an instrument somewhat similar to the above, except that the wire was of iron instead of nickel.

In order to show that the current induced in the coil is at any rate chiefly due to variations in the magnetization of the nickel wire produced by the variations of stress, and not to the relative motion of the nickel wire and the coil, the following experiment was made :—A steel wire and a nickel wire of the same dimensions were attached to similar wooden diaphragms. These wires having been magnetized by stroking them with a permanent magnet were in turn inserted into the same solenoid and clamped as before at their lower ends. The same watch telephone was used as a receiver with each. The results obtained with the weakly magnetized nickel wire were enormously better than those obtained with the strongly magnetized steel wire. If the induced currents were chiefly due to the relative motion of the coil and magnetized wire the best results would have been obtained with the strongly magnetized steel wire, since it can hardly be supposed that the relative motions of the coil and magnetized wire differed so much in the two cases as to cause such an enormous difference in the results.

V. *On the Incidence of Aerial and Electric Waves upon Small Obstacles in the form of Ellipsoids or Elliptic Cylinders, and on the Passage of Electric Waves through a circular Aperture in a Conducting Screen.* *By* Lord RAYLEIGH, *F.R.S.**

THE present paper may be regarded as a development of previous researches by the author upon allied subjects. When the character of the obstacle differs only infinitesimally from that of the surrounding medium, a solution may be obtained independently of the size and the form which it presents. But when this limitation is disregarded, when, for example, in the case of aerial vibrations the obstacle is of arbitrary compressibility and density, or in the case of electric vibrations when the dielectric constant and the permeability are arbitrary, the solutions hitherto given are confined to the case of small spheres, or circular cylinders. In the present investigation extension is made to ellipsoids, including flat circular disks and thin blades.

The results arrived at are limiting values, strictly applicable only when the dimensions of the obstacles are infinitesimal,

* Communicated by the Author.

and at distances outwards which are infinitely great in comparison with the wave-length (λ). The method proceeds by considering in the first instance what occurs in an intermediate region, where the distance (r) is at once great in comparison with the dimensions of the obstacle and small in comparison with λ. Throughout this region and within it the calculation proceeds as if λ were infinite, and depends only upon the properties of the common potential. When this problem is solved, extension is made without much difficulty to the exterior region where r is great in comparison with λ, and where the common potential no longer avails.

At the close of the paper a problem of some importance is considered relative to the escape of electric waves through small circular apertures in metallic screens. The case of narrow elongated slits has already been treated *.

Obstacle in a Uniform Field.

The analytical problem with which we commence is the same whether the flow be thermal, electric, or magnetic, the obstacle differing from the surrounding medium in conductivity, specific inductive capacity, or permeability respectively. If ϕ denote its potential, the uniform field is defined by

$$\phi = ux + vy + wz; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

u, v, w being the fluxes in the direction of fixed arbitrarily chosen rectangular axes. If ψ be the potential in the uniform medium due to the obstacle, so that the complete potential is $\phi+\psi$, ψ may be expanded in the series of spherical harmonics

$$\psi = \frac{S_0}{r} + \frac{S_1}{r^2} + \frac{S_2}{r^3} + \ldots, \quad . \quad . \quad . \quad . \quad (2)$$

the origin of r being within the obstacle. Since there is no source, S_0 vanishes. Further, at a great distance S_2, S_3, . . . may be neglected, so that ψ there reduces to

$$\psi = \frac{S_1}{r^2} = \frac{A'x + B'y + C'z}{r^3}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The disturbance (3) corresponds to (1). If we express separately the parts corresponding to u, v, w, writing $A' = A_1u + A_2v + A_3w$, &c., we have

$$\begin{aligned} r^3\psi = {} & u(A_1x + B_1y + C_1z) \\ & + v(A_2x + B_2y + C_2z) \\ & + w(A_3x + B_3y + C_3z); \quad . \quad . \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

* Phil. Mag. vol. xliii. p. 272.

but the nine coefficients are not independent. By the law of reciprocity the coefficient of the x-part due to v must be the same as that of the y-part due to u, and so on*. Thus $B_1 = A_2$, &c., and we may write (4) in the form

$$r^3\psi = u\frac{dF}{dx} + v\frac{dF}{dy} + w\frac{dF}{dz}, \quad . \quad . \quad . \quad . \quad (5)$$

where

$$F = \tfrac{1}{2}A_1x^2 + \tfrac{1}{2}B_2y^2 + \tfrac{1}{2}C_3z^2 + B_1xy + C_2yz + C_1zx. \quad . \quad (6)$$

In the case of a body, like an ellipsoid, symmetrical with respect to three planes chosen as coordinate planes,

$$B_1 = C_2 = C_1 = 0,$$

and (4) reduces to

$$r^3\psi = A_1ux + B_2vy + C_3wz. \quad . \quad . \quad . \quad . \quad (7)$$

It will now be shown that by a suitable choice of coordinates this reduction may be effected in any case. Let u, v, w originate in a source at distance R, whose coordinates are x', y', z', so that $u = x'/R^3$, &c. Then (5) becomes

$$r^3R^3\psi = x'\frac{dF}{dx} + y'\frac{dF}{dy} + z'\frac{dF}{dz} = A_1xx' + B_2yy' + C_3zz'$$

$$+ B_1(x'y + y'x) + C_2(y'z + z'y) + C_1(z'x + x'z)$$

$$= F(x + x', y + y', z + z') - F(x, y, z) - F(x', y', z').$$

Now by a suitable transformation of coordinates $F(x, y, z)$, and therefore $F(x', y', z')$ and $F(x + x', y + y', z + z')$, may be reduced to the form $A_1x^2 + B_2y^2 + C_3z^2$, &c. If this be done,

$$r^3R^3\psi = A_1xx' + B_2yy' + C_3zz',$$

or reverting to u, v, w, reckoned parallel to the new axes,

$$r^3\psi = A_1ux + B_2vy + C_3wz, \quad . \quad . \quad . \quad . \quad (8)$$

as in (7) for the ellipsoid. It should be observed that this reduction of the potential at a distance from the obstacle to the form (8) is independent of the question whether the material composing the obstacle is uniform.

For the case of the ellipsoid (a, b, c) of uniform quality the solution may be completely carried out. Thus †, if T be

* 'Theory of Sound,' § 109. u and v may be supposed to be due to point-sources situated at a great distance R along the axes of x and y respectively.

