearest-neighbor pairs. The amplitude $t = 2.57$ eV ($t_0 = 0.36$ eV) describes the in-plane (inter-plane) nearest-neighbor hopping. \hat{H}_0 can be readily diagonalized in a new basis $\gamma_{\mathbf{k}\ell\sigma}$ ($\ell = 1, \dots, 4$):

$$\hat{H}_0 = \sum_{\mathbf{k}\ell\sigma} \left(\varepsilon_{0\mathbf{k}}^{(\ell)} - \mu \right) \gamma_{\mathbf{k}\ell\sigma}^\dagger \gamma_{\mathbf{k}\ell\sigma}, \quad (2)$$

where \mathbf{k} is the momentum; the eigenenergies and eigenoperators are

$$\begin{aligned}\varepsilon_{0\mathbf{k}}^{(1)} &= -t_0 - t\zeta_{\mathbf{k}}, & \varepsilon_{0\mathbf{k}}^{(2)} &= -t_0 + t\zeta_{\mathbf{k}}, \\ \varepsilon_{0\mathbf{k}}^{(3)} &= +t_0 - t\zeta_{\mathbf{k}}, & \varepsilon_{0\mathbf{k}}^{(4)} &= +t_0 + t\zeta_{\mathbf{k}},\end{aligned}\quad (3)$$

$$d_{\mathbf{k}l\alpha\sigma} = \exp(-ai\varphi_{\mathbf{k}})[\gamma_{\mathbf{k}1\sigma} + (-1)^a\gamma_{\mathbf{k}2\sigma} + (-1)^l\gamma_{\mathbf{k}3\sigma} + (-1)^{a+l}\gamma_{\mathbf{k}4\sigma}]/2. \quad (4)$$

In Eq. (4), $\varphi_{\mathbf{k}} = \arg(f_{\mathbf{k}})$, $\zeta_{\mathbf{k}} = |f_{\mathbf{k}}|$, where $f_{\mathbf{k}} = 1 + 2\exp(3ik_x a_0/2)\cos(\sqrt{3}k_y a_0/2)$, and $a_0 = 1.42 \text{ \AA}$ is the in-plane carbon-carbon distance. The band $\ell = 2$ (band $\ell = 3$) crosses the Fermi level and forms two electron (two hole) Fermi surface sheets, one centered at the Dirac point $\mathbf{K}_1 = 2\pi(\sqrt{3}, 1)/3\sqrt{3}a_0$, and another at $\mathbf{K}_2 = 2\pi(\sqrt{3}, -1)/3\sqrt{3}a_0$. To distinguish electron and hole Fermi surface sheets, we introduce the charge flavor index $\nu = (-1)^\ell$: it equals $\nu = 1$ ($\nu = -1$) for electrons (holes). If we label [22] the graphene valley \mathbf{K}_1 (valley \mathbf{K}_2) by $\xi = +1$ (by $\xi = -1$), any sheet can be uniquely identified by values of ν and ξ . Since all sheets are circles of identical radius $k_{F0} = 2t_0/3ta_0$, we have two nesting vectors: $\mathbf{0}$ and $\mathbf{Q}_0 = \mathbf{K}_1 - \mathbf{K}_2$.

The Coulomb interaction between electrons is

$$\hat{H}_{\text{int}} = \frac{1}{2N_c} \sum_{\substack{\mathbf{k}\mathbf{k}'\mathbf{q}l\alpha \\ l'\alpha'\sigma\sigma'}} V_{aa'}^{ll'}(\mathbf{q}) d_{\mathbf{k}l\alpha\sigma}^\dagger d_{\mathbf{k}+\mathbf{q}l\alpha\sigma} d_{\mathbf{k}'l'\alpha'\sigma'}^\dagger d_{\mathbf{k}'-\mathbf{q}l'\alpha'\sigma'}, \quad (5)$$

where N_c is the number of elementary cells in the sample and $V_{aa'}^{ll'}(\mathbf{q})$ is the Fourier transform of

$$V_{aa'}^{ll'}(\mathbf{r}) = V_C \left(\sqrt{[\mathbf{r} + (a - a')\boldsymbol{\delta}_1]^2 + (l - l')^2 D^2} \right). \quad (6)$$

Here, $V_C(|\mathbf{r}|)$ is the screened Coulomb potential, $\boldsymbol{\delta}_1 = (a_0, 0)$, and $D = 3.3 \text{ \AA}$ is the inter-layer distance. The dependence of the interaction on various indices accounts for different distances between electrons at different sublattices and/or layers.

Mean field approach.— Theory predicts [28–31] that the electron repulsion converts the electronic “liquid” of the AA-BLG into a SDW insulator. The SDW order is characterized by non-zero values of $\langle \gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle$ and $\langle \gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}} \rangle$, which describe excitonic pairs with vanishing total momentum. It is possible to define a different order parameter oscillating in space with the wave vector \mathbf{Q}_0 , e.g., $\langle \gamma_{\mathbf{k}+\mathbf{Q}_0,2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle$. However, the oscillating order parameter has lower coupling constant, because it cannot interact with another oscillating order parameter unless they have opposite momenta. This condition strongly reduces the effective coupling constant. As a result, such a phase has higher energy, and we will not consider it here. Switching to band operators γ and neglecting the terms irrelevant to the mean field approximation, we transform Eq. (5) and write

$$\hat{H}_{\text{int}} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \hat{H}^{(4)}, \quad (7)$$

where

$$\begin{aligned}\hat{H}^{(1)} &= -\frac{1}{N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k}\mathbf{p}}^{(1)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}})(\gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma}) \right. \\ &\quad \left. + (\gamma_{\mathbf{k}3\bar{\sigma}}^\dagger \gamma_{\mathbf{k}2\sigma})(\gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}}) \right],\end{aligned}\quad (8)$$

$$\begin{aligned}\hat{H}^{(2)} &= -\frac{1}{2N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k}\mathbf{p}}^{(2)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}})(\gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma}) \right. \\ &\quad \left. + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}})(\gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma}) + \text{H.c.} \right],\end{aligned}\quad (9)$$

$$\begin{aligned}\hat{H}^{(3)} &= -\frac{1}{N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k}\mathbf{p}}^{(3)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}})(\gamma_{\mathbf{p}3\bar{\sigma}}^\dagger \gamma_{\mathbf{p}2\sigma}) \right. \\ &\quad \left. + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}})(\gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma}) \right],\end{aligned}\quad (10)$$

$$\begin{aligned}\hat{H}^{(4)} &= -\frac{1}{2N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k}\mathbf{p}}^{(4)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}})(\gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma}) \right. \\ &\quad \left. + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}})(\gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma}) + \text{H.c.} \right],\end{aligned}\quad (11)$$

with the coupling constants $V_{\mathbf{k}\mathbf{p}}^{(1,2,3,4)}$ defined by

$$V_{\mathbf{k}\mathbf{p}}^{(1,3)} = \frac{1}{8} [V_{AA}^{00} + V_{AA}^{10} \pm (V_{AB}^{00} + V_{AB}^{10}) e^{-i\Delta\varphi} + \text{C.c.}], \quad (12)$$

$$V_{\mathbf{k}\mathbf{p}}^{(2,4)} = \frac{1}{8} [V_{AA}^{00} - V_{AA}^{10} \mp (V_{AB}^{00} - V_{AB}^{10}) e^{-i\Delta\varphi} + \text{C.c.}]. \quad (13)$$

