INEQUALITIES FOR HOLOMORPHIC FUNCTIONS
OF SEVERAL COMPLEX VARIABLES

BY

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ABSTRACT. Sharp norm-inequalities, valid for functional Hilbert spaces of holomorphic functions on the polydisk, unit ball and $\mathbf{C}^n$ are established by using the notion of reproducing kernels. These inequalities extend earlier results of Saitoh and ours.

1. Introduction. In this paper we establish sharp norm-inequalities, valid for functional Hilbert spaces of holomorphic functions on the polydisk, unit ball and $\mathbf{C}^n$, by using the theory of reproducing kernels. Several applications, notably in the Dirichlet, Bergman-Selberg and Fischer spaces, are given. These inequalities constitute an improvement and an extension of recent results of Saitoh [7, 8], which he obtained by using methods different from ours. They also extend our earlier results in [2, 3] for the one complex variable case. The proofs given here employ elementary means and the results obtained may also be applied to various other situations where the domain in question is one of the classical Cartan domains. However, in order to avoid lengthy discussions, we shall not pursue the extensions here.

2. Preliminaries and notation. We consider the n-dimensional complex space $\mathbf{C}^n$ with the usual inner product

$$\langle z, \zeta \rangle = \sum_{j=1}^{n} z_j \bar{\zeta}_j \qquad (z, \zeta \in \mathbf{C}^n)$$

and the associated norm $|z| = \langle z, z \rangle^{1/2}$. The unit ball $B \equiv B_n$ of $\mathbf{C}^n$ is then the set of all $z \in \mathbf{C}^n$ with $|z| < 1$. The boundary of B is the sphere $S \equiv S_n$, the set of all $z \in \mathbf{C}^n$ with $|z| = 1$. By P we denote the polydisk B_1^n. Sometimes it is convenient to use the notation of D_ρ, $\rho = 0, 1, \infty$, to refer to P when $\rho = 0$, to B when $\rho = 1$ and to $\mathbf{C}^n$ when $\rho = \infty$. The class of all holomorphic functions in D_ρ is denoted by $H(D_\rho)$. It is a linear space over $\mathbf{C}$. The following standard multi-index notation is employed: If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ (i.e., an n-tuple of nonnegative integers), then $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. In addition, for $z \in \mathbf{C}^n$, the notation z^α is an abbreviation for the holomorphic monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. By $e_k \in \mathbf{Z}_+^n$ $(1 \leqslant k \leqslant n)$ we mean the n-tuple that has 1 in the kth entry and 0 everywhere else.

Any $f \in H(D_\rho)$ can be represented as

$$f(z) = \sum_\alpha a_\alpha z^\alpha; \qquad a_\alpha \in \mathbf{C}, z \in D_\rho,$$

where the series expansion converges absolutely and uniformly on compacta of D_ρ. In this way we regard $H(D_\rho)$ as the span of all z^α, $\alpha \geq 0$, in the uniform topology of D_ρ.

We introduce a "*derivation-operator*" $\partial_n\colon H(D_\rho) \to H(D_\rho)$ by

$$(2.1) \qquad \partial_n f(z) = \sum_{\alpha \geq 1} \pi(\alpha) a_\alpha z^{\alpha-1}.$$

The quantities in this sum are defined as follows: For $\alpha \geq 1$, let $\alpha_{k_1}, \ldots, \alpha_{k_m}$, $1 \leq k_1 \leq \cdots \leq k_m \leq n$ $(1 \leq m \leq n)$ be the nonzero (≥ 1) components of α. Then

$$(2.2) \qquad z^{\alpha-1} = z_{k_1}^{\alpha_{k_1}-1} \cdots z_{k_m}^{\alpha_{k_m}-1}$$

and

$$(2.3) \qquad \pi(\alpha) = \left\{ \left(\prod_{j=1}^m \alpha_{k_j} \right) \frac{(|\alpha| + n - m)!}{(|\alpha| - 1)!} \right\}^{1/2}, \qquad \alpha \geq 1.$$

This derivation ∂_n is a linear operator of $H(D_\rho)$ into itself and constitutes the n-dimensional generalization of the usual derivative $\partial \equiv \partial_1$ on $H(D_\rho)$ when $n = 1$. In fact, when $n = 1$, $\partial_1 f = f'$ for $f \in H(D_\rho)$.

By $d\sigma(z)$ we denote the Lebesgue measure on $\mathbf{C}^n = \mathbf{R}^{2n}$, and by $ds(z)$ we denote the euclidean measure on the boundary S of B. For future reference we also record the following elementary identities (see, for example, [6, pp. 16–17, 120–121]): For $q > 0$ and $\alpha, \beta \in \mathbf{Z}_+^n$ we have

$$(2.4) \qquad \frac{1}{\pi^n} \int_B z^\alpha \bar{z}^\beta \left(1 - |z|^2 \right)^{q-1} d\sigma(z) = \frac{\alpha! \Gamma(q)}{\Gamma(q + |\alpha| + n)} \delta_{\alpha\beta}$$

and

$$(2.5) \qquad \frac{1}{2\pi^n} \int_S z^\alpha \bar{z}^\beta \, ds(z) = \frac{\alpha!}{\Gamma(|\alpha| + n)} \delta_{\alpha\beta}.$$

3. Dirichlet and Bergman-Selberg spaces. For a fixed $q > 0$ we write

$$q(n) \equiv (n + 1)q/2.$$

The "(n, q)-*Dirichlet space*" is defined by

$$\mathcal{D}_{n,q} = \left\{ f \in H(B)\colon f(0) = 0, \|f\|_{\mathcal{D}_{n,q}} < \infty \right\},$$

where

$$(3.1) \qquad \|f\|_{\mathcal{D}_{n,q}}^2 = \frac{1}{\pi^n q(n)} \int_B |\partial_n f(z)|^2 \, d\sigma(z).$$

Let $f \in \mathcal{D}_{n,q}$ so that

$$f(z) = \sum_{\alpha \geq 1} a_\alpha z^\alpha, \qquad z \in B.$$

By (2.1)–(2.4) and (3.1),

$$\|f\|_{\mathcal{D}_{n,q}}^2 = \frac{1}{\pi^n q(n)} \sum_{\alpha \geq 1} |\pi(\alpha)|^2 |a_\alpha|^2 \int_B z^{\alpha-1} \bar{z}^{\alpha-1} \, d\sigma(z)$$

$$= \frac{1}{q(n)} \sum_{\alpha \geq 1} |\pi(\alpha)|^2 |a_\alpha|^2 \frac{(\alpha - 1)!}{(|\alpha| + n - m)!},$$

where $\alpha_{k_1}, \ldots, \alpha_{k_m}$, $1 \leq k_1 \leq \cdots \leq k_m \leq n$ $(1 \leq m \leq n)$ are the nonzero components of $\alpha \geq 1$. It follows that

$$(3.2) \qquad \|f\|_{\mathfrak{D}_{n,q}}^2 = \frac{1}{q(n)} \sum_{\alpha \geq 1} \frac{\alpha!}{(|\alpha|-1)!} |a_\alpha|^2.$$

Moreover, by the Cauchy-Schwarz inequality

$$|f(z)|^2 = \left| \sum_{\alpha \geq 1} a_\alpha z^\alpha \right|^2 \leq \left[q(n) \sum_{\alpha \geq 1} \frac{(|\alpha|-1)!}{\alpha!} z^\alpha \bar{z}^\alpha \right] \|f\|_{\mathfrak{D}_{n,q}}^2.$$

The quantity in the square bracket is $-q(n)\log(1-|z|^2)$, as the computation below shows, and therefore

$$|f(z)| \leq \left\{ -q(n)\log(1-|z|^2) \right\}^{1/2} \|f\|_{\mathfrak{D}_{n,q}}, \qquad z \in B.$$

This shows that $\mathfrak{D}_{n,q}$ is a functional Hilbert space of holomorphic functions f in B with $f(0) = 0$. The space $\mathfrak{D}_{n,q}$ has a reproducing kernel $k_{n,q}(\cdot, \cdot)$ which may be computed by means of any orthonormal basis of $\mathfrak{D}_{n,q}$. Clearly the sequence

$$\phi_\alpha(z) = \left\{ q(n)(|\alpha|-1)!/\alpha! \right\}^{1/2} z^\alpha, \qquad \alpha \geq 1,$$

is orthonormal in $\mathfrak{D}_{n,q}$, and by (3.2) it is also complete. Therefore,

$$k_{n,q}(z, \zeta) = \sum_{\alpha \geq 1} \phi_\alpha(z) \overline{\phi_\alpha(\zeta)} = q(n) \sum_{\alpha \geq 1} \frac{(|\alpha|-1)!}{\alpha!} z^\alpha \bar{\zeta}^\alpha$$

$$= q(n) \sum_{\alpha \geq 1} \frac{1}{\alpha!} \left(\sum_{m=|\alpha|} \frac{1}{m} m! \right) z^\alpha \bar{\zeta}^\alpha = q(n) \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} z^\alpha \bar{\zeta}^\alpha \right)$$

$$= q(n) \sum_{m=1}^{\infty} \frac{1}{m} \langle z, \zeta \rangle^m,$$

whence

$$(3.3) \qquad k_{n,q}(z, \zeta) = -q(n)\log(1 - \langle z, \zeta \rangle); \qquad z, \zeta \in B.$$

