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*Proc. R. Soc. A* 2009 **465**, 3687–3708 first published online 3 September 2009  
doi: 10.1098/rspa.2009.0239

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# Boltzmann and Fokker–Planck equations modelling opinion formation in the presence of strong leaders

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We propose a mathematical model for opinion formation in a society that is built of two groups, one group of ‘ordinary’ people and one group of ‘strong opinion leaders’. Our approach is based on an opinion formation model introduced in Toscani (Toscani 2006 *Commun. Math. Sci.* **4**, 481–496) and borrows ideas from the kinetic theory of mixtures of rarefied gases. Starting from microscopic interactions among individuals, we arrive at a macroscopic description of the opinion formation process that is characterized by a system of Fokker–Planck-type equations. We discuss the steady states of this system, extend it to incorporate emergence and decline of opinion leaders and present numerical results.

**Keywords:** Boltzmann equation; Fokker–Planck equation; opinion formation; sociophysics

## 1. Introduction

Opinion leadership is one of several sociological models trying to explain formation of opinions in a society. It is a concept that goes back to Lazarsfeld *et al.* (1944). In the course of their study of the presidential elections in the USA in 1940, Lazarsfeld *et al.* (1944) found interpersonal communication to be much more influential than direct media effects. They formulated a theory of a two-step flow of communication where so-called opinion leaders who are active media users select, interpret, modify, facilitate and finally transmit information from the media to less active parts of the population. Later, sociologists obtained a new view on opinion leader characteristics by developing the notion of public individuation. Public individuation describes how people feel the urge to differentiate themselves and act differently from other people (Maslach *et al.* 1985). This is a necessity for an opinion leader because she or he must be willing to set herself or himself apart from the ordinary people. Certain, typical personal characteristics are supposed to characterize opinion leaders: high confidence, high self-esteem, a strong need to be unique and the ability to withstand criticism. An opinion leader is socially active, highly connected and held in high esteem by

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those accepting his or her opinions. Although different from the others, opinion leaders are still related to their followers and not always easy to distinguish from them. This is because opinion leadership is specific to a subject and can change over time. Someone who is a strong opinion leader in one field may be a follower in another.

In the last decade, new communication forms, such as email, web navigation, blogs and instant messaging, have globally changed the way that information is disseminated and opinions formed in society (cf. [Rash 1997](#)). Still, opinion leadership continues to play a critical role in all these processes, independent of the underlying technology. Opinion leadership appears in such different fields as, but not limited to,

- (i) political parties and movements: a prominent example of the latter is Al Gore's initiative *The Climate Project*,
- (ii) advertisement of commercial products: product reviewers in the media who have a deeper knowledge and background than average consumers,
- (iii) dissemination of new technologies: *early adopters* play an important role, either as lighthouse customers that assist in the development or as individuals that recommend a new product to others, and
- (iv) pharmaceutical industry: companies engage with *key opinion leaders*, i.e. physicians who influence their colleagues' prescribing behaviour.

In recent years, opinion formation has received growing attention from physicists ([Galam & Zucker 2000](#); [Sznajd-Weron & Sznajd 2000](#); [Deffuant \*et al.\* 2002](#)), opening its own research field termed *sociophysics* that goes back to the pioneering work of [Galam \*et al.\* \(1982\)](#). We refer also to [Comincioli \*et al.\* \(2009\)](#) and [Galam \(2005\)](#) and references therein. Often, especially in numerical simulation studies, cellular automata are used. Another approach uses models of mean field type, which lead to systems of (ordinary or partial) differential equations. This approach has the advantage that, up to a certain extent, they can be treated analytically and help to obtain a deeper understanding of the underlying dynamics. A third approach is to introduce kinetic models of opinion formation ([Toscani 2006](#); [Boudin & Salvarani 2009](#)). The basic paradigm is that the behaviour of a sufficiently large number of interacting individuals in a society can be described by methods of statistical physics just as well as the colliding molecules of a gas in a container. Exchange of opinion between individuals in these models is defined by pairwise, *microscopic* interactions. In dependence on the specification of these interactions, the whole society develops a certain *macroscopic* opinion distribution.

Independent of the approach chosen, the prevalent literature primarily has focused on election processes, referendums or public opinion tendencies. With the exception of [Bertotti & Delitala \(2008\)](#), who propose a simple, discrete model for the influence of strong leaders in opinion formation, less attention has been paid to the important effect that opinion leaders have on the dissemination of new ideas and the diffusion of beliefs in a society. In this paper, we turn to this problem.

Our work is based on a kinetic model for opinion formation introduced in [Toscani \(2006\)](#). It is built on two main aspects of opinion formation. The first one is a *compromise process* ([Weidlich 2000](#); [Deffuant \*et al.\* 2002](#)), in which individuals tend to reach a compromise after exchange of opinions. The second one is *self-thinking*, where individuals change their opinion in a diffusive

way, possibly influenced by exogenous information sources such as the media. Based on both, Toscani (2006) introduced a kinetic model in which opinion is exchanged between individuals through pairwise interactions. In a suitable scaling limit, a partial differential equation of Fokker–Planck type was derived for the distribution of opinion in a society. Similar diffusion equations were also obtained recently in Slanina & Lavička (2003) as a mean field limit of the Sznajd model (Sznajd-Weron & Sznajd 2000). Mathematically, the model in Toscani (2006) is related to work in the kinetic theory of granular gases (Cercignani *et al.* 1994). In particular, the non-local nature of the compromise process is analogous to the variable coefficient of restitution in inelastic collisions (Toscani 2000). Similar models were used in the modelling of wealth and income distributions that show Pareto tails (cf. Düring *et al.* 2008 and references therein).

Clearly, there are some limitations to our approach, which is—as often in applied mathematics—a very simplified model of the complex reality. First, the statistical description will be expected to be valid only if the number of individuals is rather large. Second, we do not consider the structures of social networks that can play an important role in diffusing opinions. Mathematically, such networks can be expressed as graphs (cf. Sood *et al.* 2008). However, it should be noted that also in sociological models, which focus on such underlying structures of society, opinion leaders play an important role. They act as promoters of opinions across different sub-groups of the society (Burt 1999), in which opinions are easily communicated as and spread, e.g. in a group of colleagues at work, among friends and family or members of a social or sports club. Mathematically, this can be represented by scale-free networks with the opinion leaders as ‘hubs’, i.e. highest-degree nodes in the graph. In our model, although we have abstracted from the underlying social network, we can model this fact by controlling the interaction frequencies between opinion leaders and their followers. In any case, it is important to obtain a better understanding of the influence of opinion leaders on normal people. The model presented in this paper is a first step to a quantitative study of opinion leadership.

The paper is organized as follows. In the following section, we introduce the model that leads to a system of Boltzmann equations. We derive and study an associated system of Fokker–Planck-type equations in §3. Numerical examples are presented in §4 and §5 concludes.

## 2. Kinetic models for opinion formation

The goal of a kinetic model for opinion formation is to describe the evolution of the distribution of opinion by means of *microscopic* interactions among individuals in a society. Opinion is represented as a continuous variable  $w \in \mathcal{I}$  with  $\mathcal{I} = [-1, 1]$ , where  $\pm 1$  represent extreme opinions. Concerning political opinions,  $\mathcal{I}$  corresponds to the left–right political spectrum.

### (a) Toscani’s model

The study of the time evolution of the distribution of opinion among individuals in a simple, homogeneous society has been recently studied by means of kinetic collision-like models in Toscani (2006). This model is based on binary interactions.

When two individuals with pre-interaction opinion  $v$  and  $w$  meet, then their post-trade opinions  $v^*$  and  $w^*$  are given by

$$v^* = v - \gamma P(|v - w|)(v - w) + \eta_1 D(v)$$

and

$$w^* = w - \gamma P(|w - v|)(w - v) + \eta_2 D(w).$$

Here,  $\gamma \in (0, \frac{1}{2})$  is the constant *compromise parameter*. The quantities  $\eta_1$  and  $\eta_2$  are random variables with mean zero and variance  $\sigma^2$ . They model *self-thinking* that each individual performs in a random diffusion fashion through an exogenous, global access to information, e.g. through the press, television or internet. The functions  $P(\cdot)$  and  $D(\cdot)$  model the local relevance of compromise and self-thinking for a given opinion. To ensure that post-interaction opinions remain in the interval  $\mathcal{I}$ , additional assumptions need to be made on the random variables and the functions  $D(\cdot)$ .

(b) *A kinetic model with opinion leaders*

In this section, we propose a generalized model, where individuals from two different groups of individuals interact with each other. Human societies typically contain a set of individuals who, empirically speaking, strongly influence opinion through their strong personalities, financial means, access to media, etc. The sociophysical kinetic modelling of their effect on public opinion is based on the hypothesis that their own opinions are not changed through interactions with regular society members. Therefore, we consider two groups, one shall be identified with such ‘strong opinion leaders’ and the other with their followers, the ‘ordinary people’. We will adopt the hypothesis that all individuals belonging to one group share a common compromise parameter. This hypothesis can be further relaxed by assuming that the compromise parameter is a random quantity, with a statistical mean that is different for the two groups.