Here $V_{aa'}^{ll'} = V_{aa'}^{ll'}(\mathbf{k}-\mathbf{p}) = V_{a'a}^{l'l'}(\mathbf{p}-\mathbf{k})$, and $\Delta\varphi = \Delta\varphi_{\mathbf{k}\mathbf{p}} = \varphi_{\mathbf{k}} - \varphi_{\mathbf{p}}$. One can assume [32, 33] that intra-layer and inter-layer interactions in a graphene bilayer are approximately equal (at small momentum): $V_{aa'}^{00} \approx V_{aa'}^{10}$. In such a limit, we have in the first approximation

$$V_{\mathbf{k}\mathbf{p}}^{(1,3)} \approx \frac{1}{2} V_C(\mathbf{k}-\mathbf{p}) [1 \pm \cos(\Delta\varphi_{\mathbf{k}\mathbf{p}})], \quad V_{\mathbf{k}\mathbf{p}}^{(2,4)} \approx 0. \quad (14)$$

Thus, the interaction can be approximated as $\hat{H}_{\text{int}} \approx \hat{H}^{(1)} + \hat{H}^{(3)}$. We analyze this Hamiltonian using mean field theory, and the terms $\hat{H}^{(2,4)}$ will be taken into account perturbatively. The mean field version of \hat{H}_{int} is

$$\hat{H}_{\text{int}}^{\text{MF}} = -\sum_{\mathbf{p}\sigma} \tilde{\Delta}_{\mathbf{p}\sigma} \gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma} + \Delta_{\mathbf{p}\sigma} \gamma_{\mathbf{p}3\bar{\sigma}}^\dagger \gamma_{\mathbf{p}2\sigma} + \text{H.c.} + B, \quad (15)$$

where

$$\begin{aligned}\Delta_{\mathbf{k}\sigma} &= \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p}\mathbf{k}}^{(1)*} \langle \gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}} \rangle + V_{\mathbf{p}\mathbf{k}}^{(3)} \langle \gamma_{\mathbf{p}1\sigma}^\dagger \gamma_{\mathbf{p}4\bar{\sigma}} \rangle \right], \\ \tilde{\Delta}_{\mathbf{k}\sigma} &= \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p}\mathbf{k}}^{(1)} \langle \gamma_{\mathbf{p}1\sigma}^\dagger \gamma_{\mathbf{p}4\bar{\sigma}} \rangle + V_{\mathbf{p}\mathbf{k}}^{(3)*} \langle \gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}} \rangle \right], \\ B &= \sum_{\mathbf{p}\sigma} \left[\Delta_{\mathbf{p}\sigma} \langle \gamma_{\mathbf{p}3\bar{\sigma}}^\dagger \gamma_{\mathbf{p}2\sigma} \rangle + \tilde{\Delta}_{\mathbf{p}\sigma} \langle \gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma} \rangle \right].\end{aligned}\quad (16)$$

The spectrum of the mean-field Hamiltonian can be easily derived:

$$\begin{aligned}E_{\mathbf{k}\sigma}^{(2,3)} &= \mp E_{\mathbf{k}\sigma}^{\text{m}}, & E_{\mathbf{k}\sigma}^{(1,4)} &= \mp E_{\mathbf{k}\sigma}^{\text{h}}, \\ E_{\mathbf{k}\sigma}^{\text{m}} &= \sqrt{|\Delta_{\mathbf{k}\sigma}|^2 + (t_0 - t\zeta_{\mathbf{k}})^2}, & E_{\mathbf{k}\sigma}^{\text{h}} &= \sqrt{|\tilde{\Delta}_{\mathbf{k}\sigma}|^2 + (t_0 + t\zeta_{\mathbf{k}})^2}.\end{aligned}\quad (17)$$

The grand potential of the system is equal to

$$\Omega = \sum_{v=1}^4 \sum_{\mathbf{k}\sigma} (E_{\mathbf{k}\sigma}^{(v)} - \mu) \Theta(\mu - E_{\mathbf{k}\sigma}^{(v)}) + B, \quad (18)$$

where $\Theta(E)$ is the step-function. Minimization of Ω with respect to $\langle \gamma_{\mathbf{p}3\sigma}^\dagger \gamma_{\mathbf{p}2\sigma} \rangle$ and $\langle \gamma_{\mathbf{p}4\sigma}^\dagger \gamma_{\mathbf{p}1\sigma} \rangle$ gives us the system of equations for $\Delta_{\mathbf{k}\sigma}$ and $\tilde{\Delta}_{\mathbf{k}\sigma}$:

$$\Delta_{\mathbf{k}\sigma} = \sum_{\mathbf{p}} \left\{ \frac{V_{\mathbf{p}\mathbf{k}}^{(1)*} \Delta_{\mathbf{p}\sigma}}{2N_c E_{\mathbf{p}\sigma}^m} [\Theta(\mu + E_{\mathbf{p}\sigma}^m) - \Theta(\mu - E_{\mathbf{p}\sigma}^m)] + \frac{V_{\mathbf{p}\mathbf{k}}^{(3)} \tilde{\Delta}_{\mathbf{p}\sigma}}{2N_c E_{\mathbf{p}\sigma}^h} [\Theta(\mu + E_{\mathbf{p}\sigma}^h) - \Theta(\mu - E_{\mathbf{p}\sigma}^h)] \right\}, \quad (19)$$

$$\tilde{\Delta}_{\mathbf{k}\sigma} = \sum_{\mathbf{p}} \left\{ \frac{V_{\mathbf{p}\mathbf{k}}^{(1)} \tilde{\Delta}_{\mathbf{p}\sigma}}{2N_c E_{\mathbf{p}\sigma}^h} [\Theta(\mu + E_{\mathbf{p}\sigma}^h) - \Theta(\mu - E_{\mathbf{p}\sigma}^h)] + \frac{V_{\mathbf{p}\mathbf{k}}^{(3)} \Delta_{\mathbf{p}\sigma}}{2N_c E_{\mathbf{p}\sigma}^m} [\Theta(\mu + E_{\mathbf{p}\sigma}^m) - \Theta(\mu - E_{\mathbf{p}\sigma}^m)] \right\}. \quad (20)$$

The summation in Eqs. (19,20) covers the whole Brillouin zone. However, the interaction $V_{\mathbf{p}\mathbf{k}}^{(1,3)}$ is strongest when $\mathbf{p} \approx \mathbf{k}$, and decays for larger $|\mathbf{p} - \mathbf{k}|$. In the limit of vanishing backscattering

$$V_{\text{bs}}^{(1,3)} \equiv V_{\mathbf{K}_1, \mathbf{K}_2}^{(1,3)} \approx 0, \quad (21)$$

it is possible to define order parameters localized near the specific Dirac point \mathbf{K}_ξ : $\Delta_{\mathbf{k}\sigma} = \Delta_{\mathbf{k}\xi\sigma}$, when $\mathbf{k} \approx \mathbf{K}_\xi$. We see that, within our approximations, the electronic states and the order parameters can be split into four independent sectors, labeled by the multi-index $s = (\sigma, \xi)$. A sector with label $s = (\sigma, \xi)$ contains electron states with spin σ from valley ξ , and hole states with spin $-\sigma$ from the same valley. This definition implies that all states within a sector have the same value of the product $\sigma\xi$. The sectors are weakly coupled by neglected contributions proportional to V_{bs} and $V^{(2,4)}$. These corrections will be studied perturbatively.