† The magnetic problem is considered in Maxwell's 'Electricity and Magnetism,' 1873, § 437, and in Mascart's *Leçons*, 1896, §§ 52, 53, 276.

the volume, so that

$$T=\tfrac{4}{3}\pi abc, \quad \ldots \ldots \quad (9)$$

we have

$$A_1u=-AT, \quad B_2v=-BT, \quad C_3w=-CT, \quad . \quad (10)$$

$$A=\frac{\kappa u}{1+\kappa L}, \quad B=\frac{\kappa v}{1+\kappa M}, \quad C=\frac{\kappa w}{1+\kappa N}, \quad . \quad (11)$$

where

$$L=2\pi abc\int_0^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{3}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}}, \quad . \quad (12)$$

with similar expressions for M and N.

In (11) κ denotes the susceptibility to magnetization. In terms of the permeability μ, analogous to conductivity in the allied problems, we have, if μ' relate to the ellipsoid and μ to the surrounding medium,

$$1+4\pi\kappa=\mu'/\mu, \quad \ldots \ldots \quad (13)$$

so that

$$A=\frac{(\mu'-\mu)u}{4\pi\mu+(\mu'-\mu)L}, \quad \ldots \ldots \quad (14)$$

with similar equations for B and C.

Two extreme cases are worthy of especial notice. If $\mu'/\mu=\infty$, the general equation for ψ becomes

$$-\frac{r^3\psi}{T}=\frac{ux}{L}+\frac{vy}{M}+\frac{wz}{N}. \quad \ldots \quad (15)$$

On the other hand, if $\mu'/\mu=0$,

$$-\frac{r^3\psi}{T}=\frac{ux}{L-4\pi}+\frac{vy}{M-4\pi}+\frac{wz}{N-4\pi}. \quad . \quad . \quad (16)$$

In the case of the sphere (a)

$$L=M=N=\tfrac{4}{3}\pi; \quad \ldots \ldots \quad (17)$$

so that (15) becomes

$$\psi=-\frac{a^3}{r^3}(ux+vy+wz), \quad \ldots \ldots \quad (18)$$

giving, when $r=a$, $\phi+\psi=0$. This is the case of the perfect conductor.

In like manner for the non-conducting sphere (16) gives

$$\psi=\frac{a^3}{2r^3}(ux+vy+wz). \quad \ldots \ldots \quad (19)$$

If the conductivity of the sphere be finite (μ'),

$$\psi = -\frac{a^3}{r^3}\frac{\mu'-\mu}{\mu'+2\mu}(ux+vy+wz), \quad . \quad . \quad . \quad . \quad (20)$$

which includes (18) and (19) as particular cases.

If the ellipsoid has two axes equal, and is of the planetary or flattened form,

$$b=c=\frac{a}{\sqrt{(1-e^2)}}, \qquad T=\tfrac{4}{3}\pi c^3\sqrt{(1-e^2)}\,; \quad . \quad (21)$$

$$L=4\pi\left\{\frac{1}{e^2}-\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e\right\}, \quad . \quad . \quad . \quad (22)$$

$$M=N=2\pi\left\{\frac{\sqrt{(1-e^2)}}{e^3}\sin^{-1}e-\frac{1-e^2}{e^2}\right\}. \quad . \quad (23)$$

In the extreme case of a disk, when $e=1$ nearly,

$$L=4\pi-2\pi^2\sqrt{(1-e^2)}, \quad . \quad . \quad . \quad . \quad (24)$$

$$M=N=\pi^2\sqrt{(1-e^2)}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Thus in the limit from (14), (21) $TA=0$, unless $\mu'=0$; and when $\mu'=0$,

$$TA=-\frac{2c^3u}{3\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In like manner the limiting values of TB, TC are zero, unless $\mu'=\infty$, and then

$$TB=\frac{4c^3v}{3\pi}, \qquad TC=\frac{4c^3w}{3\pi}. \quad . \quad . \quad . \quad . \quad (27)$$

In all cases

$$\psi=-\frac{T(Ax+By+Cz)}{r^3} \quad . \quad . \quad . \quad . \quad (28)$$

gives the disturbance due to the ellipsoid.

If the ellipsoid of revolution be of the ovary or elongated form,

$$a=b=c\sqrt{(1-e^2)}\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

$$L=M=2\pi\left\{\frac{1}{e^2}-\frac{1-e^2}{2e^3}\log\frac{1+e}{1-e}\right\}, \quad . \quad . \quad (30)$$

$$N=4\pi\left\{\frac{1}{e^2}-1\right\}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\}. \quad (31)^*$$

* There are slight errors in the values of L, M, N recorded for this case in both the works cited.

In the case of a very elongated ovoid L and M approximate to the value 2π, while N approximates to the form

$$\mathrm{N}=4\pi\,\frac{a^2}{c^2}\left(\log\frac{2c}{a}-1\right), \quad . \quad . \quad . \quad . \quad (32)$$

vanishing when $e=1$.

In Two Dimensions.

The case of an elliptical cylinder in two dimensions may be deduced from (12) by making c infinite, when the integration is readily effected. We find

$$\mathrm{L}=\frac{4\pi b}{a+b}, \qquad \mathrm{M}=\frac{4\pi a}{a+b}. \quad . \quad . \quad . \quad (33)$$

A and B are then given by (14) as before, and finally

$$\psi=-\frac{ab(a+b)}{2r^2}\left\{\frac{(\mu'-\mu)ux}{\mu a+\mu' b}+\frac{(\mu'-\mu)vy}{\mu b+\mu' a}\right\}, \quad . \quad (34)$$

corresponding to

$$\phi=ux+vy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

In the case of circular section $\mathrm{L}=\mathrm{M}=2\pi$, so that

$$\psi=-\frac{a^2}{r^2}\,\frac{\mu'-\mu}{\mu'+\mu}\,(ux+vy). \quad . \quad . \quad . \quad . \quad (36)$$

When $b=0$, that is when the obstacle reduces itself to an infinitely thin blade, ψ vanishes unless $\mu'=0$ or $\mu'=\infty$. In the first case

$$(\mu'=0) \qquad \psi=\frac{a^2 vy}{2r^2}; \quad . \quad . \quad . \quad . \quad . \quad (37)$$

in the second

$$(\mu'=\infty) \qquad \psi=-\frac{a^2 ux}{2r^2}. \quad . \quad . \quad . \quad . \quad (38)$$

Aerial Waves.