Another related functional Hilbert space is the so-called "(n, q)-*Bergman-Selberg space*," which is defined by

$$\mathcal{K}_{n,q} = \left\{ g \in H(B) \colon \|g\|_{n,q} < \infty \right\},$$

where, for $g \in H(B)$ with

$$g(z) = \sum_{\alpha \geq 0} b_\alpha z^\alpha, \qquad z \in B,$$

$$(3.4)$$
$$\|g\|_{n,q}^2 = \Gamma(q(n)) \sum_{\alpha \geq 0} \frac{\alpha!}{\Gamma(q(n)+|\alpha|)} |b_\alpha|^2.$$

Exactly as before one shows that $\mathcal{K}_{n,q}$ is a functional Hilbert space of holomorphic functions in B whose reproducing kernel is

$$K_{n,q}(z,\zeta) = \frac{1}{\Gamma(q(n))} \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \Gamma(q(n) + |\alpha|) z^\alpha \bar{\zeta}^\alpha$$

$$= \frac{1}{\Gamma(q(n))} \sum_{m=0}^\infty \frac{1}{m!} \Gamma(q(n) + m) \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} z^\alpha \bar{\zeta}^\alpha \right)$$

$$= \sum_{m=0}^\infty \frac{\Gamma(q(n) + m)}{m! \Gamma(q(n))} \langle z, \zeta \rangle^m,$$

and, therefore,

$$(3.5) \qquad\qquad K_{n,q}(z,\zeta) = (1 - \langle z, \zeta \rangle)^{-q(n)}; \qquad z, \zeta \in B.$$

For $q(n) \geqslant n$ (i.e., for $q \geqslant 2n/(n+1)$) the norm (3.4) of $\mathcal{K}_{n,q}$ can be realized as

$$\|g\|_{n,q}^2 = \frac{1}{\pi^n} \frac{\Gamma(q(n))}{\Gamma(q(n) - n)} \int_B |g(z)|^2 (1 - |z|^2)^{q(n) - (n+1)} \, d\sigma(z), \qquad q(n) > n,$$

and

$$\|g\|_{n,q}^2 = \frac{(n-1)!}{2\pi^n} \int_S |g(z)|^2 \, ds(z), \qquad q(n) = n,$$

where in the last integral, g stands for the nontangential boundary values of the holomorphic function $g(z)$ in B. These representations follow from (2.4)–(2.5) and (3.4). It thus follows that $\mathcal{K}_{n,q}$ for $q(n) = n$ is the ordinary "*Hardy-Szegö space*" with the "*Cauchy-Szegö kernel*" $(1 - \langle z, \zeta \rangle)^{-n}$ [**6**, p. 38], and, for $q(n) = n + 1$, $\mathcal{K}_{n,q}$ is the ordinary "*Bergman-space*" with the "*Bergman-kernel*" $(1 - \langle z, \zeta \rangle)^{-(n+1)}$ [**6**, p. 36]. The space $\mathcal{K}_{n,q}$ with $q(n) > n$ is a natural generalization of the Bergman space and was first studied by Selberg [**9**] for more general homogeneous domains. For $0 < q(n) < n$, the norm of $\mathcal{K}_{n,q}$ does not admit such a simple integral representation, a fact which is not crucial, for we shall only use (3.4) as the norm of $g \in \mathcal{K}_{n,q}, q > 0$.

The Dirichlet-space $\mathcal{D}_{n,q}$ and the Bergman-Selberg space $\mathcal{K}_{n,q}$ are intimately related. In fact, from (3.3) and (3.5) we have

$$K_{n,q}(z,\zeta) = \exp\left[k_{n,q}(z,\zeta) \right]; \qquad q > 0, z, \zeta \in B,$$

which in some sense means that $\mathcal{K}_{n,q}$ is the exponentiation of $\mathcal{D}_{n,q}$. The following theorem is even more striking.

THEOREM 1. *Let* $f \in \mathcal{D}_{n,q}, q > 0$. *Then* $\exp f \in \mathcal{K}_{n,q}$ *and*

$$\|\exp f\|_{n,q}^2 \leqslant \exp\|f\|_{D_{n,q}}^2.$$

Equality holds if and only if f *is of the form* $f = k_{n,q}(\cdot, \zeta)$ *for some* $\zeta \in B$, *that is*

$$f(z) = -q(n)\log(1 - \langle z, \zeta \rangle), \qquad z \in B,$$

for some $\zeta \in B$.

The inequality and the sufficient condition in the equality statement of this theorem constitute a simple consequence of a more abstract theorem in the general theory of reproducing kernels (see Burbea [2] for details). The essence, therefore, of this theorem is in the necessity of the equality statement. In the special case that $n = 1$ and $q \geqslant 1$, a proof of Theorem 1 was given by Saitoh [7]. His proof, however, is difficult and rather involved. In [2], Burbea has extended the result of Saitoh to cover the case of $0 < q < 1$ with a proof that is substantially easier and shorter than that of [7]. The method of proof in Burbea [2] conceals in it the main ingredient for establishing the present Theorem 1. This theorem will be proved as a special case of a more general result, valid for entire functions and functions holomorphic in B or P.

While the previous assertion relates two functional Hilbert spaces whose reproducing kernels are related via the exponential function, the next theorem gives a sharp norm inequality in a functional Hilbert space whose reproducing kernel is the ordinary product of reproducing kernels of other functional Hilbert spaces. Other generalizations are possible. However, for clarity we shall confine ourselves to the following case: For $q_j > 0, j = 1, \ldots, m$, we write

$$q_j(n) \equiv (n + 1)q_j/2, \qquad j = 1, \ldots, m.$$

THEOREM 2. *Let $q_j > 0$ and $g_j \in \mathcal{K}_{n,q_j}, j = 1, \ldots, m$. Then $\prod_{j=1}^{m} g_j \in \mathcal{K}_{n,q_1 + \cdots + q_m}$ and*

$$\left\| \prod_{j=1}^{m} g_j \right\|_{n,q_1 + \cdots + q_m} \leqslant \prod_{j=1}^{m} \|g_j\|_{n,q_j}.$$

Equality holds if and only if either $\prod_{j=1}^{m} g_j = 0$ or each g_j $(1 \leqslant j \leqslant m)$ is of the form

$$g_j(z) = C_j(1 - \langle z, \zeta \rangle)^{-q_j(n)}; \qquad z \in B, \; j = 1, \ldots, m,$$

for some (common) $\zeta \in B$ and some nonzero constants C_j $(1 \leqslant j \leqslant m)$.

Again, this sharp inequality will be obtained as a special case of a more general result. The last two theorems when combined yield the following interesting sharp inequality.

THEOREM 3. *Let $q_j > 0$ and $f_j \in \mathcal{D}_{n,q_j}, j = 1, \ldots, m$. Then $\exp \sum_{j=1}^{m} f_j \in \mathcal{K}_{n, \sum_{j=1}^{m} q_j}$ and*

$$\left\| \exp \sum_{j=1}^{m} f_j \right\|_{n, \sum_{j=1}^{m} q_j}^{2} \leqslant \exp \sum_{j=1}^{m} \|f_j\|_{\mathcal{D}_{n,q_j}}^{2}.$$

Equality holds if and only if each f_j $(1 \leqslant j \leqslant m)$ is of the form

$$f_j(z) = -q_j(n)\log(1 - \langle z, \zeta \rangle); \qquad z \in B, j = 1, \ldots, m,$$

for some (common) $\zeta \in B$.

4. Fischer spaces. For fixed $p, q > 0$ the "(p, q)-*Fischer space*" is defined by

$$\mathcal{F}_{p,q} \equiv \mathcal{F}_{p,q}(n) = \left\{ f \in H(\mathbf{C}^n) : \|f\|_{\mathcal{F}_{p,q}} < \infty \right\}$$

where

$$\|f\|_{\mathscr{F}_{p,q}}^2 = \frac{q^{p+n-1}\Gamma(n)}{\Gamma(p+n-1)} \frac{1}{\pi^n} \int_{\mathbf{C}^n} |f(z)|^2 |z|^{2(p-1)} e^{-q|z|^2} \, d\sigma(z).$$

Let $f \in \mathscr{F}_{p,q}$ so that

$$f(z) = \sum_\alpha a_\alpha z^\alpha.$$

Using integration in polar coordinates [6, p. 13] and (2.5) we obtain

$$\frac{1}{\pi^n} \int_{\mathbf{C}^n} |f(z)|^2 |z|^{2(p-1)} e^{-q|z|^2} \, d\sigma(z)$$

$$= 2 \sum_\alpha |a_\alpha|^2 \frac{\alpha!}{\Gamma(|\alpha|+n)} \int_0^\infty r^{2(n+|\alpha|+p-2)} e^{-qr^2} \, r \, dr$$

$$= q^{-(p+n-1)} \sum_\alpha |a_\alpha|^2 \frac{\alpha!}{q^{|\alpha|}} \frac{\Gamma(|\alpha|+p+n-1)}{\Gamma(|\alpha|+n)},$$

and, therefore,

$$(4.1) \qquad \|f\|_{\mathscr{F}_{p,q}}^2 = \frac{\Gamma(n)}{\Gamma(p+n-1)} \sum_\alpha \frac{\alpha!}{q^{|\alpha|}} \frac{\Gamma(|\alpha|+p+n-1)}{\Gamma(|\alpha|+n)} |a_\alpha|^2.$$

