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INTERPOLATION OF PERIODICALLY CORRELATED STOCHASTIC SEQUENCES

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Abstract. We study the problem of optimal estimation of a linear functional of unknown values of a periodically correlated random sequence from observed values of a sequence with an additive noise. Formulas for calculating the mean square error and spectral characteristic of the optimal linear estimate of a functional are established in the case where the spectral densities are known. The least favorable spectral densities and minimax spectral characteristic of the optimal linear estimate of a functional are found for some classes of admissible spectral densities.

1. Introduction

Gladyshev [5] studied the spectral properties and representations of periodically correlated sequences. His results are based on relationships between periodically correlated sequences and vector stationary sequences. Following the Gladyshev approach, the estimation problem for periodically correlated sequences can be reduced to the corresponding problem for vector stationary sequences. The main results concerning representations of periodically correlated sequences in terms of simpler random sequences are given in the book by Hurd and Miamee [8].

Classical methods for solving the problems of extrapolation, interpolation, and filtration for stationary processes with known spectral densities are developed by Kolmogorov [9], Wiener [21], and Yaglom [22, 23]. The problem of prediction for vector stationary sequences is studied by Rozanov [20]. If the spectral densities are unknown, but a set of admissible spectral densities is specified instead, the minimax method can be used in the problem of estimation. This method consists in minimizing the error for all densities of a given class simultaneously. Grenander [6] is the first to use this approach for the problem of extrapolation of stationary processes. The problem of minimax extrapolation and filtration of stationary sequences is studied by Franke [2, 3] and Franke and Poor [4] with the help of methods of convex optimization. Moklyachuk [10]–[16] and Moklyachuk and Masyutka [17]–[19] studied the problems of extrapolation, interpolation, and filtration for stationary random processes and sequences.

In the current paper, we study the problem of optimal linear estimation of the functional

$$A_N\zeta = \sum_{j=0}^{N} a(j)\zeta(j)$$

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from observations $\zeta(j)+\theta(j)$ for $j \in \mathbb{Z} \setminus \{0,1,\dots,N\}$, where $\zeta(j)$ is a periodically correlated sequence and $\theta(j)$ is a periodically correlated sequence that is uncorrelated with $\zeta(j)$. We obtain formulas for calculation of the spectral characteristic and mean square error of the estimate of the functional $A_N\zeta$ in the case where the spectral densities of the sequence $\zeta(j)$ and those of the noise $\theta(j)$ are known. If the spectral densities are unknown, but a set of admissible spectral densities is specified, we propose formulas to calculate the least favorable spectral density and the minimax spectral characteristic of the optimal linear estimate of the functional.

2. Periodically correlated sequences generated by vector stationary sequences

Periodically correlated sequences are random sequences that have a periodic structure (see [5, 8]).

Definition 2.1. A sequence of complex-valued random variables $\zeta(n)$, $n \in \mathbb{Z}$, such that $\mathsf{E}\,|\zeta(n)|^2 < +\infty$, is called periodically correlated with period $T>0$ if

$$\mathsf{E}\,\zeta(n+T) = \mathsf{E}\,\zeta(n), \tag{1}$$

$$\mathsf{E}\,\zeta(n+T)\overline{\zeta(m+T)} = R(n+T,m+T) = R(n,m), \tag{2}$$

and if there is no number, smaller than $T>0$, for which equalities (1) and (2) hold.

Studies of properties of periodically correlated random sequences are initiated by Gladyshev [5]. Note also that stochastic periodic processes are considered by Bennet [1] and called the cyclostationary processes there.

Definition 2.2 ([20])**.** A complex-valued T-dimensional random sequence

$$\vec{\xi}(n) = \{\xi_k(n)\}_{k=0}^{T-1}, \qquad n \in \mathbb{Z},$$

such that $\mathsf{E}\,\|\vec{\xi}(n)\|^2 < \infty$, is called stationary if

$$\mathsf{E}\,\xi_k(n) = m_k$$

and

$$\mathsf{E}\,\xi_k(n)\overline{\xi_j(m)} = R_{kj}(n,m) = R_{kj}(n-m)$$

for all $n,m \in \mathbb{Z}$ and $j,k \in \{0,1,\dots,T-1\}$.

In this case, $R(n) = \{R_{kj}(n)\}_{k,j=0}^{T-1}$, $n \in \mathbb{Z}$, is called the covariance matrix of the T-dimensional stationary sequence $\vec{\xi}(n)$.

Theorem 2.1 (Gladyshev [5])**.** *A sequence $\zeta(n)$ is a periodically correlated random sequence with period T if and only if there exists a T-dimensional stationary sequence $\vec{\xi}(n) = \{\xi_k(n)\}_{k=0}^{T-1}$ such that $\zeta(n)$ admits the following representation:*

$$\zeta(n) = \sum_{k=0}^{T-1} e^{2\pi i n k/T}\xi_k(n), \qquad n \in \mathbb{Z}. \tag{3}$$

We say that the sequence $\vec{\xi}(n)$ generates the sequence $\zeta(n)$.

Denote by $f^{\vec{\xi}}(\lambda)$ the matrix of spectral densities of a T-dimensional stationary sequence $\vec{\xi}(n) = \{\xi_k(n)\}_{k=0}^{T-1}$. Let $f^{\vec{\zeta}}(\lambda)$ be the matrix of the spectral densities of the T-dimensional stationary sequence $\vec{\zeta}(n)$ constructed by subdividing a periodically correlated sequence $\zeta(n)$ into the blocks of length T. This means that the coordinate p of the random vector $\vec{\zeta}(n)$ is equal to

$$[\vec{\zeta}(n)]^p = \zeta(nT+p), \qquad n \in \mathbb{Z},\ p = 0,1,\dots,T-1.$$

If the spectral density $f^{\vec{\xi}}(\lambda)$ exists, then the spectral density $f^{\vec{\zeta}}(\lambda)$ exists, too. Moreover

$$f^{\vec{\zeta}}(\lambda) = T \cdot V(\lambda) f^{\vec{\xi}}(\lambda/T) V^{-1}(\lambda), \tag{4}$$

where $V(\lambda)$ is an unitary matrix with entries

$$v_{kj}(\lambda) = \frac{1}{\sqrt{T}} e^{2\pi ijk/T + ij\lambda/T}, \qquad k, j = 0, 1, \dots, T-1.$$

Since $V(\lambda)$ is continuous for $\lambda \in [-\pi, \pi)$ and since the inverse matrix exists for $V(\lambda)$, one can rewrite equality (4) as

$$f^{\vec{\xi}}(\lambda) = \frac{1}{T} \cdot V^{-1}(T\lambda) f^{\vec{\zeta}}(T\lambda) V(T\lambda). \tag{5}$$

3. Projection method for linear interpolation

Let $\zeta(n)$ and $\theta(n)$ be mutually uncorrelated T-periodically correlated random sequences. Assuming that the values of $\zeta(n)$ are unknown, consider the problem of optimal linear estimation of the functional $A_N\zeta = \sum_{j=0}^{N} a(j)\zeta(j)$ from observed values of the sequence $\zeta(j) + \theta(j)$ for $j \in \mathbb{Z} \setminus \{0, 1, \dots, N\}$.

