Global monopole solutions in Horava gravity

Tae Hoon Lee

Department of Physics and Institute of Natural Sciences,
Soongsil University,
Seoul 156-743 Korea

E-mail: `thlee@ssu.ac.kr`

Abstract.

In Horava's theory of gravity coupled to a global monopole source, we seek for static, spherically symmetric spacetime solutions for general values of λ. We obtain the explicit solutions with deficit solid angles, in the IR modified Horava gravity model, at the IR fixed point $\lambda = 1$ and at the conformal point $\lambda = 1/3$. For the other values of $1 > \lambda > 0$ we also find special solutions to the inhomogenous equation of the gravity model with detailed balance, and we discuss an possibility of astrophysical applications of the $\lambda = 1/2$ solution that has a deficit angle for a finite range.

PACS numbers: 04.70.Bw, 04.60.Bc, 04.20.Jb

Since recently Horava proposed a renormalizable gravity theory in the UV limit [1], a lot of related works have been widely circulated. Studies on Horava-Lifshitz cosmology [2], black hole solutions [3, 4], and other interesting topics [5] have been reported.

In the IR modified Horava theory of gravity [1] where the detailed balance condition is softly violated (via the term proportional to ωR in Eq. (4) below), we study geometric structures affected by gravitationally coupled global monopole(GM) source [7]. Considering static, spherically symmetric spacetimes, we obtain solutions to a set of equations derived for general values of λ. In this IR modified model, we find the explicit solutions at the IR fixed point $\lambda = 1$ and at the conformal point $\lambda = 1/3$. In both cases deficit solid angles occur.

For the other values of $1 > \lambda > 0$ in the case with detailed balance $\omega = 0$, we have new special solutions, in addition to known general solutions [3] to the corresponding homogeneous equation. By simple analysis, we show that the GM spacetime in the case $\lambda = 1/2$ can have a deficit solid angle only for a finite range and that it is asymptotically flat. We discuss an possibility of its astrophysical applications.

Using the ADM decomposition of the spacetime metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{1}$$

with the lapse N and shift fields N^i, the IR modified Horava gravity theory is described by the action

$$S_H = \int dt d^3x \sqrt{g} N\,[\mathcal{L}_K + \mathcal{L}_V], \tag{2}$$

where the kinetic term

$$\mathcal{L}_K = 2\kappa^{-2}(K_{ij}K^{ij} - \lambda K^2) \tag{3}$$

is made of the extrinsic curvature $K_{ij}(\equiv 2N^{-1}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i))$, its trace K and a parameter λ. $\mathcal{L}_V$ includes all potential terms satisfying the detailed balance condition [1], and it is given by

$$\begin{aligned}\mathcal{L}_V = \kappa^2[&-\frac{1}{2\zeta^4}C_{ij}C^{ij} + \frac{\mu}{2\zeta^2}\epsilon^{ijk}R_{il}\nabla_j R^l_k - \frac{\mu^2}{8}R_{ij}R^{ij} \\ &+ \frac{\mu^2}{8(1-3\lambda)}(\frac{1-4\lambda}{4}R^2 + \Lambda R - 3\Lambda^2) - \frac{\mu^2\omega}{8(1-3\lambda)}R],\end{aligned} \tag{4}$$

where Λ (< 0) is a cosmological constant and the last term which violates softly the detailed balance condition is added [1].

When we adopt the static, spherically symmetric metric ansatz as

$$ds^2 = -N^2(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

we can write $S_H = 4\pi \int dt dr \mathcal{L}_H$ with the Lagrangian density

$$\begin{aligned}\mathcal{L}_H = \frac{N}{q^2\sqrt{f}}[&3\Lambda^2 r^2 + 2(\omega - \Lambda)(1 - f - rf') \\ &+ (1-\lambda)\frac{f'^2}{2} + (1-2\lambda)\frac{(1-f)^2}{r^2} - 2\lambda\frac{(1-f)f'}{r}],\end{aligned} \tag{6}$$

where $q^2 = 8(3\lambda - 1)/(\kappa^2\mu^2)$.