We add and subtract Eqs. (19) and (20), use Eqs. (14), and change the summation by integration over the momentum near the Dirac point \mathbf{K}_ξ . We also assume that both Δ and $\tilde{\Delta}$ depend on $|\mathbf{k}|$ only. Finally, using the symmetry of our theory with respect to the sign of μ , we derive for $0 < \mu < t_0$

$$\Delta_{ks} + \tilde{\Delta}_{ks} = \int_p \bar{V}(k, p) \left[\frac{\Delta_{ps}}{2E_{ps}^m} \Theta(E_{ps}^m - \mu) + \frac{\tilde{\Delta}_{ps}}{2E_{ps}^h} \right],$$

$$\Delta_{ks} - \tilde{\Delta}_{ks} = \int_p \bar{U}(k, p) \left[\frac{\Delta_{ps}}{2E_{ps}^m} \Theta(E_{ps}^m - \mu) - \frac{\tilde{\Delta}_{ps}}{2E_{ps}^h} \right], \quad (22)$$

where $\int_p \dots = (2\pi/v_{\text{BZ}}) \int p dp \dots$, and the volume (area) of the Brillouin zone is $v_{\text{BZ}} = 8\pi^2/(3\sqrt{3}a_0^2)$. In Eqs. (22),

the averaged coupling constants are

$$\bar{V}(k, p) = \int \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 - 2kp \cos \phi}), \quad (23)$$

$$\bar{U}(k, p) = \int \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 - 2kp \cos \phi}) \cos \phi,$$

and the spectrum (17) in sector $s = (\sigma, \xi)$ can be approximated as

$$E_{ps}^m \cong \sqrt{|\Delta_s|^2 + t_0^2(1 - p/k_{F0})^2}, \quad (24)$$

$$E_{ps}^h \cong \sqrt{|\tilde{\Delta}_s|^2 + t_0^2(1 + p/k_{F0})^2} \cong t_0(1 + p/k_{F0}),$$

where $p = |\mathbf{p} - \mathbf{K}_\xi|$.

BCS-like approximation.— In general, we can choose some model for $V_C(q)$ and solve Eqs. (22) numerically. However, modeling the effective Coulomb interaction in graphene bilayers is notoriously difficult, and no universal and compact answer is known [34]. In this situation, finding an accurate numerical solution to the integral equations (22) is impractical. Instead, we use the simple BCS-like ansatz $\Delta_s(q) = \Delta_s \Theta(\Lambda - |q - k_{F0}|)$ and $\tilde{\Delta}_s(q) = \tilde{\Delta}_s \Theta(\Lambda - |q - k_{F0}|)$ for the order parameters (the cutoff momentum Λ satisfies $\Lambda < k_{F0}$), and assume that \bar{V} and \bar{U} are constants independent of k and p . We believe that this ansatz, despite its simplicity, captures all the necessary physics. Now the integral equations become non-linear algebraic equations

$$\Delta_s + \tilde{\Delta}_s = g \Delta_s \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta_s^2}} \right) + \tilde{g} \tilde{\Delta}_s,$$

$$\Delta_s - \tilde{\Delta}_s = \frac{g}{\alpha} \Delta_s \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta_s^2}} \right) - \frac{\tilde{g}}{\alpha} \tilde{\Delta}_s, \quad (25)$$

where the energy scale is $E^* = 2t_0\Lambda/k_{F0}$ and the coupling constants are

$$g = \frac{t_0}{\sqrt{3}\pi t^2} \bar{V}, \quad \tilde{g} = \frac{\Lambda}{2k_{F0}} g, \quad \alpha = \bar{V}/\bar{U} > 1. \quad (26)$$

It trivially follows from Eqs. (25) that $\tilde{\Delta}_s = C\Delta_s$, where $C = (\alpha - 1)/(\alpha + 1 - 2\tilde{g})$. At zero doping, which corresponds to the case $\mu = \Delta_s$, one finds

$$\Delta_s = \Delta_0 = E^* \exp \left[-\frac{1}{g} \frac{2\alpha - \tilde{g}(1 + \alpha)}{1 + \alpha - 2\tilde{g}} \right]. \quad (27)$$

This compact mean field solution is valid in the small coupling limit; that is, when g (and \tilde{g}) is small, and, consequently, Δ_0 and $\tilde{\Delta}_0$ are much less than t_0 . The doped state is characterized by $\mu > \Delta_s$. To describe the solution of Eq. (25) in such a regime, let us define the partial doping x_s : the concentration of electrons residing in sector s , per single carbon atom. It is known [35–38] that a finite x_s acts to decrease the order parameter Δ_s :

$$\Delta_s(x_s) = \Delta_0 \sqrt{1 - \frac{4x_s}{x_0}}, \quad \mu = \Delta_0 \left(1 - \frac{2x_s}{x_0} \right). \quad (28)$$

where $x_0 = \Delta_0 t_0 / (\pi \sqrt{3} t^2)$. It is easy to check that Eqs. (28) indeed guarantee that μ exceeds Δ_s , making the doping of sector s possible. At $T = 0$ the partial free energy (per unit cell) associated with doping is

$$\Delta F_s(x_s) = 4 \int_0^{x_s} \mu(x) dx = 4\Delta_0 \left(x_s - \frac{x_s^2}{x_0} \right). \quad (29)$$

Since a unit cell contains four carbon atoms, the factor 4 is required in this formula.

Fractional metal state.— The relations (28,29) describe a single sector. To determine the state of the whole system, we must understand how the total doping x is distributed between the sectors. One might expect that x is spread evenly: $x_s = x/4$. Yet such an assumption might not be most advantageous thermodynamically: we demonstrated [20, 21], for a two-sector system, that placing all the extra charge x into a single sector optimizes the system free energy relative to the state with an even distribution of x . To settle this issue for our four-sector model, we must minimize the doping-related part of the free energy for the whole system

$$\Delta F = \sum_s \Delta F_s = 4\Delta_0 x - \frac{4\Delta_0}{x_0} \sum_{\xi\sigma} x_{\xi\sigma}^2 \quad (30)$$

at fixed doping $x = \sum_s x_s$. Simple calculations demonstrate that, for $x < x_0$, the term ΔF reaches its smallest value, $\Delta F_{\text{qm}} = 4\Delta_0(x - x^2/x_0)$, when all extra electrons are placed into a specific sector s , while all other sectors are kept doping-free

$$x_s = x, \quad x_{s'} = 0 \quad \text{for } s' \neq s. \quad (31)$$

For example, ΔF_{qm} is smaller than $\Delta F_e = 4\Delta_0 x - \Delta_0 x^2/x_0$, which is the free energy of the state with $x_s = x/4$ for all four s . For the distribution (31) the Fermi surface lies entirely in sector $s = (\sigma, \xi)$. Therefore, only states with spin σ near the Dirac point \mathbf{K}_ξ reach the Fermi level. In other words, the Fermi surface is perfectly polarized in terms of both σ and ξ indices. Since the insulating gap persists in three other sectors, the state described by Eq. (31) may be called ‘a quarter-metal’, a first example of a series of ‘fractional metals’.

As in the case of the half-metal in the system with nesting [20, 21], the gap in the first sector closes when increasing doping. The doped electrons begin to enter the second sector, then to the third and fourth sectors. As a result, the system passes respectively through the states of a half-metal, 3/4-metal, and finally the gaps in all sectors close and the system occurs in the usual metallic phase. We can show that each transformation is a first-order phase transition. The analysis of the electronic states evolution with doping is quite similar to the half-metal case [20, 21].

