Branes as BIons

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October 16, 2018

Abstract

A BIon may be defined as a finite energy solution of a non-linear field theory with distributional sources. By contrast a soliton is usually defined to have no sources. I show how harmonic coordinates map the exteriors of the topologically and causally non-trivial spacetimes of extreme p-branes to BIonic solutions of the Einstein equations in a topologically trivial spacetime in which the combined gravitational and matter energy momentum is located on distributional sources. As a consequence the tension of BPS p-branes is classically unrenormalized. The result holds equally for spacetimes with singularities and for those, like the M-5-brane, which are everywhere singularity free.

One of the most striking aspects of the many recent applications of p-branes to black holes in M-theory is the extent to which they admit two almost complementary aspects. On the one hand one may view a p-branes as a flat sheet-like object of zero thickness moving in flat spacetime, described by a Dirac-Born-Infeld action, and on the other hand they may be regarded as curved spacetimes with non-trivial topology and causal structure which solve the Einstein equations [1, 4]. This second aspect has become especially prominent recently in the many papers in which the $AdS_{p+2} \times S^{d_T-1}$ geometry near the throat has played a vital role. In this paper I wish to begin to address the question of why a description of p-branes based on flat space can be so effective. This is of course part of a much wider puzzle; how is it that string theory and M-theory, based as they are on objects moving in a fixed, and usually flat, background give rise to theories like general relativity in which no particular background is preferred?

If we view the problem from the point of view of general relativity the answer is perhaps not so hard to see. In general relativity and related theories no particular coordinate system is preferred and indeed it may be impossible to find a single coordinate system which covers the entire spacetime manifold $\mathcal{M}^n$. However that does not prevent us fixing upon a *particular* set of coordinates x^α say and restricting our coordinate transformations to Poincaré transformations of the x^α. In other words we can always, locally at least, introduce an arbitrary flat spacetime with inertial coordinates x^α and metric $\eta_{\mu\nu}$ and view gravity as the manifestation of a a non-linear spin two field in flat space.

In fact precisely this procedure is often followed when one discusses the definition of energy and its conservation in general relativity [11, 12]. One assumes additionally that the coordinates x^α are asymptotically Minkowskian in the sense that at large spatial distances the curved spacetime metric $g_{\mu\nu}$ tends to the flat metric $\eta_{\mu\nu}$. Because one has a variational principle one may construct the conserved canonical Einstein energy momentum pseudo tensor ${}_E\mathfrak{t}^\mu{}_\nu$ such that the conservation equation takes the form:

$$\partial_\mu\Big(\sqrt{-g}T^\mu{}_\nu +_E \mathfrak{t}^\mu{}_\nu\Big) = 0. \tag{1}$$

The difficulty is of course that the pseudo-tensor ${}_E\mathfrak{t}^\mu{}_\nu$ depends in an essential way on the chosen coordinates x^α. To some extent this does not matter if one wishes to calculate for example, in an $E(p)$-invariant $(p+1+d_T)$-dimensional spacetime spacetime a quantities like the tension

$$T = -\int d^{d_T}y(\sqrt{-g}T^0{}_0 +_E \mathfrak{t}^0{}_0) \tag{2}$$

because, as long as the coordinates cover the whole spacetime $\mathcal{M}^n$, this is independent of the choice of coordinates. However if the spacetime is not topologically trivial $\mathcal{M}^n \not\equiv \mathbb{R}^n$ or if one seeks some more localized idea of energy one has problems.

One way out of this *impasse* is to fix the troublesome gauge freedom once and for all and to decree that although all coordinates are equal some are more equal than others. In other words, that, *at least for some problems* , a particular choice of gauge is preferred. This does not mean giving up the equivalence principle or the principle of general covariance, any more than using unitary gauge in Yang-Mills theory means giving up Yang-Mills gauge invariance, it simply means that in order to exploit to the full our flat space concepts we are going to pick the most convenient coordinate system for that purpose.