To some extent, this can be seen as an analogue to the physical problem of a mixture of gases, where the molecules of the different gases exchange momentum during collisions (Bobylyev & Gamba 2006). However, a complete analogy fails because the opinion leaders influence ordinary people in their opinion and maintain their own. A possibly better analogy is with the solid-state physics Boltzmann equation, where charged particles collide with a fixed phonon background (Markowich *et al.* 1995). If two individuals from the same group meet, the interaction shall, as in Toscani (2006), be given by ( $i = 1, 2$ )

$$v^* = v - \gamma_i P_i(|v - w|)(v - w) + \eta_{i1} D_i(v) \quad (2.1a)$$

and

$$w^* = w - \gamma_i P_i(|w - v|)(w - v) + \eta_{i2} D_i(w). \quad (2.1b)$$

If one individual from the group of ordinary people with opinion  $v$  meets a strong opinion leader with opinion  $w$ , their post-interaction opinions are given by

$$v^* = v - \gamma_3 P_3(|v - w|)(v - w) + \eta_{11} D_1(v) \quad (2.2a)$$

and

$$w^* = w. \quad (2.2b)$$

Again,  $\gamma_k \in (0, \frac{1}{2})$  ( $k = 1, 2, 3$ ) are constant *compromise parameters*, which control the ‘speed’ of attraction of two different opinions. This assumption can be further relaxed by choosing the compromise parameters as random quantities, each with a certain statistical mean. In the following, we assume, for simplicity, that all individuals in the society share a common compromise parameter  $\gamma$ . The quantities  $\eta_{ij}$  are random variables with distribution  $\Theta$ , variance  $\sigma_{ij}^2$  and zero mean, assuming values on a set  $\mathcal{B} \subset \mathbb{R}$ .

The functions  $P_i(\cdot)$  ( $i = 1, 2, 3$ ) and  $D_j(\cdot)$  ( $j = 1, 2$ ) model the local relevance of compromise and self-thinking for a given opinion, respectively. The random variable and the function  $D_j(\cdot)$  are characteristic for the particular class of individuals and are the same in both types of interaction, while the compromise function  $P_i(\cdot)$  can be different in the three types of interactions.

The first terms on the right-hand sides of equations (2.1a), (2.1b) and (2.2a) model the compromise process, the second the self-thinking process. Opinion leaders retain their opinion in equation (2.2b), when interacting with ordinary people, which reflects their high self-confidence and the ability to withstand other opinions. In our model, they can only be influenced through their peers, by interactions in equations (2.1a) and (2.1b). The pre-interaction opinion  $v$  increases (gets closer to  $w$ ) when  $v < w$  and decreases in the opposite situation. We assume that the ability to find a compromise is linked to the distance between opinions. The higher this distance is, the lower the possibility to find a compromise. Hence, functions  $P_i(\cdot)$  are assumed to be decreasing functions of their argument. We also assume that the ability to change individual opinions by self-thinking decreases as one gets closer to the extremal opinions. This reflects the fact that extremal opinions are more difficult to change. Therefore, we assume that functions  $D_j(\cdot)$  are decreasing functions of  $v^2$  with  $D_j(1) = 0$ . In equations (2.1a), (2.1b), (2.2a) and (2.2b), we will only allow interactions that guarantee  $v^*, w^* \in \mathcal{I}$ . To this end, we assume additionally

$$0 \leq P_i(|v - w|) \leq 1 \quad \text{and} \quad 0 \leq D_j(v) \leq 1.$$

We now need to choose the set  $\mathcal{B}$ , i.e. we have to specify the range of values the random variables can assume. Clearly, it depends on the particular choice for  $D_j(\cdot)$ . Let us consider the upper bound at  $w = 1$  first. To ensure that individuals’ opinions do not leave  $\mathcal{I}$ , we need

$$v^* = v - \gamma_i P_i(|v - w|)(v - w) + \eta_{ik} D_j(v) \leq 1.$$

Obviously, the worst case is  $w = 1$ , where we have to ensure

$$\eta_{ik} D_j(v) \leq 1 - v + \gamma_i(v - 1) = (1 - v)(1 - \gamma_i).$$

Hence, if  $D_j(v)/(1 - v) \leq K_+$ , it suffices to have  $|\eta_{ik}| \leq (1 - \gamma_i)/K_+$ . A similar computation for the lower boundary shows that if  $D_j(v)/(1 + v) \leq K_-$ , it suffices to have  $|\eta_{ik}| \leq (1 - \gamma_i)/K_-$ .

In this setting, we are led to study the evolution of the distribution function for each group as a function depending on the opinion  $w \in \mathcal{I}$  and time  $t \in \mathbb{R}_+$ ,  $f_i = f_i(w, t)$ . In analogous with the classical kinetic theory of mixtures of rarefied

gases, the time evolution of the distributions will obey a system of two Boltzmann-like equations, given by

$$\frac{\partial}{\partial t} f_1(w, t) = \frac{1}{\tau_{11}} \mathcal{Q}_{11}(f_1, f_1)(w) + \frac{1}{\tau_{12}} \mathcal{Q}_{12}(f_1, f_2)(w) \quad (2.3)$$

and

$$\frac{\partial}{\partial t} f_2(w, t) = \frac{1}{\tau_{22}} \mathcal{Q}_{22}(f_2, f_2)(w). \quad (2.4)$$

Herein,  $\tau_{ij}$  are suitable relaxation times that allow the control of the interaction frequencies of opinion leaders and followers. The Boltzmann-like collision operators are derived by standard methods of kinetic theory, considering that the change in time of  $f_i(w, t)$  due to binary interaction depends on a balance between the gain and loss of individuals with opinion  $w$ . The operators  $\mathcal{Q}_{11}$  and  $\mathcal{Q}_{22}$  relate to the microscopic interaction equations (2.1a) and (2.1b), whereas  $\mathcal{Q}_{12}$  relates to equations (2.2a) and (2.2b).

Let  $\langle \cdot \rangle$  denote the operation of mean with respect to the random quantities  $\eta_{ij}$ . A useful way of writing the collision operators is the so-called weak form. It corresponds to consider, for all smooth functions  $\phi(w)$ ,

$$\int_{\mathcal{I}} \mathcal{Q}_{ij}(f_i, f_j)(w) \phi(w) \, dw = \frac{1}{2} \left\langle \int_{\mathcal{I}^2} (\phi(w^*) + \phi(v^*) - \phi(w) - \phi(v)) f_i(v) f_j(w) \, dv \, dw \right\rangle. \quad (2.5)$$

### 3. Fokker–Planck limit system

In general, it is rather difficult to describe analytically the behaviour of the evolution of the densities. As is usual in kinetic theory, it is convenient to study certain asymptotics, which frequently lead to simplified models of Fokker–Planck type. By means of this approach, it is easier to identify steady states while retaining important information on the microscopic interaction at a macroscopic level. To this end, we study, by formal asymptotics, the quasi-invariant opinion limit ( $\gamma, \sigma_{ij} \rightarrow 0$  and  $\sigma_{ij}^2/\gamma = \lambda_{ij}$ ), following the path laid out in Toscani (2006). To study the situation for large times, i.e. close to the steady state, we introduce for  $\gamma \ll 1$  the transformation

$$\tau = \gamma t, \quad g_i(w, \tau) = f_i(w, t), \quad i = 1, 2,$$

which implies  $f_i(w, 0) = g_i(w, 0)$ . Denote by  $M_i = \int g_i \, dv$  ( $i = 1, 2$ ) the masses of the opinion leaders and followers, respectively. In appendix A, we derive the following system of Fokker–Planck limit equations:

$$\begin{aligned} \frac{\partial}{\partial \tau} g_1(w, \tau) = & \frac{\partial}{\partial w} \left( \left( \frac{1}{\tau_{11}} \mathcal{K}_1(w, \tau) + \frac{1}{2\tau_{12}} \mathcal{K}_3(w, \tau) \right) g_1(w, \tau) \right) \\ & + \left( \frac{\lambda_{11} M_1}{2\tau_{11}} + \frac{\lambda_{12} M_2}{4\tau_{12}} \right) \frac{\partial^2}{\partial w^2} (D_1^2(w) g_1(w, \tau)) \end{aligned} \quad (3.1a)$$

and

$$\frac{\partial}{\partial \tau} g_2(w, \tau) = \frac{\partial}{\partial w} \left( \frac{1}{\tau_{22}} \mathcal{K}_2(w, \tau) g_2(w, \tau) \right) + \frac{\lambda_{22} M_2}{2\tau_{22}} \frac{\partial^2}{\partial w^2} (D_2^2(w) g_2(w, \tau)), \quad (3.1b)$$



subject to the following no-flux boundary conditions (which result from the integration by parts)

$$\begin{aligned} & \left( \frac{1}{\tau_{11}} \mathcal{K}_1(w, \tau) + \frac{1}{2\tau_{12}} \mathcal{K}_3(w, \tau) \right) g_1(w, \tau) \\ & + \left( \frac{\lambda_{11} M_1}{2\tau_{11}} + \frac{\lambda_{12} M_2}{4\tau_{12}} \right) \frac{\partial}{\partial w} (D_1^2(w) g_1(w, \tau)) = 0 \quad \text{on } w = \pm 1, \end{aligned} \quad (3.2a)$$

$$\frac{1}{\tau_{22}} \mathcal{K}_2(w, \tau) g_2(w, \tau) + \frac{\lambda_{22} M_2}{2\tau_{22}} \frac{\partial}{\partial w} (D_2^2(w) g_2(w, \tau)) = 0 \quad \text{on } w = \pm 1 \quad (3.2b)$$

$$\text{and} \quad D_1^2(w) g_1(w) = D_2^2(w) g_2(w) = 0 \quad \text{on } w = \pm 1. \quad (3.2c)$$