Stability of Fractional metal.— Above we neglected interactions between electrons in different sectors. Then,

treating individual sectors independently, we derived Eqs. (28,29). Now we want to assess the effects of the neglected terms. There are two types of interaction terms: (i) umklapp interaction $\hat{H}^{(2,4)}$, Eqs. (9,11), which couples sectors with the same ξ but different spins, and (ii) the backscattering amplitude $V_{\text{bs}}^{(1,3)}$, which describes interactions between sectors with the same σ but different valley ξ , Eq. (21). In principle, $\hat{H}^{(2,4)}$ also contain the backscattering $V_{\text{bs}}^{(2)}$, which is even weaker, and will be neglected. If the associated coupling constants are small, we can use perturbation theory. The lowest-order perturbative correction F_{um} to the free energy due to the umklapp term $\hat{H}^{(2)}$ equals $\langle \hat{H}^{(2)} \rangle$. Thus, neglecting small contributions due to $\hat{\Delta}_s$, we determine the umklapp correction to the free energy (per unit cell)

$$F_{\text{um}} = -\frac{\mathcal{F}}{2} \sum_{\xi} \sqrt{\left(1 - \frac{4x_{\uparrow\xi}}{x_0}\right) \left(1 - \frac{4x_{\downarrow\xi}}{x_0}\right)}, \quad (32)$$

where $\mathcal{F} = 8\alpha^2 g_{\text{um}} \Delta_0 x_0 / (1 + \alpha)^2 g^2$, and the dimensionless Fermi-surface-averaged umklapp coupling constant is $g_{\text{um}} = t_0 \bar{V}_{\text{um}} / \sqrt{3\pi} t^2$. We also used the fact that $V^{(1)}$, upon averaging over the Fermi surface, becomes equal to $g(1 + \alpha)/2\alpha$. When x is low, one has $F_{\text{um}}/\mathcal{F} \approx -1 + x/x_0 + \sum_{\xi} (x_{\uparrow\xi} - x_{\downarrow\xi})^2/x_0^2$, which is smallest at $x_s = x/4$. A similar result can be derived for the backscattering interaction. Thus, both the umklapp and the backscattering favor an even distribution of doping over the sectors. However, in the limit $g_{\text{um}} \ll g^2$, $g_{\text{bs}} \ll g^2$, their contributions are small, and cannot destroy the fractional metal phase. The perturbative derivation of the stability criterion is intuitively clear and transparent. Its primary purpose is to demonstrate that the fractional metal phase can survive weak deviations from the highly idealized model neglecting any couplings between the sectors. On the other hand, this criterion is very stringent, and one may wonder if it can be satisfied in a real material. Fortunately, a more complex non-perturbative approach, which accounts for the inter-sector couplings at the mean field level, allows to relax it: we demonstrated [39] that it is sufficient to have

$$g_{\text{bs}} < g, \quad g_{\text{um}} < g \quad (33)$$

to maintain the stability of the FraM. More detailed stability analysis will be presented in future studies.

Discussion.— Using AA bilayer graphene as a test example, we argue that in a system with a nested multi-sheet Fermi surface, a peculiar state (which we call fractional metal, or FraM) can be stabilized. In the FraM phase, part of the Fermi surface is gapped and charge carriers on the remaining gapless part of the Fermi surface belong to a specific sector of the low-energy electronic states. Similar to a half-metal, the states at the Fermi energy can be characterized in terms of polarization; but, unlike the usual half-metals, this is not spin

polarization. Let us introduce the spin-flavor [40] operator $\hat{S}_f = \sum_{\sigma\xi\nu} \sigma\nu \hat{N}_{\sigma\xi\nu}$, where $\hat{N}_{\sigma\xi\nu}$ is the number operator for fermions with spin σ , charge ν , in valley ξ . Since doping enters only in one sector, all states at the Fermi surface have the same value of $\sigma\nu$. Therefore, these states are eigenstates of \hat{S}_f with the same eigenvalue $\sigma\nu$. The same is true for the valley operator $\hat{S}_v = \sum_{\sigma\xi\nu} \xi \hat{N}_{\sigma\xi\nu}$, since a given sector is localized entirely in one valley.

Thus, *the Fermi surface of the FraM is polarized in terms of two spin-like operators $S_{f,v}$* . This implies that *the electric current though the FraM carries, in addition to the electric charge, spin-flavor and valley quanta*. Finally, note that, if superconductivity arises in a FraM phase, it should obey rather peculiar properties. The superconducting order parameter might have a very unusual symmetry, classified according to a non-trivial spin and valley structure, and superconducting currents would be spin-flavor and valley polarized. However, the detailed analysis of this superconductivity requires the specifica-

tion of the symmetric properties of the electron-phonon coupling.

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APPENDIX: CAN BILAYER GRAPHENE BECOME A FRACTIONAL METAL?

Below we show the study of the stability of the quarter-metal against the umklapp interaction term.

BASIC EQUATIONS

For reader's convenience, let us recall several basic equations and facts from the main text.

Definitions

The interaction Hamiltonian is

$$\hat{H}_{\text{int}} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \hat{H}^{(4)}, \quad \text{where} \quad (34)$$

$$\hat{H}^{(1)} = -\frac{1}{N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(1)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma}) + (\gamma_{\mathbf{k}3\bar{\sigma}}^\dagger \gamma_{\mathbf{k}2\sigma}) (\gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}}) \right], \quad (35)$$

$$\hat{H}^{(2)} = -\frac{1}{2N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(2)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma}) + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}}) (\gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}}) + \text{H.c.} \right], \quad (36)$$

$$\hat{H}^{(3)} = -\frac{1}{N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(3)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}3\bar{\sigma}}^\dagger \gamma_{\mathbf{p}2\sigma}) + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}}) (\gamma_{\mathbf{p}4\bar{\sigma}}^\dagger \gamma_{\mathbf{p}1\sigma}) \right], \quad (37)$$

$$\hat{H}^{(4)} = -\frac{1}{2N_c} \sum_{\mathbf{k}\mathbf{p}\sigma} V_{\mathbf{k},\mathbf{p}}^{(4)} \left[(\gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}}) (\gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma}) + (\gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}}) (\gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma}) + \text{H.c.} \right], \quad (38)$$

with the coupling constants $V_{\mathbf{k},\mathbf{p}}^{(1,2,3,4)}$ defined as

$$V_{\mathbf{k},\mathbf{p}}^{(1,3)} = \frac{1}{8} [V_{AA}^{00} + V_{AA}^{10} \pm (V_{AB}^{00} + V_{AB}^{10}) e^{-i\Delta\varphi} + \text{C.c.}], \quad (39)$$

$$V_{\mathbf{k},\mathbf{p}}^{(2,4)} = \frac{1}{8} [V_{AA}^{00} - V_{AA}^{10} \mp (V_{AB}^{00} - V_{AB}^{10}) e^{-i\Delta\varphi} + \text{C.c.}]. \quad (40)$$

Our first-step approximation is

$$V_{\mathbf{k},\mathbf{p}}^{(1,3)} \approx \frac{1}{2} V_C(\mathbf{k} - \mathbf{p}) [1 \pm \cos(\Delta\varphi_{\mathbf{k},\mathbf{p}})], \quad V_{\mathbf{k},\mathbf{p}}^{(2,4)} \approx 0. \quad (41)$$

The interaction can be approximated as $\hat{H}_{\text{int}} \approx \hat{H}^{(1)} + \hat{H}^{(3)}$.