Applying relation (3) for periodically correlated and vector stationary sequences, we rewrite the functional $A_N\zeta$ as follows:

$$\begin{aligned} A_N\zeta &= \sum_{j=0}^{N} a(j)\zeta(j) = \sum_{j=0}^{N} a(j) \sum_{k=0}^{T-1} e^{2\pi ijk/T} \xi_k(j) \\ &= \sum_{j=0}^{N} \sum_{k=0}^{T-1} a(j) e^{2\pi ijk/T} \xi_k(j) = \sum_{j=0}^{N} \vec{a}^{\top}(j) \vec{\xi}(j) = A_N\vec{\xi}, \end{aligned}$$

where $\vec{a}(j) = (a_0(j), \dots, a_{T-1}(j))^{\top}$, $a_k(j) = a(j)e^{2\pi ijk/T}$, $k = 0, 1, \dots, T-1$, and where $\vec{\xi}(j) = \{\xi_k(j)\}_{k=0}^{T-1}$ is a T-dimensional stationary sequence generating $\zeta(j)$.

Let $\vec{\xi}(j)$ and $\vec{\eta}(j)$ be uncorrelated T-dimensional stationary random sequences whose matrices of spectral densities are given by

$$f^{\vec{\xi}}(\lambda) = \left\{ f^{\vec{\xi}}_{kl}(\lambda) \right\}_{k,l=0}^{T-1} \qquad \text{and} \qquad f^{\vec{\eta}}(\lambda) = \left\{ f^{\vec{\eta}}_{kl}(\lambda) \right\}_{k,l=0}^{T-1},$$

respectively.

Consider the problem of optimal linear estimation of the functional

$$A_N\vec{\xi} = \sum_{j=0}^{N} \vec{a}^{\top}(j) \vec{\xi}(j)$$

from observed values of the sequence $\vec{\xi}(j) + \vec{\eta}(j)$ for $j \in \mathbb{Z} \setminus \{0, 1, \dots, N\}$.

Suppose the spectral densities $f^{\vec{\xi}}(\lambda)$ and $f^{\vec{\eta}}(\lambda)$ satisfy the minimality condition (see [20]), namely,

$$\int_{-\pi}^{\pi} Tr\left[\left(f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda) \right)^{-1} \right] d\lambda < +\infty. \tag{6}$$

Condition (6) is necessary and sufficient for the property that an error-free interpolation is not possible for the sequence $\vec{\xi}(j) + \vec{\eta}(j)$ (see [20]).

Denote by $L_2(f)$ the Hilbert space of complex-valued vector functions

$$b(\lambda) = \{b_k(\lambda)\}_{k=0}^{T-1}$$

that are square integrable with respect to the measure whose density $f(\lambda) = \{f_{kl}(\lambda)\}_{k,l=0}^{T-1}$ satisfies

$$\int_{-\pi}^{\pi} b^{\top}(\lambda) f(\lambda) \overline{b(\lambda)}\, d\lambda = \int_{-\pi}^{\pi} \sum_{k,l=0}^{T-1} b_k(\lambda) \overline{b_l(\lambda)} f_{kl}(\lambda)\, d\lambda < +\infty.$$

The subspace of $L_2(f)$ generated by the functions

$$e^{ij\lambda}\delta_k, \qquad k = 0, 1, \ldots, T-1,\ j \in \mathbb{Z} \setminus \{0, 1, \ldots, N\},$$

is denoted by $L_2^{N-}(f)$, where $\delta_k = \{\delta_{kl}\}_{l=0}^{T-1}$ and

$$\delta_{kl} = \begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

Every linear estimate $\hat{A}_N\vec{\xi}$ of the functional $A_N\vec{\xi}$ constructed from observed values of the sequence $\vec{\xi}(j) + \vec{\eta}(j)$ for $j \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$ can be written as

$$\hat{A}_N\vec{\xi} = \int_{-\pi}^{\pi} h^{\top}\left(e^{i\lambda}\right)\left(Z^{\xi}(d\lambda) + Z^{\eta}(d\lambda)\right) = \int_{-\pi}^{\pi} \sum_{k=0}^{T-1} h_k\left(e^{i\lambda}\right)\left(Z_k^{\xi}(d\lambda) + Z_k^{\eta}(d\lambda)\right),$$

where $Z^{\xi}(\Delta) = \left\{Z_k^{\xi}(\Delta)\right\}_{k=0}^{T-1}$ and $Z^{\eta}(\Delta) = \{Z_k^{\eta}(\Delta)\}_{k=0}^{T-1}$ are random orthogonal measures corresponding to the sequences $\vec{\xi}(j)$ and $\vec{\eta}(j)$ and where $h(e^{i\lambda}) = \left\{h_k(e^{i\lambda})\right\}_{k=0}^{T-1}$ is the spectral characteristic of the estimate $\hat{A}_N\vec{\xi}$. Note that $h(e^{i\lambda}) \in L_2^{N-}(f^{\vec{\xi}} + f^{\vec{\eta}})$.

The mean square error $\Delta(h; f^{\vec{\xi}}, f^{\vec{\eta}})$ of the estimate $\hat{A}_N\vec{\xi}$ is given by

$$\begin{aligned}\Delta\left(h; f^{\vec{\xi}}, f^{\vec{\eta}}\right) &= \mathsf{E}\left|A_N\vec{\xi} - \hat{A}_N\vec{\xi}\right|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\left[A_N(e^{i\lambda}) - h(e^{i\lambda})\right]^{\top} f^{\vec{\xi}}(\lambda)\overline{\left[A_N(e^{i\lambda}) - h(e^{i\lambda})\right]} \\ &\qquad\qquad + h^{\top}(e^{i\lambda}) f^{\vec{\eta}}(\lambda)\overline{h(e^{i\lambda})}\Big)\, d\lambda,\end{aligned}$$

$$A_N(e^{i\lambda}) = \sum_{j=0}^{N} \vec{a}(j) e^{ij\lambda}.$$

The spectral characteristic $h(f^{\vec{\xi}}, f^{\vec{\eta}})$ of the optimal linear estimate $A_N\vec{\xi}$ minimizes the mean square error

$$\begin{aligned}\Delta\left(f^{\vec{\xi}}, f^{\vec{\eta}}\right) &= \Delta\left(h\left(f^{\vec{\xi}}, f^{\vec{\eta}}\right); f^{\vec{\xi}}, f^{\vec{\eta}}\right) \\ &= \min_{h \in L_2^{N-}\left(f^{\vec{\xi}} + f^{\vec{\eta}}\right)} \Delta\left(h; f^{\vec{\xi}}, f^{\vec{\eta}}\right) = \min_{\hat{A}_N\vec{\xi}} \mathsf{E}\left|A_N\vec{\xi} - \hat{A}_N\vec{\xi}\right|^2.\end{aligned} \tag{7}$$

The optimal linear estimate $\hat{A}_N\vec{\xi}$ is a solution of the optimization problem (7). Using the classical Kolmogorov projection method [9], we obtain

$$\begin{aligned}h^{\top}\left(f^{\vec{\xi}}, f^{\vec{\eta}}\right) &= \left(A_N^{\top}\left(e^{i\lambda}\right) f^{\vec{\xi}}(\lambda) - C_N^{\top}\left(e^{i\lambda}\right)\right)\left[f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda)\right]^{-1} \\ &= A_N^{\top}\left(e^{i\lambda}\right) - \left(A_N^{\top}\left(e^{i\lambda}\right) f^{\vec{\eta}}(\lambda) + C_N^{\top}\left(e^{i\lambda}\right)\right)\left[f^{\vec{\xi}}(\lambda) + f^{\vec{\eta}}(\lambda)\right]^{-1}\end{aligned} \tag{8}$$

and

$$\Delta\left(f^{\vec{\xi}}, f^{\vec{\eta}}\right) = \langle \vec{a}_N, R_N\vec{a}_N \rangle + \langle \vec{c}_N, B_N\vec{c}_N \rangle, \tag{9}$$

where

$$C_N(e^{i\lambda})=\sum_{j=0}^{N}\vec{c}(j)e^{ij\lambda},\qquad \vec{a}_N=\{\vec{a}(k)\}_{k=0}^{N},\qquad \vec{c}_N=\{\vec{c}(k)\}_{k=0}^{N}=B_N^{-1}D_N\vec{a}_N,$$