Note that, if the solutions  $g_1$  and  $g_2$  are sufficiently regular, the third condition (3.2c) holds automatically because  $D_1(w) = D_2(w) = 0$  for  $w = \pm 1$ . The operators appearing in the drift term are defined as

$$\mathcal{K}_i(w, \tau) = \int_{\mathcal{I}} P_i(|w - v|)(w - v) g_i(v, \tau) \, dv \quad \text{for } i = 1, 2 \quad (3.3)$$

and

$$\mathcal{K}_3(w, \tau) = \int_{\mathcal{I}} P_3(|w - v|)(w - v) g_2(v, \tau) \, dv. \quad (3.4)$$

#### (a) Stationary solutions of the Fokker–Planck system

Next, we analyse explicitly computable stationary states of the Fokker–Planck system. Steady states are particular solutions of the time-dependent problem, which are candidates for the long-time limit of the Fokker–Planck system. In this section, we consider the special case  $P_i(|w - v|) \equiv 1$  ( $i = 1, 2, 3$ ), which implies conservation of the average opinion and the first momentum for equation (3.1b). From the application point of view, this case is less realistic; however, it allows us to explicitly solve for the steady states and to show their integrability. The analysis presented here for the special case combined with the numerical results in §4 for the general situation strongly suggests that the Fokker–Planck system admits integrable stationary states also in the general case, although they are not explicitly computable. For the sake of simplicity, we choose

$$D_1(w) = D_2(w) = D(w) := (1 - w^2)^\alpha, \quad (3.5)$$

with  $\alpha > \frac{1}{2}$  as a model for the diffusion, which is consistent with the requirement that post-collisional opinions have to be in  $\mathcal{I}$ , at least when the ranges of the random variables  $\eta_{ij}$  are sufficiently small. This function has been introduced in Toscani (2006) and includes that extremal opinions are more difficult to change than moderate ones. The choice of  $\alpha$  is directly related to the regions where diffusion of opinions is prevalent.



The steady state of equations (3.1a) and (3.1b) is a solution of

$$0 = \left( \frac{wM_1 - m_1}{\tau_{11}} + \frac{wM_2 - m_2}{2\tau_{12}} \right) g_{1,\infty}(w) + \left( \frac{\lambda_{11}M_1}{2\tau_{11}} + \frac{\lambda_{12}M_2}{4\tau_{12}} \right) (D^2(w)g_{2,\infty}(w))_w \quad (3.6a)$$

and

$$0 = \frac{wM_2 - m_2}{\tau_{22}} g_{2,\infty} + \frac{\lambda_{22}M_2}{2\tau_{22}} (D^2(w)g_{2,\infty})_w. \quad (3.6b)$$

We denote the masses of the opinion leaders and followers by  $M_i = \int g_{i,\infty} dv$ , with  $i = 1, 2$ , and their first-order moments by  $m_i = \int v g_{i,\infty} dv$ ,  $i = 1, 2$ .

Equation (3.6b) can be written as

$$-\frac{wM_2 - m_2}{D^2(w)} f_2 = \frac{\lambda_{22}M_2}{2} \frac{d}{dw} f_2,$$

with  $f_2 = D^2(w)g_{2,\infty}$ . Therefore,

$$f_2 = c_2 \exp \left[ -\frac{2}{\lambda_{22}M_2} \int_0^w \frac{vM_2 - m_2}{(1-v^2)^{2\alpha}} dv \right]$$

and hence

$$g_{2,\infty} = \frac{c_2}{(1-w^2)^{2\alpha}} \exp \left[ -\frac{2}{\lambda_{22}M_2} \int_0^w \frac{vM_2 - m_2}{(1-v^2)^{2\alpha}} dv \right], \quad (3.7)$$

where  $c_2$  is chosen such that the mass of  $g_{2,\infty}$  is equal to  $M_2$ . Note that since  $|m_2| < M_2$  and  $\alpha > \frac{1}{2}$ ,  $g_{2,\infty}(\pm 1) = 0$ . The solution of equation (3.6a) can be calculated using the same arguments

$$g_{1,\infty} = \frac{c_1}{(1-w^2)^{2\alpha}} \exp \left[ -k \int_0^w \left[ \frac{vM_1 - m_1}{\tau_{11}(1-v^2)^{2\alpha}} + \frac{vM_2 - m_2}{2\tau_{12}(1-v^2)^{2\alpha}} \right] dv \right], \quad (3.8)$$

with  $k = 4\tau_{11}\tau_{12}/(2\lambda_{11}M_1\tau_{12} + \lambda_{12}M_2\tau_{11})$ . The integrability of these steady states is discussed in appendix B. Integration of equation (3.6a) leads to

$$M_2m_1 - m_2M_1 = 0.$$

Therefore, we can fix  $m_1$  and rewrite equation (3.8) as

$$g_{1,\infty} = \frac{c_1}{(1-w^2)^{2\alpha}} \exp \left[ -k \left( \frac{M_1}{\tau_{11}} + \frac{M_2}{2\tau_{12}} \right) \int_0^w \frac{v}{(1-v^2)^{2\alpha}} dv \right] \times \exp \left[ km_2 \left( \frac{1}{2\tau_{12}} + \frac{M_1}{\tau_{11}M_2} \right) \int_0^w \frac{1}{(1-v^2)^{2\alpha}} dv \right].$$

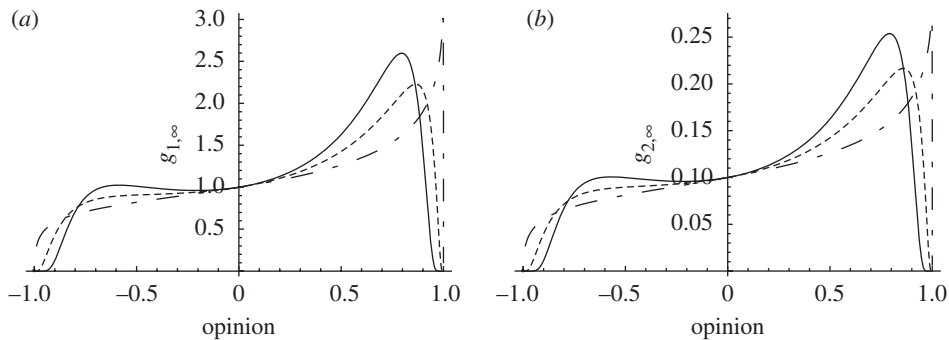


Figure 1. Illustration of stationary solutions for different values of  $\alpha$ . (a) Stationary solution  $g_{1,\infty}$  and (b) stationary solution  $g_{2,\infty}$ . Solid line,  $\alpha = 1$ ; dashed line,  $\alpha = 0.75$ ; dashed-dotted line,  $\alpha = 0.5025$ .

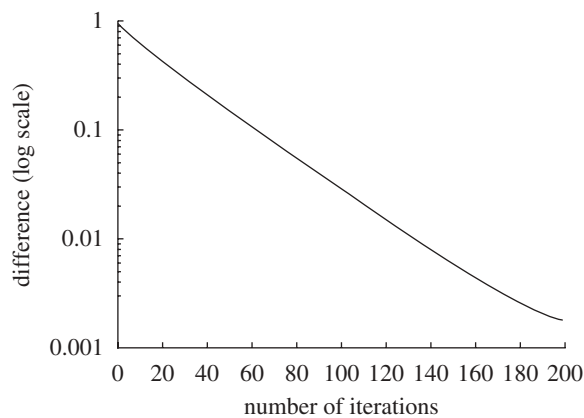


Figure 2. Difference between approximate solution  $g_2$  and exact solution  $g_{2,\infty}$  in  $L^1$ -norm.

From  $|m_2| < M_2$ , we conclude that if  $\alpha > 1/2$  then  $c_1$  can be determined such that the mass of  $g_{1,\infty}$  equals  $M_1$ . Figure 1 illustrates the behaviour of the stationary solutions for different values of  $\alpha$ . Here, we chose the following parameters:

$$M_1 = 1, \quad M_2 = 0.05, \quad m_2 = 0.01, \quad m_1 = \frac{m_2 M_1}{M_2} = 0.2,$$

$$\tau_{11} = 1, \quad \tau_{12} = 10 \quad \text{and} \quad \lambda_{ij} = 1 \quad \text{for all } i, j = 1, 2.$$

Note that the stationary solutions are symmetric with respect to the  $y$ -axis if the first-order moments vanish.

