LOWER BOUNDS FOR SLOSHING FREQUENCIES*

By J. R. KUTTLER AND V. G. SIGILLITO (*The Johns Hopkins University*)

In a recent paper [3] the authors have used a simple device to give lower bounds for Stekloff and free membrane eigenvalues. The idea of the device can also be used in problems where there is a Stekloff condition on part of the boundary. One such problem is that of the sloshing of a liquid in a tank. Mathematically, the problem is to find the frequencies λ satisfying

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad V, \\ \partial u/\partial n &= 0 \quad \text{on} \quad \Sigma, \\ \partial u/\partial n &= \lambda u \quad \text{on} \quad S, \end{aligned} \tag{1}$$

where u is the time-independent velocity potential of an incompressible, inviscid fluid, subject to gravity, in a rigid tank V with free surface S and walls Σ. We order the frequencies

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots .$$

This problem has been considered, among others, by Rayleigh [7], Lamb [4], Lawrence et al. [5], Budiansky [1], Ehrlich et al. [2] and Troesch [8], [9]. Upper bounds for the frequencies have been estimated by the Ritz method applied to the Rayleigh quotient for (1),

$$\frac{D(u)}{\int_S u^2 \, d\sigma}, \tag{2}$$

minimized subject to the condition $\int_S u \, d\sigma = 0$. Here $D(u)$ is the Dirichlet integral. Troesch has obtained results by an inverse method [8] and also by a "shallow water" assumption [9]. No shallow water assumption is made in this paper.

For simplicity, we consider a two-dimensional problem, corresponding to an infinite canal of uniform cross section. The method for higher dimensions will be similar. Suppose V is symmetric about the y-axis with side walls given by $x = \pm f(y)$, $-l \leq y \leq 0$, where f is a smooth, positive function. The free surface is the portion of the x-axis between $-f(0)$ and $f(0)$ (see Fig. 1).

We can write the Dirichlet integral as

$$D(u) = \int_{-l}^{0} dy \int_{-f(y)}^{f(y)} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right] dx.$$

*Received May 13, 1968; revised version received June 19, 1968. Supported by Contract No. AR-8-0001, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force.

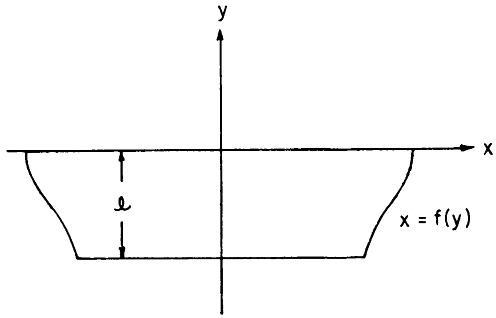

FIG. 1

We make the change of variables $\xi = kx/f(y)$, $\eta = y$, where k is a positive parameter. In these coordinates

$$D(u) = \int_{-l}^{0} d\eta \int_{-k}^{k} \left[\left(\frac{k}{f}\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial u}{\partial \eta} - \xi \frac{f'}{f}\frac{\partial u}{\partial \xi}\right)^2\right] \frac{f}{k}\, d\xi$$

$$\geq \int_{-l}^{0} d\eta \int_{-k}^{k} \left\{\left[\left(\frac{k}{f}\right)^2 + \left(1 - \frac{1}{\alpha}\right)\left(\xi \frac{f'}{f}\right)^2\right]\left(\frac{\partial u}{\partial \xi}\right)^2 + (1-\alpha)\left(\frac{\partial u}{\partial \eta}\right)^2\right\} \frac{f}{k}\, d\xi,$$

by the arithmetic-geometric mean inequality, where $0 < \alpha < 1$. This in turn is greater than or equal to

$$\int_{-l}^{0} d\eta \int_{-k}^{k} \left\{\left[\left(\frac{k}{f}\right)^2 + \left(1 - \frac{1}{\alpha}\right)\left(k \frac{f'}{f}\right)^2\right]\left(\frac{\partial u}{\partial \xi}\right)^2 + (1-\alpha)\left(\frac{\partial u}{\partial \eta}\right)^2\right\} \frac{f}{k}\, d\xi.$$

We choose α so that

$$\left(\frac{k}{f}\right)^2 + \left(1 - \frac{1}{\alpha}\right)\left(k\frac{f'}{f}\right)^2 = (1 - \alpha),$$

and it follows that

$$D(u) \geq \min_{-l \leq \eta \leq 0} \frac{1}{2}\left\{\left(\frac{f}{k}\right) + (1 + f'^2)\left(\frac{k}{f}\right) - \left(\left[\left(\frac{f}{k}\right) + (1 + f'^2)\left(\frac{k}{f}\right)\right]^2 - 4\right)^{1/2}\right\}$$
$$\cdot \int_{-l}^{0} d\eta \int_{-k}^{k} \left[\left(\frac{\partial u}{\partial \xi}\right)^2 + \left(\frac{\partial u}{\partial \eta}\right)^2\right] d\xi.$$

We also have

$$\int_S u^2\, d\sigma = \int_{-f(0)}^{f(0)} u^2\, dx = \frac{f(0)}{k}\int_{-k}^{k} u^2\, d\xi.$$

Therefore, using the max-min characterization of the frequencies (see [8]),

$$\lambda_i \geq \min_{-l \leq \eta \leq 0} \frac{k}{2f(0)}\left\{\left(\frac{f}{k}\right) + (1 + f'^2)\left(\frac{k}{f}\right) - \left(\left[\left(\frac{f}{k}\right) + (1 + f'^2)\left(\frac{k}{f}\right)\right]^2 - 4\right)^{1/2}\right\}\lambda_i^*,$$

where λ_i^* is the ith frequency of problem (1) for the rectangle of width $2k$ and depth l.