$\langle a,b\rangle$ denotes the scalar product, and where B_N, D_N, and R_N are matrices with entries equal to the following $T\times T$ block matrices:

$$B_N(j,k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(f^{\vec{\xi}}(\lambda)+f^{\vec{\eta}}(\lambda)\right)^{-1}\right]^{\top}e^{i(k-j)\lambda}\,d\lambda,$$

$$D_N(j,k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f^{\vec{\xi}}(\lambda)\left(f^{\vec{\xi}}(\lambda)+f^{\vec{\eta}}(\lambda)\right)^{-1}\right]^{\top}e^{i(k-j)\lambda}\,d\lambda,$$

$$R_N(j,k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f^{\vec{\xi}}(\lambda)\left(f^{\vec{\xi}}(\lambda)+f^{\vec{\eta}}(\lambda)\right)^{-1}f^{\vec{\eta}}(\lambda)\right]^{\top}e^{i(k-j)\lambda}d\lambda$$

for $k,j=0,1,\dots,N$. Thus the following theorem holds for the interpolation of a T-dimensional stationary sequence [17].

Theorem 3.1. *Let $\vec{\xi}(j)=\{\xi_k(j)\}_{k=0}^{T-1}$ and $\vec{\eta}(j)=\{\eta_k(j)\}_{k=0}^{T-1}$ be mutually uncorrelated T-dimensional stationary sequences whose matrices of spectral densities are given by*

$$f^{\vec{\xi}}(\lambda)=\left\{f^{\vec{\xi}}_{kj}(\lambda)\right\}_{k,j=0}^{T-1}\quad\text{and}\quad f^{\vec{\eta}}(\lambda)=\left\{f^{\vec{\eta}}_{kj}(\lambda)\right\}_{k,j=0}^{T-1},$$

respectively.

Assume that the matrices $f^{\vec{\xi}}(\lambda)$ and $f^{\vec{\eta}}(\lambda)$ satisfy the minimality condition (6). Then the spectral characteristic $h\big(f^{\vec{\xi}},f^{\vec{\eta}}\big)$ and the mean square error $\Delta\big(f^{\vec{\xi}},f^{\vec{\eta}}\big)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ constructed from observed values of the sequence $\vec{\xi}(j)+\vec{\eta}(j)$, $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$, are given by equalities (8) and (9), respectively.

Corollary 3.1. *Let $\vec{\xi}(j)=\{\xi_k(j)\}_{k=0}^{T-1}$ be a T-dimensional stationary sequence whose matrix of spectral densities $f^{\vec{\xi}}(\lambda)$ satisfies the minimality condition*

$$\int_{-\pi}^{\pi}Tr\left[\left(f^{\vec{\xi}}(\lambda)\right)^{-1}\right]d\lambda<+\infty. \tag{10}$$

Then the spectral characteristic $h(f^{\vec{\xi}})$ and the mean square error $\Delta(f^{\vec{\xi}})$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ constructed from observed values of the sequence $\vec{\xi}(j)$, $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$, are given by

$$h^{\top}\left(f^{\vec{\xi}}\right)=A_N^{\top}\left(e^{i\lambda}\right)-C_N^{\top}\left(e^{i\lambda}\right)\left[f^{\vec{\xi}}(\lambda)\right]^{-1}, \tag{11}$$

$$\Delta\left(f^{\vec{\xi}}\right)=\langle\vec{c}_N,\vec{a}_N\rangle, \tag{12}$$

where $\vec{a}_N=\{\vec{a}(k)\}_{k=0}^{N}$, $\vec{c}_N=\{\vec{c}(k)\}_{k=0}^{N}=B_N^{-1}\vec{a}_N$, and where B_N is the matrix with entries equal to the following $T\times T$ block matrices:

$$B_N(j,k)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\left(f^{\vec{\xi}}(\lambda)\right)^{-1}\right]^{\top}e^{i(k-j)\lambda}\,d\lambda$$

for $k,j=0,1,\dots,N$.

Using the latter result, one can solve the problem of estimation of the functional $A_N\zeta$ constructed from a T-periodically correlated sequence.

Theorem 3.2. *Let $\zeta(j)$ and $\theta(j)$ be uncorrelated T-periodically correlated random sequences. Then the optimal linear estimate of the functional $A_N\zeta$ constructed from observed values of the sequence $\zeta(j)+\theta(j)$, $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$, is given by*

$$\hat{A}_N\zeta=\int_{-\pi}^{\pi}h^{\top}\left(f^{\vec{\xi}},f^{\vec{\eta}}\right)\left(Z^{\xi}(d\lambda)+Z^{\eta}(d\lambda)\right)=\int_{-\pi}^{\pi}\sum_{k=0}^{T-1}h_k\left(f^{\vec{\xi}},f^{\vec{\eta}}\right)\left(Z^{\xi}_k(d\lambda)+Z^{\eta}_k(d\lambda)\right),$$

where $\vec{\xi}(j)$ and $\vec{\eta}(j)$ are vector sequences generating the T-periodically correlated random sequences $\zeta(j)$ and $\theta(j)$, respectively. The spectral characteristic $h\big(f^{\vec{\xi}},f^{\vec{\eta}}\big)$ and mean square error $\Delta\big(f^{\vec{\xi}},f^{\vec{\eta}}\big)$ of the estimate $\hat{A}_N\zeta$ are given by equalities (8) *and* (9)*, where $\vec{a}(j)=(a_0(j),\dots,a_{T-1}(j))^{\top}$ and $a_k(j)=a(j)e^{2\pi ijk/T}$, $k=0,1,\dots,T-1$. The matrices of spectral densities $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ of the T-dimensional stationary sequences $\vec{\zeta}(j)$ and $\vec{\theta}(j)$ obtained by subdividing the one-dimensional periodically correlated sequences $\zeta(j)$ and $\theta(j)$ into blocks of length T are related to the corresponding matrices of spectral densities $f^{\vec{\xi}}(\lambda)$ and $f^{\vec{\eta}}(\lambda)$ of the sequences $\vec{\xi}$ and $\vec{\eta}$ by* (4).