Numerical simulations provide strong evidence that solutions converge exponentially fast to their steady state (see figure 2); here, all parameters are set to 1, except  $m_1 = m_2 = 0$ . The mathematical analysis of solutions of the Fokker–Planck system (3.1a) and (3.1b) is the subject of a forthcoming paper.

### (b) Emergence and decline of opinion leaders

Opinion leadership is not constant over time. Someone who is an opinion leader today may lose this role, or a follower may become a leader in the near future. Hence, the emergence and decline of opinion leaders is an important process in a

society, which we would like to include in the limiting Fokker–Planck system (3.1*a*) and (3.1*b*). The proposed mathematical model for the emergence and decline of leaders is based on the following assumptions.

(A1) The overall mass of opinion leaders and followers is constant in time, i.e.

$$\frac{d}{d\tau} \int_{\mathcal{I}} (g_1(w, \tau) + g_2(w, \tau)) dw = 0.$$

(A2) The society has a certain characteristic percentage of strong opinion leaders in the long-run average, e.g. 5 per cent of the whole population may typically be opinion leaders. The society is assumed to approach this level of opinion leaders in the long run.

(A3) The exchange of information between followers causes the formation of ‘groups’ sharing a similar opinion, even if no strong leaders are present. If such a ‘group’ is sufficiently large, it is likely for somebody to step up and take the lead. Hence, if the density of followers sharing a similar opinion exceeds a certain minimal threshold  $\underline{c}$  and the overall number of leaders is less than the typical 5 per cent, then a leader promoting this opinion emerges.

(A4) If leaders promoting a certain opinion do not have enough followers, i.e. less than a particular threshold  $\bar{c}$ , and if there are more than the typical 5 per cent of leaders present in the whole society, then the leaders promoting this opinion decline.

Based on the assumptions stated above, we propose the following model:

$$\begin{aligned} \frac{\partial}{\partial \tau} g_1(w, \tau) = & \frac{\partial}{\partial w} \left( \left( \frac{1}{\tau_{11}} \mathcal{K}_1(w, \tau) + \frac{1}{2\tau_{12}} \mathcal{K}_3(w, \tau) \right) g_1(w, \tau) \right) \\ & + \left( \frac{\lambda_{11} M_1}{2\tau_{11}} + \frac{\lambda_{12} M_2}{4\tau_{12}} \right) \frac{\partial^2}{\partial w^2} (D^2(w) g_1(w, \tau)) - a(g_1) g_1 + b(g_1) g_2 \end{aligned} \quad (3.9a)$$

and

$$\begin{aligned} \frac{\partial}{\partial \tau} g_2(w, \tau) = & \frac{\partial}{\partial w} \left( \frac{1}{\tau_{22}} \mathcal{K}_2(w, \tau) g_2(w, \tau) \right) \\ & + \frac{\lambda_{22} M_2}{2\tau_{22}} \frac{\partial^2}{\partial w^2} (D^2(w) g_2(w, \tau)) + a(g_1) g_1 - b(g_1) g_2. \end{aligned} \quad (3.9b)$$

The function  $a(g_1)$  models the emergence of leaders, see assumption (A3), by

$$a(g_1) = \mathbf{1}_{\{g_1(w) \geq \underline{c}\}} e^{-(M_1 - M)^2 / \sqrt{2\pi} \sigma_1},$$

where  $\mathbf{1}_A$  is the indicator function of the set  $A$  and  $M = M_1 + M_2$  is the overall mass of followers and opinion leaders. If the number of followers sharing the same opinion is greater than the threshold  $\underline{c}$ , then leaders can emerge with a rate depending on the mass of leaders  $M_2$ . The parameter  $\sigma_1$  is chosen such that the exponential function assumes very small values on the interval  $[0, 0.95M]$ , i.e. leaders can only emerge if they make up less than 5 per cent of the overall

population. The function  $b$  corresponds to assumption (A4), i.e. the decline of leaders

$$b(g_1) = \mathbf{1}_{\{g_1(w) \leq \bar{c}\}} e^{-M_1^2/\sqrt{2\pi}\sigma_2}.$$

If the density of normal people sharing a particular opinion is below a certain threshold  $\bar{c}$ , the number of leaders promoting this opinion declines (depending on the overall mass of followers  $M_1$ ). Here, the parameter  $\sigma_2$  is chosen such that the exponential function assumes very small values on the interval  $[0.95M, M]$ , i.e. leaders can only decline if they make up more than 5 per cent of the overall population. With this extension of our model, we shall obtain first insights in the emergence and decline of strong leaders.

#### 4. Numerical simulations

In this section, we illustrate the behaviour of the kinetic model and the limiting Fokker–Planck system with various simulations. We assume that the diffusion of opinion is given by equation (3.5) and the compromise propensity  $P_i(\cdot)$  ( $i = 1, 2, 3$ ) by

$$P_i(|v - w|) = \mathbf{1}_{\{|v - w| \leq r_i\}}. \quad (4.1)$$

The following parameters are fixed throughout this section, if not mentioned otherwise:

- relaxation times:  $\tau_{11} = \tau_{12} = \tau_{22} = 1$ ;
- ratio of normal people to opinion leaders:  $M_1 = 0.95$  and  $M_2 = 0.05$ ;
- diffusion parameters:  $\lambda_{11} = \lambda_{12} = \lambda_{22} = \lambda := 5 \times 10^{-3}$ ; and
- exponent of the diffusion function in equation (3.5):  $\alpha = 2$ .

The initial distribution of normal people is given by a Gaussian

$$g_1(w, 0) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(w-\sigma_1)^2/2}, \quad (4.2)$$

with  $\sigma_1 = 0.4$ . The initial distribution of opinion leaders is

$$g_2(w, 0) = \sum_{i=1}^n \frac{q_i}{\sqrt{2\pi}\sigma_i} e^{-(w-\sigma_i)^2/2}, \quad (4.3)$$

with weights  $\sum_{i=1}^n q_i = 1$ .

##### (a) Monte Carlo simulations

To illustrate the relaxation behaviour and to study the influence of the different model parameters, we have performed a series of kinetic Monte Carlo simulations for the Boltzmann model presented in the previous section. Generally, in this kind of simulations, known as direct simulation Monte Carlo (DSMC) or Bird's scheme, pairs of individuals are randomly and non-exclusively selected for binary collisions, and exchange opinion according to the rule under consideration. Let us denote by  $N_i$  ( $i = 1, 2$ ) the number of individuals in the groups we consider in our simulation. One *time step* in our simulation corresponds to  $N_1 + N_2$

interactions. The average of  $M = 10$  simulations is used as an approximate steady opinion distribution. To compute a good approximation of the steady state, each simulation is carried out for about  $10^6$  time steps, and then the opinion distribution is averaged over another  $10^3$  time steps. We choose  $N_1 = 1900$ ,  $N_2 = 100$  and  $\gamma = 0.02$ . The random variables are chosen such that  $\eta_{ij}$  assume only values  $\pm v = \pm 0.01$  with equal probability. The initial distributions are chosen as discrete analogues of equations (4.2) and (4.3).

(b) *Numerical solution of the Fokker–Planck system*

To illustrate the long-time behaviour of the proposed model, we discretize the non-linear Fokker–Planck system (3.1a) and (3.1b) using a hybrid discontinuous Galerkin (DG) method introduced by Egger & Schöberl (in press). This hybrid DG method was initially developed for convection diffusion equations and yields stable discretizations for convection-dominated problems as well as hyperbolic ones. In addition, the method is conservative, which is consistent with the assumption that the initial mass of the Fokker–Planck system is preserved in time.

We choose a partition of the time interval  $[0, T]$ ,  $0 = t_0 < t_1 < \dots < t_j < \dots < t_m = T$ , and define  $\Delta t_j = t_{j+1} - t_j$ . We consider the following linearization of the Fokker–Planck equations (3.1a) and (3.1b), which fits into the framework of Egger & Schöberl (in press):

$$\begin{aligned} \frac{g_1^{j+1} - g_1^j}{\Delta t_j} = & \frac{\partial}{\partial w} \left( \left( \frac{1}{\tau_{11}} \mathcal{K}_1(g_1^j; w, t) + \frac{1}{\tau_{12}} \mathcal{K}_3(g_2^j; w, t) \right) g_1^{j+1}(w, t) \right) \\ & + \left( \frac{\lambda_{11} M_1}{2\tau_{22}} + \frac{\lambda_{12} M_2}{4\tau_{12}} \right) \frac{\partial^2}{\partial w^2} \left( D^2(w) g_1^{j+1}(w, t) \right) \end{aligned} \quad (4.4a)$$

and

$$\frac{g_2^{j+1} - g_2^j}{\Delta t_j} = \frac{\partial}{\partial w} \left( \frac{1}{\tau_{22}} \mathcal{K}_2(g_2^j; w, t) g_2^{j+1}(w, t) \right) + \frac{\lambda_{22} M_2}{2\tau_{22}} \frac{\partial^2}{\partial w^2} \left( D^2(w) g_2^{j+1}(w, t) \right). \quad (4.4b)$$

Here,  $g_i^j$ ,  $i = 1, 2$ , denotes the solution at time  $t = t_j$ . We choose an equidistant mesh of mesh size  $h = 1/400$  to discretize the interval  $[-1, 1]$ . The time steps  $\Delta t_j$  are set to 0.01.