This leads us to the question: what is the most convenient coordinate choice for studying p-branes? The suggestion of this paper is that the answer is *harmonic coordinates* for which:

$$\left(\sqrt{-g}g^{\mu\nu}\right)_{,\mu} = 0. \tag{3}$$

Moreover I suggest that the most convenient choice of variables to describe the gravitational field is

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}, \tag{4}$$

in terms of which the harmonic gauge condition becomes

$$\mathfrak{g}^{\mu\nu}{}_{,\mu} = 0. \tag{5}$$

Before discussing p-branes I will review some of the (largely well known) properties of harmonic coordinates and the variables $\mathfrak{g}^{\mu\nu}$. Firstly the name harmonic means just that: the n-functions x^α are solutions of the curved space wave equation

$$\nabla^2 x^\alpha \frac{1}{\sqrt{-g}} \partial_\mu \big(\mathfrak{g}^{\mu\nu} x^\alpha_{,\nu} \big) = 0. \tag{6}$$

Another way of saying the same thing is that the identity map is a harmonic map from $\{\mathcal{M}^n, g_{\mu\nu}\}$ to $\{\mathbb{R}^n, \eta_{\mu\nu}\}$. In linear theory the harmonic condition coincides with the De-Donder gauge frequently used in perturbation theory. This is because if $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ then to lowest order

$$\mathfrak{g}^{\mu\nu} \approx \eta^\mu - h^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} h^\alpha_\alpha. \tag{7}$$

Perhaps one of the most useful properties of the gothic variables $\mathfrak{g}^{\mu\nu}$ is that they behave nicely under dimensional reduction. It is well known that if a metric g is the metric on a product manifold :

$$g = g_1 \oplus g_2 \tag{8}$$

one must Weyl rescale the metrics g_1 and g_2 to put them in Einstein conformal gauge. This is because the actions don't add, in other words even though

$$R_{\mu\nu}(g) = R_{\mu\nu}(g_1) \oplus R_{\mu\nu}(g_2) \tag{9}$$

it is not true that

$$\sqrt{-g} g^{\mu\nu} R_{\mu\nu}(g) \neq \sqrt{-g_1} g_1^{\mu\nu} R_{\mu\nu}(g_1) + \sqrt{-g_2} g_2^{\mu\nu} R_{\mu\nu}(g_2) \tag{10}$$

However for the gothic variables if the metric is a product and if in addition

$$\mathfrak{g}^{\mu\nu} = \mathfrak{g}_1^{\mu\nu} \oplus \mathfrak{g}_2^{\mu\nu} \tag{11}$$

then necessarily

$$\mathfrak{g}^{\mu\nu} R_{\mu\nu}(g) \neq \mathfrak{g}^{\mu\nu} R_{\mu\nu}(g_1) + \mathfrak{g}^{\mu\nu} R_{\mu\nu}(g_2). \tag{12}$$

Of course

$$\sqrt{-g_1} g_1^{\mu\nu} \neq \mathfrak{g}_1{}^{\mu\nu}. \tag{13}$$

In string theory products of metrics correspond to tensor products of conformal field theories so the moral seems to be that the use of the gothic variables better respects that tensor product structure. As a practical matter it is certainly easier to use (11) rather than to remember the formulae for the Weyl rescaling.

It is of course standard that using the gothic variables $\mathfrak{g}^{\mu\nu}$ simplifies the Lagrangian formulation. If $\Gamma_\alpha{}^\beta{}_\gamma = \Gamma_\gamma{}^\beta{}_\alpha$ are the Christoffel symbols of the metric $g_{\mu\nu}$ then one has the identity

$$\sqrt{-g} g^{\mu\nu} R_{\mu\nu} = \mathfrak{g}^{\mu\nu} \big(\Gamma_\mu{}^\beta{}_\alpha \Gamma_\nu{}^\alpha{}_\beta - \Gamma_\mu{}^\alpha{}_\nu \Gamma_\alpha{}^\beta{}_\beta \big) - \partial_\alpha \mathfrak{W}^\alpha \tag{14}$$

where

$$\mathfrak{W}^\mu = \mathfrak{g}^{\alpha\sigma} \Gamma_\sigma{}^\beta{}_\beta - \mathfrak{g}^{\mu\nu} \Gamma_\mu{}^\alpha{}_\nu \tag{15}$$

Now if we define

$$\mathfrak{L} = \frac{1}{16\pi} \mathfrak{g}^{\mu\nu} \big(\Gamma_\mu{}^\beta{}_\alpha \Gamma_\nu{}^\alpha{}_\beta - \Gamma_\mu{}^\alpha{}_\nu \Gamma_\alpha{}^\beta{}_\beta \big), \tag{16}$$

then we find that that $\mathfrak{L}$ contains no second derivatives of and is a homogeneous function of degree -1 in $\mathfrak{g}^{\mu\nu}$ and a homogeneous function of degree 2 in $\mathfrak{g}^{\mu\nu}{}_{,\lambda}$. Moreover

$$\mathfrak{W}^\alpha = \frac{1}{16\pi} \mathfrak{g}^{\mu\nu} \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{\mu\nu}{}_{,\alpha}}. \tag{17}$$