Corollary 3.2. *Let $\zeta(j)$ be a T-periodically correlated random sequence. Then the optimal linear estimate of the functional $A_N\zeta$ constructed from observations of the sequence $\zeta(j)$, $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$, is given by*

$$\hat{A}_N\zeta=\int_{-\pi}^{\pi}h^{\top}\left(f^{\vec{\xi}}\right)Z^{\xi}(d\lambda)=\int_{-\pi}^{\pi}\sum_{k=0}^{T-1}h_k\left(f^{\vec{\xi}}\right)Z^{\xi}_k(d\lambda),\tag{13}$$

where $\vec{\xi}(j)$ is a stationary sequence generating the T-periodically correlated random sequence $\zeta(j)$. The spectral characteristic $h\big(f^{\vec{\xi}}\big)$ and the mean square error $\Delta\big(f^{\vec{\xi}}\big)$ of the estimate $\hat{A}_N\zeta$ are given by equalities (11) *and* (12)*, respectively, where*

$$\vec{a}(j)=(a_0(j),\dots,a_{T-1}(j))^{\top},\qquad a_k(j)=a(j)e^{2\pi ijk/T},\quad k=0,1,\dots,T-1.$$

The matrix of the spectral density $f^{\vec{\zeta}}$ of the T-dimensional stationary sequence $\vec{\zeta}(j)$ obtained by subdividing the one-dimensional periodically correlated sequence $\zeta(j)$ into blocks of length T is related to the matrix of the spectral density $f^{\vec{\xi}}$ of the sequence $\vec{\xi}$ by (4).

Example 3.1. Consider a 2-periodically correlated sequence $\zeta(n)=\xi_0(n)+e^{\pi in}\xi_1(n)$, where $\vec{\xi}(n)=\left(\begin{smallmatrix}\xi_0(n)\\ \xi_1(n)\end{smallmatrix}\right)$ is a 2-dimensional stationary sequence. Let $\xi_0(n)=\eta(n)$ be a one-dimensional stationary sequence with spectral density $f(\lambda)=(2\pi)^{-1}$ (white noise) and let $\xi_1(n)=\gamma(n)$ be a one-dimensional stationary sequence, uncorrelated with $\eta(n)$ and whose spectral density is $g(\lambda)=(5+4\cos\lambda)/(2\pi)=\left|2+e^{i\lambda}\right|^2/(2\pi)$.

We estimate the functional

$$A_1\zeta=2\zeta(0)-3\zeta(1)=(2,2)\begin{pmatrix}\xi_0(0)\\ \xi_1(0)\end{pmatrix}+(-3,3)\begin{pmatrix}\xi_0(1)\\ \xi_1(1)\end{pmatrix}=A_1\vec{\xi}$$

from the observations $\zeta(n)$, $n\in\mathbb{Z}\setminus\{0,1\}$. Here $a(0)=2$, $a(1)=-3$.

In this case, the matrix of the spectral densities $\vec{\xi}(n)$ is such that

$$f^{\vec{\xi}}(\lambda)=\begin{pmatrix}f(\lambda)&0\\ 0&g(\lambda)\end{pmatrix}$$

and the inverse matrix $[f^{\vec{\xi}}(\lambda)]^{-1}$ satisfies the minimality condition (10). The matrix B_1, its inverse B_1^{-1}, and the vector of unknown coefficients $\vec{c}_1$ are given by

$$B_1 = \frac{2\pi}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 4 \end{pmatrix}, \qquad B_1^{-1} = \frac{1}{4\pi}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \qquad \vec{c}_1 = \frac{1}{4\pi}\begin{pmatrix} 4 \\ 7 \\ -6 \\ 8 \end{pmatrix},$$

respectively.

Then the spectral characteristic defined by equality (11) is equal to

$$h^{\top}(f^{\vec{\xi}}) = \left(0, -\frac{2}{3}\sum_{\substack{j=-\infty,\\ j\neq 0,1}}^{\infty} (-1)^j \left(\frac{7}{2^{|j|}} - \frac{8}{2^{|j-1|}} \right) e^{ij\lambda} \right).$$

The optimal linear estimate $A_1\zeta$ defined by (13) is

$$\hat{A}_1\zeta = -\frac{2}{3}\sum_{\substack{j=-\infty,\\ j\neq 0,1}}^{\infty} (-1)^j \left(\frac{7}{2^{|j|}} - \frac{8}{2^{|j-1|}} \right) \xi_1(j).$$

The mean square error of the estimate $\hat{A}_1\zeta$ determined from (12) equals $\Delta(f^{\vec{\xi}}) = \frac{16}{\pi} \approx 5.09$. Then the matrix of the spectral densities of the T-dimensional stationary sequence $\vec{\zeta}(j)$ defined by (4) is equal to

$$f^{\vec{\zeta}}(\lambda) = \frac{1}{\pi}\begin{pmatrix} 3 + 2\cos(\frac{\lambda}{2}) & -2e^{\frac{-i\lambda}{2}} - 1 - e^{-i\lambda} \\ -2e^{\frac{i\lambda}{2}} - 1 - e^{i\lambda} & 3 + 2\cos(\frac{\lambda}{2}) \end{pmatrix}.$$

Example 3.2. Let $\zeta(n) = \xi_0(n) + e^{\pi i n}\xi_1(n)$, where $\xi_0(n) = \eta(n)$ is a one-dimensional Ornstein–Uhlenbeck stationary sequence with spectral density

$$f(\lambda) = \frac{q_1}{2\pi|1 - be^{-i\lambda}|^2}$$

and let $\xi_1(n) = \eta(n) + \gamma(n)$, where $\gamma(n)$ is an uncorrelated with $\eta(n)$ one-dimensional stationary sequence with spectral density $g(\lambda) = q_2/(2\pi)$. Let $q_1, q_2 \geq 0$ and $|b| < 1$. We estimate the functional $A_1\zeta = a(0)\zeta(0) + a(1)\zeta(1)$ with $a(0) = \alpha$ and $a(1) = \beta$.

In this case, the matrix of the spectral densities $\vec{\xi}(n)$ is given by

$$f^{\vec{\xi}}(\lambda) = \begin{pmatrix} f(\lambda) & f(\lambda) \\ f(\lambda) & f(\lambda) + g(\lambda), \end{pmatrix}$$

and the inverse matrix $[f^{\vec{\xi}}(\lambda)]^{-1}$ satisfies the minimality condition (10). The spectral characteristic of the linear optimal estimate $A_1\zeta$ defined by (11) is equal to

$$h^{\top}(f^{\vec{\xi}}) = \left(\frac{2\alpha b}{1 + b^2 + b^4}\left[\left(1 + b^2\right)e^{-i\lambda} + be^{2i\lambda}\right], 0 \right).$$

The optimal linear estimate of $A_1\zeta$ defined by (13) has the form

$$\hat{A}_1\zeta = \frac{2\alpha b}{1 + b^2 + b^4}\left(1 + b^2\right)\xi_0(-1) + \frac{2\alpha b^2}{1 + b^2 + b^4}\xi_0(2).$$

Then the mean square error of this estimate defined by (12) is equal to

$$\Delta(f^{\vec{\xi}}) = \frac{q_2}{2\pi}\left(\alpha^2 + \beta^2\right) + \frac{4\alpha^2 q_1}{1\pi(1 + b^2 + b^4)}\left(1 + b^2\right),$$

and the matrix of spectral densities of the T-dimensional stationary sequence $\vec{\zeta}(j)$ defined by (4) is given by

$$f^{\vec{\zeta}}(\lambda) = \frac{q_2}{2\pi}\begin{pmatrix} \frac{4q_1}{q_2|1-be^{-i\lambda/2}|^2}+1 & e^{-i\lambda/2} \\ e^{i\lambda/2} & 1 \end{pmatrix}.$$

The following results are expressed in terms of the matrices of spectral densities $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ of the T-dimensional stationary sequences $\vec{\zeta}(j)$ and $\vec{\theta}(j)$, respectively. These sequences are obtained by subdividing the one-dimensional periodically correlated sequences $\zeta(j)$ and $\theta(j)$ into blocks of length T.