We may now proceed to investigate the disturbance of plane aerial waves by obstacles whose largest diameter is small in comparison with the wave-length (λ). The volume occupied by the obstacle will be denoted by T; as to its shape we shall at first impose no restriction beyond the exclusion of very special cases, such as would involve resonance in spite of the small dimensions. The compressibilities and densities of the medium and of the obstacle are denoted by m, m'; σ, σ'; so that if V, V′ be the velocities of propagation

$$V^2=m/\sigma, \quad V'^2=m'/\sigma'. \quad . \quad . \quad . \quad . \quad (39)$$

The velocity-potential of the undisturbed plane waves is represented by

$$\phi=e^{ikVt}.e^{ikx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in which $k=2\pi/\lambda$. The time factor e^{ikVt}, which operates throughout, may be omitted for the sake of brevity.

The velocity-potential (ψ) of the disturbance propagated outwards from T may be expanded in spherical harmonic terms*

$$r\psi=e^{-ikr}\{S_0+S_1f_1(ikr)+S_2f_2(ikr)+..\}, \quad . \quad (41)$$

where

$$f_n(ikr)=1+\frac{n(n+1)}{2\,.\,ikr}+\frac{(n-1)\ldots(n+2)}{2\,.\,4\,.\,(ikr)^2}$$

$$+\ldots\ldots+\frac{1\,.\,2\,.\,3\ldots 2n}{2\,.\,4\,.\,6\ldots 2n\,(ikr)^n}. \quad . \quad . \quad . \quad (42)$$

At a great distance from the obstacle $f_n(ikr)=1$; and the relative importance of the various harmonic terms decreases in going outwards with the order of the harmonic. For the present purpose we shall need to regard only the terms of order 0 and 1. Of these the term of order 0 depends upon the variation of compressibility, and that of order 1 upon the variation of density.

The relation between the variable part of the pressure δp, the condensation s, and ϕ is

$$V^2 s=-\frac{d\phi}{dt}=\frac{\delta p}{\sigma};$$

so that during the passage of the undisturbed primary waves the rate at which fluid enters the volume T (supposed for the moment to be of the same quality as the surrounding medium) is

$$T\frac{ds}{dt}=-\frac{T}{V^2}\frac{d^2\phi}{dt^2}=k^2T. \quad . \quad . \quad . \quad . \quad . \quad (43)$$

If the obstacle present an unyielding surface, its effect is to prevent the entrance of the fluid (43); that is, to superpose upon the plane waves such a disturbance as is caused by the *introduction* of (43) into the medium. Thus, if the potential of this disturbance be

$$\psi=S_0\frac{e^{-ikr}}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

* 'Theory of Sound,' §§ 323, 324.

S_0 is to be determined by the condition that when $r=0$

$$4\pi r^2 d\psi/dr = k^2 T,$$

so that $S_0 = -k^2 T/4\pi$, and

$$\psi = -\frac{k^2 T}{4\pi}\frac{e^{-ikr}}{r} = -\frac{\pi T}{\lambda^2}\frac{e^{-ikr}}{r}. \quad . \quad . \quad . \quad (45)$$

This result corresponds with $m' = \infty$ representing absolute incompressibility. The effect of finite compressibility, differing from that of the surrounding medium, is readily inferred by means of the pressure relation ($\delta p = ms$). The effect of the variation of compressibility at the obstacle is to increase the rate of introduction of fluid into T from what it would otherwise be in the ratio $m : m'$; and thus (45) now becomes

$$\psi = -\frac{\pi T}{\lambda^2}\frac{m'-m}{m'}\frac{e^{-ikr}}{r}; \quad . \quad . \quad . \quad . \quad (46)$$

or if we restore the factor e^{ikVt} and throw away the imaginary part of the solution,

$$\psi = -\frac{\pi T}{\lambda^2 r}\frac{m'-m}{m'}\cos k(Vt-r). \quad . \quad . \quad (47)$$

This is superposed upon the primary waves

$$\phi = \cos k(Vt+x). \quad . \quad . \quad . \quad . \quad . \quad (48)$$

When $m'=0$, *i. e.*, when the material composing the obstacle offers no resistance to compression, (47) fails. In this case the condition to be satisfied at the surface of T is the evanescence of δp, or of the total potential $(\phi+\psi)$. In the neighbourhood of the obstacle $\phi=1$; and thus if M′ denote the electrical "capacity" of a conducting body of form T situated in the open, $\psi = -M'/r$, r being supposed to be large in comparison with the linear dimension of T but small in comparison with λ. The latter restriction is removed by the insertion of the factor e^{-ikr}, and thus, in place of (46), we now have

$$\psi = -\frac{M' e^{-ikr}}{r}. \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The value of M′ may be expressed when T is in the form of an ellipsoid. For a sphere of radius R,

$$M' = R; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

for a circular plate of radius R,

$$M' = 2R/\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

When the density of the obstacle (σ') is the same as that of the surrounding medium, (47) constitutes the complete solution. Otherwise the difference of densities causes an interference with the flow of fluid, giving rise to a disturbance of order 1 in spherical harmonics. This disturbance is independent of that already considered, and the flow in the neighbourhood of the obstacle may be calculated as if the fluid were incompressible. We thus fall back upon the problem considered in the earlier part of this paper, and the results will be applicable as soon as we have established the correspondence between density and conductivity.

In the present problem, if χ denote the whole velocity-potential, the conditions to be satisfied at any part of the surface of the obstacle are the continuity of $d\chi/dn$ and of $\sigma\chi$, the latter of which represents the pressure. Thus, if we regard $\sigma\chi$ as the variable, the conditions are the continuity of $(\sigma\chi)$ and of $\sigma^{-1}d(\sigma\chi)/dn$. In the conductivity problem the conditions to be satisfied by the potential (χ') are the continuity of χ' and of $\mu\, d\chi'/dn$.

In an expression relating only to the external region where σ is constant, it makes no difference whether we are dealing with $\sigma\chi$ or with χ; and accordingly there is correspondence between the two problems provided that we suppose the ratio of μ's in the one problem to be the reciprocal of the ratio of the σ's in the other.

We may now proceed to the calculation of the disturbance due to an obstacle, based upon the assumption that there is a region over which r is large compared with the linear dimension of T, but small in comparison with λ. Within this region ψ is given by (8) if the motion be referred to certain principal axes determined by the nature and form of the obstacle, the quantities u, v, w being the components of flow in the primary waves. By (41), (42), this is to be identified with

$$\psi = \mathrm{S}_1 \frac{e^{-ikr}}{r}\left(1+\frac{1}{ikr}\right), \quad . \quad . \quad . \quad . \quad (52)$$

when r is small in comparison with λ; so that

$$\mathrm{S}_1 = \frac{ik(\mathrm{A}_1\, ux + \mathrm{B}_2\, vy + \mathrm{C}_3\, wz)}{r}. \quad . \quad . \quad . \quad . \quad (53)$$

At a great distance from T, (52) reduces to

$$\psi = \frac{ik(\mathrm{A}_1\, ux + \mathrm{B}_2\, vy + \mathrm{C}_3\, wz)e^{-ikr}}{r^2}, \quad . \quad . \quad . \quad (54)$$

—a term of order 1, to be added to that of zero order given in (46).