It follows from (17) that the current density $\mathfrak{W}^\alpha$ has the interpretation of the Noether current associated to dilations or homotheties. Rescaling the coordinates x^α is equivalent to rescaling the gothic variable $\mathfrak{g}^{\mu\nu}$. The associated Noether charge is of course closely related to the surface term in the gravitational action. We shall not pursue that avenue here but instead remark that the Einstein pseudo-tensor $\mathfrak{t}^\mu{}_\nu$ which appears in (1) is just the canonical energy-momentum tensor associated to $\mathfrak{L}$, i.e.

$${}_E\mathfrak{t}^\mu{}_\nu = \delta^\mu_\nu \mathfrak{L} - \mathfrak{g}^{\alpha\beta}{}_{,\nu} \frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{\alpha\beta}{}_{,\mu}}. \tag{18}$$

Its conservation (1) may be ascribed to the invariance of the action associated to the Lagrangian $\mathfrak{L}$ under translations along the directions of the arbitrary coordinate chart $\{x^\alpha\}$. Of course to obtain something like the conventional idea of energy one chooses the chart $\{x^\alpha\}$ to be asymptotically Minkowskian. If the spacetime manifold M^n is not topologically trivial then one will not be able to define the translations globally all over M^n. This is of course the problem one faces if black holes or branes are present. In the case of extreme branes however, and for the particular choice of harmonic coordinates, we shall see that this problem is somewhat alleviated since the exterior region of the branes is mapped to $\mathbb{R}^{p+1} \times (\mathbb{R}^{d_T} \setminus \{\mathbf{y}_i\})$ where $\{\mathbf{y}_i\}$ are the positions of the branes.

The Einstein pseudo-tensor ${}_E\mathfrak{t}^\mu_\nu$ has a number of counter-intuitive properties. For example Schrödinger [13] has pointed out that if one calculates it for the Schwarzschild solution using the coordinates $x^\alpha = (t, r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, where r is the usual Schwarzshild radial coordinate defined so that the area of a sphere is $4\pi r^2$, one finds that it vanishes, at least away from the horizon at $r = 2M$ and the singularity $r = 0$, where it is not defined. The coordinates chosen by Schrödinger have the property that $\sqrt{-g}g = 1$ but they are not in fact harmonic. One obtains harmonic coordinates by taking $x^\alpha = (t, (r-M)\sin\theta\cos\phi, (r-M)\sin\theta\sin\phi, (r-M)\cos\theta)$. In fact this formulae gives harmonic coordinates for the entire Reissner-Nordstrom family. For later use we note that isotropic coordinates $x^\alpha = (t, \rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)$ where

$$r = \rho + M + \frac{M^2 - Z^2}{4\rho} \tag{19}$$

and $Z^2 = q^2 + p^2$ are harmonic if and only if

$$M = \pm|Z|. \tag{20}$$

Of course we usually take the plus sign to get a solution with a horizon rather than one with a naked singularity but as with the condition for the existence of Killing spinors either sign is actually allowed.

Another disadvantage of the Einstein pseudo-tensor is that it does not give rise to simple expressions for the total angular momentum. This is because if one raises an index with $g^{\mu\nu}$ it is not necessarily symmetric. Thus if one introduces a flat metric $\eta_{\mu\nu}$ and uses it to define a set of lorentz transformations with respect to the chart $\{x^\alpha\}$ one will not obtain directly form ${}_E t^\mu_\nu$ a set of conserved currents. It is clear however by Noether's theorem that such currents can always be constructed. Indeed associated with any one parameter family of diffeomorphisms there exists such a current. In fact these currents are not unique. One possible choice is that of Landau and Lifshitz. They introduce a symmetric 'complex" ${}_{LL}\mathfrak{t}^{\mu\nu} ={}_{LL}\, \mathfrak{t}^{\nu\mu}$ in terms of the the symmetric quantity

$$\begin{aligned} {}_{LL}\Theta^{\mu\nu} &= \sqrt{-g}\big(\mathfrak{T}^{\mu\nu} + \mathfrak{t}^{\mu\nu}\big) \\ &= \partial_\beta\partial_\alpha\Big(\frac{1}{16\pi}\big(\mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\mu\alpha}\mathfrak{g}^{\beta\nu}\big)\Big) \end{aligned}$$

It follows that

$$\partial_\mu {}_{LL}\Theta^{\mu\nu} = 0. \tag{21}$$

One may therefore, with the same caveats as before, regard ${}_{LL}\Theta^{\mu\nu}$ as giving the distribution of energy and momentum for the combined gravitational and matter fields. As with the Einstein-pseudo-tensor, so with the Landua-Lifshitz complex, it is reallly only integral quantities for asymptotically flat spacetimes which are invariant under change of the coordinate chart $\{x^\alpha\}$.