Let us consider a GM source which has the action written upto $\mathcal{O}((\partial_j\vec{\Phi})^2)$ [6]

$$S_{matter} = -\int dtd^3x\sqrt{g}N[-\frac{1}{2N^2}\partial_t\vec{\Phi}\cdot\partial_t\vec{\Phi}+\frac{1}{2}g^{ij}\partial_i\vec{\Phi}\cdot\partial_j\vec{\Phi}+\frac{\chi}{4}(\vec{\Phi}^2-\eta^2)^2], \quad (7)$$

with a dimensionless coupling constant χ. We can write $S_{matter} = 4\pi\int dtdr\ \mathcal{L}_{matter}$ with

$$\mathcal{L}_{matter} = -\frac{Nr^2}{\sqrt{f}}[\frac{1}{2}(fh'^2 + \frac{2h^2}{r^2}) + \frac{\chi}{4}(h^2 - \eta^2)^2], \quad (8)$$

where a hedgehog ansatz for the GM, $\vec{\Phi} = h(r)\vec{x}/r$, is assumed.

Performing variation of the total action $S_{total} = S_H + S_{matter}$ with respect to $h(r)$, $N(r)$, and $f(r)$ respectively, we obtain the following equations:

$$\frac{\sqrt{f}}{Nr^2}(\frac{Nr^2}{\sqrt{f}}fh')' = \frac{2h}{r^2} + \chi(h^2 - \eta^2)h, \quad (9)$$

$$(1-\lambda)\frac{f'^2}{2} + (1-2\lambda)\frac{(1-f)^2}{r^2} - 2\lambda\frac{(1-f)f'}{r} + 2(\omega - \Lambda)(1 - f - rf') + 3\Lambda^2r^2$$
$$= q^2r^2[\frac{fh'^2}{2} + \frac{h^2}{r^2} + \frac{\chi}{4}(h^2 - \eta^2)^2], \quad (10)$$

$$(\frac{N}{\sqrt{f}})'[-2\lambda\frac{(1-f)}{r} + (1-\lambda)f' - 2r(\omega - \Lambda)] \quad (11)$$
$$= -\frac{N}{\sqrt{f}}[2(1-\lambda)\frac{(1-f)}{r^2} + (1-\lambda)f'' + q^2\frac{r^2h'^2}{2}].$$

With the solution to Eq. (9)

$$h(r) = \eta \quad (12)$$

valid for the outside of the GM core, $r > \chi^{-1/2}\eta^{-1}$ [7], the solutions to the other equations (10) and (11) for various values of λ are given as follow.

1. $\lambda = 1$ case

In this $\lambda = 1$ case where Horava's theory coincides with Einstein's general theory of relativity in IR limit, Eq. (12) gives us the simple solution to Eq. (11) as $N/\sqrt{f} = 1$. Putting $1 - f \equiv -(\omega - \Lambda)r^2 + X^{1/2}$, we can rewrite the remaining equation (10) as

$$3\omega(\omega - 2\Lambda)r^2 + q^2\eta^2 = \frac{X'}{r} - \frac{X}{r^2}, \quad (13)$$

whose solution is

$$f = 1 + (\omega - \Lambda)r^2 - \sqrt{\omega(\omega - 2\Lambda)r^4 + q^2\eta^2r^2 + \beta r}. \quad (14)$$

Note that Eq. (14) would be the same as the result of Ref. [8] if there were not the additional new term $q^2\eta^2r^2$. In the limit $r >> \sqrt{q^2\eta^2/[\omega(\omega - 2\Lambda)]}$ and $r >> [\beta/\{\omega(\omega - 2\Lambda)\}]^{1/3}$, Eq. (14) can be approximated as

$$f = 1 - \frac{q^2\eta^2}{2\sqrt{\omega(\omega - 2\Lambda)}} + \frac{\Lambda_{eff}}{2}r^2 - \frac{\beta}{2\sqrt{\omega(\omega - 2\Lambda)}\,r}, \quad (15)$$

which can be compared with the Schwarzschild-AdS black hole carrying a GM charge $q\eta$ and a mass $M \simeq \beta/[4\sqrt{\omega(\omega-2\Lambda)}]$. Here a effective cosmological constant $\Lambda_{eff} \equiv 2[(\omega-\Lambda)-\sqrt{\omega(\omega-2\Lambda)}](\simeq \Lambda^2/\omega$ for $-\Lambda<\omega)$ and a deficit angle $q^2\eta^2/[2\sqrt{\omega(\omega-2\Lambda)}] = 8\eta^2/[\kappa^2\mu^2\sqrt{\omega(\omega-2\Lambda)}]$.