Theorem 3.3. *Let $\zeta(j)$ and $\theta(j)$ be uncorrelated T-periodically correlated sequences and let $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ be the matrices of spectral densities of the T-dimensional stationary sequences $\vec{\zeta}(j)$ and $\vec{\theta}(j)$, respectively, obtained by subdividing the one-dimensional periodically correlated sequences $\zeta(j)$ and $\theta(j)$ into blocks of length T. Assume that $f^{\vec{\zeta}}(\lambda)$ and $f^{\vec{\theta}}(\lambda)$ satisfy the minimality condition (6). Then the optimal linear estimate of the functional $A_N\zeta$ constructed from the observations $\zeta(j)+\theta(j)$, $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$, is given by*

$$\hat{A}_N\zeta = \int_{-\pi}^{\pi} h^\top\left(f^{\vec{\zeta}}, f^{\vec{\theta}}\right)\left(Z^{\xi}(d\lambda)+Z^{\eta}(d\lambda)\right) = \int_{-\pi}^{\pi}\sum_{k=0}^{T-1} h_k\left(f^{\vec{\zeta}}, f^{\vec{\theta}}\right)\left(Z^{\xi}_k(d\lambda)+Z^{\eta}_k(d\lambda)\right),$$

where $\vec{\xi}(j)$ and $\vec{\eta}(j)$ are stationary sequences generating $\zeta(j)$ and $\theta(j)$, respectively. The spectral characteristic $h\big(f^{\vec{\zeta}}, f^{\vec{\theta}}\big)$ and the mean square error $\Delta\big(f^{\vec{\zeta}}, f^{\vec{\theta}}\big)$ of the estimate $\hat{A}_N\zeta$ are given by

$$\begin{aligned} h^\top\left(f^{\vec{\zeta}}, f^{\vec{\theta}}\right) &= \left(A_N^\top\left(e^{i\lambda}\right)V^{-1}(T\lambda)f^{\vec{\zeta}}(T\lambda) - T\cdot C_N^\top(e^{i\lambda})V^{-1}(T\lambda)\right) \\ &\quad\times\left[f^{\vec{\zeta}}(T\lambda)+f^{\vec{\theta}}(T\lambda)\right]^{-1}V(T\lambda) \\ &= A_N^\top\left(e^{i\lambda}\right) \\ &\quad -\left(A_N^\top\left(e^{i\lambda}\right)V^{-1}(T\lambda)f^{\vec{\theta}}(T\lambda)+T\cdot C_N^\top\left(e^{i\lambda}\right)V^{-1}(T\lambda)\right) \\ &\quad\times\left[f^{\vec{\zeta}}(T\lambda)+f^{\vec{\theta}}(T\lambda)\right]^{-1}V(T\lambda), \end{aligned} \tag{14}$$

$$\Delta\left(f^{\vec{\zeta}}, f^{\vec{\theta}}\right) = \left\langle \vec{a}_N, R^{\zeta}_N\vec{a}_N\right\rangle + \left\langle \vec{c}^{\,\zeta}_N, B^{\zeta}_N\vec{c}^{\,\zeta}_N\right\rangle, \tag{15}$$

where $\vec{c}^{\,\zeta}_N = \left\{\vec{c}^{\,\zeta}(k)\right\}_{k=0}^{N} = \left(B^{\zeta}_N\right)^{-1}D^{\zeta}_N\vec{a}_N$ and where B^{ζ}_N, D^{ζ}_N, and R^{ζ}_N are the matrices whose entries are the following $T\times T$ block matrices:

$$B^{\zeta}_N(j,k) = \frac{T}{2\pi}\int_{-\pi}^{\pi} V^\top(T\lambda)\left[\left(f^{\vec{\zeta}}(T\lambda)+f^{\vec{\theta}}(T\lambda)\right)^{-1}\right]^\top \overline{V}(T\lambda)e^{i(k-j)\lambda}\,d\lambda,$$

$$D^{\zeta}_N(j,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V^\top(T\lambda)\left[f^{\vec{\zeta}}(T\lambda)\left(f^{\vec{\zeta}}(T\lambda)+f^{\vec{\theta}}(T\lambda)\right)^{-1}\right]^\top \overline{V}(T\lambda)e^{i(k-j)\lambda}\,d\lambda,$$

$$\begin{aligned} D^{\zeta}_N(j,k) = \frac{1}{T\cdot 2\pi}\int_{-\pi}^{\pi} V^\top(T\lambda)&\left[f^{\vec{\zeta}}(T\lambda)\left(f^{\vec{\zeta}}(T\lambda)+f^{\vec{\theta}}(T\lambda)\right)^{-1}f^{\vec{\theta}}(T\lambda)\right]^\top \\ &\times\overline{V}(T\lambda)e^{i(k-j)\lambda}\,d\lambda \end{aligned}$$

for $k,j=0,1,\dots,N$.

Corollary 3.3. *Let $\zeta(j)$ be a T-periodically correlated sequence and let $f^{\vec{\zeta}}(\lambda)$ be the matrix spectral density of the T-dimensional stationary sequence $\vec{\zeta}(j)$. Assume that $f^{\vec{\zeta}}(\lambda)$ satisfies the minimality condition* (10). *Then the optimal linear estimate of $A_N\zeta$ constructed from the observed values of the sequence $\zeta(j)$, $j \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$, is given by*

$$\hat{A}_N\zeta = \int_{-\pi}^{\pi} h^\top \left(f^{\vec{\zeta}}\right) Z^\xi(d\lambda) = \int_{-\pi}^{\pi} \sum_{k=0}^{T-1} h_k \left(f^{\vec{\zeta}}\right) Z_k^\xi(d\lambda),$$

where $\vec{\xi}(j)$ is the stationary sequence generating $\zeta(j)$.

The spectral characteristic $h(f^{\vec{\zeta}})$ and mean square error $\Delta(f^{\vec{\zeta}})$ of the estimate $\hat{A}_N\zeta$ are given by

$$h^\top \left(f^{\vec{\zeta}}\right) = A_N^\top \left(e^{i\lambda}\right) - T \cdot C_N^\top \left(e^{i\lambda}\right) V^{-1}(T\lambda) \left[f^{\vec{\zeta}}(T\lambda)\right]^{-1} V(T\lambda), \tag{16}$$

$$\Delta \left(f^{\vec{\zeta}}\right) = \left\langle \vec{c}_N^{\,\zeta}, \vec{a}_N \right\rangle, \tag{17}$$

where $\vec{c}_N^{\,\zeta} = \left\{\vec{c}^{\,\zeta}(k)\right\}_{k=0}^{N} = \left(B_N^\zeta\right)^{-1} \vec{a}_N$ and where B_N^ζ is the matrix whose entries are the following $T \times T$ block matrices:

$$B_N^\zeta(j,k) = \frac{T}{2\pi} \int_{-\pi}^{\pi} V^\top(T\lambda) \left[\left(f^{\vec{\zeta}}(T\lambda)\right)^{-1}\right]^\top \overline{V}(T\lambda) e^{i(k-j)\lambda}\, d\lambda,$$

$$k, j = 0, 1, \ldots, N.$$

Example 3.3. Let $\zeta(n)$ be a 2-periodically correlated sequence. Let $\zeta(2n) = \eta(n)$ be a one-dimensional white noise sequence with spectral density $f(\lambda) = a/(2\pi)$, $a \geq 0$, and let $\zeta(2n+1) = \gamma(n)$ be an uncorrelated with $\eta(n)$ one-dimensional stationary Ornstein–Uhlenbeck sequence with the spectral density

$$g(\lambda) = \frac{b}{2\pi|1 - ce^{i\lambda}|^2}$$

with $b \geq 0$ and $|c| < 1$. We estimate the functional $A_1\zeta$ with the coefficients $a(0) = \alpha$ and $a(1) = \beta$.