As before, (4.1) shows that $\mathscr{F}_{p,q}$ is a functional Hilbert space of entire functions whose reproducing kernel is

$$k_{p,q}(z, \zeta) = \frac{\Gamma(p+n-1)}{\Gamma(n)} \sum_\alpha \frac{q^{|\alpha|}\Gamma(|\alpha|+n)}{\Gamma(|\alpha|+p+n-1)} \frac{1}{\alpha!} z^\alpha \bar{\zeta}^\alpha$$

$$= \frac{\Gamma(p+n-1)}{\Gamma(n)} \sum_{m=0}^\infty \frac{1}{m!} \frac{q^m \Gamma(m+n)}{\Gamma(m+p+n-1)} \left(\sum_{|\alpha|=m} \frac{m!}{\alpha!} z^\alpha \bar{\zeta}^\alpha \right)$$

$$= \frac{\Gamma(p+n-1)}{\Gamma(n)} \sum_{m=0}^\infty \frac{1}{m!} \frac{\Gamma(m+n)}{\Gamma(m+p+n-1)} \{q\langle z, \zeta\rangle\}^m.$$

Therefore

$$(4.2) \qquad k_{p,q}(z, \zeta) = {}_1F_1(n; p+n-1; q\langle z, \zeta\rangle)$$

where ${}_1F_1(\alpha; \beta; x)$ is the well-known confluent hypergeometric function [10, p. 35]. In particular,

$$(4.3) \qquad k_{1,q}(z, \zeta) = {}_1F_1(n; n; q\langle z, \zeta\rangle) = e^{q\langle z, \zeta\rangle}.$$

The space $\mathscr{F}_{1,1} \equiv \mathscr{F}_{1,1}(n)$ is the ordinary Fischer space, a space studied by several authors (cf. [1] and [5]) from various points of view.

Closely related to $\mathscr{F}_{p,q} \equiv \mathscr{F}_{p,q}(n)$ is another functional Hilbert space $\mathscr{E}_{p,q} \equiv \mathscr{E}_{p,q}(n)$ of entire functions g in $\mathbf{C}^n$ with norm $\|g\|_{p,q}$ and whose reproducing kernel is

$$(4.4) \qquad K_{p,q}(z, \zeta) = e^{-1}\exp\{{}_1F_1(n; p+n-1; q\langle z, \zeta\rangle)\}.$$

In particular,

$$(4.5) \qquad K_{1,q}(z, \zeta) = e^{-1}\exp\{\exp(q\langle z, \zeta\rangle)\}.$$

For clarity of the exposition we shall here confine ourselves to the case $p = 1$. The more general case is deferred to §9. We first introduce the "*Stirling numbers of the second kind*":

$$(4.6) \qquad s_k = \sum_{m=0}^{\infty} \frac{m^k}{m!}, \qquad k = 0, 1, \ldots;$$

thus, $s_0 = e$, $s_1 = e$, $s_2 = 2e$, $s_3 = 5e$ and so on. The space $\mathscr{E}_{1,q}$ is defined by

$$\mathscr{E}_{1,q} = \left\{ g \in H(\mathbf{C}^n) \colon \|g\|_{1,q} < \infty \right\},$$

where, for $g \in H(\mathbf{C}^n)$ with

$$g(z) = \sum_{\alpha \geqslant 0} b_\alpha z^\alpha, \qquad z \in \mathbf{C}^n,$$

$$(4.7)$$

$$\|g\|_{1,q}^2 = e \sum_{\alpha \geqslant 0} \frac{\alpha!}{q^{|\alpha|} s_{|\alpha|}} |a_\alpha|^2.$$

The expression (4.5) for the reproducing kernel of $\mathscr{E}_{1,q}$ may be verified with the aid of (4.6) and (4.7) as follows:

$$K_{1,q}(z, \zeta) = e^{-1} \sum_{\alpha \geqslant 0} q^{|\alpha|} s_{|\alpha|} \frac{1}{\alpha!} z^\alpha \bar{\zeta}^\alpha = e^{-1} \sum_{\alpha \geqslant 0} \left(\sum_{k=|\alpha|} q^k s_k \right) \frac{1}{\alpha!} z^\alpha \bar{\zeta}^\alpha$$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} q^k s_k \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} z^\alpha \bar{\zeta}^\alpha \right) = e^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} s_k q^k \langle z, \zeta \rangle^k$$

$$= e^{-1} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=0}^{\infty} \frac{m^k}{m!} \right) q^k \langle z, \zeta \rangle^k = e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{k=0}^{\infty} \frac{1}{k!} m^k q^k \langle z, \zeta \rangle^k \right)$$

$$= e^{-1} \sum_{m=0}^{\infty} \frac{1}{m!} e^{mq\langle z, \zeta \rangle}$$

and (4.5) follows.

Analogous to Theorems 1 and 2 we have the following theorems.

THEOREM 4. *Let* $f \in \mathscr{F}_{1,q}(n)$, $q > 0$, *with* $f(0) = 0$. *Then* $\exp f \in \mathscr{E}_{1,q}(n)$ *and*

$$\|\exp f\|_{1,q}^2 \leqslant \exp \|f\|_{\mathscr{F}_{1,q}}^2.$$

Equality holds if and only if f *is of the form*

$$f(z) = e^{q\langle z, \zeta \rangle} - 1, \qquad z \in \mathbf{C}^n,$$

for some $\zeta \in \mathbf{C}^n$.

THEOREM 5. *Let* $q_j > 0$ *and* $f_j \in \mathscr{F}_{1,q_j}(n)$, $j = 1, \ldots, m$. *Then* $\prod_{j=1}^{m} f_j \in \mathscr{F}_{1, q_1 + \cdots + q_m}(n)$ *and*

$$\left\| \prod_{j=1}^{m} f_j \right\|_{1, q_1 + \cdots + q_m} \leqslant \prod_{j=1}^{m} \|f_j\|_{1, q_j};$$

equivalently,

$$\int_{\mathbf{C}^n} \left| \prod_{j=1}^m f_j(z) \right|^2 e^{-(q_1 + \cdots + q_m)|z|^2} \, d\sigma(z)$$

$$\leq \left(\frac{q_1 \cdots q_m}{q_1 + \cdots + q_m} \right)^n \frac{1}{\pi^{n(m-1)}} \prod_{j=1}^m \left\{ \int_{\mathbf{C}^n} |f_j(z)|^2 e^{-q_j|z|^2} \, d\sigma(z) \right\}.$$

Equality holds if and only if either $\prod_{j=1}^m f_j = 0$ or each $f_j \, (1 \leq j \leq m)$ is of the form

$$f_j(z) = C_j e^{q_j \langle z, \zeta \rangle}, \qquad z \in \mathbf{C}^n,$$

for some (common) $\zeta \in \mathbf{C}^n$ and some nonzero constants $C_j \, (1 \leq j \leq m)$.

A special case of Theorem 5, namely when $n = 1$ and $q_1 = \cdots = q_m = 1$, appears also in Saitoh [8]. The proof of Saitoh for this special case is rather difficult and is based on the theory of tensor products of reproducing kernel spaces of entire functions. Also, it appears that the equality statement in [8] contains an error or a misprint. The last two theorems will also be obtained as special cases of more general results on spaces of square summable series.

5. Square summable series spaces. We consider the space $H(D_\rho)$, $\rho = 0, 1, \infty$. We shall also use the notation of $z\bar{\zeta} = (z_1\bar{\zeta}_1, \ldots, z_n\bar{\zeta}_n)$ for $z, \zeta \in \mathbf{C}^n$. By $\mathscr{P}(D_\rho)$ we denote the class of all functions ϕ with an expansion

$$(5.1) \qquad \phi(z\bar{z}) = \sum_\alpha c_\alpha(z\bar{z})^\alpha; \qquad c_\alpha > 0, \alpha \in \mathbf{Z}_+^n, z \in D_\rho,$$

which converges on compacta of D_ρ, but $\phi(z\bar{z}) = \infty$ when z lies on the boundary of D_ρ. Evidently, for any $\zeta \in D_\rho$, $\phi(\cdot\bar{\zeta}) \in H(D_\rho)$. Associated with any $\phi \in \mathscr{P}(D_\rho)$ we consider the space

$$\mathcal{K}_\phi = \left\{ f \in H(D_\rho) : \|f\|_\phi < \infty \right\},$$

where, for

$$f(z) = \sum_\alpha a_\alpha z^\alpha; \qquad f \in H(D_\rho), z \in D_\rho,$$

$$(5.2) \qquad \|f\|_\phi^2 = \sum_\alpha c_\alpha^{-1} |a_\alpha|^2.$$

For $\zeta \in D_\rho$, we have

$$f(\zeta) = \sum_\alpha a_\alpha \zeta^\alpha = \sum_\alpha c_\alpha^{-1}(a_\alpha c_\alpha \zeta^\alpha) = \left(f(\cdot), \phi(\cdot\bar{\zeta}) \right)_\phi$$

and, therefore,

$$|f(\zeta)| \leq \left\{ \phi(\zeta\bar{\zeta}) \right\}^{1/2} \|f\|_\phi; \qquad f \in \mathcal{K}_\phi, \zeta \in D_\rho.$$

This shows that $\mathcal{K}_\phi$ is a functional Hilbert space of holomorphic functions in D_ρ with the reproducing kernel

$$(5.3) \qquad k_\phi(z, \zeta) = \phi(z\bar{\zeta}); \qquad z, \zeta \in D_\rho.$$

The norm of $\mathcal{K}_\phi$ is given by (5.2) which also shows that $\{\phi_\alpha\}_{\alpha \geqslant 0}$ with $\phi_\alpha(z) = \sqrt{c_\alpha}\, z^\alpha$ is an orthonormal basis for $\mathcal{K}_\phi$.