(c) *Numerical results*

(i) *Influence of interaction radii  $r_i$  and distribution of opinion leaders*

We choose a symmetric initial distribution of opinion leaders with  $w_i = \pm 0.5$ ,  $q_i = 0.5$  and  $\sigma_i = 0.05$ , where  $i = 1, 2$ . The interaction radii take the same value,  $r_i = 0.5$ , for  $i = 1, \dots, 3$ . The behaviour of both species is illustrated in figure 3. For the followers, the results of the numerical solution of the Fokker–Planck system and the Monte Carlo simulation agree well. For the opinion leaders, the peaked, high densities that are obtained from the numerical solution of the Fokker–Planck system cannot be as well resolved by the Monte Carlo simulation.

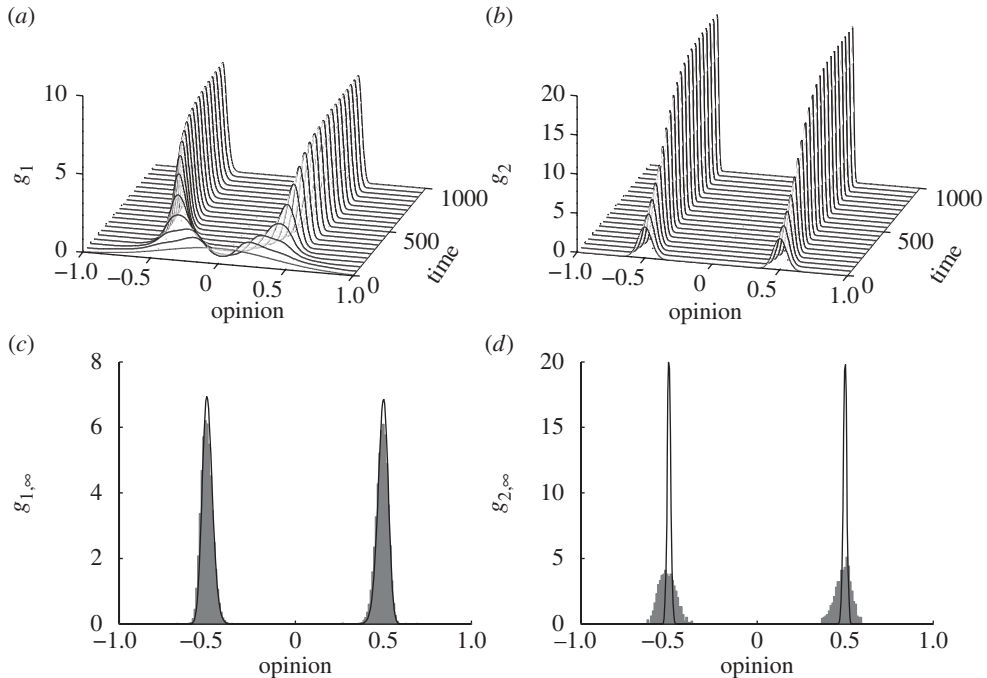


Figure 3. Numerical solutions of the Fokker–Planck system (*a–d*) and density histograms of the Monte Carlo simulation (*c,d*) with  $r_1 = r_2 = r_3 = 0.5$ . Evolution of (*a*) normal people and (*b*) opinion leaders. Stationary solution of (*c*) normal people and (*d*) opinion leaders.

The number of leaders is fixed to be 5 per cent of the total population and hence the number of realizations is rather small. Increasing  $M$  and  $N_1$ ,  $N_2$  will lead to a better resolution, but will render the method to be computationally infeasible. Therefore, in our more involved examples, we will rely on the numerical solution of the Fokker–Planck system.

If we change the interaction radius to  $r_i = 0.3$ ,  $i = 1, \dots, 3$ , the formation of a small group centred at  $w = 0$ , which is not attracted by the opinion leader, can be observed (see figure 4).

If  $r_1 = 0.6$  and  $r_2 = r_3 = 0.3$ , we observe that the interaction of the normal people with each other dominate the opinion formation process and results in an aggregation at  $w = 0$  (see figure 5).

Next, we illustrate the opinion formation process with a non-symmetric initial distribution of opinion leaders. We choose  $w_1 = -0.7$  and  $w_2 = 0.5$  with interaction radii  $r_i = 0.5$  for  $i = 1, 2, 3$ . The behaviour is illustrated in figure 6.

## (ii) *Understanding Carinthia*

In our next example, we would like to illustrate the behaviour of our model for opinion formation under extreme conditions, such as in Carinthia. Carinthia is the southernmost state of Austria. Carinthia's landscape of political parties shows an interesting peculiarity. In 1999, the right-wing Freedom Party of Austria (FPÖ) became the strongest party in Carinthia. Since then, their results in elections

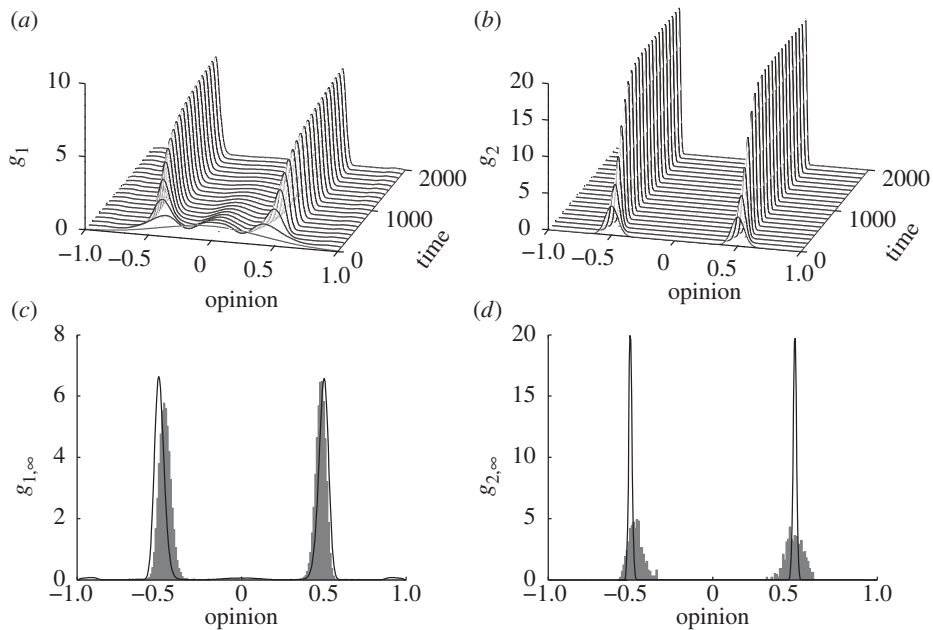


Figure 4. Numerical solutions of the Fokker–Planck system (*a–d*) and density histograms of the Monte Carlo simulation (*c,d*) with  $r_1 = r_2 = r_3 = 0.3$ . Evolution of (*a*) normal people and (*b*) opinion leaders. Stationary solution of (*c*) normal people and (*d*) opinion leaders.

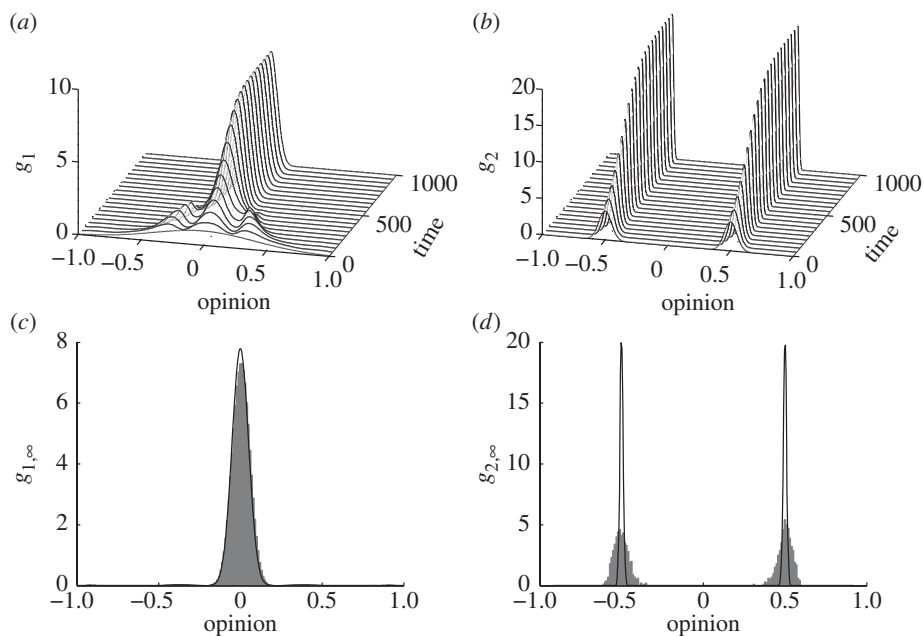


Figure 5. Numerical solutions of the Fokker–Planck system (*a–d*) and density histograms of the Monte Carlo simulation (*c,d*) with  $r_1 = 0.6$  and  $r_2 = r_3 = 0.3$ . Evolution of (*a*) normal people and (*b*) opinion leaders. Stationary solution of (*c*) normal people and (*d*) opinion leaders.