In general, the axis of the harmonic in (54) is inclined to the direction of propagation of the primary waves; but there are certain cases of exception. For example, v and w vanish if the primary propagation be parallel to x (one of the principal axes). Again, as for a sphere or a cube, A_1, B_2, C_3 may be equal.

We will now limit ourselves to the case of the ellipsoid, and for brevity will further suppose that the primary waves move parallel to x, so that $v=w=0$. The terms corresponding to u and v, if existent, are simply superposed. If, as hitherto, $\phi=e^{ikx}$, $u=ik$; so that by (14), σ being substituted for μ' and σ' for μ,

$$A=\frac{ik(\sigma-\sigma')}{4\pi\sigma'+(\sigma-\sigma')L}. \quad . \quad . \quad . \quad . \quad (55)$$

In the intermediate region by (28) $\psi=-TAx/r^3$, and thus at a great distance

$$\psi=-\frac{ikx\,TAe^{-ikr}}{r^2}; \quad . \quad . \quad . \quad . \quad . \quad (56)$$

or on substitution of the values of A and k,

$$\psi=-\frac{\pi Tx\,e^{-ikr}}{\lambda^2 r^2}\,\frac{4\pi(\sigma'-\sigma)}{4\pi\sigma'+(\sigma-\sigma')L}. \quad . \quad . \quad (57)$$

Equations (46), (57) express the complete solution in the case supposed.

For an obstacle which is rigid and fixed, we may deduce the result by supposing in our equations $m'=\infty$, $\sigma'=\infty$. Thus

$$\psi=-\frac{\pi Te^{-ikr}}{\lambda^2 r}\left\{1+\frac{x}{r}\,\frac{4\pi}{4\pi-L}\right\}. \quad . \quad . \quad (58)$$

Certain particular cases are worthy of notice. For the sphere $L=\frac{4}{3}\pi$, and

$$\psi=-\frac{\pi Te^{-ikr}}{\lambda^2 r}\left\{1+\frac{3x}{2r}\right\}. \quad . \quad . \quad . \quad (59)\,*$$

If the ellipsoid reduce to an infinitely thin circular disk of radius c, $T=0$ and the term of zero order vanishes. The term of the first order also vanishes if the plane of the disk be parallel to x. If the plane of the disk be perpendicular to

* 'Theory of Sound,' § 334.

x, $4\pi - \mathrm{L}$ is infinitesimal. By (21), (24) we get in this case

$$\frac{4\pi \mathrm{T}}{4\pi - \mathrm{L}} = \frac{8c^3}{3};$$

so that

$$\psi = -\frac{8\pi c^3}{3\lambda^2} \frac{x}{r} \frac{e^{-ikr}}{r}. \quad . \quad . \quad . \quad . \quad (60)$$

If the axis of the disk be inclined to that of x, ψ retains its symmetry with respect to the former axis, and is reduced in magnitude in the ratio of the cosine of the angle of inclination to unity.

In the case of the sphere the general solution is

$$\psi = -\frac{\pi \mathrm{T} e^{-ikr}}{\lambda^2 r} \left\{ \frac{m' - m}{m'} + \frac{3x}{r} \frac{\sigma' - \sigma}{2\sigma' + \sigma} \right\}. \quad . \quad (61)^*$$

Waves in Two Dimensions.

In the case of two dimensions (x, y) the waves diverging from a cylindrical obstacle have the expression, analogous to (41),

$$\psi = \mathrm{S}_0 \, \mathrm{D}_0(kr) + \mathrm{S}_1 \, \mathrm{D}_1(kr) + \ldots, \quad . \quad . \quad (62)\dagger$$

where S_0, $\mathrm{S}_1 \ldots$ are the plane circular functions of the various orders, and

$$\mathrm{D}_0(kr) = -\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr} \left\{ 1 - \frac{1^2}{1 \,.\, 8ikr} + \ldots \right\}$$

$$= \left(\gamma + \log \frac{ikr}{2}\right) \left\{ 1 - \frac{k^2 r^2}{2^2} + \ldots \right\} + \frac{k^2 r^2}{2^2} - \frac{3}{2} \frac{k^4 r^4}{2^2 \,.\, 4^2} + \ldots,$$

$$. \quad . \quad . \quad (63)$$

$$\mathrm{D}_1(kr) = \frac{d\mathrm{D}_0(kr)}{d(kr)} = \left(\frac{\pi i}{2kr}\right)^{\frac{1}{2}} e^{-ikr} \left\{ 1 - \frac{-1 \,.\, 3}{1 \,.\, 8ikr} + \ldots \right\}$$

$$= \frac{1}{kr} \left\{ 1 - \frac{k^2 r^2}{2^2} + \ldots \right\} + \left(\gamma + \log \frac{ikr}{2}\right) \left\{ \frac{kr}{2} - \frac{k^3 r^3}{2^2 \,.\, 4} + \ldots \right\}$$

$$+ \frac{kr}{2} - \frac{3}{2} \frac{k^3 r^3}{2^2 \,.\, 4} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

As in the case of three dimensions already considered, the term of zero order in ψ depends upon the variation of compressibility. If we again begin with the case of an unyielding

* 'Theory of Sound,' § 335.

† See 'Theory of Sound,' § 341; Phil. Mag. April 1897, p. 266.

boundary, the constant S_0 is to be found from the condition that when $r=0$

$$2\pi r\, d\psi/dr = k^2 \mathrm{T},$$

T denoting now the area of cross-section. When r is small,

$$\frac{d\mathrm{D}_0(kr)}{dr} = \frac{1}{r};$$

and thus $S_0 = k^2\mathrm{T}/2\pi$,

$$\psi = \frac{k^2\mathrm{T}}{2\pi}\mathrm{D}_0(kr) = -\frac{k^2\mathrm{T}}{2\pi}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} e^{-ikr}, \ldots \quad . \quad (65)$$

when r is very great. This corresponds to (45).

