The Gravitational Hamiltonian, Action, Entropy and Surface Terms

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Abstract

We give a general derivation of the gravitational hamiltonian starting from the Einstein-Hilbert action, keeping track of all surface terms. The surface term that arises in the hamiltonian can be taken as the definition of the 'total energy', even for spacetimes that are not asymptotically flat. (In the asymptotically flat case, it agrees with the usual ADM energy.) We also discuss the relation between the euclidean action and the hamiltonian when there are horizons of infinite area (e.g. acceleration horizons) as well as the usual finite area black hole horizons. Acceleration horizons seem to be more analogous to extreme than nonextreme black holes, since we find evidence that their horizon area is not related to the total entropy.

1. Introduction

Traditionally, the gravitational hamiltonian has been studied in the context of either spatially closed universes or asymptotically flat spacetimes (see e.g. [1]). In the latter case, the effect of black hole horizons has been investigated [2]. However in recent years, there has been interest in more general boundary conditions. One example involves the possibility of a negative cosmological constant, resulting in spacetimes which asymptotically approach anti-de Sitter space. Perhaps of greater interest is the study of the pair creation of black holes in a background magnetic field [3]. This involves spacetimes such as the Ernst solution [4] which asymptotically approach the Melvin metric [5], and have a noncompact acceleration horizon as well as the familiar black hole horizons. We will give a general derivation of the gravitational hamiltonian which can be applied to all spacetimes regardless of their asymptotic behavior or type of horizons.

In most field theories, the hamiltonian can be derived from the covariant action in a straightforward way. In general relativity the situation is complicated by the fact that the Einstein-Hilbert action includes a surface term. In most derivations of the gravitational hamiltonian, the surface term is ignored. This results in a hamiltonian which is just a multiple of a constraint. One must then add to this constraint appropriate surface terms so that its variation is well defined [1]. We will show that the boundary terms in H come directly from the boundary terms in the action, and do not need to be added "by hand".

Since the value of the hamiltonian on a solution is the total energy, we obtain a definition of the total energy for spacetimes with general asymptotic behavior. We will show that this definition agrees with previous definitions in special cases where they are defined. In particular, for asymptotically flat spacetimes, the energy agrees with the usual ADM definition [6], and for asymptotically anti-de Sitter spacetimes it agrees with the definition proposed by Abbott and Deser [7].

The relation between the action and the hamiltonian is of special interest in the euclidean context where it is related to thermodynamic properties of the spacetime. For an ordinary field theory, the euclidean action for a static configuration whose imaginary time is identified with period β is simply $\tilde{I} = \beta H$. It is well known that in general relativity, if there is a (nonextreme) black hole horizon present, this relation is modified to include a factor of one quarter of the area of the horizon on the right hand side. It is clear that an acceleration horizon must enter this formula differently, since its area is infinite. We will derive the general relation between the euclidean action and the hamiltonian which applies to acceleration horizons as well as black hole horizons.

The fact that the naive relation $\tilde{I} = \beta H$ can be modified by black holes leads to a simple argument that the entropy of nonextremal black holes is S = A/4, where A is the horizon area [8]. It has recently been shown [9] that a similar argument applied directly to extreme Reissner-Nordström black holes yields S = 0, even though the horizon area is nonzero (see also [10,11]). We will argue that acceleration horizons are similar to extreme horizons in that they also do not contribute to the total entropy, although for a different reason.

We begin in section 2 by deriving the canonical hamiltonian from the covariant Einstein-Hilbert action, keeping track of all surface terms. This discussion applies to spacetimes that can be foliated by complete, nonintersecting spacelike surfaces. Thus, there are no inner boundaries, and horizons play no special role at this point. In section 3 we show that the surface term that arises in the hamiltonian is a reasonable definition of the total energy for a general spacetime: It agrees with previous definitions when they are defined. In section 4 we consider the effect of horizons, and derive the general relation between the hamiltonian and the euclidean action. We then discuss the entropy, and point out the differences between horizons of finite and infinite area.