In this case, the matrix of spectral densities of $\vec{\zeta}(n)$ is given by

$$f^{\vec{\zeta}}(\lambda) = \begin{pmatrix} f(\lambda) & 0 \\ 0 & g(\lambda) \end{pmatrix},$$

and its inverse $\left[f^{\vec{\zeta}}(\lambda)\right]^{-1}$ satisfies the minimality condition (6). The spectral characteristic of the optimal estimate of $A_1\zeta$ defined by (16) is equal to

$$h^\top(f^{\vec{\zeta}}) = \left(\frac{c\beta}{1+c}\left[e^{-i\lambda} + e^{3i\lambda}\right], -\frac{c\beta}{1+c}\left[e^{-i\lambda} + e^{3i\lambda}\right]\right).$$

The optimal linear estimate $A_1\zeta$ is of the form

$$\hat{A}_1\zeta = \frac{c\beta}{1+c}\left(\xi_0(-1) + \xi_0(3)\right) - \frac{c\beta}{1+c}\left(\xi_1(-1) + \xi_1(3)\right)$$

and the mean square error of the estimate defined by (17) is

$$\Delta(f^{\vec{\xi}}) = \frac{1}{2\pi}\left(a\alpha^2 + \frac{\beta^2 b}{1+c}\right).$$

4. Minimax (robust) interpolation method

Relations (14)–(17) can be used for finding the spectral characteristic and the mean square error of the optimal linear estimate of the functional $A_N\zeta$ if the matrices of spectral densities $f(\lambda)$ and $g(\lambda)$ of the T-dimensional stationary sequences obtained by subdividing the initial one-dimensional periodically correlated sequences into blocks of length T are known. If the matrices of densities are not known, but a family $D = D_f \times D_g$ of admissible spectral densities is specified, then one can apply the minimax approach to solve the problems of estimation of the functionals depending on unknown values of stationary sequences. We search for an estimate that minimizes the mean square error for all spectral densities belonging to a given class D.

Definition 4.1. Given a set $D = D_f \times D_g$ of pairs of spectral densities, the matrices of spectral densities $f^0(\lambda) \in D_f$ and $g^0(\lambda) \in D_g$ are called the least favorable in D for the optimal linear interpolation of the functional $A_N\zeta$ if

$$\Delta(f^0, g^0) = \Delta\left(h(f^0, g^0); f^0, g^0\right) = \max_{(f,g)\in D} \Delta(h(f,g); f, g).$$

Definition 4.2. Given a set $D = D_f \times D_g$ of pairs of spectral densities, a spectral characteristic $h^0(\lambda)$ of the optimal linear interpolation of the functional $A_N\zeta$ is called minimax (robust) if

$$h^0(\lambda) \in H_D = \bigcap_{(f,g)\in D} L_2^{N-}(f+g), \qquad \min_{h\in H_D} \max_{(f,g)\in D} \Delta(h; f, g) = \max_{(f,g)\in D} \Delta\left(h^0; f, g\right).$$

Using these definitions and the above relations (14)–(17) one can prove the following results (see [17]).

Lemma 4.1. *The matrices of spectral densities $f^0(\lambda) \in D_f$ and $g^0(\lambda) \in D_g$ satisfying condition* (6) *are least favorable in a class D for the optimal linear interpolation of the functional $A_N\zeta$ if the Fourier coefficients of the matrix-valued functions*

$$T \cdot V^{-1}(T\lambda)\left(f^0(T\lambda) + g^0(T\lambda)\right)^{-1} V(T\lambda),$$
$$V^{-1}(T\lambda) f^0(T\lambda)\left(f^0(T\lambda) + g^0(T\lambda)\right)^{-1} V(T\lambda),$$
$$\frac{1}{T} \cdot V^{-1}(T\lambda) f^0(T\lambda)\left(f^0(T\lambda) + g^0(T\lambda)\right)^{-1} g^0(T\lambda) V(T\lambda)$$

generate the matrices B_N^0, D_N^0, and R_N^0 that determine the solution of the following extremal problem:

$$\max_{(f,g)\in D} \left(\left\langle \vec{a}_N, R_N^\zeta \vec{a}_N \right\rangle + \left\langle (B_N^\zeta)^{-1} D_N^\zeta \vec{a}_N, D_N^\zeta \vec{a}_N \right\rangle\right)$$
$$= \left\langle \vec{a}_N, R_N^0 \vec{a}_N \right\rangle + \left\langle \left(B_N^0\right)^{-1} D_N^0 \vec{a}_N, D_N^0 \vec{a}_N \right\rangle.$$

The minimax spectral characteristic $h^0 = h(f^0, g^0)$ is evaluated by (14) *if*

$$h\left(f^0, g^0\right) \in H_D.$$

Lemma 4.2. *The matrix of the spectral density $f^0(\lambda) \in D_f$ that satisfies condition* (10) *is least favorable in the class D_f for the optimal linear interpolation of $A_N\zeta$ if the Fourier coefficients of the matrix-valued function $T \cdot V^{-1}(T\lambda)(f^0(T\lambda))^{-1} V(T\lambda)$ generate the matrix B_N^0 that determines the solution of the extremal problem*

$$\max_{f\in D_f} \left\langle \left(B_N^\zeta\right)^{-1} \vec{a}_N, \vec{a}_N \right\rangle = \left\langle \left(B_N^0\right)^{-1} \vec{a}_N, \vec{a}_N \right\rangle.$$

The minimax spectral characteristic $h^0 = h(f^0)$ is given by equality (16) *if $h(f^0) \in H_D$.*

The least favorable spectral densities $f^0(\lambda) \in D_f$ and $g^0(\lambda) \in D_g$ and the minimax spectral characteristic $h^0 = h(f^0, g^0)$ form a saddle point of the function $\Delta(h; f, g)$ in the set $H_D \times D$. The conditions for a saddle point

$$\Delta\left(h^0; f, g\right) \leq \Delta\left(h^0; f^0, g^0\right) \leq \Delta\left(h; f^0, g^0\right), \qquad \forall h \in H_D,\ \forall f \in D_f,\ \forall g \in D_g$$

hold if $h^0 = h(f^0, g^0)$, $h(f^0, g^0) \in H_D$, and if (f^0, g^0) is a solution of the conditional extremum problem

$$\begin{aligned}
&\Delta\left(h\left(f^0, g^0\right); f, g\right)\\
&= \frac{1}{2\pi T}\int_{-\pi}^{\pi}\left(A\left(e^{i\lambda}\right)V^{-1}(T\lambda)g^0(T\lambda)V(T\lambda) + TC^0\left(e^{i\lambda}\right)\right)^{\top}\\
&\qquad \times V^{-1}(T\lambda)\left(f^0(T\lambda) + g^0(T\lambda)\right)^{-1} f(T\lambda)\left(f^0(T\lambda) + g^0(T\lambda)\right)^{-1} V(T\lambda)\\
&\qquad\qquad \times \overline{\left(A(e^{i\lambda})V^{-1}(T\lambda)g^0(T\lambda)V(T\lambda) + TC^0(e^{i\lambda})\right)}\, d\lambda\\
&\quad + \frac{1}{2\pi T}\int_{-\pi}^{\pi}\left(A\left(e^{i\lambda}\right)V^{-1}(T\lambda)f^0(T\lambda)V(T\lambda) - T\cdot C^0\left(e^{i\lambda}\right)\right)^{\top}\\
&\qquad \times V^{-1}(T\lambda)\left(f^0(T\lambda) + g^0(T\lambda)\right)^{-1} g(T\lambda)\left(f^0(T\lambda) + g^0(T\lambda)\right)^{-1} V(T\lambda)\\
&\qquad \times \left(\overline{A(e^{i\lambda})V^{-1}(T\lambda)f^0(T\lambda)V(T\lambda) - T\cdot C^0(e^{i\lambda})}\right) d\lambda\\
&\to \sup, \qquad (f, g) \in D.
\end{aligned}$$