In the definition of $\phi \in \mathcal{P}(D_\rho)$ we may allow some of the coefficients c_α to equal zero. This corresponds to $\phi(z\bar\zeta)$ being the reproducing kernel of a functional Hilbert space of holomorphic functions in D_ρ with certain (mixed) partial derivatives vanishing at the origin. For example, if $c_0 = 0$ then $\phi(0) = 0$ and $\mathcal{K}_\phi = \{f \in H(D_\rho): f(0) = 0, \|f\|_\phi < \infty\}$. Such a ϕ is said to belong to class $\mathcal{P}_0(D_\rho)$.

We note that $\phi,\ \psi \in \mathcal{P}(D_\rho)$ implies $\phi\psi \in \mathcal{P}(D_\rho)$. With this observation we state our basic lemma.

LEMMA 1. *Let $\phi,\ \psi \in \mathcal{P}(D_\rho)$ and $f \in \mathcal{K}_\phi$, $g \in \mathcal{K}_\psi$. Then $fg \in \mathcal{K}_{\phi\psi}$ and*

$$\|fg\|_{\phi\psi} \leqslant \|f\|_\phi \|g\|_\psi.$$

Equality holds if and only if either $fg = 0$ or f and g are of the forms $f(z) = C_1\phi(z\bar\zeta)$, $g(z) = C_2\psi(z\bar\zeta)$; $z \in D_\rho$, for some (common) $\zeta \in D_\rho$ and some nonzero constants C_1 and C_2.

PROOF. We assume that ϕ, f and $\|f\|_\phi$ are as in (5.1)–(5.2). The corresponding quantities for $\mathcal{K}_\psi$ will be given by

$$\psi(z) = \sum_\alpha d_\alpha z^\alpha; \qquad d_\alpha > 0, \alpha \in \mathbf{Z}_+^n, z \in D_\rho,$$

$$g(z) = \sum_\alpha b_\alpha z^\alpha, \qquad z \in D_\rho,$$

and

$$\|g\|_\psi^2 = \sum_\alpha d_\alpha^{-1}|b_\alpha|^2.$$

Under these circumstances,

$$\phi(z)\psi(z) = \sum_\alpha M_\alpha z^\alpha, \quad M_\alpha = \sum_{\beta \leqslant \alpha} c_\beta d_{\alpha-\beta},$$

$$f(z)g(z) = \sum_\alpha A_\alpha z^\alpha, \quad A_\alpha = \sum_{\beta \leqslant \alpha} a_\beta b_{\alpha-\beta},$$

and

$$\|fg\|_{\phi\psi}^2 = \sum_\alpha M_\alpha^{-1}|A_\alpha|^2.$$

The lemma, therefore, is completely equivalent to the following sharp inequality:

$$(5.4) \qquad \sum_\alpha M_\alpha^{-1}|A_\alpha|^2 \leqslant \left(\sum_\alpha c_\alpha^{-1}|a_\alpha|^2 \right)\left(\sum_\alpha d_\alpha^{-1}|b_\alpha|^2 \right)$$

with equality if and only if either (i) $a_\alpha = 0$ or $b_\alpha = 0$ for all $\alpha \in \mathbf{Z}_+^n$, or (ii) $a_\alpha = C_1 c_\alpha \bar\zeta^\alpha$ and $b_\alpha = C_2 d_\alpha \bar\zeta^\alpha$ for all $\alpha \in \mathbf{Z}_+^n$, for some $\zeta \in D_\rho$ and for some

nonzero constants C_1 and C_2. In order to prove this inequality we let $r = (r_1, \ldots, r_n) \in [0, 1)^n$ and introduce

$$M(r) = \sum_\alpha M_\alpha^{-1} |A_\alpha|^2 r^\alpha, \qquad C(r) = \sum_\alpha c_\alpha^{-1} |a_\alpha|^2 r^\alpha,$$

$$D(r) = \sum_\alpha d_\alpha^{-1} |b_\alpha|^2 r^\alpha.$$

By the Cauchy-Schwarz inequality,

$$|A_\alpha|^2 = \left| \sum_{\beta < \alpha} a_\beta b_{\alpha - \beta} \right|^2 = \left| \sum_{\beta < \alpha} \frac{a_\beta b_{\alpha - \beta}}{\left(c_\beta d_{\alpha - \beta} \right)^{1/2}} \left(c_\beta d_{\alpha - \beta} \right)^{1/2} \right|^2$$

$$\leq \left(\sum_{\beta < \alpha} \frac{|a_\beta|^2}{c_\beta} \frac{|b_{\alpha - \beta}|^2}{d_{\alpha - \beta}} \right) \left(\sum_{\beta < \alpha} c_\beta d_{\alpha - \beta} \right) = \left(\sum_{\beta < \alpha} \frac{|a_\beta|^2}{c_\beta} \frac{|b_{\alpha - \beta}|^2}{d_{\alpha - \beta}} \right) M_\alpha,$$

and so

$$(5.5) \qquad M_\alpha^{-1} |A_\alpha|^2 \leq \sum_{\beta \leq \alpha} \frac{|a_\beta|^2}{c_\beta} \cdot \frac{|b_{\alpha - \beta}|^2}{d_{\alpha - \beta}}, \qquad \alpha \in \mathbf{Z}_+^n.$$

This shows that

$$(5.6) \qquad M(r) \leq C(r) D(r), \qquad r = (r_1, \ldots, r_n) \in [0, 1)^n,$$

as formal power series. Letting $r = (r_1, \ldots, r_n) \to 1^- = (1^-, \ldots, 1^-)$ in (5.6), inequality (5.4) is obtained. In view of (5.4)–(5.6) and the fact that $M(0) = C(0)D(0)$, equality in (5.4) holds if and only if $M(r) = C(r)D(r)$ for each $r = (r_1, \ldots, r_n) \in [0, 1)^n$, which is equivalent to having equality in (5.5) for every $\alpha \in \mathbf{Z}_+^n$. This is, obviously, equivalent to an existence of $\lambda_\alpha \in \mathbf{C}$ so that

$$(5.7) \qquad a_\beta b_{\alpha - \beta} = \lambda_\alpha c_\beta d_{\alpha - \beta}; \qquad \beta \leq \alpha, \alpha \in \mathbf{Z}_+^n.$$

Putting $\beta = 0$ and $\beta = \alpha$ in (5.7) results in

$$(5.8) \qquad \lambda_\alpha c_0 d_\alpha = a_0 b_\alpha, \quad \lambda_\alpha c_\alpha d_0 = a_\alpha b_0; \qquad \alpha \in \mathbf{Z}_+^n.$$

On the other hand, summing up (5.7) from $\beta = 0$ through $\beta = \alpha$ yields

$$(5.9) \qquad A_\alpha = \lambda_\alpha M_\alpha, \qquad \alpha \in \mathbf{Z}_+^n.$$

If $a_0 b_0 = 0$, then by (5.8), $\lambda_\alpha = 0$ for all $\alpha \in \mathbf{Z}_+^n$. Therefore, by (5.9), $A_\alpha = 0$ for all $\alpha \in \mathbf{Z}_+^n$ which means $a_\alpha = 0$ or $b_\alpha = 0$ for all $\alpha \in \mathbf{Z}_+^n$. This covers item (i) of the equality statement. We now assume that $a_0 b_0 \neq 0$ and define

$$(5.10) \qquad C_1 = a_0 c_0^{-1}, \qquad C_2 = b_0 d_0^{-1}.$$

We also define

$$(5.11) \qquad \zeta_k = c_0 \bar{a}_{e_k} \left(c_{e_k} \bar{a}_0 \right)^{-1}, \qquad 1 \leq k \leq n.$$

It follows from (5.8) that also

$$(5.12) \qquad \zeta_k = d_0 \bar{b}_{e_k} \left(d_{e_k} \bar{b}_0 \right)^{-1}, \qquad 1 \leq k \leq n.$$