2. Derivation of the Hamiltonian: No inner boundaries

2.1. The action

We start with the covariant Lorentzian action for a metric g and generic matter fields ϕ :

$$I(g,\phi) = \int_{M} \left[\frac{R}{16\pi} + L_m(g,\phi) \right] + \frac{1}{8\pi} \oint_{\partial M} K$$
 (2.1)

where R is the scalar curvature of g, L_m is the matter lagrangian, and K is the trace of the extrinsic curvature of the boundary. The surface term is required so that the action yields the correct equations of motion subject only to the condition that the induced three metric and matter fields on the boundary are held fixed. (We assume that L_m includes at most first order derivatives.) The action (2.1) is well defined for spatially compact geometries, but diverges for noncompact ones. To define the the action for noncompact geometries, one must choose a reference background g_0 , ϕ_0 . We require that this background be a static solution to the field equations. The physical action is then the difference

$$I_P(g,\phi) \equiv I(g,\phi) - I(g_0,\phi_0) ,$$
 (2.2)

so the physical action of the reference background is defined to be zero. I_P is finite for a class of fields g, ϕ which asymptotically approach g_0, ϕ_0 in the following sense. We fix a boundary near infinity Σ^{∞} , and require that g, ϕ induce the same fields on this boundary as g_0, ϕ_0^{-1} .

For asymptotically flat spacetimes, the appropriate background is flat space with zero matter fields, and (2.2) reduces to the familiar form of the gravitational action

$$I_P(g,\phi) = \int_M \left[\frac{R}{16\pi} + L_m \right] + \frac{1}{8\pi} \oint_{\partial M} (K - K_0)$$
 (2.3)

where K_0 is the trace of the extrinsic curvature of the boundary embedded in flat spacetime. However, when matter (or a cosmological constant) is included, one may wish to consider spacetimes which are not asymptotically flat. In this case one cannot use flat space as the background, and one must use the more general form of the action (2.2).

¹ This condition can be weakened so that the induced fields agree to sufficient order so that their difference does not contribute to the action in the limit that Σ^{∞} recedes to infinity.

2.2. The hamiltonian

Since the physical action is given by (2.2), the physical hamiltonian is the difference between the hamiltonian computed from (2.1) and the one computed for the background. To cast the action (2.1) into hamiltonian form we follow the discussion in [12] except that all surface terms are retained. To begin, we introduce a family of spacelike surfaces Σ_t labeled by t, and a timelike vector field t^{μ} satisfying $t^{\mu}\nabla_{\mu}t = 1$. In terms of the unit normal n^{μ} to the surfaces, we can decompose t^{μ} into the usual lapse function and shift vector $t^{\mu} = Nn^{\mu} + N^{\mu}$. In this section we assume that there are no inner boundaries, so the surfaces Σ_t do not intersect and are complete. This does not rule out the existence of horizons, but it implies that if horizons form, one continues to evolve the spacetime inside the horizon as well as outside. It is convenient to choose the surfaces Σ_t so that they meet the boundary near infinity Σ^{∞} orthogonally. (This is not essential, but it simplifies the analysis. Notice that we do not require that t^{μ} be tangent to Σ^{∞} .) Thus the boundary ∂M consists of an initial and final surface with unit normal n^{μ} , and a surface near infinity Σ^{∞} on which n^{μ} is tangent.

The four dimensional scalar curvature can be related to the three dimensional one \mathcal{R} and the extrinsic curvature $K_{\mu\nu}$ of the surfaces Σ_t by writing

$$R = 2(G_{\mu\nu} - R_{\mu\nu})n^{\mu}n^{\nu} . {(2.4)}$$

From the usual initial value constraints, the first term can be expressed

$$2G_{\mu\nu}n^{\mu}n^{\nu} = \mathcal{R} - K_{\mu\nu}K^{\mu\nu} + K^2 \ . \tag{2.5}$$

The second term can be evaluated by commuting covariant derivatives on n^{μ} with the result

$$R_{\mu\nu}n^{\mu}n^{\nu} = K^2 - K_{\mu\nu}K^{\mu\nu} - \nabla_{\mu}(n^{\mu}\nabla_{\nu}n^{\nu}) + \nabla_{\nu}(n^{\mu}\nabla_{\mu}n^{\nu})$$
 (2.6)