Lemma 4.3. *Suppose $f^0(\lambda)$ satisfies the minimality condition* (10) *and is a solution of the following conditional extremum problem:*

$$\begin{aligned}
\Delta\left(h(f^0); f\right) = \frac{T}{2\pi}\int_{-\pi}^{\pi}\left(C_N^0\left(e^{i\lambda}\right)\right)^{\top} V^{-1}(T\lambda)\left(f^0(T\lambda)\right)^{-1} f(T\lambda)\left(f^0(T\lambda)\right)^{-1}&\\
\times V(T\lambda)\overline{\left(C_N^0\left(e^{i\lambda}\right)\right)}\, d\lambda&\\
\to \sup, \qquad f(\lambda) \in D_f.&
\end{aligned} \tag{18}$$

Then $f^0(\lambda)$ is the least favorable matrix of the spectral densities for the optimal linear interpolation of $A_N\zeta$ constructed from observed values of the sequence

$$\zeta(j), \qquad j \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}.$$

The spectral characteristic $h^0 = h(f^0)$ given by (16) *is minimax if $h(f^0) \in H_D$.*

5. Least favorable spectral densities for the set D_0^-

Consider the minimax estimation problem for the functional $A_N\zeta$ from observations $\zeta(j)$, $j \in \mathbb{Z} \setminus \{0, 1, \ldots, N\}$, for which the matrix of spectral densities $f(\lambda)$ belongs to the set

$$D_0^- = \left\{ f(\lambda) \,\middle|\, \frac{1}{2\pi\cdot T}\int_{-\pi}^{\pi} V^{-1}(T\lambda) f^{-1}(T\lambda) V(T\lambda)\, d\lambda = P \right\},$$

where $P = \{p_{ij}\}_{i,j=0}^{T-1}$ is a given positive definite matrix such that $\det P \neq 0$. Using Lemma 4.3 and the Lagrange multipliers method we prove that the solution $f^0(\lambda)$ of the conditional extremum problem (18) satisfies the equation

$$\frac{1}{T}\cdot\overline{V(T\lambda)}\Big[\left(f^0(T\lambda)\right)^{-1}\Big]^{\top} V^{\top}(T\lambda) C_N^0\left(e^{i\lambda}\right) = \overline{V(T\lambda)}\Big[\left(f^0(T\lambda)\right)^{-1}\Big]^{\top} V^{\top}(T\lambda)\vec{\alpha}, \tag{19}$$

where $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{T-1})^{\top}$ is the vector of Lagrange multipliers,

$$C_N^0(e^{i\lambda}) = \sum_{j=0}^{N} \vec{c^0}(j) e^{ij\lambda},$$

$\vec{c^0}_N = \{\vec{c^0}(k)\}_{k=0}^N = (B_N^0)^{-1}\vec{a}_N$, and where B_N^0 is the matrix constructed from the Fourier coefficients of the matrix-valued function $\overline{V(T\lambda)}\left[(f^0(T\lambda))^{-1}\right]^\top V^\top(T\lambda)$,

$$B_N^0(k,j) = R^\top(k-j) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{V(T\lambda)}\Big[\big(f^0(T\lambda)\big)^{-1}\Big]^\top V^\top(T\lambda)e^{i(j-k)\lambda}\,d\lambda,$$
$$k,j = 0,1,\dots,N.$$

The Fourier coefficients $R(k) = R^*(-k)$, $k = 0,1,\dots,N$, found from the equation

$$B_N^0\vec{\alpha}_N = \vec{a}_N$$

with $\vec{\alpha}_N = (\vec{\alpha},\vec{0},\dots,\vec{0})^\top$, satisfy relation (19) and equality $B_N^0\vec{c^0}_N = \vec{a}_N$. The equations obtained above imply that

$$R(k) = P(\vec{a}(0))^{-1}\vec{a}^\top(k), \qquad k = 0,1,\dots,N,$$

where $\left[(\vec{a}(0))^{-1}\right]^\top\cdot\vec{a}(0) = 1$. This implies that $R(0) = P$.

Let $\vec{a}(k)$, $k = 0,1,\dots,N$, be a vector sequence such that the matrix-valued function

$$T\cdot V^{-1}(T\lambda)(f^0(T\lambda))^{-1}V(T\lambda) = \sum_{k=-N}^{N} R^\top(k)e^{ik\lambda}$$

is positive definite and nonsingular. Then $T\cdot V^{-1}(T\lambda)(f^0(T\lambda))^{-1}V(T\lambda)$ is represented as follows:

$$T\cdot V^{-1}(T\lambda)\left(f^0(T\lambda)\right)^{-1}V(T\lambda) = \left(\sum_{k=0}^{N}A_k e^{-ik\lambda}\right)\cdot\left(\sum_{k=0}^{N}A_k e^{-ik\lambda}\right)^*$$

(see [7]). Thus $T\cdot V^{-1}(T\lambda)(f^0(T\lambda))^{-1}V(T\lambda)$ is the spectral density of the multivariate random autoregressive sequence of order N given by

$$\sum_{k=0}^{N}A_k\vec{\xi}(n-k) = \vec{\varepsilon}(n), \tag{20}$$

where $\vec{\xi}(n)$ is the sequence generating $\zeta(n)$ and where $\vec{\varepsilon}(n)$ is a white noise vector sequence. Then the minimax spectral characteristic $h(f^0)$ is given by

$$h(f^0) = -\sum_{k=1}^{N}\overline{R(k)}\left(P^T\right)^{-1}\vec{a}(0)e^{-ik\lambda}. \tag{21}$$

Thus the following result holds.