2. $1>\lambda$ case

With Eq. (12) in the case where $1>\lambda$, we can rewrite Eq. (10) as

$$\frac{1-\lambda}{2}(\frac{dY}{du})^2+\frac{1-3\lambda}{1-\lambda}Y^2 = 3\omega(\omega-2\Lambda)e^{(\frac{4\lambda}{1-\lambda}+4)u}+q^2\eta^2e^{(\frac{4\lambda}{1-\lambda}+2)u} \tag{16}$$

with $u=\ln r$ and $Y(u(r)) = r^{2\lambda/(1-\lambda)}(1-f(r))+(\omega-\Lambda)r^{2/(1-\lambda)}$

2.1. $\lambda=\frac{1}{3}$ *case*

In the case $\lambda=1/3$ where it is allowed for us to get a nontrivial conformal limit [9], the set of equations in Eqs. (9)-(11) obtained from Eqs. (4) and (8) can be replaced by the same form with only substitution $q^2 \to q^2/(3\lambda-1) = 8/(\kappa^2\mu^2)$, and we have their solutions

$$f = 1+(\omega-\Lambda)r^2-\frac{2M}{r}-\frac{\sqrt{3}}{9\omega(\omega-2\Lambda)\,r}[\frac{8\eta^2}{\kappa^2\mu^2}+3\omega(\omega-2\Lambda)r^2]^{\frac{3}{2}}, \tag{17}$$

and $N^2 = r^2 f(r)$. Eq. (17) goes to

$$f = 1-\frac{4\eta^2}{\kappa^2\mu^2\sqrt{\omega(\omega-2\Lambda)}}+\frac{\Lambda_{eff}}{2}r^2-\frac{2M}{r}, \tag{18}$$

in the region $r >> \sqrt{8}\eta/[\kappa\mu\sqrt{\omega(\omega-2\Lambda)}]$. In this large r limit, $f(r)$ (of this case $\lambda=1/3$) is almost the same as one of the $\lambda=1$ case (in Eq. (15)) except different values of the deficit angle, while the lapse functions $N(r)$ in these cases are very different from each other.

When there is no GM source, we have

$$f = 1+\frac{\Lambda_{eff}}{2}r^2-\frac{2M}{r}, \tag{19}$$

and $N^2 = f(r)$.

2.2. $1>\lambda>\frac{1}{3}$ *case*

From now on, we consider the case with detailed balance condition (*i.e.* $\omega=0$) and without q^2 rescaling which is done in the case $\lambda=1/3$. Eq. (16) can be written as a simple inhomogeneous equation

$$(\frac{dY}{dU})^2 = AY^2+Be^U, \tag{20}$$

where $U \equiv \gamma u$, $A = 2(3\lambda-1)/(\gamma^2(1-\lambda)^2)$, $B = 2q^2\eta^2/(\gamma^2(1-\lambda))$, and $\gamma = 2(1+\lambda)/(1-\lambda)$.

The solution is

$$f = 1 - \Lambda r^2 - q\eta\sqrt{\frac{1-\lambda}{3\lambda-1}}\frac{r}{R(r)}, \tag{21}$$

where with a constant c_R

$$\ln r(R)\,(=\frac{U}{\gamma}) = c_R - \frac{\sqrt{2(3\lambda-1)}}{\lambda-3}\ln\frac{\sqrt{1+R^2}-1}{R} + \frac{1+\lambda}{\lambda-3}\ln\frac{|1-\frac{\sqrt{2(3\lambda-1)(1+R^2)}}{1+\lambda}|}{R}. \tag{22}$$

From the last equation we can estimate the asymptotic behavior of Eq. (21) as; $r/R \simeq 0$ for large R, while, for small R, $r/R \propto r^{1-1/n(\lambda)}$ with $n(\lambda) \equiv -1+(4+\sqrt{2(3\lambda-1)})/(3-\lambda)$ and $-1 < 1-1/n(\lambda) < 1/2$. Since especially $r/R = constant$ when $n(\lambda = 1/2) = 1$, we may have a deficit angle for a finite range $r < r_0$ (with a constant r_0) in the case $\lambda = 1/2$.