When substituted into the action (2.1), the two total derivative terms in (2.6) give rise to boundary contributions. The first is proportional to n^{μ} and hence contributes only on the initial and final boundary. It completely cancels the $\oint K$ term on these surfaces. The second term is orthogonal to n^{μ} and only contributes to the surface integral near infinity. If r^{μ} is the unit normal to Σ^{∞} , then the integral over this surface becomes

$$\frac{1}{8\pi} \int_{\Sigma^{\infty}} \nabla_{\mu} r^{\mu} + r_{\nu} n^{\mu} \nabla_{\mu} n^{\nu} = \frac{1}{8\pi} \int_{\Sigma^{\infty}} (g^{\mu\nu} - n^{\mu} n^{\nu}) \nabla_{\mu} r_{\nu}$$
 (2.7)

This surface integral has a simple geometric interpretation. The surface Σ^{∞} is foliated by a family of two surfaces S_t^{∞} coming from its intersection with Σ_t . The integrand in (2.7) is simply the trace of the two dimensional extrinsic curvature 2K of S_t^{∞} in Σ_t . Thus the action (2.1) takes the form

$$I = \int Ndt \left[\frac{1}{16\pi} \int_{\Sigma^{\infty}} \sqrt{{}^{3}g} (\mathcal{R} + K_{\mu\nu}K^{\mu\nu} - K^{2} + 16\pi L_{m}) + \frac{1}{8\pi} \int_{S_{t}^{\infty}} {}^{2}K \right]$$
(2.8)

where 3g is the induced metric on Σ_t .

We now introduce the canonical momenta $p^{\mu\nu}$, p conjugate to ${}^3g_{\mu\nu}$, ϕ and rewrite the action in hamiltonian form. We first consider the case when the matter does not contain gauge fields. Since the extrinsic curvature $K_{\mu\nu}$ is related to the time derivative of the three metric ${}^3\dot{g}_{\mu\nu}$ by

$$K_{\mu\nu} = \frac{1}{2N} [^{3} \dot{g}_{\mu\nu} - 2D_{(\mu}N_{\nu)}]$$
 (2.9)

where D_{μ} is the covariant derivative associated with ${}^3g_{\mu\nu}$, when we write the action in a form that does not contain derivatives of the shift vector, we obtain another surface term $-2\int_{S_{+}^{\infty}} N^{\mu}p_{\mu\nu}r^{\nu}$. So the action takes the form

$$I = \int dt \left[\int_{\Sigma_t} (p^{\mu\nu} \, {}^3\dot{g}_{\mu\nu} + p\dot{\phi} - N\mathcal{H} - N^{\mu}\mathcal{H}_{\mu}) + \frac{1}{8\pi} \int_{S_t^{\infty}} (N^2 K - N^{\mu} p_{\mu\nu} r^{\nu}) \right]$$
(2.10)

where \mathcal{H} is the Hamiltonian constraint, and \mathcal{H}_{μ} is the momentum constraint. Both of these constraints contain contributions from the matter as well as the gravitational field. The hamiltonian is thus

$$H = \int_{\Sigma_t} (N\mathcal{H} + N^{\mu}\mathcal{H}_{\mu}) - \frac{1}{8\pi} \int_{S_*^{\infty}} (N^2 K - N^{\mu} p_{\mu\nu} r^{\nu}) . \tag{2.11}$$

This expression for the hamiltonian diverges in general, but recall that physically we are not interested in the action (2.1) but in (2.2). We must therefore derive the hamiltonian for the reference background. Since this background is a stationary solution to the field equations, when we repeat the above analysis using the stationary slices we find that the momenta $p_0^{\mu\nu}$, p_0 vanish and the constraints vanish. If we label the static slices so that $N_0 = N$ on Σ^{∞} , the reference hamiltonian is simply

$$H_0 = -\frac{1}{8\pi} \int_{S_{\infty}^{\infty}} N^2 K_0 \tag{2.12}$$

The physical Hamiltonian is the difference

$$H_P \equiv H - H_0 = \int_{\Sigma_t} (N\mathcal{H} + N^{\mu}\mathcal{H}_{\mu}) - \frac{1}{8\pi} \int_{S_t^{\infty}} \left[N(^2K - ^2K_0) - N^{\mu}p_{\mu\nu}r^{\nu} \right]$$
 (2.13)

Given a solution, one can define its total energy associated with the time translation $t^{\mu} = Nn^{\mu} + N^{\mu}$ to be simply the value of the physical hamiltonian²

$$E = -\frac{1}{8\pi} \int_{S_s^{\infty}} \left[N(^2K - {}^2K_0) - N^{\mu} p_{\mu\nu} r^{\nu} \right]$$
 (2.14)

² Choosing N=1 and $N^{\mu}=0$, our expression is similar to the one proposed in [13] for a quasilocal energy. However, the choice of reference background seems highly ambiguous for a general finite two sphere, while it is fixed in our approach from the beginning by the asymptotic behavior of the fields.