Theorem 5.1. *Suppose the sequence of coefficients* $\vec{a}(k) = (a_0(k),a_1(k),\dots,a_{T-1}(k))^\top$, $a_j(k) = a(k)e^{2\pi ijk/T}$, $j = 0,1,\dots,T-1$, *determining the linear functional* $A_N\zeta$ *of a* T*-periodically correlated sequence* ζ *is such that the matrix-valued function* $\sum_{k=-N}^{N}R^\top(k)e^{ik\lambda}$, *where*

$$R(k) = R^*(-k) = P(\vec{a}(0))^{-1}\vec{a}^\top(k), \qquad k = 0,1,\dots,N,$$

is positive definite and nonsingular. Then the least favorable spectral density in the class D_0^- *for the optimal linear interpolation of* $A_N\zeta$ *is given by*

$$f^0(T\lambda) = T\cdot V(T\lambda)\bigg(\sum_{k=-N}^{N}R(k)^\top e^{ik\lambda}\bigg)^{-1}V^{-1}(T\lambda). \tag{22}$$

The minimum spectral characteristic $h(f^0)$ *is defined by* (21). *The maximum value of the mean square error of the estimate* $\hat{A}_N\zeta$ *is given by*

$$\Delta(f^0) = \langle\vec{c^0}_N,\vec{a}_N\rangle. \tag{23}$$

Example 5.1. Let $\zeta(n)$ be a 2-periodically correlated sequence. Consider the problem of the minimax interpolation of $A_0\zeta = \kappa\zeta(0)$, $\kappa \in \mathbb{R}$, in the set D_0^- with

$$P = \begin{pmatrix} 17 & 11 \\ 11 & 13 \end{pmatrix}.$$

The matrix of the least favorable spectral density in the class D_0^- for the estimate of $A_0\zeta$ is defined by (22) and thus is equal to

$$f^0(\lambda) = \frac{1}{25}\begin{pmatrix} 1 & -e^{-i\lambda/2} \\ -e^{i\lambda/2} & 13 \end{pmatrix}.$$

The matrix of the least favorable spectral density of the two-dimensional stationary sequence $\vec{\xi}(n)$ generating $\zeta(n)$ equals

$$f^{\vec{\xi},0} = \frac{1}{100}\begin{pmatrix} 13 & -11 \\ -11 & 17 \end{pmatrix}.$$

The two-dimensional stationary sequence itself admits the representation

$$\vec{\xi}(n) = \frac{1}{10}\begin{pmatrix} 3 & -1 \\ -2 & 4 \end{pmatrix}\vec{\varepsilon}(n)$$

(see (20)). The maximum value of the mean square error of this estimate is calculated by (23) and thus equals $\Delta(f^0) = \frac{2}{25}\kappa^2$.

6. Concluding remarks

Formulas for calculating the mean square error and spectral characteristic in the problem of optimal linear interpolation of the functional

$$A_N\zeta = \sum_{j=0}^{N} a(j)\zeta(j)$$

from observed values of the sequence $\zeta(j) + \theta(j)$, $j \in \mathbb{Z} \setminus \{0, 1, \dots, N\}$, are proposed, where $\zeta(j)$ is a periodically correlated random sequence (its values are unknown) and $\theta(j)$ is a uncorrelated with $\zeta(j)$ periodically correlated random sequence. The problem is considered for two cases, namely for the case where the matrices of spectral densities $f(\lambda)$ and $g(\lambda)$ of the signal $\zeta(n)$ and of the noise $\theta(n)$, respectively, are known, and for the case where the matrices of spectral densities are unknown but a family $D = D_f \times D_g$ of admissible spectral densities is specified. The results are obtained by using the relationship between periodically correlated and vector stationary sequences and by the method of estimation of vector stationary sequences.

Bibliography

1. W. R. Bennett, *Statistics of regenerative digital transmission*, Bell Syst. Tech. **37** (1958), 1501–1542. MR0102138 (21:932)
2. J. Franke, *On the robust prediction and interpolation of time series in the presence of correlated noise*, J. Time Series Analysis **5** (1984), 227–244. MR782077 (86i:62192)
3. J. Franke, *Minimax robust prediction of discrete time series*, Z. Wahrsch. Verw. Gebiete **68** (1985), 337–364. MR771471 (86f:62164)
4. J. Franke and H. V. Poor, *Minimax-robust filtering and finite-length robust predictors*, Robust and Nonlinear Time Series Analysis, Lecture Notes in Statistics, vol. 26, Springer-Verlag, 1984, pp. 87–126. MR786305 (86i:93058)
5. E. G. Gladyshev, *Periodically correlated random sequences*, Doklady Akad. Nauk SSSR **137** (1961), no. 5, 1026–1029; English transl. in Soviet Math. Dokl. **2** (1961), 385–388. MR0126873 (23:A4167)
6. U. Grenander, *A prediction problem in game theory*, Ark. Mat. **3** (1957), 371–379. MR0090486 (19:822g)

7. E. J. Hannan, *Multiple Time Series*, Wiley, New York–London–Sydney–Toronto, 1974. MR0279952 (43:5673)
8. H. L. Hurd and A. Miamee, *Periodically Correlated Random Sequences: Spectral Theory and Practice*, John Wiley & Sons, Inc., 2007. MR2348769 (2009e:62002)
9. A. N. Kolmogorov, *Probability Theory and Mathematical Statistics*, Selected works, vol. II, Nauka, Moscow, 1986; English. transl., Kluwer Academic Publishers, Dordrecht, 1992. MR1153022 (92j:01071)
10. M. P. Moklyachuk, *Stochastic autoregressive sequence and minimax interpolation*, Teor. Imovirnost. Matem. Statyst. **48** (1993), 135–146; English transl. in Theor. Probab. and Math. Statist. **48** (1994), 95–103. MR1445234 (98b:60081)
11. M. P. Moklyachuk, *Estimates of stochastic processes from observations with noise*, Theory Stoch. Process. **3(19)** (1997), 330–338.
12. M. P. Moklyachuk, *Extrapolation of stationary sequences from observations with noise*, Teor. Imovirnost. Matem. Statyst. **57** (1997), 125–133; English transl. in Theor. Probab. and Math. Statist. **57** (1998), 133–141. MR1806892 (2003b:60054)
13. M. P. Moklyachuk, *Robust procedures in time series analysis*, Theory Stoch. Process. **6(22)** (2000), no. 3–4, 127–147.
14. M. P. Moklyachuk, *Game theory and convex optimization methods in robust estimation problems*, Theory Stoch. Process. **7(23)** (2001), no. 1–2, 253–264.
15. M. P. Moklyachuk, *Robust Estimates of Functionals of Stochastic Processes*, Kyiv University, Kyiv, 2008. (Ukrainian)
16. M. P. Moklyachuk, *Robust prediction problem for periodically correlated stochastic sequences*, 5th Conference in Actuarial Science and Finance on Samos, Proceedings, 2009, pp. 51–65.
17. M. P. Moklyachuk and O. Yu. Masyutka, *Interpolation of multidimensional stationary sequences*, Theory Probab. Math. Stat. **73** (2006), 125–133. MR2213847 (2006m:60061)
18. M. P. Moklyachuk and O. Yu. Masyutka, *Extrapolation of multidimensional stationary processes*, Random Operators Stoch. Equ. **14** (2006), 233–244. MR2264364
19. M. P. Moklyachuk and O. Yu. Masyutka, *Minimax prediction of stochastic sequences*, Theory Stoch. Process. **14(30)** (2008), no. 3–4, 89–103. MR2498607 (2010c:60131)
20. Yu. A. Rozanov, *Stationary Random Processes*, Second edition, Nauka, Moscow, 1990; English transl. of the first Russian edition, Holden-Day, Inc., San Francisco–London–Amsterdam, 1967. MR1090826 (92d:60046)
21. N. Wiener, *Extrapolation, interpolation and smoothing of stationary time series. With engineering applications*, M.I.T. Press, Massachusetts Institute of Technology, Cambridge, Mass., 1966. MR0031213 (11:118j)
22. A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions*, vol. 1, Basic Results, Springer Series in Statistics, Springer-Verlag, New York, 1987. MR893393 (89a:60105)
23. A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions*, vol. 2, Supplementary Notes and References, Springer Series in Statistics, Springer-Verlag, New York, 1987. MR915557 (89a:60106)

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