Clearly, $C_1, C_2 \neq 0$. From (5.7)–(5.10) we have

$$(5.13) \qquad b_\alpha = C_2 C_1^{-1}(a_\alpha/c_\alpha)d_\alpha, \qquad \alpha \in \mathbf{Z}_+^n,$$

$$a_0 b_\alpha + \sum_{0 < \beta < \alpha} a_\beta b_{\alpha-\beta} + a_\alpha b_0 = \frac{a_\alpha b_0}{c_\alpha d_0} \sum_{\beta \leq \alpha} c_\beta d_{\alpha-\beta}, \qquad \alpha \in \mathbf{Z}_+^n - \{0\},$$

and, therefore,

$$(5.14) \qquad a_\alpha b_0 (c_\alpha d_0 + c_0 d_\alpha) + c_\alpha d_0 \sum_{0 < \beta < \alpha} a_\beta b_{\alpha-\beta} = a_\alpha b_0 \sum_{\beta \leq \alpha} c_\beta d_{\alpha-\beta},$$

$$\alpha \in \mathbf{Z}_+^n - \{0\}.$$

We use induction on the weight $|\alpha|$ to show that with $\zeta = (\zeta_1, \ldots, \zeta_n)$,

$$(5.15) \qquad a_\alpha = C_1 c_\alpha \bar\zeta^\alpha, \quad b_\alpha = C_2 d_\alpha \bar\zeta^\alpha; \qquad \alpha \in \mathbf{Z}_+^n.$$

Clearly, by (5.10)–(5.13), (5.15) is true for $|\alpha| = 0$ and $|\alpha| = 1$. Assuming (5.15) is true for α, $0 \leq |\alpha| \leq m - 1$, $m \geq 2$, we find by (5.14) that for $|\alpha| = m$,

$$a_\alpha b_0 (c_\alpha d_0 + c_0 d_\alpha) + c_\alpha d_0 C_1 C_2 \bar\zeta^\alpha \sum_{0 < \beta < \alpha} c_\beta d_{\alpha-\beta} = a_\alpha b_0 \sum_{\beta \leq \alpha} c_\beta d_{\alpha-\beta},$$

and so, by (5.10),

$$a_\alpha b_0 \sum_{0 < \beta < \alpha} c_\beta d_{\alpha-\beta} = C_1 c_\alpha \bar\zeta^\alpha b_0 \sum_{0 < \beta < \alpha} c_\beta d_{\alpha-\beta}, \qquad |\alpha| = m \geq 2.$$

This shows that $a_\alpha = C_1 c_\alpha \bar\zeta^\alpha$, and, by (5.13), also $b_\alpha = C_2 d_\alpha \bar\zeta^\alpha$ and (5.15) is proved. Finally, ζ must lie in D_ρ because, for example, the value of the first factor on the right-hand side of (5.4) for the solution in (5.15) is $|C_1|^2 \phi(\zeta\bar\zeta)$ and, since the domain of convergence of $\phi(\zeta\bar\zeta)$ is D_ρ ($\rho = 0, 1, \infty$), the assertion follows. This concludes the proof.

An immediate consequence of this lemma is

THEOREM 6. *Let $\phi_j \in \mathcal{P}(D_\rho)$ and $f_j \in \mathcal{K}_{\phi_j}$, $j = 1, \ldots, m$. Then $\prod_{j=1}^m f_j \in \mathcal{K}_{\phi_1 \cdots \phi_m}$ and*

$$\left\| \prod_{j=1}^m f_j \right\|_{\phi_1 \cdots \phi_m} \leq \prod_{j=1}^m \|f_j\|_{\phi_j}$$

with equality if and only if either $\prod_{j=1}^m f_j = 0$ or each f_j is of the form $f_j(z) = C_j \phi_j(z\bar\zeta)$, $z \in D_\rho$, for some (common) $\zeta \in D_\rho$ and some nonzero constants C_j ($1 \leq j \leq m$).

An analogous sharp inequality with respect to the ordinary sum of reproducing kernels is also available. In fact, one can directly verify the following theorem.

THEOREM 7. *Let $\phi_j \in \mathcal{P}(D_\rho)$ and $f_j \in \mathcal{K}_{\phi_j}$, $j = 1, \ldots, m$. Then $\sum_{j=1}^m f_j \in \mathcal{K}_{\Sigma \phi_j}$ and*

$$\left\| \sum_{j=1}^m f_j \right\|_{\Sigma \phi_j}^2 \leq \sum_{j=1}^m \|f_j\|_{\phi_j}^2,$$

with equality if and only if each f_j is of the form $f_j(z) = C\phi_j(z\bar\zeta)$, $z \in D_\rho$, for some (common) $\zeta \in D_\rho$ and some constant C.

6. Exponentiation. Let $\alpha \in \mathbf{Z}_+^n - \{0\}$ and let α_k, $1 \leqslant k \leqslant n$, be a nonzero component of α. We then write

$$\alpha - 1_k = (\alpha_1, \ldots, \alpha_k - 1, \ldots, \alpha_n); \qquad \alpha_k \geqslant 1, 1 \leqslant k \leqslant n.$$

Given an $f \in H(D_\rho)$, $f(0) = 0$, with

$$(6.1) \qquad\qquad f(z) = \sum_{\alpha \geqslant 1} a_\alpha z^\alpha \qquad (f(0) = 0)$$

we consider its exponential transform

$$(6.2) \qquad\qquad g(z) = \exp\{f(z)\} = \sum_{\alpha \geqslant 0} b_\alpha z^\alpha,$$

and hence $b_0 = 1$. For an arbitrary k, $1 \leqslant k \leqslant n$, (6.2) shows that

$$(\partial/\partial z_k)g(z) = g(z)(\partial/\partial z_k)f(z)$$

and therefore, by (6.1)–(6.2),

$$\sum_{\alpha_k \geqslant 1} \alpha_k b_\alpha z^{\alpha - 1_k} = \left(\sum_{\alpha \geqslant 0} b_\alpha z^\alpha \right)\left(\sum_{\alpha_k \geqslant 1} \alpha_k a_\alpha z^{\alpha - 1_k} \right) = \sum_{\alpha_k \geqslant 1} \left(\sum_{\beta \leqslant \alpha} \beta_k a_\beta b_{\alpha - \beta} \right) z^{\alpha - 1_k}.$$

This shows that

$$(6.3) \qquad\qquad b_\alpha = \frac{1}{\alpha_k} \sum_{\beta \leqslant \alpha} \beta_k a_\beta b_{\alpha - \beta}; \qquad \alpha_k \geqslant 1 \ (b_0 = 1),$$

a formula which is independent of k, $1 \leqslant k \leqslant n$, with $\alpha_k \geqslant 1$.

Let $\phi \in \mathcal{P}_0(D_\rho)$ so that

$$(6.4) \quad \phi(z\bar{z}) = \sum_{\alpha \geqslant 1} c_\alpha(z\bar{z})^\alpha; \qquad c_\alpha > 0, \alpha \in \mathbf{Z}_+^n - \{0\}, z \in D_\rho \ (\phi(0) = 0),$$

and consider its exponential transform

$$(6.5) \qquad\qquad \psi(z\bar{z}) = \exp\{\phi(z\bar{z})\}, \qquad z \in D_\rho.$$

Clearly, $\psi \in \mathcal{P}(D_\rho)$ and, in view of (6.3), the coefficients of its expansion

$$(6.6) \qquad\qquad \psi(z\bar{z}) = \sum_{\alpha \geqslant 0} d_\alpha(z\bar{z})^\alpha$$

satisfy

$$(6.7) \qquad\qquad d_\alpha = \frac{1}{\alpha_k} \sum_{\beta \leqslant \alpha} \beta_k c_\beta d_{\alpha - \beta}; \qquad \alpha_k \geqslant 1 \ (d_0 = 1).$$

This shows that $d_{e_k} = c_{e_k}$ $(1 \leqslant k \leqslant n)$ and, moreover,

LEMMA 2. *Let the notation of (6.4)–(6.7) apply. Then $d_\alpha = 1$ for $\alpha = 0$, $d_\alpha = c_\alpha$ for $|\alpha| = 1$ and $d_\alpha > c_\alpha$ for $|\alpha| \geqslant 2$.*

PROOF. The lemma is obvious for $|\alpha| \leqslant 1$. For $|\alpha| \geqslant 2$ we have, by (6.7),

$$d_\alpha = c_\alpha + \frac{1}{\alpha_k} \sum_{\beta < \alpha} \beta_k c_\beta d_{\alpha - \beta}, \qquad \alpha_k \geqslant 1 \ (1 \leqslant k \leqslant n).$$

In particular, since $d_\alpha > 0$ for any $\alpha \geqslant 0$ and since the above sum contains the $\beta = e_k$ term we have

$$d_\alpha \geqslant c_\alpha + (1/\alpha_k) c_{e_k} d_{\alpha - e_k} > c_\alpha,$$

and the lemma follows.

With the above notation, we may consider the functional Hilbert space

$$\mathcal{K}_\phi^0 = \left\{ f \in H(D_\rho) : f(0) = 0, \|f\|_\phi < \infty \right\}$$

where for $f \in H(D_\rho)$, $f(0) = 0$, given as in (6.1),

$$(6.8) \qquad \|f\|_\phi^2 = \sum_{\alpha \geqslant 1} c_\alpha^{-1} |a_\alpha|^2.$$

This space has

$$(6.9) \qquad k_\phi(z, \zeta) = \phi(z\bar\zeta) = \sum_{\alpha \geqslant 1} c_\alpha (z\bar\zeta)^\alpha; \qquad z, \zeta \in D_\rho,$$

as its reproducing kernel. Also, $\{\phi_\alpha\}_{\alpha \geqslant 1}$ with $\phi_\alpha(z) = \sqrt{c_\alpha}\, z^\alpha$ is an orthonormal basis for $\mathcal{K}_\phi^0$.