For static $E(p,1)$-invariant asymptotically flat brane configurations multiplication of (21) by x^α and integration by parts leads to some non-trivial Virial relations for the integrals of the non-zero components of ${}_{LL}\Theta^{\mu\nu}$ over the transverse dimensions. Thus

$$\int d^{d_T} y_{LL}\Theta^{ij} = 0, \tag{22}$$

and

$$\int d^{d_T} y_{LL}\Theta^{ab} = \eta^{ab} T, \tag{23}$$

where $i, j = i, \dots, d_T$ and $a, b = 0, \dots, p$ and T is the tension, i.e. the energy per unit p-volume.

In harmonic coordinates the formula for ${}_{LL}\Theta^{\mu\nu}$ simplifies

$${}_{LL}\Theta^{\mu\nu} = \frac{1}{16\pi}\Big(\mathfrak{g}^{\alpha\beta}\partial_\alpha\partial_\beta\mathfrak{g}^{\mu\nu} - \partial_\alpha\mathfrak{g}^{\beta\nu}\partial_\beta\mathfrak{g}^{\mu\alpha}\Big). \tag{24}$$

The harmonic function rule for orthogonally 'intersecting" or 'overlapping " branes depending on a set of harmonic functions $H_i(\mathbf{y})$ on $\mathbb{E}^{d_T}$ states that

- the metric is diagonal,
- that while the transverse space is not flat nevertheless

$$\mathfrak{g}^{ij} = \delta^{ij}. \tag{25}$$

- The time direction is common to all the branes and

$$\mathfrak{g}^{00} = -\prod H_i \tag{26}$$

- In a direction in one or more branes

$$\mathfrak{g}^{aa} = \prod_a H_i, \tag{27}$$

  where $\prod_a$ denotes a product over all the harmonic functions associated with branes sharing the direction a.

A simple calculation reveals that the coordinates $x^\alpha = (x^a, y^i)$ are harmonic. Moreover in the case of a single type of brane, with just one harmonic function, corresponding to branes located at positions $\mathbf{y} = \mathbf{y}_i \in \mathbb{R}^{d_T}$, the transverse stresses vanish point-wise

$${}_{LL}\Theta^{ij} = 0., \tag{28}$$

while the energy-momentum is strictly localized on the branes

$${}_{LL}\Theta^{ab} = -\eta^{ab}\sum_i T_i\delta(\mathbf{y}_i). \tag{29}$$

For more than one type of brane the property (28) that the stresses vanish remains true but the distribution of energy-momentum is more complicated than (29). This is presumably not unrelated to the fact while the single branes cary no entropy, sytems of intersecting branes can.

These results apply in particular to the both the 'elementary" M-2-brane and the 'solitonic" M-5-brane. This is quite surprising. Both have a non-trivial topological and causal structure [1, 4, 2]. The former has Reissner-Nordstrom like geometry with a singularity inside an event horizon, the latter is everywhere non singular [4]. Harmonic coordinates map the exteriors of both to $\mathbb{R}^{11}$ with distributional sources, a fact first noted for the M-2-brane by Duff and Stelle [3]. As they observed, double-dimensional reduction yields the fundamental string in 10 dimensions [7, 8] which is truly singular but has a distributional source. As noted in [7] and [8] the string tension is not renormalized. This is of course consistent with the present analysis and gibes with some recent work on cosmic strings [9, 10]. These papers consider time-dependent cosmic strings in first order perturbation theory. In the present paper I have considered static branes in the exact theory. They find that self-interactions do not result in a classical renormalization of the string tension and that in four spacetime dimensions gravitational self-interactions vanish. At the linear level this cancellation is a direct consequence of the BPS condition and is closely related to the antigravitating properties of the solutions, since the sum of the relevant propagators vanishes as a consequence the antigravity condition cf [14]. Similar observations may be found in the old literature on self-energies [15]. The present paper establishes the classical non-renormalization property for general p in the fully non-linear theory.