The lapse function $N = \sqrt{f(r)}M$ with

$$\begin{aligned} &ln\frac{M}{\sqrt{2(1-\lambda)q^2\eta^2 + 2(3\lambda-1)r^{-2}(1-f-\Lambda r^2)^2}} \\ &= \int dr[\frac{2\lambda}{(1-\lambda)r} \\ &- \frac{2(3\lambda-1)(1-f-\Lambda r^2)}{(1-\lambda)r^2\sqrt{2(1-\lambda)q^2\eta^2+2(3\lambda-1)(1-f-\Lambda r^2)^2/r^2}}]. \end{aligned} \tag{23}$$

When $q\eta = 0$, the lapse [3]

$$N = \sqrt{f(r)}\,r^{\frac{1+3\lambda\pm2\sqrt{2(3\lambda-1)}}{1-\lambda}} \tag{24}$$

is obtained from Eq. (23) with the substitution of the term $1-f-\Lambda r^2$ by $r^{\frac{-2\lambda\pm\sqrt{2(3\lambda-1)}}{1-\lambda}}$ [3] which is a solution to the corresponding homogeneous equation of (20) (instead of the last term in Eq. (21)).

2.3. $\lambda < \frac{1}{3}$ case

When $\lambda < \frac{1}{3}$, Eq. (20) is replaced by

$$(\frac{dY}{dU})^2 = -\alpha Y^2 + Be^U, \tag{25}$$

with $\alpha = 2(1-3\lambda)/(\gamma^2(1-\lambda)^2) > 0$ and $B > 0$ given below Eq. (20). This inhomogeneous equation has a (special) solution

$$f = 1 - \Lambda r^2 - q\eta\sqrt{\frac{1-\lambda}{1-3\lambda}}r\,I(r), \tag{26}$$

where

$$\ln r(I) = \frac{U}{\gamma} = c_I - \frac{\sqrt{2(1-3\lambda)}}{\lambda - 3}\arctan\frac{I}{\sqrt{1-I^2}} + \frac{1+\lambda}{\lambda-3}\ln\left|\frac{\sqrt{2(1-3\lambda)(1-I^2)}}{1+\lambda} - I\right|. \tag{27}$$

The lapse function in this case can be obtained by the similar method as we have done in Eq. (23).

In summary, we have studied the IR modified Horava theory of gravity. In static, spherically symmetric spacetimes, we obtain exact solutions (valid outside a GM core) for general values of λ to the equations of gravity coupled to the GM. As we can see from Eqs. (15) and (18) obtained in the large r limit, in the cases $\lambda = 1$ and $\lambda = 1/3$ we have deficit angles as in Einstein's theory of gravity coupled to the GM [7]. $f(r)$ of the case $\lambda = 1/3$ (in Eq. (18)) is almost the same as one of the $\lambda = 1$ case (in Eq. (15)) except different values of the deficit angle. We also have the explicit solutions of the lapse function in both cases.

In the case $1 > \lambda > 1/3$ we have studied the Horava model with detailed balance and obtained special solutions, in addition to known general solutions [3] to its homogeneous equation. When especially $\lambda = 1/2$, $r/R \simeq$ *constant* for $r < r_0$ as seen in Eqs. (21), (22) and below, and we can have a GM spacetime that has a deficit solid angle for a finite range and is asymptotically flat, which is different from the GM spacetime in Einstein's theory of gravity [7]. This might be more helpful for us, with the GM as Refs. [10, 11], to explain near flatness of rotation curves of galaxies, which appears over a finite range $0 << r < r_0$.

To explain the near flatness of rotation curves in preceding models using GM with an energy density proportional to r^{-2}, we need nonlinear coupling between gravity and the GM as nonminimal coupling in Ref. [10] or Brans-Dicke field coupling. In the latter case, as discussed below Eq. (4) of Ref. [12], it can be yielded by the finite range, logarithmic gravitational potential that is derived from the Brans-Dicke field equation. For the rotation velocity formula to be valid only for the finite range given by the galactic halo radius r_0, the responsible GM field should vanish at distance larger than r_0 due to interactions with the nearest topological defect such as anti-monopole, in the way that the GM field lines can be absorbed into the anti-monopole core, as argued in Ref. [12].