The new function ψ of (6.5) determines another functional Hilbert space $\mathcal{K}_\psi$ of functions g,

$$g(z) = \sum_{\alpha \geqslant 0} b_\alpha z^\alpha, \qquad z \in D_\rho,$$

holomorphic in D_ρ, with $\|g\|_\psi < \infty$ where, in view of (6.6)–(6.7),

$$(6.10) \qquad \|g\|_\psi^2 = \sum_{\alpha \geqslant 0} d_\alpha^{-1} |b_\alpha|^2.$$

As before, this space has

$$k_\psi(z, \zeta) = \psi(z\bar\zeta) = \exp\{\phi(z\bar\zeta)\}$$

as its reproducing kernel.

The main theorem in this direction is

THEOREM 8. *Let $\phi \in \mathcal{P}_0(D_\rho)$ and thus $\psi = \exp\phi \in \mathcal{P}(D_\rho)$. Assume $f \in \mathcal{K}_\phi^0$; then $\exp f \in \mathcal{K}_\psi$ and*

$$\|\exp f\|_\psi^2 \leqslant \exp\|f\|_\phi^2.$$

Equality holds if and only if f is of the form $f(z) = k_\phi(z, \zeta) = \phi(z\bar\zeta)$, $z \in D_\rho$, for some $\zeta \in D_\rho$.

This theorem is completely equivalent to the following lemma.

LEMMA 3. *Let the notation of (6.1)–(6.10) apply. Then*

$$(6.11) \qquad \sum_{\alpha \geqslant 0} d_\alpha^{-1} |b_\alpha|^2 \leqslant \exp\left\{ \sum_{\alpha \geqslant 1} c_\alpha^{-1} |a_\alpha|^2 \right\}$$

if the right-hand side is finite. Equality holds if and only if $a_\alpha = c_\alpha \bar\zeta^\alpha$ ($\alpha \geqslant 1$) and $b_\alpha = d_\alpha \bar\zeta^\alpha$ ($\alpha \geqslant 0$) for some (common) $\zeta \in D_\rho$.

PROOF. For $r = (r_1, \ldots, r_n) \in [0, 1)^n$, define

$$A(r) = \sum_{\alpha \geq 1} c_\alpha^{-1} |a_\alpha|^2 r^\alpha, \qquad B(r) = \sum_{\alpha \geq 0} d_\alpha^{-1} |b_\alpha|^2 r^\alpha.$$

Let $\alpha \in \mathbf{Z}_+^n - \{0\}$ with $\alpha_k \geq 1$, $1 \leq k \leq n$. By (6.3), (6.7) and the Cauchy-Schwarz inequality,

$$\alpha_k^2 |b_\alpha|^2 = \left| \sum_{\beta \leq \alpha} \left(\frac{\beta_k}{c_\beta d_{\alpha-\beta}} \right)^{1/2} a_\beta b_{\alpha-\beta} (\beta_k c_\beta d_{\alpha-\beta})^{1/2} \right|^2$$

$$\leq \left(\sum_{\beta \leq \alpha} \beta_k \frac{|a_\beta|^2}{c_\beta} \frac{|b_{\alpha-\beta}|^2}{d_{\alpha-\beta}} \right) \left(\sum_{\beta \leq \alpha} \beta_k c_\beta d_{\alpha-\beta} \right)$$

$$= \left(\sum_{\beta \leq \alpha} \beta_k \frac{|a_\beta|^2}{c_\beta} \frac{|b_{\alpha-\beta}|^2}{d_{\alpha-\beta}} \right) \alpha_k d_\alpha$$

and thus

$$(6.12) \qquad \alpha_k d_\alpha^{-1} |b_\alpha|^2 \leq \sum_{\beta \leq \alpha} \beta_k \frac{|a_\beta|^2}{c_\beta} \frac{|b_{\alpha-\beta}|^2}{d_{\alpha-\beta}}.$$

This shows that

$$(\partial/\partial r_k) B(r) \leq B(r) (\partial/\partial r_k) A(r)$$

or that

$$(6.13) \qquad (\partial/\partial r_k) \log B(r) \leq (\partial/\partial r_k) A(r), \qquad 1 \leq k \leq n,$$

as a formal power series. Note that if the right-hand side of (6.11) is finite then $A(r)$ converges for $r \in [0, 1)^n$ and, therefore, by (6.13), $B(r)$ also converges for $r \in [0, 1)^n$. Let $\delta = (\delta_1, \ldots, \delta_n) \in [0, 1)^n$ and choose a path $r_k = r_k(t)$, where $r_k(t)$ is increasing in $t \in [0, 1]$ and $r_k(0) = 0$, $r_k(1) = \delta_k$ for $1 \leq k \leq n$. Since $A(0) = 0$ and $B(0) = 1$, it follows from (6.13) that

$$\log B(\delta) = \int_0^1 \frac{d}{dt} \log B[r(t)] \, dt \leq \int_0^1 \frac{d}{dt} A[r(t)] \, dt = A(\delta)$$

and (6.11) follows by letting $\delta \to 1^-$. In view of the last inequality or (6.13), equality in (6.11) holds if and only if for any $r \in [0, 1)^n$ and any k, $1 \leq k \leq n$, (6.13) is an equality. This is equivalent to having equality in (6.12) for every component $\alpha_k \geq 1$. This, in turn, is equivalent to an existence of $\lambda_\alpha \in \mathbf{C}$ so that

$$(6.14) \qquad a_\beta b_{\alpha-\beta} = \lambda_\alpha c_\beta d_{\alpha-\beta}, \qquad 1 \leq \beta \leq \alpha.$$

Putting $\beta = \alpha$ in (6.14) results in $a_\alpha = \lambda_\alpha c_\alpha$. On the other hand summing up (6.14) from all β with $|\beta| = 1$ through β with $\beta = \alpha$, and, using (6.3) and (6.7), yields $b_\alpha = \lambda_\alpha d_\alpha$. Consequently,

$$(6.15) \qquad a_\alpha = c_\alpha d_\alpha^{-1} b_\alpha, \qquad \alpha \geq 1,$$

and with (6.3) this also shows

$$(6.16) \qquad a_\alpha (d_\alpha - c_\alpha) = c_\alpha \frac{1}{\alpha_k} \sum_{\beta < \alpha} \beta_k a_\beta b_{\alpha-\beta}, \qquad \alpha_k \geq 1 \ (1 \leq k \leq n).$$

We define $\zeta \in (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$ by

$$(6.17) \qquad \zeta_k = \bar{a}_{e_k} \bar{c}_{e_k}^{-1}, \qquad 1 \leqslant k \leqslant n.$$

We shall show that the only solution of (6.15) subject to (6.3) and (6.7) is

$$(6.18) \qquad a_\alpha = c_\alpha \bar{\zeta}^\alpha \quad (\alpha \geqslant 1), \qquad b_\alpha = d_\alpha \bar{\zeta}^\alpha \quad (\alpha \geqslant 0).$$

This will be accomplished by induction on the weight $|\alpha|$. The case of $|\alpha| = 0$ is trivial and for $|\alpha| = 1$, (6.18) is true by virtue of the definition (6.17) and (6.15). We assume that (6.18) is true for α, $1 \leqslant |\alpha| < m$. When $|\alpha| = m$, there is a component $\alpha_k \geqslant 1$, $1 \leqslant k \leqslant n$, of α so that (6.3), (6.7) and (6.16) apply. By (6.16) and the inductive assumption,

$$a_\alpha(d_\alpha - c_\alpha) = c_\alpha \bar{\zeta}^\alpha \frac{1}{\alpha_k} \sum_{\beta < \alpha} \beta_k c_\beta d_{\alpha-\beta}.$$

In view of (6.7) this may be written as

$$a_\alpha(d_\alpha - c_\alpha) = c_\alpha \bar{\zeta}^\alpha (d_\alpha - c_\alpha) \qquad (|\alpha| = m \geqslant 1).$$

However, by Lemma 1 we have $d_\alpha - c_\alpha > 0$ for $|\alpha| > 1$ and, therefore, $a_\alpha = c_\alpha \bar{\zeta}^\alpha$. With this and (6.15), (6.18) follows at once. Finally, the right-hand side of (6.11) for this solution

$$\exp\left\{ \sum_{\alpha \geqslant 1} c_\alpha (\zeta \bar{\zeta})^\alpha \right\} = \exp\{\phi(\zeta \bar{\zeta})\}$$

is assumed to be finite. Since $0 \leqslant \phi(\zeta \bar{\zeta}) < \infty$ for every $\zeta \in D_\rho$ ($\rho = 0, 1, \infty$) and $\phi(\zeta \bar{\zeta}) = \infty$ when ζ is on the boundary of D_ρ, we must have $\zeta \in D_\rho$. This concludes the proof.