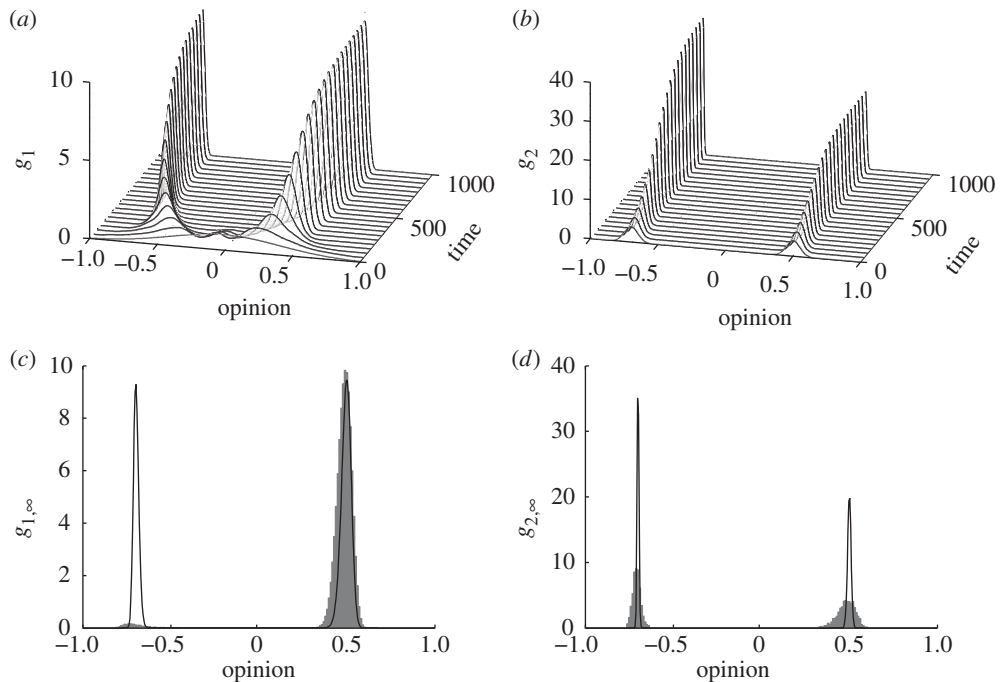


Figure 6. Numerical solutions of the Fokker–Planck system (*a–d*) and density histograms of the Monte Carlo simulation (*c,d*) with non-symmetric initial data for the opinion leaders and with  $r_1 = r_2 = r_3 = 0.5$ . Evolution of (*a*) normal people and (*b*) opinion leaders. Stationary solution of (*c*) normal people and (*d*) opinion leaders.

Table 1. Results of the state elections (%) in Carinthia.

	Grüne	SPÖ	ÖVP	FPÖ	BZÖ
2004	6.7	38.4	11.6	42.5	—
2009	5.2	28.8	16.8	3.8	44.9

for the state assembly (Landtagswahlen) continually improved, holding almost 45 per cent of the votes in 2008. This outcome was strongly influenced by the popularity of their party leader Jörg Haider. Haider, a controversial figure, was frequently criticized in Austria and abroad, being considered populist, extreme-right or even antisemitic. On the other hand, he was strongly acclaimed by his followers. Haider had been elected Carinthian governor in 1989, but was forced to step down 2 years later after his remarks about a ‘proper employment policy’ in the Third Reich. He was elected again as Carinthian governor in 1999 and re-elected in 2004. Haider, who practically led the FPÖ single-handedly, was able to unite the political spectrum from conservatives to extreme-right and establish a governing party whose success was less founded on political ideologies rather than on the authority of one opinion leader—Haider himself.

Table 1 shows the results of the state elections in 2004 and 2009, respectively.

We set the initial distribution of normal people to

$$g_1(w, 0) = \frac{0.07}{\sigma_1 \sqrt{2\pi}} e^{-(w+0.75)^2/2\sigma_1^2} + \frac{0.385}{\sigma_1 \sqrt{2\pi}} e^{-(w+0.25)^2/2\sigma_1^2} \\ + \frac{0.115}{\sigma_1 \sqrt{2\pi}} e^{-(w-0.25)^2/2\sigma_1^2} + \frac{0.45}{\sigma_1 \sqrt{2\pi}} e^{-(w-0.8)^2/2\sigma_1^2}, \quad (4.5)$$

where the weights of the Gaussian distributions are chosen in accordance with the results of the Landtagswahlen in 2004 (see [table 1](#)). Here,  $w = -0.75$  corresponds to the Greens (Grüne),  $w = -0.25$  to the Social Democratic Party of Austria (SPÖ),  $w = 0.25$  to the Austrian People's Party (ÖVP) and  $w = 0.8$  to the FPÖ. We assume that there are several opinion leaders present in the system associated with the different parties, but with different weights representing their influence. The initial distribution of opinion leaders is given by

$$g_2(w, 0) = \frac{0.1}{\sigma_2 \sqrt{2\pi}} e^{-(w+0.75)^2/2\sigma_2^2} + \frac{0.15}{\sigma_2 \sqrt{2\pi}} e^{-(w+0.2)^2/2\sigma_2^2} \\ + \frac{0.3}{\sigma_2 \sqrt{2\pi}} e^{-(w-0.25)^2/2\sigma_2^2} + \frac{0.45}{\sigma_2 \sqrt{2\pi}} e^{-(w-0.8)^2/2\sigma_2^2}. \quad (4.6)$$

We choose the following parameters:

$$\alpha = 1.5, \quad \lambda = 3 \times 10^{-3}, \quad r_1 = r_2 = 0.2, \quad r_3 = 0.45, \\ \tau_{11} = \tau_{12} = 1, \quad \tau_{22} = 10, \quad \sigma_1 = 0.1 \quad \text{and} \quad \sigma_2 = 0.05.$$

The behaviour of the solution is depicted in [figure 7](#). We observe that, in the presence of the stronger ÖVP leader, people move from the SPÖ to the ÖVP, while the people with an extreme opinion accumulate around the strong leader at  $w = 0.8$ . Note that a small group of people splits from the initial density at  $w = 0.8$  (initially attracted by the strong leader at  $w = 0.5$ ) and form a new group at  $w = 0.7$ . This formation can be interpreted as the separation of two parties associated with the formation of a new opinion leader. This is an interesting similarity with the real situation in Carinthia. In April 2005, Haider formed a new party, the Alliance for the Future of Austria (BZÖ), with himself as the leader, thereby de facto splitting the FPÖ into two parties. Haider died in a car crash in October 2008. In the elections in March 2009, the BZÖ, strongly referring to its deceased leader, managed to enlarge its share of votes to 44.9 per cent, while the FPÖ failed to enter the Landtag.

### (iii) *Emergence and decline of opinion leaders*

In our final example, we would like to show the emergence and decline of opinion leaders. We now solve the system (3.9a) and (3.9b) with equation (4.2) as the initial distribution of the followers and set  $g_2(w, 0) = 0$ . Furthermore, we assume that the leaders make up 5 per cent of the population in equilibrium. The interaction radii are  $r_i = 0.3$ , for  $i = 1, 2, 3$ , and the upper and lower thresholds are given by

$$\underline{c} = 1.0 \quad \text{and} \quad \bar{c} = 10^{-3}.$$

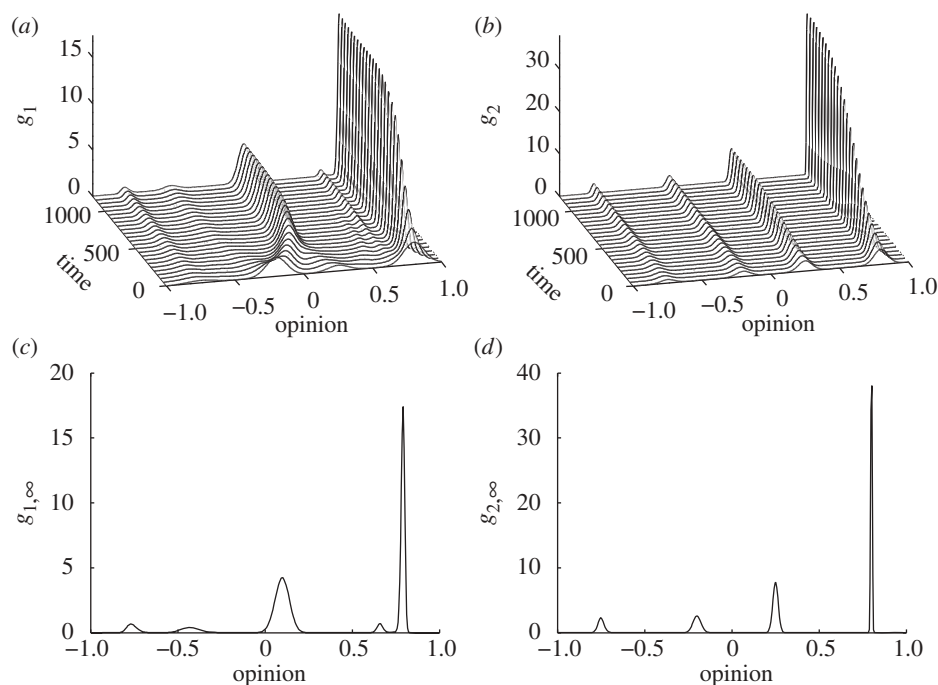


Figure 7. Opinion formation in Carinthia. Evolution of (a) normal people and (b) opinion leaders. Stationary solution of (c) normal people and (d) opinion leaders.