Mean field approximation

The mean field version of \hat{H}_{int} is

$$\hat{H}_{\text{int}}^{\text{MF}} = \frac{1}{N_c} (B_{\uparrow} + B_{\downarrow}) - \sum_{\mathbf{p}\sigma} \left(\tilde{\Delta}_{\mathbf{p}\sigma} \gamma_{\mathbf{p}4\bar{\sigma}}^{\dagger} \gamma_{\mathbf{p}1\sigma} + \Delta_{\mathbf{p}\sigma} \gamma_{\mathbf{p}3\bar{\sigma}}^{\dagger} \gamma_{\mathbf{p}2\sigma} + \text{H.c.} \right), \quad (42)$$

where

$$\Delta_{\mathbf{k}\sigma} = \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p},\mathbf{k}}^{(1)} \langle \gamma_{\mathbf{p}2\sigma}^{\dagger} \gamma_{\mathbf{p}3\bar{\sigma}} \rangle + V_{\mathbf{p},\mathbf{k}}^{(3)} \langle \gamma_{\mathbf{p}1\sigma}^{\dagger} \gamma_{\mathbf{p}4\bar{\sigma}} \rangle \right], \quad (43)$$

$$\tilde{\Delta}_{\mathbf{k}\sigma} = \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p},\mathbf{k}}^{(1)} \langle \gamma_{\mathbf{p}1\sigma}^{\dagger} \gamma_{\mathbf{p}4\bar{\sigma}} \rangle + V_{\mathbf{p},\mathbf{k}}^{(3)} \langle \gamma_{\mathbf{p}2\sigma}^{\dagger} \gamma_{\mathbf{p}3\bar{\sigma}} \rangle \right], \quad (44)$$

$$B_{\sigma} = \frac{1}{N_c} \sum_{\mathbf{k}} \left[\Delta_{\mathbf{k}\sigma} \langle \gamma_{\mathbf{k}3\bar{\sigma}}^{\dagger} \gamma_{\mathbf{k}2\sigma} \rangle + \tilde{\Delta}_{\mathbf{k}\sigma} \langle \gamma_{\mathbf{k}4\bar{\sigma}}^{\dagger} \gamma_{\mathbf{k}1\sigma} \rangle \right]. \quad (45)$$

The spectrum of the mean-field Hamiltonian can be easily derived

$$E_{\mathbf{k}\sigma}^{(2,3)} = \mp E_{\mathbf{k}\sigma}^{\text{m}}, \quad E_{\mathbf{k}\sigma}^{(1,4)} = \mp E_{\mathbf{k}\sigma}^{\text{h}}, \quad (46)$$

where

$$E_{\mathbf{k}\sigma}^{\text{m}} = \sqrt{|\Delta_{\mathbf{k}\sigma}|^2 + (t_0 - t\zeta_{\mathbf{k}})^2}, \quad E_{\mathbf{k}\sigma}^{\text{h}} = \sqrt{|\tilde{\Delta}_{\mathbf{k}\sigma}|^2 + (t_0 + t\zeta_{\mathbf{k}})^2}.$$

The total energy of the system is

$$E = \sum_{\nu=1}^4 \sum_{\mathbf{k}\sigma} (E_{\mathbf{k}\sigma}^{(\nu)} - \mu) \Theta(\mu - E_{\mathbf{k}\sigma}^{(\nu)}), \quad (47)$$

where $\Theta(E)$ is the step-function. Using the Hellmann-Feynman theorem, we obtain

$$\begin{aligned} \langle \gamma_{\mathbf{k}3\bar{\sigma}}^{\dagger} \gamma_{\mathbf{k}2\sigma} \rangle &= \frac{\Delta_{\mathbf{k}\sigma}}{2E_{\mathbf{k}\sigma}^{\text{m}}} [\Theta(\mu + E_{\mathbf{k}\sigma}^{\text{m}}) - \Theta(\mu - E_{\mathbf{k}\sigma}^{\text{m}})], \\ \langle \gamma_{\mathbf{k}4\bar{\sigma}}^{\dagger} \gamma_{\mathbf{k}1\sigma} \rangle &= \frac{\tilde{\Delta}_{\mathbf{k}\sigma}}{2E_{\mathbf{k}\sigma}^{\text{h}}} [\Theta(\mu + E_{\mathbf{k}\sigma}^{\text{h}}) - \Theta(\mu - E_{\mathbf{k}\sigma}^{\text{h}})]. \end{aligned} \quad (48)$$

Formally, the summation in Eq. (43) covers the whole Brillouin zone. However, the interaction $V_{\mathbf{p},\mathbf{k}}^{(1,3)}$ is the strongest when $\mathbf{p} \approx \mathbf{k}$, and decays for larger $|\mathbf{p} - \mathbf{k}|$. In the limit of vanishing backscattering

$$V_{\text{bs}}^{(1,3)} \equiv V_{\mathbf{K}_1, \mathbf{K}_2}^{(1,3)} \approx 0, \quad (49)$$

it is possible to define order parameters localized near a specific Dirac point \mathbf{K}_{ξ} : $\Delta_{\mathbf{k}\sigma\xi} = \Delta_{\mathbf{k}\sigma}$, when $\mathbf{k} \approx \mathbf{K}_{\xi}$. Combining Eqs. (43) and (48), we obtain the self-consistent equations in the form

$$\Delta_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_{\xi}} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^{\text{m}}} [\Theta(\mu + E_{\mathbf{p}\sigma}^{\text{m}}) - \Theta(\mu - E_{\mathbf{p}\sigma}^{\text{m}})] + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^{\text{h}}} [\Theta(\mu + E_{\mathbf{p}\sigma}^{\text{h}}) - \Theta(\mu - E_{\mathbf{p}\sigma}^{\text{h}})] \right\}, \quad (50)$$

$$\tilde{\Delta}_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_{\xi}} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^{\text{h}}} [\Theta(\mu + E_{\mathbf{p}\sigma}^{\text{h}}) - \Theta(\mu - E_{\mathbf{p}\sigma}^{\text{h}})] + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^{\text{m}}} [\Theta(\mu + E_{\mathbf{p}\sigma}^{\text{m}}) - \Theta(\mu - E_{\mathbf{p}\sigma}^{\text{m}})] \right\}. \quad (51)$$

Simplifying the latter equations in the regime $\mu > 0$, we derive

$$\Delta_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_\xi} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^m} \Theta(E_{\mathbf{p}\sigma}^m - \mu) + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^h} \Theta(E_{\mathbf{p}\sigma}^h - \mu) \right\}, \quad (52)$$

$$\tilde{\Delta}_{\mathbf{k}\sigma\xi} = \frac{1}{N_c} \sum_{\mathbf{p} \in \mathbf{K}_\xi} \left\{ \frac{V_{\mathbf{p},\mathbf{k}}^{(1)} \tilde{\Delta}_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^h} \Theta(E_{\mathbf{p}\sigma}^h - \mu) + \frac{V_{\mathbf{p},\mathbf{k}}^{(3)} \Delta_{\mathbf{p}\sigma\xi}}{2E_{\mathbf{p}\sigma}^m} \Theta(E_{\mathbf{p}\sigma}^m - \mu) \right\}. \quad (53)$$

We see that, within our approximations, the electronic states and the order parameters can be split into four independent sectors, which can be labeled by the multi-index $s = (\sigma, \xi)$. Our derivation implies that the sectors are not entirely independent: neglected contributions proportional to V_{bs} and $V^{(2,4)}$ couple them. Due to the smallness of these couplings, they can be treated perturbatively.

We add and subtract Eqs. (52) and (53), use Eqs. (41), and change the summation over momentum by an integration. We also assume that both Δ and $\tilde{\Delta}$ only depend on $|\mathbf{k}|$. Finally, using the symmetry of our theory with respect to the sign of μ , we derive for $0 < \mu < t_0$

$$\begin{aligned} \Delta_{ks} + \tilde{\Delta}_{ks} &= \int_p \bar{V}_Q^{(0)} \left[\frac{\Delta_{ps}}{2E_{ps}^m} \Theta(E_{ps}^m - \mu) + \frac{\tilde{\Delta}_{ps}}{2E_{ps}^h} \right], \\ \Delta_{ks} - \tilde{\Delta}_{ks} &= \int_p \bar{V}_Q^{(1)} \left[\frac{\Delta_{ps}}{2E_{ps}^m} \Theta(E_{ps}^m - \mu) - \frac{\tilde{\Delta}_{ps}}{2E_{ps}^h} \right], \end{aligned} \quad (54)$$

where the integration symbol stands for $\int_p \dots = (2\pi p/v_{\text{BZ}}) \int dp \dots$, and the volume (area) of the Brillouin zone is $v_{\text{BZ}} = 8\pi^2/(3\sqrt{3}a^2)$. In Eqs. (54), the averaged coupling constants are

$$\begin{aligned} \bar{V}_Q^{(0)}(k, p) &= \int \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 + 2kp \cos \phi}), \\ \bar{V}_Q^{(1)}(k, p) &= \int \frac{d\phi}{2\pi} V_C(\sqrt{k^2 + p^2 + 2kp \cos \phi}) \cos \phi, \end{aligned} \quad (55)$$

and the spectrum (46) in sector $s = (\sigma, \xi)$ can be approximated as

$$\begin{aligned} E_{ps}^m &\approx \sqrt{|\Delta_s|^2 + t_0^2(1 - p/k_{F0})^2}, \\ E_{ps}^h &\approx \sqrt{|\tilde{\Delta}_s|^2 + t_0^2(1 + p/k_{F0})^2} \approx t_0(1 + p/k_{F0}), \end{aligned} \quad (56)$$

where $p = |\mathbf{p} - \mathbf{K}_\xi|$.