In like manner, if the compressibility of the obstacle be finite,

$$\psi = -\frac{k^2\mathrm{T}}{\pi}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} \frac{m'-m}{2m'} e^{-ikr} \ldots \quad . \quad . \quad (66)$$

The factor $i^{-\frac{1}{2}} = e^{-\frac{1}{4}i\pi}$; and thus if we restore the time-factor $e^{ik\mathrm{V}t}$, and reject the imaginary part of the solution, we have

$$\psi = -\frac{2\pi\mathrm{T}}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}} \frac{m'-m}{2m'} \cos\frac{2\pi}{\lambda}(\mathrm{V}t - r - \tfrac{1}{8}\lambda), \quad . \quad . \quad (67)$$

corresponding to the plane waves

$$\phi = \cos\frac{2\pi}{\lambda}(\mathrm{V}t + x). \quad . \quad . \quad . \quad . \quad . \quad (68)$$

In considering the term of the first order we will limit ourselves to the case of the cylinder of elliptic section, and suppose that one of the principal axes of the ellipse is parallel to the direction (x) of primary wave-propagation. Thus in (34), which gives the value of ψ at a distance from the cylinder which is great in comparison with a and b, but small in comparison with λ, we are to suppose $u=ik$, $v=0$, at the same time substituting σ, σ' for μ', μ respectively. Thus for the region in question

$$\psi = \frac{ab \,.\, ikx}{2r^2} \frac{(\sigma'-\sigma)(a+b)}{\sigma' a + \sigma b}; \quad . \quad . \quad . \quad (69)$$

and this is to be identified with $S_1 D_1(kr)$ when kr is small, *i. e.* with S_1/kr. Accordingly

$$S_1 = \frac{x}{r} \frac{ik^2 ab}{2} \frac{(\sigma'-\sigma)(a+b)}{\sigma' a + \sigma b};$$

so that, at a distance r great in comparison with λ, ψ becomes

$$\psi = -\frac{k^2 \mathrm{T}}{\pi}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} \frac{(\sigma'-\sigma)(a+b)}{2(\sigma' a+\sigma b)} \frac{x}{r} e^{-ikr}, \ldots \quad (70)$$

T being written for πab. The complete solution for a great distance is given by addition of (66) and (70), and corresponds to $\phi = e^{ikx}$.

In the case of circular section ($b=a$) we have altogether *

$$\psi = -k^2 a^2 e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}} \left\{\frac{m'-m}{2m'} + \frac{\sigma'-\sigma}{\sigma'+\sigma}\frac{x}{r}\right\}, \quad (71)$$

which may be realized as in (67). If the material be unyielding, the corresponding result is obtained by making $m'=\infty$, $\sigma'=\infty$ in (71). The realized value is then †

$$\psi = -\frac{2\pi \,.\, \pi a^2}{r^{\frac{1}{2}}\lambda^{\frac{3}{2}}}\left(\frac{1}{2}+\frac{x}{r}\right)\cos\frac{2\pi}{\lambda}(\mathrm{V}t - r - \tfrac{1}{8}\lambda). \quad . \quad (72)$$

In general, if the material be unyielding, we get from (66), (70)

$$\psi = -k^2 ab\, e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left(\frac{1}{2}+\frac{a+b}{2a}\frac{x}{r}\right). \quad . \quad . \quad (73)$$

The most interesting case of a difference between a and b is when one of them vanishes, so that the cylinder reduces to an infinitely thin blade. If $b=0$, ψ vanishes as to both its parts; but if $a=0$, although the term of zero order vanishes, that of the first order remains finite, and we have

$$\psi = -\tfrac{1}{2}k^2 b^2 e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\frac{x}{r}, \quad . \quad . \quad . \quad (74)$$

in agreement with the value formerly obtained ‡.

It remains to consider the extreme case which arises when $m'=0$. The term of zero order in circular harmonics, as given in (66), then becomes infinite, and that of the first order (70) is relatively negligible. The condition to be satisfied at the surface of the obstacle is now the evanescence of the total potential $(\phi+\psi)$, in which $\phi=1$.

It will conduce to clearness to take first the case of the circular cylinder (a). By (62), (63) the surface condition is

$$\mathrm{S}_0\{\gamma + \log(\tfrac{1}{2}ika)\} + 1 = 0. \quad . \quad . \quad . \quad . \quad (75)$$

* 'Theory of Sound,' § 343.

† *Loc. cit.* equation (17).

‡ Phil. Mag. April 1897, p. 271. The primary waves are there supposed to travel in the direction of $+x$, but here in the direction of $-x$.

Thus at a distance r great in comparison with λ we have

$$\psi=\frac{e^{-ikr}}{\gamma+\log\left(\frac{1}{2}ika\right)}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad (76)$$

When the section of the obstacle is other than circular, a less direct process must be followed. Let us consider a circle of radius ρ concentric with the obstacle, where ρ is large in comparison with the dimensions of the obstacle but small in comparison with λ. Within this circle the flow may be identified with that of an incompressible fluid. On the circle we have

$$\phi+\psi=1+S_0\{\gamma+\log\left(\tfrac{1}{2}ik\rho\right)\}, \quad . \quad . \quad (77)$$

$$2\pi\, d(\phi+\psi)/dr=2\pi S_0, \quad . \quad . \quad . \quad . \quad (78)$$

of which the latter expresses the flow of fluid across the circumference. This flow in the region between the circle and the obstacle corresponds to the potential-difference (77). Thus, if R denote the electrical resistance between the two surfaces (reckoned of course for unit length parallel to z),

$$S_0\{\gamma+\log\left(\tfrac{1}{2}ik\rho\right)-2\pi R\}=1, \quad . \quad . \quad . \quad (79)$$

and $\psi=S_0 D_0(kr)$, as usual.

The value of S_0 in (79) is of course independent of the actual value of ρ, so long as it is large. If the obstacle be circular,

$$2\pi R=\log(\rho/a).$$

The problem of determining R for an elliptic section (a, b) can, as is well known, be solved by the method of conjugate functions. If we take

$$x=c\cosh\xi\cos\eta, \qquad y=c\sinh\xi\sin\eta, \quad . \quad . \quad . \quad (80)$$

the confocal ellipses

$$\frac{x^2}{\cosh^2\xi}+\frac{y^2}{\sinh^2\xi}=c^2 \quad . \quad . \quad . \quad . \quad . \quad (81)$$

are the equipotential curves. One of these, for which ξ is large, can be identified with the circle of radius ρ, the relation between ρ and ξ being

$$\xi=\log\left(2\rho/c\right).$$

An inner one, for which $\xi=\xi_0$, is to be identified with the ellipse (a, b), so that

$$a=c\cosh\xi_0, \qquad b=c\sinh\xi_0,$$

whence

$$c^2=a^2-b^2, \qquad \tanh \xi_0=b/a.$$

Thus

$$2\pi R=\xi-\xi_0=\log\frac{2\rho}{a+b}; \quad . \quad . \quad . \quad . \quad (82)$$

and then (79) gives as applicable at a great distance

$$\psi=\frac{e^{-ikr}}{\gamma+\log\{\frac{1}{4}ik(a+b)\}}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}. \quad . \quad . \quad (83)$$

The result for an infinitely thin blade is obtained by merely putting $b=0$ in (83).