Notice that the energy of the reference background is automatically zero. In the next section we will show that (2.14) agrees with previous definitions of the energy in special cases where they have been defined.

There is a well known generalization of the above discussion to the case where the matter lagrangian contains gauge fields. For example, suppose we start with the Maxwell lagrangian $L_M = -\frac{1}{16\pi}F^2$ where F = dA is the Maxwell field. Then the canonical variables are the spatial components of A_μ and their conjugate momenta E^μ , while the time component A_t acts like a Lagrange multiplier. Using the fact that the inverse spacetime metric can be written $g^{\mu\nu} = {}^3g^{\mu\nu} - n^\mu n^\nu$ with $n^\mu = (t^\mu - N^\mu)/N$ one can rewrite the Maxwell action in Hamiltonian form. The usual energy density $\frac{1}{8\pi}(E^2 + B^2)$ is multiplied by the lapse N and contributes to the Hamiltonian constraint \mathcal{H} . The usual momentum density $\frac{1}{4\pi}\epsilon_{\mu\nu\rho\sigma}n^\nu E^\rho B^\sigma$ is multiplied by the shift N^μ and contributes to the momentum constraint \mathcal{H}_μ . The net result is that the Hamiltonian for the combined Einstein-Maxwell theory again takes the form (2.11) except for an additional term $\frac{1}{4\pi}E^\mu D_\mu A_t$ in the volume integral. This can be integrated by parts to yield $-A_t/4\pi$ times the Gauss constraint, $D_\mu E^\mu = 0$, and another surface term $\frac{1}{4\pi}\oint_{S_t^\infty}A_t E^\mu r_\mu$. This term vanishes for asymptotically flat spacetimes without horizons and for any purely magnetic field configuration, but it may be nonzero in general. We shall ignore it in this paper but it is important in electrically charged black holes [14].

3. Agreement with previous expressions for the total energy

3.1. Asymptotically flat spacetimes

In this section we show that the expression for the total energy obtained directly from the action in the previous section (2.14) agrees with earlier expressions whenever they are defined. We first consider asymptotically flat spacetimes. Here, the ADM energy is given by

$$E_{ADM} = \frac{1}{16\pi} \oint_{S} (D^{i}h_{ij} - D_{j}h)r^{j}$$
 (3.1)

where the indices i, j run over the three spatial dimensions, $h_{ij} = {}^3g_{ij} - {}^3g_{0ij}$ (${}^3g_{0ij}$ being the background three-metric 3), D_i is the background covariant derivative, and r^i is the unit normal to the large sphere S. The energy obtained from the action (2.14) depends on a choice of lapse and shift. Taking N = 1 and $N^{\mu} = 0$ (which is appropriate for a unit time translation) yields

$$E = -\frac{1}{8\pi} \int_{S} (^{2}K - {}^{2}K_{0}) \tag{3.2}$$

Both (3.2) and (3.1) are coordinate invariant but depend on a choice of reference background. We want to show that they are equal whenever the induced metrics on S agree.⁴

³ For asymptotically flat spacetimes, the background three-metric is usually chosen to be flat, but for later applications it is convenient to keep the notation general.

⁴ This was also noted in [15].

To this end, it is convenient to choose a particular set of coordinates. Given a large sphere S in the original spacetime, one can choose coordinates in a neighborhood of S so that the metric 3g is

$$ds^2 = dr^2 + q_{ab}dx^a dx^b (3.3)$$

where a, b run over the two angular variables, r = 0 on S, and the two dimensional metric q_{ab} is a function of r and x^a . Similarly, for the background metric we can choose coordinates in a neighborhood of S so that the metric 3g_0 is

$$ds^2 = d\rho^2 + q_{0ab}dy^a dy^b \tag{3.4}$$

We now choose a diffeomorphism from the original spacetime to the background so that $r = \rho, x^a = y^a$. This identification insures that the unit normal to S in the two metrics agree. Since we are assuming the intrinsic metric also agrees, $h_{ab} = q_{ab} - q_{0ab} = 0$ on S.