To solve the integral equations (54) we use the simple BCS-like ansatz

$$\Delta_a(q) = \Delta_s \Theta(\Lambda - |q - k_{F0}|) \quad \text{and} \quad \tilde{\Delta}_{\xi\sigma}(q) = \tilde{\Delta}_s \Theta(\Lambda - |q - k_{F0}|) \quad (57)$$

for the order parameters (the cutoff momentum Λ satisfies $\Lambda \ll Q_0$), and assume that $\bar{V}_Q^{(0,1)}$ are constants independent of k and p . This allows us to convert the integral equations into non-linear algebraic equations

$$\begin{aligned} \Delta_s + \tilde{\Delta}_s &= g \Delta_s \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta_s^2}} \right) + \tilde{g} \tilde{\Delta}_s, \\ \Delta_s - \tilde{\Delta}_s &= \frac{g}{\alpha} \Delta_s \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta_s^2}} \right) - \frac{\tilde{g}}{\alpha} \tilde{\Delta}_s, \end{aligned} \quad (58)$$

where the energy scale is $E^* = 2t_0\Lambda/k_{F0}$, and the coupling constants are

$$g = \frac{t_0}{\sqrt{3}\pi t^2} \bar{V}^{(0)}, \quad \tilde{g} = \frac{\Lambda}{2k_{F0}} g, \quad \alpha = \frac{\bar{V}^{(0)}}{\bar{V}^{(1)}} > 1. \quad (59)$$

Solutions of the mean field equations

At zero doping, which corresponds to the case $\mu = \Delta_s$, the order parameters are

$$\Delta_s = \Delta_0 = E^* \exp \left[-\frac{1}{g} \frac{2\alpha - \tilde{g}(1 + \alpha)}{1 + \alpha - 2\tilde{g}} \right], \quad \tilde{\Delta}_s = \frac{\alpha - 1}{\alpha + 1 - 2\tilde{g}} \Delta_s. \quad (60)$$

This mean-field solution is valid in the weak-coupling limit, that is, when g is small, and, consequently, Δ_0 and $\tilde{\Delta}_0$ are much less than t_0 . The doped state is characterized by $\mu > \Delta_s$. To describe the solution of Eq. (58) in such a regime, let us define the partial doping x_s for the concentration of electrons residing in sector s . It is known that a finite x_s decreases the order parameter Δ_s :

$$\Delta_s(x_s) = \Delta_0 \sqrt{1 - \frac{4x_s}{x_0}}, \quad \mu = \Delta_0 \left(1 - \frac{2x_s}{x_0} \right), \quad (61)$$

where $x_0 = \Delta_0 t_0 / (\pi \sqrt{3} t^2)$. It is easy to check that Eqs. (61) indeed guarantee that μ exceeds Δ_s , making it possible to dope sector s . At zero temperature, the partial free energy (per unit cell) associated with doping becomes

$$\Delta F_s(x_s) = 4 \int_0^{x_s} \mu(x) dx = 4\Delta_0 \left(x_s - \frac{x_s^2}{x_0} \right). \quad (62)$$

As in the main text, the factor 4 accounts for the four carbon atoms in a single unit cell.

STABILITY AGAINST THE UMKLAPP INTERACTION

Self-consistent equations

The next step is to add the inter-sector interaction. We will use $H^{(2)}$ as an example of the inter-sector interaction. The other example is the backscattering $V_{\text{bs}}^{(1,3)}$. The term $H^{(2)}$ is a type of umklapp scattering: such term is non-zero only when the nesting vector is either zero or half of the elementary reciprocal lattice vector. If we average $H^{(2)}$ we obtain

$$\begin{aligned} \langle \hat{H}^{(2)} \rangle &= -\frac{1}{2N_c} \sum_{\mathbf{k}, \mathbf{p}} V_{\mathbf{k}, \mathbf{p}}^{(2)} \left[\langle \gamma_{\mathbf{k}1\sigma}^\dagger \gamma_{\mathbf{k}4\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}1\bar{\sigma}}^\dagger \gamma_{\mathbf{p}4\sigma} \rangle + \langle \gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle + \text{C.c.} \right] \\ &\approx -\frac{1}{2N_c} \sum_{\mathbf{k}, \mathbf{p}} V_{\mathbf{k}, \mathbf{p}}^{(2)} \left[\langle \gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle + \text{C.c.} \right] \approx -\frac{\bar{V}_{\text{um}}}{2N_c} \sum_{\mathbf{k}, \mathbf{p}} \left[\langle \gamma_{\mathbf{k}2\sigma}^\dagger \gamma_{\mathbf{k}3\bar{\sigma}} \rangle \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle + \text{C.c.} \right], \end{aligned} \quad (63)$$

where \bar{V}_{um} is the averaged value of $V_{\mathbf{k}, \mathbf{p}}^{(2)}$. Using the definition of the order parameter in terms of the anomalous operator averages, Eq. (43), we derive

$$\langle \hat{H}^{(2)} \rangle \approx -N_c \frac{\bar{V}_{\text{um}}}{V_1^2} \Delta_{\uparrow\xi} \Delta_{\downarrow\xi} + \text{C.c.} \quad (64)$$

This suggests that the self-consistent equations for $s = (\uparrow, \xi)$ and $s' = (\downarrow, \xi)$ become coupled. To account for this, we take the first of the two equations (43) and add a term $V_{\mathbf{p}, \mathbf{k}}^{(2)} \langle \gamma_{\bar{\sigma}}^\dagger \gamma_{\sigma} \rangle$ to its right-hand side

$$\Delta_{\mathbf{k}\sigma} = \frac{1}{N_c} \sum_{\mathbf{p}} \left[V_{\mathbf{p}, \mathbf{k}}^{(1)} \langle \gamma_{\mathbf{p}2\sigma}^\dagger \gamma_{\mathbf{p}3\bar{\sigma}} \rangle + V_{\mathbf{p}, \mathbf{k}}^{(2)} \langle \gamma_{\mathbf{p}2\bar{\sigma}}^\dagger \gamma_{\mathbf{p}3\sigma} \rangle \right], \quad (65)$$

where we discarded the term with bands 1 and 4. Finally, using Eq. (48), we derive

$$\Delta_{\uparrow} = g \Delta_{\uparrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\uparrow})} \right] + g_{\text{um}} \Delta_{\downarrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\downarrow})} \right], \quad (66)$$

$$\Delta_{\downarrow} = g \Delta_{\downarrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\downarrow})} \right] + g_{\text{um}} \Delta_{\uparrow} \ln \left[\frac{E^*}{M(\mu, \Delta_{\uparrow})} \right]. \quad (67)$$