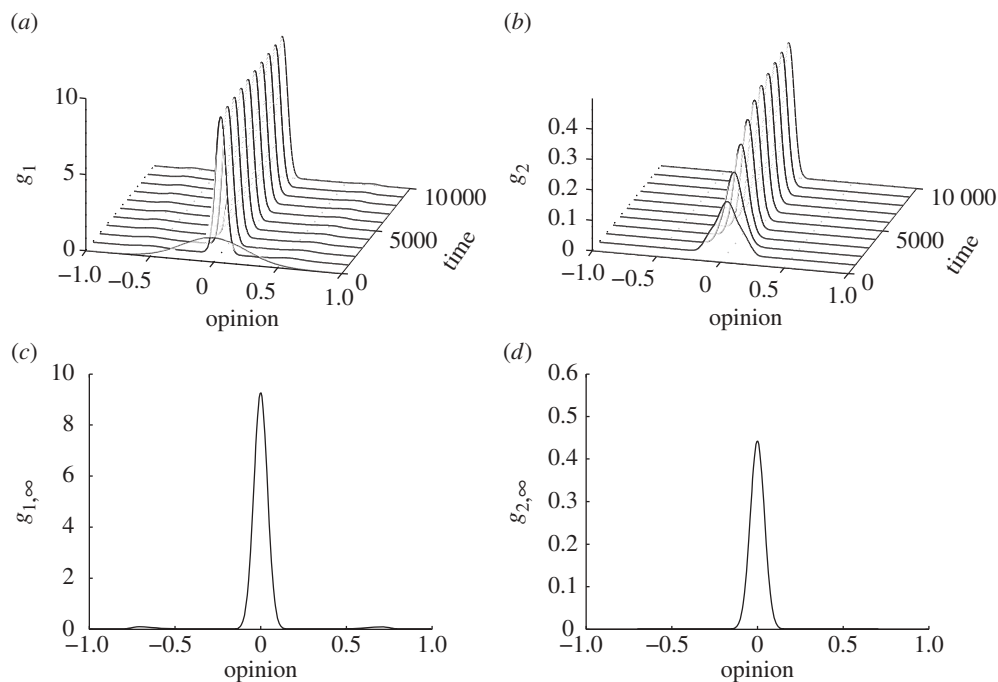


Figure 8. Emergence of opinion leaders. Evolution of (a) normal people and (b) opinion leaders. Stationary solution of (c) normal people and (d) opinion leaders.

The evolution of the normal people and the emergence of an opinion leader at  $w = 0$  is illustrated in figure 8. Note that the emergence of leaders stops when they make up 5 per cent of the overall population and that no leaders can emerge at  $w = \pm 0.7$  because the density of followers does not exceed the threshold  $\underline{c}$ .

## 5. Conclusions

We introduced and discussed a nonlinear kinetic model for a society that is built of two social groups, a group of strong opinion leaders and a group of ordinary people. The evolution of opinion is described by a system of Boltzmann-like equations in which collisions describe binary exchanges of opinion and self-thinking. We showed that at suitably large times, in the presence of a large number of interactions in each of which individuals change their opinions only slightly, the nonlinear system of Boltzmann-type equations is well approximated by a system of Fokker–Planck-type equations, which admits different, non-trivial steady states that depend on the specific choice of the compromise and self-thinking functions and parameters. We extended this model by allowing for emergence and decline of opinion leaders.

This work was supported by award no. KUK-I1-007-43, made by King Abdullah University of Science and Technology (KAUST) and by the Leverhulme Trust through the research grant entitled ‘Kinetic and mean field partial differential models for socio-economic processes’ (P.M.). P.M. also acknowledges the support from the Royal Society through his Wolfson Research Merit Award. B.D. was partly supported by the Deutsche Forschungsgemeinschaft (DFG), grant JU 359/6 (Forschergruppe 518). B.D. thanks the Department of Applied Mathematics and Theoretical Physics of University of Cambridge, where part of this research has been carried out, for their kind hospitality.

## Appendix A. Derivation of the Fokker–Planck limit system

Let us introduce some notation, analogous to Toscani (2006). First, restrict the test functions  $\phi$  to  $\mathcal{C}^{2,\delta}([-1, 1])$  for some  $\delta > 0$ . We use the usual Hölder norms

$$\|\phi\|_\delta = \sum_{|\alpha| \leq 2} \|D^\alpha \phi\|_C + \sum_{\alpha=2} [D^\alpha \phi]_{\mathcal{C}^{0,\delta}},$$

where

$$[h]_{\mathcal{C}^{0,\delta}} = \sup_{v \neq w} \frac{|h(v) - h(w)|}{|v - w|^\delta}. \quad (\text{A } 1)$$

Denoting by  $\mathcal{M}_0(A)$ ,  $A \subset \mathbb{R}$  the space of probability measures on  $A$ , we define

$$\mathcal{M}_p(A) = \left\{ \Theta \in \mathcal{M}_0 \left| \int_A |\eta|^p d\Theta(\eta) < \infty, p \geq 0 \right. \right\}, \quad (\text{A } 2)$$

the space of measures with finite  $p$ th momentum. In the following, all our probability densities belong to  $\mathcal{M}_{2+\delta}$  and we assume that the density  $\Theta$  is obtained from a random variable  $Y$  with zero mean and unit variance. We then obtain

$$\int_{\mathcal{I}} |\eta|^p \Theta(\eta) d\eta = \mathbb{E}[|\sigma Y|^p] = \sigma^p \mathbb{E}[|Y|^p], \quad (\text{A } 3)$$

where  $E[|Y|^p]$  is finite. The weak form of equation (2.3) is given by

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{I}} f_1(w, t) \phi(w) dw &= \int_{\mathcal{I}} \frac{1}{\tau_{11}} \mathcal{Q}_{11}(f_1, f_1)(w) \phi(w) dw \\ &+ \int_{\mathcal{I}} \frac{1}{\tau_{12}} \mathcal{Q}_{12}(f_1, f_2)(w) \phi(w) dw \end{aligned} \quad (\text{A } 4a)$$

and

$$\frac{d}{dt} \int_{\mathcal{I}} f_2(w, t) \phi(w) dw = \int_{\mathcal{I}} \frac{1}{\tau_{22}} \mathcal{Q}_{22}(f_2, f_2)(w) \phi(w) dw, \quad (\text{A } 4b)$$

where the terms on the right-hand sides are given by equation (2.5). To study the situation for large times, i.e. close to the steady state, we introduce, for  $\gamma \ll 1$ , the transformation

$$\tau = \gamma t, \quad g_i(w, \tau) = f_i(w, t), \quad i = 1, 2.$$

This implies  $f_i(w, 0) = g_i(w, 0)$  and the evolution of the scaled densities  $g_i(w, \tau)$  follows

$$\begin{aligned} \frac{d}{d\tau} \int_{\mathcal{I}} g_1(w, \tau) \phi(w) dw &= \frac{1}{\gamma} \int_{\mathcal{I}} \frac{1}{\tau_{11}} \mathcal{Q}_{11}(f_1, f_1)(w) \phi(w) dw \\ &+ \frac{1}{\gamma} \int_{\mathcal{I}} \frac{1}{\tau_{12}} \mathcal{Q}_{12}(f_1, f_2)(w) \phi(w) dw \end{aligned} \quad (\text{A } 5a)$$

and

$$\frac{d}{d\tau} \int_{\mathcal{I}} g_2(w, \tau) \phi(w) dw = \frac{1}{\gamma} \int_{\mathcal{I}} \frac{1}{\tau_{22}} \mathcal{Q}_{22}(f_2, f_2)(w) \phi(w) dw. \quad (\text{A } 5b)$$