In these coordinates, we have

$$\oint_{S} {}^{2}K = \frac{1}{2} \oint_{S} q^{ab}(q_{ab,r}) \tag{3.5}$$

So

$$E = -\frac{1}{8\pi} \oint_{S} (^{2}K - {}^{2}K_{0}) = -\frac{1}{16\pi} \oint_{S} q^{ab}(h_{ab,r})$$
(3.6)

In the ADM expression (3.1), the first term can be written $r^j D^i h_{ij} = D^i (r^j h_{ij}) - h_{ij} D^i r^j$. The first term on the right is zero since h_{ij} is always orthogonal to r^j , and the second term is zero since h_{ij} vanishes on S. So

$$E_{ADM} = -\frac{1}{16\pi} \oint_{S} h_{,r} = -\frac{1}{16\pi} \oint_{S} q^{ab}(h_{ab,r})$$
 (3.7)

where we have again used the fact that h_{ij} vanishes on S. Comparing (3.6) and (3.7) we see that the two expressions for the total energy are equal in this case.

For asymptotically flat spacetimes, one can also define a total momentum. By taking constant lapse and shift in (2.14) and considering how the energy changes under boosts of t^{μ} , one can read off the momentum

$$P_i N^i = \frac{1}{8\pi} \oint_S p_{ij} N^i r^j \tag{3.8}$$

which again agrees with the standard ADM result.

3.2. Asymptotically anti-de Sitter spacetimes

Abbott and Deser [7] have given a definition of the total energy for spacetimes which asymptotically approach a static solution to Einstein's equation with negative cosmological constant (see also [16,17]). If g_0 is the static background with timelike Killing vector ξ^{μ} , and $h = g - g_0$, then their definition of the energy is

$$E_{AD} = \frac{1}{8\pi} \oint_{S} dS_{\alpha} n_{\mu} [\xi_{\nu} D_{\beta} K^{\mu\alpha\nu\beta} - K^{\mu\beta\nu\alpha} D_{\beta} \xi_{\nu}]$$
 (3.9)

where

$$K^{\mu\alpha\nu\beta} \equiv g_0^{\mu[\beta} H^{\nu]\alpha} - g_0^{\alpha[\beta} H^{\nu]\mu} \tag{3.10}$$

and

$$H^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2} g_0^{\mu\nu} h_\alpha^\alpha \tag{3.11}$$

We will again show that E_{AD} agrees with the energy derived from the action (2.14) when the induced metrics on the surface S agree. Choosing synchronous gauge for both the physical metric and the background insures that $h_{0\mu} = 0$. In the spatial gauge described above, $h_{ij} = 0$ on S which implies $K^{\mu\alpha\nu\beta} = 0$ on S, so the second term in (3.9) vanishes. If we choose the surface near infinity so that $\xi^{\mu} = Nn^{\mu}$, then the first term reduces to

$$E_{AD} = \frac{1}{16\pi} \oint_{S} N(D^{i}h_{ij} - D_{j}h)r^{j}$$
 (3.12)

In other words, it is identical to the usual ADM expression except that the background metric is not flat and the lapse is not one. Since the above comparison between the ADM expression and (2.14) did not use any special properties of the flat background and did not involve integration by parts on the two sphere, it can be repeated in the present context to show that (3.12) agrees with (2.14) for general lapse N (and $N^{\mu} = 0$). It also agrees with the limit of the quasilocal mass considered in [18].