To describe two remaining sectors, $(\uparrow, \bar{\xi})$ and $(\downarrow, \bar{\xi})$, the identical set of equations should be used. In Eqs. (66) and (67), the quantity $M(\mu, \Delta)$ effectively functions as the low-energy cutoff: if in a given sector $\Delta > \mu$, this sector remains undoped, and $M(\mu, \Delta) = \Delta$; when a sector accommodates finite doping $\mu > \Delta$, in such a situation $M(\mu, \Delta) = \mu + \sqrt{\mu^2 - \Delta^2}$. Formally, this can be expressed as

$$M(\mu, \Delta) = (\mu + \sqrt{\mu^2 - \Delta^2})\Theta(\mu - \Delta) + \Delta\Theta(\Delta - \mu). \quad (68)$$

Note also that in Eqs. (66) and (67) we used the simplified notation $\Delta_{\uparrow} \equiv \Delta_{\uparrow\xi}$ and $\Delta_{\downarrow} \equiv \Delta_{\downarrow\xi}$. The coupling constant is $g_{\text{um}} = \beta\bar{V}_{\text{um}}\nu(\varepsilon_{\text{F}})$, where $\nu(\varepsilon_{\text{F}})$ is the density of states, and β is a numerical coefficient of order unity.

When the system is undoped, we can introduce Δ_0 as follows $\mu = \Delta_{\uparrow} = \Delta_{\downarrow} \equiv \Delta_0$ [note that this is a redefinition of Δ_0 initially given by Eq. (60)]. In such a limit, both equations become identical

$$\Delta_0 = g(1 + \gamma)\Delta_0 \ln \left(\frac{E^*}{\Delta_0} \right), \quad \text{where} \quad \gamma = \frac{g_{\text{um}}}{g}. \quad (69)$$

This equation has one non-zero solution

$$\Delta_0 = E^* \exp \left(-\frac{1}{g(1 + \gamma)} \right). \quad (70)$$

We can see that the umklapp coupling increases Δ_0 .

Doped state

Now we discuss the doped system. Below we will consider two possibilities: (i) all four sectors are doped equally, and (ii) three sectors remain undoped, and all doping only enters a single sector. Let us start with (i). In such a situation $\mu > \Delta_s = \Delta(x)$ for all four s . Equations (66) and (67) become identical

$$\Delta = g(1 + \gamma)\Delta \ln \left(\frac{E^*}{\mu + \sqrt{\mu^2 - \Delta^2}} \right), \quad (71)$$

valid in all four sectors. The solution to this equation is similar to Eq. (61)

$$\Delta(x) = \Delta_0 \sqrt{1 - \frac{x}{x_0}}, \quad \mu = \Delta_0 \left(1 - \frac{x}{2x_0} \right), \quad (72)$$

where we took into account that partial dopings equal to half of the total doping: $x_s = x/4$. The expression for $\mu(x)$ allows us to calculate $\Delta F(x)$

$$\Delta F(x) = 4 \int_0^x \mu(x) dx = 4\Delta_0 x - \Delta_0 \frac{x^2}{x_0}. \quad (73)$$

This free energy is denoted as ΔF_e in the main text.

For case (ii), the calculations are more complicated. We define $\delta_{\sigma}(x)$ as follows $\Delta_{\sigma}(x) = \Delta_0[1 - \delta_{\sigma}(x)]$. For definiteness, we assume that the sector $s = (\uparrow, \xi)$ is undoped, while $s = (\downarrow, \xi)$ is doped. This means that $\Delta_{\uparrow} > \mu > \Delta_{\downarrow}$. Two other sectors, $(\uparrow, \bar{\xi})$ and $(\downarrow, \bar{\xi})$, are undoped, and decoupled from s and s' . Therefore, they are characterized by the order parameter Δ_0 , given by Eq. (70)

$$0 < \delta_{\uparrow} < m < \delta_{\downarrow}, \quad \text{where} \quad m = \frac{\Delta_0 - \mu}{\Delta_0}. \quad (74)$$

Let us introduce yet another quantity, δS , as follows

$$\mu + \sqrt{\mu^2 - \Delta_{\downarrow}^2} = \Delta_0 \left[1 - m + \sqrt{(1 - m)^2 - (1 - \delta_{\downarrow})^2} \right] = \Delta_0(1 + \delta S), \quad (75)$$

$$\delta S = \sqrt{(1 - m)^2 - (1 - \delta_{\downarrow})^2} - m. \quad (76)$$

The parameters δ_σ , δS , and m are small in the limit of small doping x . However, they have different degrees of smallness. Indeed, as we will see later

$$\delta_\sigma = O(m), \quad \delta S = O(m^{1/2}). \quad (77)$$

These relations become important when we solve the self-consistent equations in the limit of small doping.

Our goal is to solve the following equations

$$(1 - \delta_\uparrow) = g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 - \delta_\uparrow) \right] + \gamma g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 + \delta S) \right], \quad (78)$$

$$(1 - \delta_\downarrow) = g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 + \delta S) \right] + \gamma g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} - \ln(1 - \delta_\uparrow) \right], \quad (79)$$

to find δ_σ as a function of m , and then determine m versus x . In the limit of small x , we expand the self-consistent equations and, keeping in mind Eq. (77), we derive

$$(1 - \delta_\uparrow) = g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} + \delta_\uparrow \right] + \gamma g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \delta S + \frac{\delta S^2}{2} \right] + O(m^{3/2}), \quad (80)$$

$$(1 - \delta_\downarrow) = g(1 - \delta_\downarrow) \left[\frac{1}{g(1 + \gamma)} - \delta S + \frac{\delta S^2}{2} \right] + \gamma g(1 - \delta_\uparrow) \left[\frac{1}{g(1 + \gamma)} + \delta_\uparrow \right] + O(m^{3/2}). \quad (81)$$

Simplifying, we obtain

$$\delta_\uparrow \approx \left[\frac{\delta_\uparrow}{1 + \gamma} - g\delta_\uparrow \right] + \gamma \left[\frac{1}{1 + \gamma} \delta_\downarrow + g \left(\delta S - \frac{\delta S^2}{2} \right) \right], \quad (82)$$

$$\delta_\downarrow \approx \left[\frac{\delta_\downarrow}{1 + \gamma} + g \left(\delta S - \frac{\delta S^2}{2} \right) \right] + \gamma \left[\frac{1}{1 + \gamma} \delta_\uparrow - g\delta_\uparrow \right], \quad (83)$$

Next step:

$$\left(\frac{\gamma}{1 + \gamma} + g \right) \delta_\uparrow = \frac{\gamma}{1 + \gamma} \delta_\downarrow + g\gamma \left(\delta S - \frac{\delta S^2}{2} \right), \quad (84)$$

$$\left(\frac{\gamma}{1 + \gamma} - g\gamma \right) \delta_\uparrow = \frac{\gamma}{1 + \gamma} \delta_\downarrow - g \left(\delta S - \frac{\delta S^2}{2} \right). \quad (85)$$

Subtracting these two equations we derive

$$g(1 + \gamma)\delta_\uparrow = g(1 + \gamma) \left(\delta S - \frac{\delta S^2}{2} \right) \Leftrightarrow \delta_\uparrow = \delta S - \frac{\delta S^2}{2}. \quad (86)$$

Now δ_\uparrow can be eliminated

$$\left[\frac{\gamma}{1 + \gamma} + g(1 - \gamma) \right] \left(\delta S - \frac{\delta S^2}{2} \right) = \frac{\gamma}{1 + \gamma} \delta_\downarrow. \quad (87)$$