Consider the first term on the right-hand side of equation (A 5a). Due to the collision rule (2.1a) and (2.1b), it holds that

$$w^* - w = -\gamma P_1(|w - v|)(w - v) + \eta_{11} D_1(w).$$

Taylor expansion of  $\phi$  up to second order around  $w$  in the first term of the right-hand side of equation (A 5a) leads to

$$\begin{aligned} &\left\langle \frac{1}{\gamma \tau_{11}} \int_{\mathcal{I}^2} \phi'(w) [-\gamma P_1(|w - v|)(w - v) + \eta_{11} D_1(w)] g_1(w) g_1(v) dv dw \right\rangle \\ &+ \left\langle \frac{1}{2\gamma \tau_{11}} \int_{\mathcal{I}^2} \phi''(\tilde{w}) [-\gamma P_1(|w - v|)(w - v) + \eta_{11} D_1(w)]^2 g_1(w) g_1(v) dv dw \right\rangle \\ &= \frac{1}{\gamma \tau_{11}} \int_{\mathcal{I}^2} \phi'(w) [-\gamma P_1(|w - v|)(w - v)] g_1(w) g_1(v) dv dw \\ &+ \left\langle \frac{1}{2\gamma \tau_{11}} \int_{\mathcal{I}^2} \phi''(w) [\gamma P_1(|w - v|)(w - v) + \eta_{11} D_1(w)]^2 g_1(w) g_1(v) dv dw \right\rangle \\ &+ R(\gamma, \sigma_{11}) \\ &= -\frac{1}{\tau_{11}} \int_{\mathcal{I}} \phi'(w) \mathcal{K}_1(w) g_1(w) dw \\ &+ \frac{1}{2\gamma \tau_{11}} \int_{\mathcal{I}^2} \phi''(w) [\gamma^2 P_1^2(|w - v|)(w - v)^2 + \gamma \lambda_{11} D_1^2(w)] g_1(w) g_1(v) dv dw \\ &+ R(\gamma, \sigma_{11}), \end{aligned}$$

with  $\tilde{w} = \kappa w^* + (1 - \kappa)w$  for some  $\kappa \in [0, 1]$  and

$$R(\gamma, \sigma_{11}) = \left\langle \frac{1}{2\gamma\tau_{11}} \int_{\mathcal{I}^2} (\phi''(\tilde{w}) - \phi''(w)) \times [-\gamma P_1(|w - v|)(w - v) + \eta_{11} D_1(w)]^2 g_1(w) g_1(v) dv dw \right\rangle.$$

Here, we defined

$$\mathcal{K}_i(w, \tau) = \int_{\mathcal{I}} P_i(|w - v|)(w - v) g_i(v, \tau) dv \quad \text{for } i = 1, 2 \quad (\text{A } 6)$$

and

$$\mathcal{K}_3(w, \tau) = \int_{\mathcal{I}} P_3(|w - v|)(w - v) g_2(v, \tau) dv. \quad (\text{A } 7)$$

Now we consider the formal limit  $\gamma, \sigma_{11} \rightarrow 0$  while keeping  $\lambda_{11} = \sigma_{11}^2/\gamma$  fixed. We will later argue that the remainder vanishes in this limit. Then, the first term on the right-hand side of equation (A 5a) converges to

$$\begin{aligned} & -\frac{1}{\tau_{11}} \int_{\mathcal{I}} \phi'(w) \mathcal{K}_1(w) g_1(w) dw + \frac{1}{2\tau_{11}} \int_{\mathcal{I}^2} \phi''(w) [\lambda_{11} D_1^2(w)] g_1(w) g_1(v) dv dw \\ & = -\frac{1}{\tau_{11}} \int_{\mathcal{I}} \phi'(w) \mathcal{K}_1(w) g_1(w) dw + \frac{\lambda_{11} M_1}{2\tau_{11}} \int_{\mathcal{I}} \phi''(w) D_1^2(w) g_1(w) dw. \end{aligned}$$

For the second term on the right-hand side of equation (A 5a), we obtain in the same way

$$-\frac{1}{2\tau_{12}} \int_{\mathcal{I}} \phi'(w) \mathcal{K}_2(w) g_1(w) dw + \frac{\lambda_{12} M_2}{4\tau_{12}} \int_{\mathcal{I}} \phi''(w) D_1^2(w) g_1(w) dw.$$

Performing a similar analysis for the right-hand side of equation (A 5b), we obtain, after integration by parts, the system of Fokker–Planck equations (3.1a) and (3.1b) subject to the no-flux boundary conditions (3.2a)–(3.2c) that result from the integration by parts. Note, however, that if the solutions  $g_1$  and  $g_2$  are sufficiently regular, the third condition holds automatically because  $D_1(w) = D_2(w) = 0$  for  $w = \pm 1$ . What is left is to show that the remainder terms  $R(\gamma, \sigma_{ij})$  vanish in the above limit. We consider only  $R(\gamma, \sigma_{11})$ , as the argument is similar for all the remainder terms occurring in the limit process of equation (A 5b). Note first that as  $\phi \in \mathcal{F}_{2+\delta}$ , by equations (2.1a) and (2.1b) and the definition of  $\tilde{w}$ , we have

$$\begin{aligned} |\phi''(\tilde{w}) - \phi''(w)| & \leq \|\phi''\|_{\delta} |\tilde{w} - w|^{\delta} \leq \|\phi''\|_{\delta} |w^* - w|^{\delta} \\ & = \|\phi''\|_{\delta} |\gamma P_1(|w - v|)(w - v) + \eta_{11} D_1(w)|^{\delta}. \end{aligned}$$

Thus, we obtain

$$R(\gamma, \sigma_{11}) \leq \frac{\|\phi''\|_{\delta}}{2\gamma\tau_{11}} \left\langle \int_{\mathcal{I}^2} [-\gamma P_1(|w - v|)(w - v) + \eta_{11} D_1(w)]^{2+\delta} g_1(w) g_1(v) dv dw \right\rangle.$$

Furthermore, we note that

$$\begin{aligned} & [\eta_{11} D_1(w) - \gamma P_1(|w - v|)(w - v)]^{2+\delta} \\ & \leq 2^{1+\delta} (|\gamma P_1(|w - v|)(w - v)|^{2+\delta} + |\eta_{11} D_1(w)|^{2+\delta}) \\ & \leq 2^{3+2\delta} |\gamma|^{2+\delta} + 2^{1+\delta} |\eta_{11}|^{2+\delta}. \end{aligned}$$

Here, we used the convexity of  $f(s) := |s|^{2+\delta}$  and the fact that  $w, v \in \mathcal{I}$  and thus is bounded. We conclude

$$\begin{aligned} |R(\gamma, \sigma_{11})| & \leq \frac{C \|\phi''\|_\delta}{\tau_{11}} \left( \gamma^{1+\delta} + \frac{1}{2\gamma} \langle |\eta_{11}|^{2+\delta} \rangle \right) \\ & = \frac{C \|\phi''\|_\delta}{\tau_{11}} \left( \gamma^{1+\delta} + \frac{1}{2\gamma} \int_{\mathcal{I}} |\eta_{11}|^{2+\delta} \Theta(\eta_{11}) d\eta_{11} \right). \end{aligned}$$

Since  $\Theta \in \mathcal{M}_{2+\delta}$  and  $\eta_{11}$  has variance  $\sigma_{11}^2$ , we have (see equation (A 3))

$$\int_{\mathcal{I}} |\eta_{11}|^{2+\delta} \Theta(\eta_{11}) d\eta_{11} = \mathbb{E} \left[ \left| \sqrt{\lambda_{11} \gamma} Y \right|^{2+\delta} \right] = (\lambda_{11} \gamma)^{1+\delta/2} \mathbb{E} [|Y|^{2+\delta}]. \quad (\text{A } 8)$$

Thus, we conclude that the terms  $R(\gamma, \sigma_{ij})$  vanish in the limit  $\gamma, \sigma_{ij} \rightarrow 0$  while keeping  $\lambda_{ij} = \sigma_{ij}^2/\gamma$  fixed.

## Appendix B. Integrability of the steady states

Consider the steady state of  $g_2$  given by equation (3.7). To enhance readability, we set ‘non-critical’ constants equal to 1 in the sequel. So we have to decide on the integrability of

$$\int_{-1}^1 \frac{1}{D^2(w)} \exp \left( - \int_0^w \frac{v M_2 - m_2}{D^2(v)} dv \right) dw. \quad (\text{B } 1)$$

The behaviour close to  $w = \pm 1$  is decisive for the existence of the integral equation (B 1). Consider first the behaviour at  $w = 1$ . We make use of the substitution  $u = \int_0^w dv/D^2(v)$  such that  $du = (1/D^2(w))dw$ . Choosing  $\zeta = \int_0^\beta dv/D^2(v)$  such that  $v M_2 - m_2 \geq \zeta M_2 - m_2 > \chi > 0$ , we can estimate

$$\int_\zeta^1 \exp \left( - \int_0^w \frac{v M_2 - m_2}{D^2(v)} dv \right) \frac{dw}{D^2(w)} \leq \int_\beta^\infty \exp(-\chi u) du < \infty.$$

In the same way, the behaviour close to  $-1$  is analysed. Hence, the steady-state  $g_{2,\infty}$  is integrable. The integrability of equation (3.8) is shown using similar arguments.

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