3.3. Asymptotically conical spacetimes

As a final comparison of our formula for the energy we consider the energy per unit length of a cosmic string.⁵ Outside the string, the spacetime takes the form of Minkowski space minus a wedge

$$ds^{2} = -dt^{2} + dz^{2} + dr^{2} + a^{2}r^{2}d\varphi^{2}$$
(3.13)

where φ has period 2π and the deficit angle is $2\pi(1-a)$. The reference background is flat spacetime without a wedge removed

$$ds^{2} = -dt^{2} + dz^{2} + d\rho^{2} + \rho^{2}d\varphi^{2}$$
(3.14)

Since we are interested in the energy per unit length, we consider a large cylinder at $r=r_o$ in the cosmic string spacetime. To match the intrinsic geometry, the corresponding cylinder in the background has $\rho=\rho_o$ where $\rho_o=ar_o$. The extrinsic curvatures are $^2K=1/r_o$ and $^2K_0=1/\rho_o$. Taking N=1 and $N^{\mu}=0$ in (2.14) yields

$$E = -\frac{1}{8\pi} \int (^2K - ^2K_0) = -\frac{1}{8\pi} L \int \left[\frac{1}{r_o} - \frac{1}{\rho_o} \right] \rho_o d\varphi = \frac{L}{4} (1 - a)$$
 (3.15)

where L is the length of the cylinder. So E/L = (1-a)/4, which agrees with the standard result that the energy per unit length is equal to the deficit angle divided by 8π [19].

 $^{^{5}\,}$ We thank J. Traschen and D. Kastor for suggesting this example.

4. Horizons and the Euclidean Action

In section 2 we considered the case where the only boundary of the surfaces Σ_t was at infinity. However one often has to deal with cases where the surfaces have an inner boundary as well. We shall consider two situations:

- 1. The surfaces Σ_t all intersect on a two surface S_h .
- 2. The surfaces Σ_t have an internal infinity. In this case one has to introduce another asymptotic boundary surface $\Sigma^{-\infty}$.

The first case arises in spacetimes containing a bifurcate Killing horizon, when the surfaces Σ_t are adapted to the time translation symmetry. The second case arises both for an extreme horizon, where the intersection between the past and future horizons has receded to an internal infinity, or for spacetimes having more than one asymptotic region (such as the maximally extended Schwarzschild solution). Since we are using a form of the action that requires the metric and matter fields to be fixed on the boundary, we shall take them to be fixed on S_h and $\Sigma^{-\infty}$.

We shall consider first case (1) where the surfaces of constant time all intersect on a two surface S_h . The lapse will be zero on S_h which will be an inner boundary to the surfaces Σ_t . We can also choose the shift vector to vanish on this boundary. One can now repeat the derivation of the hamiltonian given in section 2. The only difference is that the surface term $\frac{1}{8\pi} \oint N^{-2}K$ will now appear on the inner boundary as well as at infinity. However, this term vanishes since the lapse N goes to zero at S_h . If the reference background also has a horizon, there will be an extra surface term $\frac{1}{8\pi} \oint N_0^{-2} K_0$ coming from the inner boundary there. But this will also vanish since N_0 vanishes at the horizon. Thus the hamiltonian generating evolution outside a horizon S_h is again given by (2.11) with only a surface term at infinity.

If the surfaces Σ_t do not intersect but have an internal infinity, there will be a surface term $\frac{1}{8\pi} \oint N^{-2}K$ on $\Sigma^{-\infty}$. For spacetimes like extreme Reissner-Nordström this will be zero because 2K will go to zero as one goes down the throat, as will the lapse N corresponding to the time translation Killing vector. However in the case of the maximally extended Schwarzschild solution, the surface term (including the background contribution) is $\frac{1}{8\pi} \oint N({}^2K - {}^2K_0)$ which can contribute to the value of the hamiltonian.

We now consider the euclidean action

$$\tilde{I} = -\frac{1}{16\pi} \int_{M} (R + 16\pi L_m) - \frac{1}{8\pi} \oint_{\partial M} K \tag{4.1}$$

⁶ We do not require that the fields on S_h agree with those in the background solution. Indeed in many cases the background solution will not possess a two surface of intersection S_h . Similarly, for an internal infinity with finite total action, e.g. resulting from the fact that the time difference between the initial and final surface decreases to zero as one moves along an infinite throat (as in extreme Reissner-Nordström), the background need not contain an analogous surface $\Sigma^{-\infty}$. However, in cases where the internal infinity has infinite action, the background solution must also contain a surface $\Sigma^{-\infty}$ on which the fields agree.

⁷ If one does not keep the metric on the boundary fixed, the hamiltonian picks up a surface term proportional to the derivative of the lapse [2].