For some purposes the imaginary part of the logarithmic term may be omitted. The realized solution is then

$$\psi=\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}}\frac{\cos k(Vt-r-\frac{1}{8}\lambda)}{\gamma+\log\{\frac{1}{4}k(a+b)\}}, \quad . \quad . \quad . \quad (84)$$

corresponding, as usual, to

$$\phi=\cos k(Vt+x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Electrical Applications.

The problems in two dimensions for aerial waves incident upon an obstructing cylinder of small transverse dimensions are analytically identical with certain electric problems which will now be specified. The general equation $(\nabla^2+k^2)=0$ is satisfied in all cases. In the ordinary electrical notation $V^2=1/K\mu$, $V'^2=1/K'\mu'$; while in the acoustical problem $V^2=m/\sigma$, $V'^2=m'/\sigma'$. The boundary conditions are also of the same general form. Thus if the primary waves be denoted by $\gamma=e^{ikx}$, γ being the magnetic force parallel to z, the conditions to be satisfied at the surface of the cylinder are the continuity of γ and of $K^{-1}\,d\gamma/dn$. Comparing with the acoustical conditions we see that K replaces σ, and consequently (by the value of V^2) μ replaces $1/m$. These substitutions with that of γ, or c (the magnetic induction), for ψ and ϕ suffice to make (66), (70) applicable to the electrical problem. For example, in the case of the circular cylinder, we have for the dispersed wave

$$c=-k^2a^2e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left\{\frac{\mu-\mu'}{2\mu}+\frac{K'-K}{K'+K}\frac{x}{r}\right\}, \quad . \quad (86)$$

corresponding to the primary waves

$$c=e^{ikx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

An important particular case is obtained by making $K'=\infty$, $\mu'=0$, in such a way that V' remains finite. This is equivalent to endowing the obstacle with the character of a perfect conductor, and we get

$$c = -k^2a^2e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left\{\frac{1}{2}+\frac{x}{r}\right\}, \quad . \quad . \quad (88)$$

which, when realized, coincides with (72).

The other two-dimensional electrical problem is that in which everything is expressed by means of R, the electromotive intensity parallel to z. The conditions at the surface are now the continuity of R and of $\mu^{-1}dR/dn$. Thus K and μ are simply interchanged, μ replacing σ and K replacing $1/m$ in (66), (70), ϕ and ψ also being replaced by R. In the case of the circular cylinder

$$R = -k^2a^2e^{-ikr}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}\left\{\frac{K-K'}{2K}+\frac{\mu'-\mu}{\mu'+\mu}\frac{x}{r}\right\}, \quad (89)$$

corresponding to the primary waves

$$R = e^{ikx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

If in order to obtain the solution for a perfectly conducting obstacle we make $K'=\infty$, $\mu'=0$, (89) becomes infinite, and must be replaced by the analogue of (83). Thus for the perfectly conducting circular obstacle

$$R = \frac{e^{-ikr}}{\gamma+\log\left(\frac{1}{2}ika\right)}\left(\frac{\pi}{2ikr}\right)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (91)$$

which may be realized as in (84).

The problem of a conducting cylinder is treated by Prof. J. J. Thomson in his valuable 'Recent Researches in Electricity and Magnetism,' § 364; but his result differs from (84), not only in respect to the sign of $\frac{1}{8}\lambda$, but also in the value of the denominator *. The values here given are those which follow from the equations (9), (17) of § 343 'Theory of Sound.'

Electric Waves in Three Dimensions.

In the problems which arise under this head the simple acoustical analogue no longer suffices, and we must appeal to the general electrical equations of Maxwell. The components of electric polarization (f, g, h) and of magnetic force (α, β, γ),

* It should be borne in mind that γ here is the same as Prof. Thomson's $\log\gamma$.

being proportional to e^{ikVt}, all satisfy the fundamental equation

$$(\nabla^2+k^2)=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

and they are connected together by such relations as

$$4\pi\frac{df}{dt}=\frac{d\gamma}{dy}-\frac{d\beta}{dz}, \quad . \quad . \quad . \quad . \quad . \quad (93)$$

or

$$\frac{d\alpha}{dt}=4\pi V^2\left(\frac{dg}{dz}-\frac{dh}{dy}\right), \quad . \quad . \quad . \quad . \quad (94)$$

in which any differentiation with respect to t is equivalent to the introduction of the factor ikV. Further

$$\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}=0, \qquad \frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0. \qquad (95)$$

The electromotive intensity (P, Q, R) and the magnetization (a, b, c) are connected with the quantities already defined by the relations

$$f, g, h=K(P, Q, R)/4\pi; \qquad a, b, c=\mu(\alpha, \beta, \gamma); \quad (96)$$

in which K denotes the specific inductive capacity and μ the permeability; so that $V^{-2}=K\mu$.

The problem before us is the investigation of the disturbance due to a small obstacle (K', μ') situated at the origin, upon which impinge primary waves denoted by

$$f_0=0, \qquad g_0=0, \qquad h_0=e^{ikx}, \quad . \quad . \quad . \quad . \quad (97)$$

or, as follows from (94),

$$\alpha_0=0, \qquad \beta_0=4\pi Ve^{ikx}, \qquad \gamma_0=0. \quad . \quad . \quad (98)$$

The method of solution, analogous to that already several times employed, depends upon the principle that in the neighbourhood of the obstacle and up to a distance from it great in comparison with the dimensions of the obstacle but small in comparison with λ, the condition at any moment may be identified with a steady condition such as is determined by the solution of a problem in conduction. When this is known, the disturbance at a distance from the obstacle may afterwards be derived.

We will commence with the case of the *sphere*, and consider first the magnetic functions as disturbed by the change of permeability from μ to μ'. Since in the neighbourhood of the sphere the problem is one of steady distribution, α, β, γ are derivable from a potential. By (98), in which we may write

$e^{ikx}=1$, the primary potential is $4\pi Vy$; so that in (1) we are to take $u=0$, $v=4\pi V$, $w=0$. Hence by (20) α, β, γ for the *disturbance* are given by

$$\alpha=d\psi/dx, \qquad \beta=d\psi/dy, \qquad \gamma=d\psi/dz,$$

where

$$\psi=-4\pi V\frac{\mu'-\mu}{\mu'+2\mu}\frac{a^3y}{r^3}. \quad . \quad . \quad . \quad . \quad (99)$$

In like manner f, g, h are derivable from a potential χ. The primary potential is z simply, so that in (1), $u=0$, $v=0$, $w=1$. Hence by (20)

$$\chi=-\frac{K'-K}{K'+2K}\frac{a^3z}{r^3}, \quad . \quad . \quad . \quad . \quad (100)$$

from which f, g, h for the disturbance are derived by simple differentiations with respect to x, y, z respectively.