By separation of variables we see that

$$\lambda_i^* = \frac{(i-1)\pi}{2k}\tanh\frac{(i-1)\pi l}{2k}, \qquad i = 2, 3, \cdots.$$

So, since k is arbitrary, we have

$$\lambda_i \geq \frac{(i-1)\pi}{4f(0)} \max_k \left\{ \min_{-l \leq \eta \leq 0} \left[\left(\frac{f}{k}\right) + (1 + f'^2)\left(\frac{k}{f}\right) - \left(\left[\left(\frac{f}{k}\right) + (1 + f'^2)\left(\frac{k}{f}\right) \right]^2 - 4 \right)^{1/2} \right] \tanh \frac{(i-1)\pi l}{2k} \right\} \qquad i = 2, 3, \cdots \tag{3}$$

(Notice that $x - (x^2 - 4)^{1/2}$ is monotone decreasing for $x \geq 2$.) This bound becomes equality in the case of a rectangular tank.

The assumption of symmetry was made for convenience. The nonsymmetric case is treated analogously.

Applying a similar procedure to a tank with free surface of width s, vertical side walls, and a bottom given by $y = -f(x)$, where f is a smooth, positive function (see Fig. 2), we have the inequality

$$\lambda_i \geq \frac{(i-1)\pi}{2s} \max_l \left(\min_\xi \left[\frac{f}{l} + (1 + f'^2)\,\frac{l}{f} - \left(\left[\left(\frac{f}{l}\right) + (1 + f'^2)\left(\frac{l}{f}\right) \right]^2 - 4 \right)^{1/2} \right] \right\} \cdot \tanh \frac{(i-1)\pi l}{s}, \qquad i = 2, 3, \cdots,$$

again with equality for a rectangle.

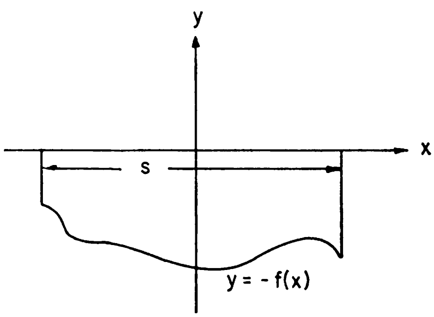

FIG. 2

As a simple example of the application of the bound, consider a symmetric, bowl-shaped region with vertical sides given by

$$x = \pm f(y) = \pm(1 - y^2)^{1/2}, \qquad -\tfrac{1}{2} = l \leq y \leq 0,$$

(see Fig. 3). The lower bound is then $\lambda_2 \geq .71$.

To obtain a crude upper bound we observe from (2) that if two tanks V_1, V_2 have the same free surface S and V_1 is contained in V_2 then the λ_i of V_2 is an upper bound for the λ_i of V_1. In our example an upper bound for λ_2 is λ_2^* for a rectangle of width 2 and depth $\frac{1}{2}$. Hence $\lambda_2 \leq 1.03$.

We can also use this observation to apply our method to regions without flat bottoms by shaving off a part of the bottom. Indeed shaving the bottom may sometimes improve the bound.

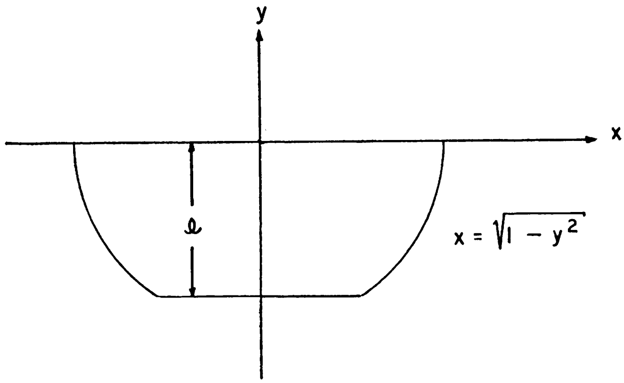

FIG. 3

REFERENCES

[1] B. Budiansky, *Sloshing of liquids in circular canals and spherical tanks*, J. Aero/Sp. Sci. (3) **27**, 161–173 (1960)

[2] L. Ehrlich, J. Riley, W. Strang and B. A. Troesch, *Finite-difference techniques for a boundary problem with an eigenvalue in a boundary condition*, J. Soc. Indust. Appl. Math. **9**, 149–164 (1961)

[3] J. R. Kuttler and V. G. Sigillito, *Lower bounds for Stekloff and free membrane eigenvalues*, SIAM Rev. **10**, 368–370 (1968)

[4] H. Lamb, *Hydrodynamics*, 6th ed. Dover, New York, 1945, 444–450

[5] H. R. Lawrence, C. H. Wang and R. B. Reddy, *Variational solution of fuel sloshing modes*, Jet Propulsion **28**, 729–736 (1958)

[6] G. Pólya and G. Szegö, *Isoperimetric inequalities in mathematical physics*, Ann. of Math. Studies No. 27, Princeton Univ. Press, Princeton, N. J., 1951

[7] Lord Rayleigh, Philos. Mag. **5**, 566 (1899)

[8] B. A. Troesch, *Free oscillations of a fluid in a container*, in *Boundary problems in differential equations*, R. Langer, ed., Univ. of Wisconsin Press, Madison, Wisc., 1960, pp. 279–299

[9] B. A. Troesch, *An isoperimetric sloshing problem*, Comm. Pure Appl. Math. **18**, 319–338 (1965)