In a static or stationary solution the time derivatives $({}^3\dot{g}_{\mu\nu},\dot{\phi})$ are zero. Thus the action for a region between surfaces Σ_t an imaginary time distance β apart is

$$\tilde{I} = \beta H \tag{4.2}$$

If the stationary time surfaces Σ_t do not intersect, then the imaginary time coordinate can be periodically identified with any period β . This is the case for the extreme Reissner-Nordström black hole since the horizon is infinitely far away. For such periodically identified solutions, the total action will be given by (4.2). However when the stationary time surfaces intersect at a horizon S_h , the periodicity β is fixed by regularity of the euclidean solution at S_h . The action of the region swept out by the surfaces Σ_t between their inner and outer boundaries is again $\tilde{I} = \beta H$. However this is not the action of the full four dimensional solution [20], but only of the solution with the two surface S_h removed. The contribution to the action from a little tubular neighborhood surrounding the two surface S_h is just -A/4 (see also [21]) where A is the area of S_h . We thus obtain

$$\tilde{I} = \beta H - \frac{1}{4}A\tag{4.3}$$

As they stand, (4.2) and (4.3) are meaningless since we have not yet taken into account the reference background. Consider first the case where the background does not contain a two surface S_h on which the stationary time surfaces intersect. The background must be identified with the same period in imaginary time at infinity as the solution under consideration in order for the induced metrics on Σ^{∞} to agree. One thus obtains $\tilde{I}_0 = \beta H_0$ for the background which leads to the familiar result

$$\tilde{I}_P = \beta H_P - \frac{1}{4} A_{bh} \tag{4.4}$$

for the case of nonextreme black holes but

$$\tilde{I}_P = \beta H_P \tag{4.5}$$

in the extreme case.

As is now well known [8], the path integral over all euclidean metrics and matter fields that are periodic with period β at infinity gives the partition function at temperature $T = \beta^{-1}$

$$Z = \sum_{\text{states}} e^{-\beta E_n} = \int D[g]D[\phi]e^{-\tilde{I}_P}$$
(4.6)

In the semiclassical approximation, the dominant contribution to the path integral will come from the neighborhood of saddle points of the action, that is, of classical solutions. The zeroth order contribution to $\log Z$ will be $-\tilde{I}_P$. All thermodynamic properties can be deduced from the partition function. For instance, the expectation value of the energy is

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z \tag{4.7}$$

By (4.4) or (4.5) the zeroth order contribution to $\langle E \rangle$ will be H_P , as one might expect. The entropy can be defined by

$$S = -\sum p_n \log p_n = -\left(\beta \frac{\partial}{\partial \beta} - 1\right) \log Z \tag{4.8}$$

where $p_n = Z^{-1}e^{-\beta E_n}$ is the probability of being in the nth state. If one applies this to the expressions for the action (4.4) and (4.5), one sees that the zeroth order contribution to the entropy of an extreme black hole is zero [9]. On the other hand, the entropy of a nonextreme black hole is $A_{bh}/4$.

So far we have assumed implicitly that the horizon two surface S_h is compact so that its area is finite. We now consider the case when the area of S_h is infinite, such as for acceleration horizons. The main difference between this case and the previous one comes from the fact that the horizon now extends out to infinity. One could try to keep the surface Σ^{∞} away from the horizon, but then the space between Σ^{∞} and the horizon would still be noncompact, so the action would be ill-defined. If the spacetime has continuous spacelike symmetries, one could compute all quantities per unit area. Alternatively, if the spacetime has appropriate discrete symmetries, one could periodically identify to make the action (and horizon area) finite. If either of these two options is adopted, then the previous discussion applies essentially unchanged. However, in general, neither option is available. One must then choose Σ^{∞} to intersect the horizon "at infinity". Thus, instead of the intersections of Σ^{∞} and the surfaces Σ_t having topology S^2 , they will now have topology D^2 . Since the metric induced on Σ^{∞} from the background spacetime must agree with that from the original spacetime, it follows that the background metric must also have a horizon that intersects Σ^{∞} .