Since f, g, h, α, β, γ all satisfy (92), the values at a distance can be derived by means of (41). The terms resulting from (99), (100) are of the second order in spherical harmonics. When r is small,

$$r^{-1}e^{-ikr}f_2(ikr)=-3/k^2r^3,$$

and when r is great

$$r^{-1}e^{-ikr}f_2(ikr)=r^{-1}e^{-ikr};$$

so that, as regards an harmonic of the second order, the value at a distance will be deduced from that in the neighbourhood of the origin by the introduction of the factor $-\frac{1}{3}k^2r^2e^{-ikr}$. Thus, for example, f in the neighbourhood of the origin is

$$f=\frac{d\chi}{dx}=\frac{K'-K}{K'+2K}\frac{3a^3xz}{r^5}; \quad . \quad . \quad . \quad (101)$$

so that at a great distance we get

$$f=-\frac{K'-K}{K'+2K}\frac{k^2a^3xz\,e^{-ikr}}{r^3}. \quad . \quad . \quad . \quad (102)$$

In this way the terms of the second order in spherical harmonics are at once obtained, but they do not constitute the complete solution of the problem. We have also to consider the possible occurrence of terms of other orders in spherical harmonics. Terms of order higher than the second are indeed excluded, because in the passage from r small to r great they suffer more than do the terms of the second order. But for a like reason it may happen that terms of order zero and 1 in spherical harmonics rise in relative importance so as to be comparable at a distance with the term of the second order, although relatively negligible in the neighbourhood of the obstacle. The factor, analogous to $-\frac{1}{3}k^2r^2e^{-ikr}$ for the second order, is for the first order $ikre^{-ikr}$, and for zero order e^{-ikr}. Thus, although (101) gives the value of f with sufficient completeness for the neighbourhood of the obstacle, (102) may need to be supplemented by terms of the first and zero orders in spherical harmonics of the same importance as itself. The supplementary terms may be obtained without much difficulty from those already arrived at by means of the relations (93), (94), (95); but the process is rather cumbrous, and it seems better to avail ourselves of the forms deduced by Hertz * for electric vibrations radiated from a centre.

If we write $\Pi = Ae^{-ikr}/r$, the solution corresponding to an impressed electric force acting at the origin parallel to z is

$$f = -\frac{d^2\Pi}{dx\,dz}, \qquad g = -\frac{d^2\Pi}{dy\,dz}, \qquad h = \frac{d^2\Pi}{dx^2} + \frac{d^2\Pi}{dy^2}; \quad (103)$$

$$\alpha = -4\pi\frac{d^2\Pi}{dy\,dt}, \qquad \beta = 4\pi\frac{d^2\Pi}{dx\,dt}, \qquad \gamma = 0. \quad . \quad (104)$$

These values evidently satisfy (92) since Π does so, and they harmonize with (93), (94), (95).

In the neighbourhood of the origin, where kr is small, e^{-ikr} may be identified with unity, so that $\Pi = A/r$. In this case (103) may be written

$$f = -\frac{d^2\Pi}{dx\,dz}, \qquad g = -\frac{d^2\Pi}{dx\,dz}, \qquad h = -\frac{d^2\Pi}{dz^2},$$

and all that remains is to identify $-d\Pi/dz$ with χ in (100). Accordingly

$$A = -a^3\frac{K'-K}{K'+2K}. \quad . \quad . \quad . \quad . \quad (105)$$

The values of f, g, h in (103) are now determined. Those of α, β, γ are relatively negligible in the neighbourhood of the origin. At a great distance we have

$$f = -A\frac{d^2}{dx\,dz}\left(\frac{e^{-ikr}}{r}\right) = -\frac{A}{r}\frac{d^2e^{-ikr}}{dx\,dz} = \frac{k^2Ae^{-ikr}}{r}\frac{xz}{r^2};$$

* *Ausbreitung der electrischen Kraft*, Leipzig, 1892, p. 150. It may be observed that the solution for the analogous but more difficult problem relating to an elastic solid was given much earlier by Stokes (Camb. Trans. vol. ix. p. 1, 1849). Compare 'Theory of Sound,' 2nd ed. § 378.

so that (103), (104) may be written

$$f,\ g,\ h = \frac{K' - K}{K' + 2K}\,\frac{k^2 a^3 e^{-ikr}}{r}\left(-\frac{xz}{r^2},\ -\frac{yz}{r^2},\ \frac{x^2+y^2}{r^2}\right), \quad (106)$$

$$\frac{\alpha,\ \beta,\ \gamma}{4\pi V} = \frac{K' - K}{K' + 2K}\,\frac{k^2 a^3 e^{-ikr}}{r}\left(\frac{y}{r},\ -\frac{x}{r},\ 0\right). \quad . \quad (107)$$

These equations give the values of the functions for a disturbance radiating from a small spherical obstacle, so far as it depends upon $(K'-K)$. We have to add a similar solution dependent upon the change from μ to μ'. In this (103), (104) are replaced by

$$\frac{\alpha}{V^2} = -\frac{d^2\Pi}{dx\,dy}, \qquad \frac{\beta}{V^2} = \frac{d^2\Pi}{dx^2} + \frac{d^2\Pi}{dz^2}, \qquad \frac{\gamma}{V^2} = -\frac{d^2\Pi}{dz\,dy}; \quad (108)$$

$$4\pi f = -\frac{d^2\Pi}{dz\,dt}, \qquad g = 0, \qquad 4\pi h = \frac{d^2\Pi}{dx\,dt}, \quad (109)$$

where $\Pi = Be^{-ikr}/r$, corresponding to an impressed magnetic force parallel to y. In the neighbourhood of the origin (108) becomes

$$\frac{\alpha}{V^2} = -\frac{d^2\Pi}{dx\,dy}, \qquad \frac{\beta}{V^2} = -\frac{d^2\Pi}{dy^2}, \qquad \frac{\gamma}{V^2} = -\frac{d^2\Pi}{dz\,dy},$$

so that ψ in (99) is to be identified with $-V^2 d\Pi/dy$. Thus

$$B = -\frac{4\pi a^3}{V}\,\frac{\mu' - \mu}{\mu' + 2\mu}. \quad . \quad . \quad . \quad . \quad (110)$$