The last theorem ties in with a more abstract theorem, proved in Burbea [2], in the general theory of reproducing kernels. In fact, let $k = k(\cdot, \cdot)$ be a positive definite kernel on $\Lambda \times \Lambda$ where Λ is an arbitrary nonvoid set. As is well known, there exists a uniquely determined functional Hilbert space H_k of functions on Λ such that k is its reproducing kernel. Also, let F be a nonconstant entire function with nonnegative coefficients. Clearly, $K(\cdot, \cdot) = F[k(\cdot, \cdot)]$, being itself a positive definite kernel on $\Lambda \times \Lambda$, uniquely determines another functional Hilbert space H_K of functions on Λ such that K is its reproducing kernel. It follows that if $f \in H_k$ then $F \circ f \in H_K$ and

$$\|F \circ f\|_K^2 \leqslant F\left(\|f\|_k^2\right)$$

and equality holds if f is of the form $f(z) = k(z, \zeta)$ for all $z \in \Lambda$ and some $\zeta \in \Lambda$.

As a particular case of this result we take $\Lambda = D_\rho$ and $k = k_\phi$ as in (6.9) with $\phi \in \mathcal{P}_0(D_\rho)$. Then $H_k = \mathcal{H}_\phi^0$ with norm as in (6.8). We also let

$$F(t) = \sum_{m=0}^{\infty} \lambda_m t^m \qquad (\lambda_m > 0, m = 0, 1, \ldots),$$

and hence $\psi \equiv F \circ \phi \in \mathcal{P}(D_\rho)$, $K = K_\psi$ and $H_K = \mathcal{H}_\psi$. Let the expansion of ψ be given as in (6.6). Consequently, if $f \in \mathcal{H}_\phi^0$ then $F \circ f \in \mathcal{H}_\psi^0$ and $\|F \circ f\|_\psi^2 \leqslant F(\|f\|_\phi^2)$ or

$$\sum_{\alpha \geqslant 0} d_\alpha^{-1} |b_\alpha|^2 \leqslant F\left(\sum_{\alpha \geqslant 0} c_\alpha^{-1} |a_\alpha|^2 \right),$$

with equality if $f = \phi(\cdot \bar{\zeta})$ for some $\zeta \in D_\rho$ (see also [2]). The inequality of Theorem 8 and (6.11) are obtained when choosing $F(t) = \exp(t)$.

Theorems 6 and 8 when combined yield the following sharp inequality.

THEOREM 9. *Let* $\phi_j \in \mathcal{P}_0(D_\rho)$ *and, thus,* $\psi_j = \exp \phi_j \in \mathcal{P}(D_\rho)$, $j = 1, \ldots, m$. *Assume that* $f_j \in \mathcal{K}^0_{\phi_j}$, $j = 1, \ldots, m$. *Then* $\exp \sum_{j=1}^m f_j \in \mathcal{K}_{\prod_{j=1}^m \psi_j}$ *and*

$$\left\| \exp \sum_{j=1}^m f_j \right\|^2_{\prod_{j=1}^m \psi_j} \leqslant \exp \sum_{j=1}^m \|f_j\|^2_{\phi_j}.$$

Equality holds if and only if each f_j $(1 \leqslant j \leqslant m)$ *is of the form* $f_j(z) = \phi_j(z\bar{\zeta})$; $z \in D_\rho$, $j = 1, \ldots, m$, *for some (common)* $\zeta \in D_\rho$.

7. Applications. Many interesting norm-inequalities are deductible from Theorems 6–9 by specifying the functions of $\mathcal{P}(D_\rho)$ and D_ρ with $\rho = 0, 1, \infty$. For example, Theorems 1–5 are obtained by such a specification. Before so doing, however, several remarks are in order on the class $\mathcal{P}(D_\rho)$ (or $\mathcal{P}_0(D_\rho)$) and reproducing kernels $k(\cdot, \cdot)$ on $D_\rho \times D_\rho$.

We first recall a couple of definitions (see, for example, Donoghue [4]), valid for any domain D in $\mathbf{C}^n$. A kernel $k(\cdot, \cdot)$ defined on $D \times D$ is called *sesquiholomorphic* in D if for any $\zeta \in D$, $k(\cdot, \zeta)$ and $\overline{k(\zeta, \cdot)}$ are holomorphic in D. This means, in view of Hartog's theorem, that the function $F(z, \bar{\zeta}) \equiv k(z, \zeta)$ is holomorphic in both variables $(z, \bar{\zeta})$; $(z, \zeta) \in D \times D$. A kernel $k(\cdot, \cdot)$ which is positive-definite on $D \times D$ and sesquiholomorphic in D is called a *Bergman kernel*. Such a kernel gives rise to a unique functional Hilbert space $\mathcal{K}(D)$ of functions holomorphic in D for which $k(\cdot, \cdot)$ is its reproducing kernel. Moreover, for any orthonormal basis $\{h_m\}_{m=1}^\infty$ of $\mathcal{K}(D)$,

$$k(z, \zeta) = \sum_{m=1}^\infty h_m(z) \overline{h_m(\zeta)}; \qquad z, \zeta \in D,$$

where the bilinear sum converges absolutely and uniformly on compacta of D. In particular, when $D = D_\rho$ $(\rho = 0, 1, \infty)$,

$$h_m(z) = \sum_\alpha d_\alpha^{(m)} z^\alpha; \qquad z \in D_\rho, \alpha \in \mathbf{Z}_+^n,$$

and, therefore,

$$(7.1) \qquad k(z, \zeta) = \sum_{\alpha, \beta} \left(\sum_{m=1}^\infty d_\alpha^{(m)} \overline{d_\beta^{(m)}} \right) z^\alpha \bar{\zeta}^\beta; \qquad z, \zeta \in D_\rho.$$

We now further assume that the nonnegative function $k(z, z)$, $z \in D_\rho$, depends only on $(|z_1|, \ldots, |z_n|) \in D_\rho$. This implies, in view of (7.1), that

$$\sum_{m=1}^\infty d_\alpha^{(m)} \overline{d_\beta^{(m)}} = c_\alpha \delta_{\alpha\beta}, \qquad c_\alpha \geqslant 0,$$

and so

$$(7.2) \qquad k(z, \zeta) = \sum_\alpha c_\alpha (z\bar{\zeta})^\alpha; \qquad z, \zeta \in D_\rho.$$

To avoid degenerate cases we also assume that $k(z, z) = \infty$ for z on the boundary of D_ρ. This corresponds to requiring that $c_\alpha > 0$ for infinitely many α in an indexed set Λ of $\mathbf{Z}^n_+$ and thus the function ϕ with $\phi(z\bar{z}) = k(z, z)$ is in $\mathcal{P}(D_\rho)$. In an obvious way this means that $\mathcal{K}_\phi = \mathcal{K}(D_\rho)$ and that the monomials $\{z^\alpha\}$ are orthogonal in $\mathcal{K}_\phi$ with c_α^{-1}, $\alpha \in \Lambda$, as their moments. The reproducing kernel of $\mathcal{K}_\phi$ is then

$$(7.3) \qquad k(z, \zeta) = \phi(z\bar{\zeta}); \qquad z, \zeta \in D_\rho.$$

We note that $D_1 \subset D_0 \subset D_\infty$ (recall that $D_0 = P$, $D_1 = B$ and $D_\infty = \mathbf{C}^n$) and that (7.3) means that $\phi(z\bar{z})$, $z \in D_\rho$, depends only on $(|z_1|, \ldots, |z_n|)$. It is, therefore, natural to also require

$$(7.4) \qquad \phi(z\bar{z}) = F_1(|z_1|^2) \cdots F_n(|z_n|^2), \qquad z = (z_1, \ldots, z_n) \in D_0,$$

and

$$(7.5) \qquad \phi(z\bar{z}) = F(|z|^2), \qquad z \in D_1,$$

where $F_1, \ldots, F_n$ and F are holomorphic functions in the unit disk B_1 of $\mathbf{C}$ and have this disk as the domain of thier convergence. In the case of $D_\infty = \mathbf{C}^n$ the two representations in (7.4) and (7.5) are possible, but now the corresponding functions are required to be entire in $\mathbf{C}$. It is now also clear that the corresponding reproducing kernels for the cases of (7.4) and (7.5) (with D_∞ included) are now

$$(7.6) \qquad k(z, \zeta) = F_1(z_1\bar{\zeta}_1) \cdots F_n(z_n\bar{\zeta}_n); \qquad z, \zeta \in D_\rho \ (\rho = 0, \infty),$$

and

$$(7.7) \qquad k(z, \zeta) = F(\langle z, \zeta \rangle); \qquad z, \zeta \in D_\rho \ (\rho = 1, \infty),$$

respectively.

We now discuss Theorems 1–5 as special cases of Theorems 6, 8 and 9 and in light of the above remarks.

That Theorem 1 follows from Theorem 8 is quite straightforward. This is accomplished by taking

$$\phi(z\bar{z}) = q(n) \sum_{\alpha \geqslant 1} \frac{(|\alpha|-1)!}{\alpha!} (z\bar{z})^\alpha, \qquad z \in D_1 = B,$$

and thus $\phi \in \mathcal{P}_0(D_1)$. Also, $\phi(z\bar{z}) = -q(n)\log(1 - |z|^2)$ and we are in case (7.7). Further, the space $\mathcal{K}_\phi^0$ is now $\mathcal{D}_{n,q}$, $\psi(z\bar{z}) = \exp\{\phi(z\bar{z})\} = (1 - |z|^2)^{-q(n)}$ and $\mathcal{K}_\psi$ is $\mathcal{K}_{n,q}$. Theorem 2 is obtained from Theorem 6 by taking

$$\phi_j(z\bar{z}) = (1 - |z|^2)^{-q_j(n)}; \qquad z \in D_1 \ (1 \leqslant j \leqslant m),$$

and thus $\mathcal{K}_{\phi_j}$ is $\mathcal{K}_{n,q_j}$. In a similar fashion Theorem 3 follows from Theorem 9.