As a simple example, consider Rindler space

$$ds^{2} = -\xi^{2}d\eta^{2} + d\xi^{2} + dy^{2} + dz^{2}$$
(4.9)

If one does not periodically identify y and z (or compute quantities per unit area), one must take Σ^{∞} to be given by fixing a large value of $R^2 = \xi^2 + y^2 + z^2$, which intersects the horizon $\xi = 0$. The surfaces of constant η intersect Σ^{∞} in a disk D^2 since $\xi \geq 0$.

We now consider the euclidean version of solutions with acceleration horizons. The argument above (4.3) can be applied to show that (4.3) holds in this case also. Since the periodicity in imaginary time is determined by regularity of the euclidean spacetime on the axis (which now extends out to infinity) the periodicity in the background β_0 must again agree with that in the original spacetime β . Repeating the argument above (4.3) one finds that the background satisfies a similar relation

$$\tilde{I}_0 = \beta H_0 - \frac{1}{4} A_0 \tag{4.10}$$

Thus, the physical euclidean action is related to the physical hamiltonian by

$$\tilde{I}_P = \beta H_P - \frac{1}{4} \Delta A \tag{4.11}$$

where ΔA is the difference between the area of S_h in the original spacetime and its area in the reference background. This general formula includes the familiar result (4.4) as a special case, since for black hole horizons, one can choose a background which does not have a horizon. If several horizons S_h are present, ΔA is the increase in area of the acceleration horizon plus the area of any nonextreme black hole horizons. It does not however include the area of extreme horizons because they do not meet at a two surface in the spacetime.

Since the area of an acceleration horizon is infinite, one might think that the difference ΔA is ill-defined. However, it can be given a precise meaning by examining how it enters into the above argument. The main point is that the surface near infinity Σ^{∞} intersects the acceleration horizon at a large but finite circle C. ΔA is defined to be the difference between the (finite) area of the acceleration horizon inside C in the original spacetime and the area inside the analogous circle C_0 in the reference background. Since the fields induced on Σ^{∞} from the original spacetime agree with those induced from the reference background, one can rephrase this prescription as follows: One fixes a large circle C in the acceleration horizon in the original spacetime and then chooses a circle C_0 in the reference background which has the same proper length and the same value of the matter fields. ΔA is then the difference in area inside these two circles. This procedure was used in [9] to analyze the Ernst instanton.

If one naively substitutes the euclidean action (4.11) into the expression for the entropy (4.8) using the zeroth order contribution $\log Z \approx -I_P$, one might conclude that an acceleration horizon should have an entropy $\Delta A/4$. However, the periodicity of the imaginary time coordinate on the boundary is fixed by the requirement of regularity where the acceleration horizon meets Σ^{∞} . Thus one cannot take the derivative of the partition function with respect to β and so cannot use (4.8) to calculate the entropy. This differs from the black hole case where β is not fixed by regularity at infinity. Instead, we shall use a different argument. Physically, a key difference between acceleration and black hole horizons is that the former are observer dependent. The information behind an acceleration horizon can be recovered by observers who simply stop accelerating. Another way to say this is that acceleration horizons are not associated with a change in the topology of spacetime. For example, consider a spacetime like the Ernst solution where there are both acceleration and black hole horizons. One could imagine replacing the black holes by something like magnetic monopoles that have no horizons. One could make the monopole solution away from the black hole horizons arbitrarily close to the solution with black holes. The monopole solution would have the same R^4 topology as the Melvin reference background. Thus one could choose a different family Σ'_t of time surfaces that cover the region within a large three sphere without intersections or inner boundaries. One would therefore expect the monopole solution to have a unitary hamiltonian evolution and zero entropy.

However, the area of the acceleration horizon in the monopole solution will still be different from that of the background. Since $H_P = 0$ [9], this difference ΔA_{acc} is directly related to the euclidean action (4.11) and thus will correspond to the tunneling probability to create a monopole-antimonopole pair (assuming there is only one species of monopole). However the instanton representing the pair creation of nonextremal black holes will have a lower action because there is an extra contribution to ΔA from the black hole horizon

area A_{bh} . One can interpret the increased pair creation probability as corresponding to the possibility of producing $N = \exp(A_{bh}/4)$ different species of black hole pairs. Thus pair creation arguments confirm the connection between entropy and (nonextreme) black hole horizon area, but suggest that there is no analogous connection with acceleration horizon area.

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