Instead, if we study further (possibly considering Brans-Dicke field coupling to Horava gravity [13]) the $\lambda = 1/2$ Horava gravity solution given below Eq. (22) having a finite range deficit angle, more natural explanations for the near flatness can be possible. This kind of r-dependent, deficit solid angle was obtained in Brans-Dicke gravity theory [14], by studying the quantum effects [15] due to the GM, which can be expressed as quadratic in curvature as if the Horava gravity with detailed balance. When we almost complete our study, we see Ref. [16] that has reported results including some information consistent with ours in the section 1. We have not considered higher derivative terms of GM fields in Eq. (7) for simplicity. Even if we add these terms $(\partial_j(\partial_k\partial^k)^{(z-1)/2}\vec{\Phi})^2$

$(1 < z \leq 3)$ [6, 16], with the vacuum solution Eq. (12) our main results are not changed in the leading $1/r$ approximation.

Acknowledgments

We thank Professor C. Lee, colleagues P. Oh and J. Lee, J. G. Lee, and others for helpful discussions. This work was supported by the Soongsil University Research Fund.

[1] Horava P 2009 *Phys. Rev.* D **79** 084008, arXiv:0901.3775 [hep-th]
[2] Calcagni G 2009 *JHEP* **0909** 112, arXiv:0904.0829 [hep-th]
Kiritsis E and Kofinas G 2009 *Nucl. Phys.* B **821** 467, arXiv:0904.1334 [hep-th]
Brandenberger R 2009 *Phys. Rev.* D **80** 043516, arXiv:0904.2835 [hep-th]
[3] Lu H, Mei J and Pope C N 2009 *Phys. Rev. Lett.* **103** 091301, arXiv:0904.1595 [hep-th]
[4] Cai R-G, Cao L-M and Ohta N 2009 *Phys. Rev.* D **80** 024003, arXiv:0904.3670 [hep-th]
[5] Dutta S and Saridakis E N 2010 *JCAP* **1005** 013, arXiv:1002.3373 [hep-th]
Takahashi T and Soda J 2009 *Phys. Rev. Lett.* **102** 231301, arXiv:0904.0554 [hep-th]
Mukohyama S 2009 *JCAP* **0906** 001, arXiv:0904.2190 [hep-th]
Kehagias A and Sfetsos K 2009 *Phys. Lett.* B **678** 123, arXiv:0905.0477 [hep-th]
Charmousis C, Niz G, Padilla A and Saffin P A 2009 *JHEP* **0908** 070, arXiv:0905.2579 [hep-th]
[6] Tang J-Z 2009 arXiv:0911.3849 [hep-th]
Chen B and Huang Q-G 2010 *Phys. Lett.* B **683** 108, arXiv: 0904.4565 [hep-th]
Horava P 2008 arXiv:0811.2217 [hep-th]
[7] Barriola M and Vilenkin A 1989 *Phys. Rev. Lett.* **63** 341
[8] Park M-I 2009 *JHEP* **0909** 123, arXiv:0905.4480 [hep-th]
[9] Capasso D and Polychronakos A P 2010 *Phys. Rev.* D **81** 084009, arXiv:0911.1535 [hep-th]
[10] Nucamendi U, Salgado M and Sudarsky D 2000 *Phy. Rev. Lett.* **84** 3037
[11] Lee T H and Lee B J 2004 *Phys. Rev.* D **69** 127502
[12] Lee T H and Lee B J 2006 *Phys. Rev.* D **73** 128502, gr-qc/0604046
[13] Lee J, Lee T H and Oh P, arXiv:1003.2840 [hep-th]
[14] Rahaman F and Ghosh P 2008 *Mod. Phys. Lett.* A **23** 2763, arXiv:0801.2622 [gr-qc]
[15] Hiscock W A 1990 *Class. Quant. Grav.* **7** 6235
[16] Kim T and Lee C O, arXiv:1002.0784 [hep-th]
Kim S-S, Kim T and Kim Y 2009 *Phys. Rev.* D **80** 124002, arXiv:0907.3093 [hep-th]