This relation is equivalent to

$$\delta S - \frac{\delta S^2}{2} = \alpha \delta_\downarrow, \quad \text{where} \quad \alpha = [1 + g(\gamma^{-1} - \gamma)]^{-1}. \quad (88)$$

Let us express δS in the limit of small doping

$$\delta S = \sqrt{(1 - m)^2 - (1 - \delta_\downarrow)^2} - m = \sqrt{(2 - m - \delta_\downarrow)(\delta_\downarrow - m)} - m = \sqrt{2(\delta_\downarrow - m)} - m + O(m^{3/2}), \quad (89)$$

$$\delta S^2 = 2(\delta_\downarrow - m) + O(m^{3/2}). \quad (90)$$

Therefore

$$\delta S - \frac{\delta S^2}{2} = \sqrt{2(\delta_\downarrow - m)} - \delta_\downarrow + O(m^{3/2}). \quad (91)$$

The self-consistent equation becomes

$$\alpha\delta_{\downarrow} = \sqrt{2(\delta_{\downarrow} - m)} - \delta_{\downarrow} + O(m^{3/2}). \quad (92)$$

Its solution is

$$\delta_{\downarrow} \approx m + \frac{(1+\alpha)^2}{2}m^2, \quad \delta_{\uparrow} = \alpha\delta_{\downarrow} \approx \alpha m + \frac{\alpha(1+\alpha)^2}{2}m^2. \quad (93)$$

Let us check the consistency of these relations with known results in the $\alpha = 0$ limit. In this case

$$\delta_{\downarrow} \approx m + \frac{1}{2}m^2, \quad \delta_{\uparrow} = 0. \quad (94)$$

At the same time, Eqs. (61) in the regime of small x can be written as

$$m = \frac{x}{2x_0}, \quad \delta_{\downarrow}(x) = \frac{\Delta_0 - \Delta(x)}{\Delta_0} \approx \frac{x}{2x_0} + \frac{x^2}{8x_0^2}. \quad (95)$$

We can now exclude x to obtain

$$\delta_{\downarrow}(x) \approx m + \frac{m^2}{2}, \quad (96)$$

which coincides with Eq. (94).

The final step is to add doping into the formalism. To this end, we write

$$4x = 2\nu_F \int_{\Delta_{\downarrow}}^{\mu} d\varepsilon \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta_{\downarrow}^2}}, \quad (97)$$

where $4x$ is the doping per unit cell, $\nu_F = t_0/(\sqrt{3}\pi t^2)$ is the density of states per unit cell for each single Fermi surface sheet (there are four Fermi surface sheets),

$$x = \frac{\nu_F}{2} \sqrt{\mu^2 - \Delta_{\downarrow}^2} = \frac{\nu_F \Delta_0}{2} \sqrt{(1-m)^2 - (1-\delta_{\downarrow})^2} = \frac{x_0}{2} \sqrt{(1-m)^2 - (1-\delta_{\downarrow})^2}, \quad (98)$$

where $x_0 = \nu_F \Delta_0$. It is possible to show that

$$4x^2 = x_0^2(\delta_{\downarrow} - m)(2 - m - \delta_{\downarrow}) \Rightarrow 4x^2 = (1+\alpha)^2 x_0^2 m^2 + O(m^3). \quad (99)$$

Deriving the latter relation we used Eq. (93), which, among other things, demonstrates that $\delta_{\downarrow} - m = O(m^2)$. Equation (99) allows us to establish the following connection between doping and the chemical potential

$$m = \frac{2x}{(1+\alpha)x_0} + O(x^2) \Leftrightarrow \mu = \Delta_0 \left(1 - \frac{2x}{(1+\alpha)x_0} \right) + O(x^2). \quad (100)$$

Integrating $\mu(x)$, we obtain

$$\Delta F_{\text{qm}} = 4\Delta_0 x - \left(\frac{4\Delta_0}{1+\alpha} \right) \frac{x^2}{x_0}. \quad (101)$$

In the limit $\alpha \rightarrow 0$ we recover the expression for ΔF_{qm} given in the main text [see after Eq. (30)]. The free energy (101) must be compared against the free energy given by Eq. (73). We see that the quarter-metal is stable if $(1+\alpha)^{-1} > 1/4$. Equivalently,

$$\text{quarter-metal is stable when } \alpha(\gamma) < 3. \quad (102)$$

To understand what the latter requirement entails, let us examine Fig. 1, which shows $\alpha(\gamma)$ for $g = 0.1$. We see that $\alpha < 3$ as long as $\gamma = g_{\text{um}}/g < 6.8$. That is, for $g = 0.1$, the umklapp satisfying

$$g_{\text{um}} < 0.68, \quad (103)$$

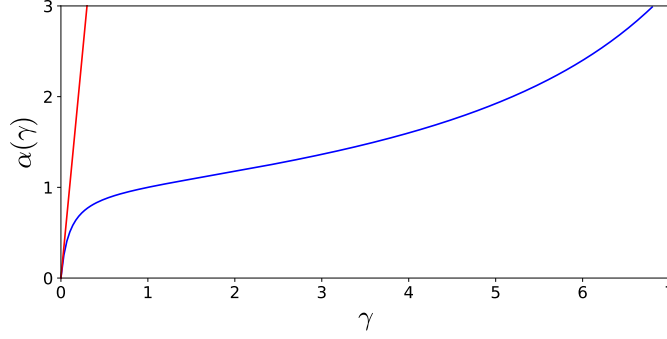


FIG. 1: The function $\alpha(\gamma)$ for $g = 0.1$ is shown by the blue curve. The straight (red) line is γ/g .

does not violate the stability of the quarter-metal.

We note that Eq. (102) is not the absolute stability criterion, rather it describes the stability of the quarter-metal against the transition into an ordinary metal, when all four sectors are doped equally. A comprehensive investigation of the stability goes well beyond the present study, and, most likely, requires input from experiments.

It is interesting to note that perturbation theory in powers of small γ strongly underestimates the stability range of the quarter-metal. To demonstrate this, we expand the expression (101) for ΔF_{qm} in powers of α

$$\Delta F_{\text{qm}} \approx 4\Delta_0 x - 4\Delta_0 \frac{x^2}{x_0} + 4\Delta_0 \frac{\alpha x^2}{x_0}. \quad (104)$$

Since at $\gamma \rightarrow 0$, the following holds $\alpha \approx \gamma/g = g_{\text{um}}/g^2$, the expression for ΔF_{qm} can be approximated as

$$\Delta F_{\text{qm}} \approx 4\Delta_0 x - 4\Delta_0 \frac{x^2}{x_0} \left(1 - \frac{g_{\text{um}}}{g^2}\right), \quad (105)$$

If we use this expression, instead of the more accurate Eq. (101), we could (erroneously) conclude that the quarter-metal is stable when $(1 - g_{\text{um}}/g^2) > 1/4$. This inequality can be transformed to

$$\frac{\gamma}{g} = \frac{g_{\text{um}}}{g^2} < \frac{3}{4} \quad \Leftrightarrow \quad g_{\text{um}} < \frac{3g^2}{4}. \quad (106)$$

In Fig. 1 we can see the low- γ approximation $\alpha(\gamma) \approx \gamma/g$ as a (red) straight line. We see that, at low g , this approximation works only at very small γ ; while for larger γ (larger g_{um}) it is completely useless. Thus, we conclude that the replacement $[1 + \alpha(\gamma)]^{-1} \rightarrow (1 - \gamma/g)$ artificially shrinks the stability range of the quarter-metal. Indeed, the requirement (106) is very strict: at $g = 0.1$, as in Fig. 1, Eq. (106) demand that $g_{\text{um}} < 0.0075$, cf. Eq. (103). This is the origin of the serious disparity between the stability condition derived in the main text using simple perturbation theory and more the sophisticated criterion (102).

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