In addition to branes, one frequently considers waves in $\mathbb{R}^2 \times \mathbb{R}^{d_T}$ with coordinates $x^+, x^-, \mathbf{y}$. Suppose that the wave moves along the x^- direction then

$$\mathfrak{g}^{ij} = \delta^{ij} \tag{30}$$

$$\mathfrak{g}^{+-} = 1 \tag{31}$$

$$\mathfrak{g}^{++} = -F(x^-, \mathbf{y}). \tag{32}$$

Evidently the coordinates $\{x^+, x^-, \mathbf{y}\}$ are harmonic irrespective of the precise form of the function $F(x^-, \mathbf{y})$ as is the vanishing stress condition (28). In fact the only non-vanishing component of ${}_{LL}\Theta^{\mu\nu}$ is

$${}_{LL}\Theta^{++} = \frac{1}{16\pi}\nabla^2 F. \tag{33}$$

For a non-singular pure gravitational wave, the function $F(x^-, \mathbf{y})$ is everywhere non-singular and the right hand side of (33) vanishes. This does not invalidate the idea of using non-tensorial measures of energy and momentum because a plane wave spacetime is not asymptotically flat but it does illustrate the potential hazards. For the singular waves that give rise on dimensional reduction to 0-branes, or indeed higher dimensional branes, one choses F to be independent of x^- and finds that there is is a distributional source just as in (29).

One may how the results of this paper are altered if one chooses a different expression for the local density of energy and momentum. The present observations were in fact stimulated by reading a paper of Papapetrou [11] who also used harmonic coordinates and who explicitly introduced a flat metric $\eta^{\alpha\beta}$. His expression, call it ${}_P\Theta^{\mu\nu}$, for the local density of energy and momentum reduces in harmonic coordinates to

$$ {}_P\Theta^{\mu\nu} = \Box \mathfrak{g}^{\mu\nu}, \tag{34} $$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ is the flat space D'Alembertian operator. In his later textbook [12] he abandons his earlier approach and instead treats the Landau-Lifshitz formalism. Applied to M-branes one gets the same results with either formalism.

In a recent paper on Born-Infeld theory [6] the concept of a BIon was introduced. This is a finite energy solution of a non-linear theory with a distributional source. In the case of the standard Born-Infeld theory, there is a source of electric charge and the analogue of (29) is

$$ J^0 = \frac{1}{4\pi}\mathrm{div}\mathbf{D} = \sum_i e_i \delta(\mathbf{y}_i), \tag{35} $$

where the zeroth componet of the current J^0 is the analogue of ${}_{LL}\Theta^{00}$. In a sense the results of the present paper may be paraphrased by saying that harmonic coordinates map extreme p-branes into a special kind of BIon associated to a non-linear spin two field. In the case of the non-linear theory of spin one, Gauss's theorem says that despite the polarization of the vacuum, the electric charge of a BIon is not classically renormalized. In the case of the gravity the tension is not classically renormalized.

References

[1] G W Gibbons and P K Townsend, Vacuum Interpolation in Supergravity via Super p-branes, *Phys Rev Lett* **23** (1993) 3754-3753 hep-th/9307049

[2] M J Duff, G W Gibbons and P K Townsend. Macroscopic Superstrings as Interpolating Solitons, *Phys Lett* **B332**(1994) 321-328 hep-th/9405124

[3] M J Duff and K S Stelle, Multimembrane solutions of D=11 Supergravity, *Phys Lett* **B253**(1991) 113-118

[4] G W Gibbons, G T Horowitz and P K Townsend, Higher Dimsnional Resolution of Dilatonic Black Hole Singularities, *Class Quant Grav* **12** (1995) 297-318 hep-th/9410073

[5] A A Tseytlin, Harmonic superpositions of M-branes, *Nucl Phys* **B276** (1996) 149 hep-th/9604035

[6] G W Gibbons, Born-Infeld particles and Dirichlet p-branes, *Nucl Phys* **B** in press hep-th/9709027

[7] A Dabholkar and J A Harvey, Nonrenormalization of the superstring tension *Phys Rev Lett* **63** (1989) 478

[8] A Dabholkar, G W Gibbons, J A Harvey and F Ruiz-Ruiz, Superstrings and Solitons *Nucl Phys* **B340** (1990) 33-55

[9] B Carter and R A Battye, Non-divergence of gravitational self interactions for Goto-Nambu strings, hep-th/9803012

[10] A Bounanno and T Damour, Effective action and tension renormaliztion for cosmic and fundamental strings, hep-the/9803025

[11] A Papapetrou, Einstein's theory of gravitation and flat space, *Proc Roy Irish Acad* **52**(1948) 11-22

[12] A Papapetrou, *Lectures on General Relativity* D Reidel, (1974)

[13] E Schrödinger, Die Energiekomponenten des Gravitationsfeldes *Phys Zeit* **18** (1918) 4

[14] J W van Holten, Stability and mass of point particles, hep-th/9709141

[15] S S Schweber, An Introduction to Relativistic Field Theory, Row Petersen and Co. (1961) section 15a , pp 513-524