Theorem 4 follows from Theorem 8. However, the latter implies the following stronger version of the former.

THEOREM 4*. *Let* $f \in \mathcal{F}_{p,q}(n)$, $p, q > 0$, *with* $f(0) = 0$. *Then* $\exp f \in \mathcal{E}_{p,q}(n)$ *and*

$$\|\exp f\|^2_{p,q} \leqslant \exp\|f\|^2_{\mathcal{F}_{p,q}}.$$

Equality holds if and only if f *is of the form*

$$f(z) = {}_1F_1(n; p + n - 1; q\langle z, \zeta \rangle) - 1$$

for some $\zeta \in \mathbf{C}^n$.

This follows on taking

$$\phi(z\bar{z}) = {}_1F_1\left(n;\ p+n-1;\ q|z|^2\right) - 1, \qquad z \in \mathbf{C}^n,$$

so that $\phi \in \mathscr{P}_0(D_\infty)$ and we are in case (7.7). Also, $\mathcal{K}_\phi^0 = \{f \in \mathscr{F}_{p,q}(n): f(0) = 0\}$. Further,

$$\psi(z\bar{z}) = \exp\{\phi(z\bar{z})\} = e^{-1}\exp\left\{{}_1F_1\left(n;\ p+n-1;\ q|z|^2\right)\right\}$$

and $\mathcal{K}_\psi = \mathscr{E}_{p,q}(n)$. This also shows that $K_{p,q}(\cdot,\ \cdot)$ in (4.4) is indeed the reproducing kernel for $\mathscr{E}_{p,q}(n)$.

A stronger version of Theorem 5, where $\mathscr{F}_{1,q_j}(n)$ is replaced by $\mathscr{F}_{p_j,q_j}(n)$, $p_j, q_j > 0$ ($j = 1,\ldots,m$), follows from Theorem 6. Note that the reproducing kernels in Theorem 5 are of both cases (7.6) and (7.7). In a similar manner the following version, analogous to Theorem 3, is at once deduced from Theorem 9. Let $q_j > 0$ and $f_j \in \mathscr{F}_{1,q_j}(n)$ with $f_j(0) = 0, j = 1,\ldots,m$. Then $\exp \sum_{j=1}^m f_j \in H_\Psi$ with

$$\Psi(z\bar{z}) = e^{-m}\prod_{j=1}^m \exp\left(\exp q_j |z|^2\right)$$

and

$$\left\|\exp \sum_{j=1}^m f_j\right\|_\Psi^2 \leqslant \exp \sum_{j=1}^m \|f_j\|_{\mathscr{F}_{1,q_j}}^2.$$

Equality holds if and only if each f_j ($1 \leqslant j \leqslant m$) is of the form

$$f_j(z) = e^{q_j\langle z, \zeta\rangle} - 1; \qquad z \in \mathbf{C}^n, j = 1,\ldots,m,$$

for some (common) $\zeta \in \mathbf{C}^n$.

Other norm-inequalities valid for functions holomorphic in D_ρ are obtainable from Theorems 6–9. We shall conclude the present discussion with a polydisk version of Theorem 1 which may also be regarded as a corollary of Theorem 8. The proof is quite straightforward and therefore omitted.

Let $A(P)$ be the subclass of $H(P)$ consisting of all functions f of the form

$$(7.8) \quad f(z) = \sum_{j=1}^n f_j(z_j); \qquad z = (z_1,\ldots,z_n) \in P = B_1^n,$$

$$f_j \in H(B_1), j = 1,\ldots,n.$$

We shall write $\mathbf{q} > 0$ for $\mathbf{q} = (q_1,\ldots,q_n)$; $q_j > 0, j = 1,\ldots,n$, and we introduce the "$\mathbf{q}$-Dirichlet space"

$$\mathscr{D}_\mathbf{q} = \left\{f \in A(P): f_j(0) = 0,\ 1 \leqslant j \leqslant n,\ \|f\|_{\mathscr{D}_\mathbf{q}} < \infty\right\},$$

where for $f \in A(P)$ as in (7.8),

$$\|f\|_{\mathscr{D}_\mathbf{q}}^2 = \frac{1}{\pi}\sum_{j=1}^n \frac{1}{q_j}\int_{B_1} \left|f_j'(z)\right|^2 d\sigma(z).$$

This is a functional Hilbert space with the reproducing kernel

$$k_\mathbf{q}(z, \zeta) = -\sum_{j=1}^n q_j \log\left(1 - z_j\bar{\zeta}_j\right); \qquad z, \zeta \in P.$$

Analogously, we define the "$\mathbf{q}$-*Bergman-Selberg space*"

$$\mathcal{K}_{\mathbf{q}} = \left\{ g \in H(P) \colon \|g\|_{\mathbf{q}} < \infty \right\},$$

where for $g \in H(P)$ with $g(z) = \sum_{\alpha \geqslant 0} b_{\alpha} z^{\alpha}$, $z \in P$,

$$(7.9) \qquad \|g\|_{\mathbf{q}}^{2} = \left(\prod_{j=1}^{n} \Gamma(q_{j}) \right) \sum_{\alpha \geqslant 0} \frac{\alpha!}{\left(\prod_{j=1}^{n} \Gamma(q_{j} + \alpha_{j}) \right)} |b_{\alpha}|^{2}.$$

Again, this is a functional Hilbert space of holomorphic functions in P whose reproducing kernel is

$$K_{\mathbf{q}}(z, \zeta) = \prod_{j=1}^{n} \left(1 - z_{j} \bar{\zeta}_{j} \right)^{-q_{j}}; \qquad z, \zeta \in P.$$

As before, when $\mathbf{q} \geqslant 1$ (i.e., when $q_{j} \geqslant 1$ for $j = 1, \ldots, n$) the norm (7.9) of $\mathcal{K}_{\mathbf{q}}$ can be realized as

$$\|g\|_{\mathbf{q}}^{2} = \frac{1}{\pi^{n}} \left(\prod_{j=1}^{n} (q_{j} - 1) \right) \int_{P} |g(z)|^{2} \left(\prod_{j=1}^{n} \left(1 - |z_{j}|^{2} \right)^{q_{j}-2} \right) d\sigma(z), \qquad \mathbf{q} \geqslant 1,$$

and

$$\|g\|_{\mathbf{q}}^{2} = \frac{1}{(2\pi)^{n}} \int_{S_{1}^{n}} |g(z)|^{2} \, ds_{1}(z_{1}) \cdots ds_{1}(z_{n}), \qquad \mathbf{q} = 1,$$

where in the last inegral, g stands for the nontangential distinguished-boundary values of the holomorphic function $g(z)$ in P.

Under these circumstances, the polydisk version of Theorem 1 is

THEOREM 10. *Let* $f \in \mathfrak{D}_{\mathbf{q}}$, $\mathbf{q} > 0$. *Then* $\exp f \in \mathcal{K}_{\mathbf{q}}$ *and*

$$\|\exp f\|_{\mathbf{q}}^{2} \leqslant \exp\|f\|_{\mathfrak{D}_{\mathbf{q}}}^{2}.$$

Equality holds if and only if f *is of the form*

$$f(z) = -\sum_{j=1}^{n} q_{j} \log\left(1 - z_{j} \bar{\zeta}_{j} \right), \qquad z \in P,$$

for some $\zeta \in P$.

BIBLIOGRAPHY

1. V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*. I, Comm. Pure Appl. Math. **14** (1961), 187–214.

2. J. Burbea, *A Dirichlet norm inequality and some inequalities for reproducing kernel spaces*, Proc. Amer. Math. Soc. **83** (1981), 279–285.

3. _____, *Inequalities for reproducing kernel spaces*, Illinois J. Math. **27** (1983).

4. W. F. Donoghue, Jr., *Reproducing kernel spaces and analytic continuation*, Rocky Mountain J. Math. **10** (1980), 85–97.

5. D. J. Newman and H. S. Shaprio, *Certain Hilbert spaces of entire functions*, Bull. Amer. Math. Soc. **72** (1966), 971–977.

6. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^{n}$, Springer-Verlag, New York, 1980.

7. S. Saitoh, *Some inequalities for analytic functions with a finite Dirichlet integral on the unit disk*, Math. Ann. **246** (1979), 69–77.

8. _____, *Some inequalities for entire functions*, Proc. Amer. Math. Soc. **80** (1980), 254–258.

9. A. Selberg, *Automorphic functions and integral operators*, Seminars on Analytic Functions, Vol. II, Institute for Advanced Study, Princeton, N. J., 1957, 152–161.

10. I. N. Sneddon, *Special functions of mathematical physics and chemistry*, Oliver and Boyd, Edinburgh, 1961.

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