At a great distance we have

$$f,\ g,\ h = \frac{\mu' - \mu}{\mu' + 2\mu}\,\frac{k^2 a^3 e^{-ikr}}{r}\left(\frac{z}{r},\ 0,\ -\frac{x}{r}\right); \quad . \quad (111)$$

$$\frac{\alpha,\ \beta,\ \gamma}{4\pi V} = \frac{\mu' - \mu}{\mu' + 2\mu}\,\frac{k^2 a^3 e^{-ikr}}{r}\left(-\frac{xy}{r^2},\ \frac{x^2+z^2}{r^2},\ -\frac{zy}{r^2}\right). \quad (112)$$

By addition of (111) to (106) and of (112) to (107) we obtain the complete values of f, g, h, α, β, γ when both the dielectric constant and the permeability undergo variation. The disturbance corresponding to the primary waves $h = e^{ikx}$ is thus determined.

When the changes in the electric constants are small, (106), (111) may be written

$$f=\frac{\pi T}{\lambda^2 r}e^{-ikr}\left(-\frac{\Delta K}{K}\frac{xz}{r^2}+\frac{\Delta\mu}{\mu}\frac{z}{r}\right), \quad . \quad . \quad (113)$$

$$g=\frac{\pi T}{\lambda^2 r}e^{-ikr}\left(-\frac{\Delta K}{K}\frac{yz}{r^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (114)$$

$$h=\frac{\pi T}{\lambda^2 r}e^{-ikr}\left(\frac{\Delta K}{K}\frac{x^2+y^2}{r^2}-\frac{\Delta\mu}{\mu}\frac{x}{r}\right), \quad . \quad . \quad (115)$$

where $T=\frac{4}{3}\pi a^3$, $k=2\pi/\lambda$. These are the results given formerly * as applicable in this case to an obstacle of volume T and of arbitrary form. When the obstacle is spherical and $\Delta K/K$ is not small, it was further shown that $\Delta K/K$ should be replaced by $(K'-K)/(K'+2K)$, and similar reasoning would have applied to $\Delta\mu/\mu$.

The solution for the case of a spherical obstacle having the character of a perfect conductor may be derived from the general expressions by supposing that $K'=\infty$, and (in order that V' may remain finite) $\mu'=0$. We get from (106), (111),

$$f=-\frac{k^2a^3e^{-ikr}}{r}\left(\frac{xz}{r^2}+\frac{z}{2r}\right), \quad . \quad . \quad . \quad (116)$$

$$g=-\frac{k^2a^3e^{-ikr}}{r}\frac{yz}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad (117)$$

$$h=+\frac{k^2a^3e^{-ikr}}{r}\left(\frac{x^2+y^2}{r^2}+\frac{x}{2r}\right), \quad . \quad . \quad (118)$$

in agreement with the results of Prof. J. J. Thomson †. As was to be expected, in every case the vectors (f, g, h), (α, β, γ), (x, y, z) are mutually perpendicular.

Obstacle in the Form of an Ellipsoid.

The case of an ellipsoidal obstacle of volume T, whose principal axes are parallel to those of x, y, z, *i. e.* parallel to the directions of propagation and of vibration in the primary waves, is scarcely more complicated. The passage from the values of the disturbance in the neighbourhood of the obstacle to that at a great distance takes place exactly as in the case of the sphere. The primary magnetic potential in the neighbourhood of the obstacle is $4\pi Vy$, and thus, as before, $u=0$, $v=4\pi V$, $w=0$ in (1). Accordingly, by (14), $A=0$, $C=0$; and (28) gives

$$\psi=-4\pi V\frac{\mu'-\mu}{4\pi\mu+(\mu'-\mu)M}\frac{Ty}{r^3}, \quad . \quad . \quad (119)$$

* "Electromagnetic Theory of Light," Phil. Mag. vol. xii. p. 90 (1881).

† 'Recent Researches,' § 377.

corresponding to (99) for the sphere. In like manner the electric potential is

$$\chi = -\frac{K'-K}{4\pi K+(K'-K)N}\frac{Tz}{r^3}. \quad . \quad . \quad . \quad (120)$$

These potentials give by differentiation the values of α, β, γ and f, g, h respectively in the neighbourhood of the ellipsoid. Thus at a great distance we obtain for the part dependent on $(K'-K)$, as generalizations of (106), (107),

$$f, g, h = \frac{K'-K}{4\pi K+(K'-K)N}\frac{k^2Te^{-ikr}}{r}\left(-\frac{xz}{r^2}, -\frac{yz}{r^2}, \frac{x^2+y^2}{r^2}\right); \quad . \quad . \quad . \quad (121)$$

$$\frac{\alpha, \beta, \gamma}{4\pi V} = \frac{K'-K}{4\pi K+(K'-K)N}\frac{k^2Te^{-ikr}}{r}\left(\frac{y}{r}, -\frac{x}{r}, 0\right). \quad . \quad (122)$$

To these are to be added corresponding terms dependent upon $(\mu'-\mu)$, viz.:—

$$f, g, h = \frac{\mu'-\mu}{4\pi\mu+(\mu'-\mu)M}\frac{k^2Te^{-ikr}}{r}\left(\frac{z}{r}, 0, -\frac{x}{r}\right); \quad . \quad . \quad (123)$$

$$\frac{\alpha, \beta, \gamma}{4\pi V} = \frac{\mu'-\mu}{4\pi\mu+(\mu'-\mu)M}\frac{k^2Te^{-ikr}}{r}\left(-\frac{xy}{r^2}, \frac{x^2+z^2}{r^2}, -\frac{zy}{r^2}\right). \quad . \quad . \quad . \quad (124)$$

The sum gives the disturbance at a distance due to the impact of the primary waves,

$$h_0 = e^{ikx}, \qquad \beta_0 = 4\pi Ve^{ikx}, \quad . \quad . \quad . \quad . \quad (125)$$

upon the ellipsoid T of dielectric capacity K' and of permeability μ'.

As in the case of the sphere, the result for an ellipsoid of perfect conductivity is obtained by making $K'=\infty$, $\mu'=0$. Thus

$$f = -\frac{k^2e^{-ikr}}{r}\left(\frac{T}{N}\frac{xz}{r^2} + \frac{T}{4\pi-M}\frac{z}{r}\right), \quad . \quad (126)$$

$$g = -\frac{k^2e^{-ikr}}{r}\frac{T}{N}\frac{yz}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

$$h = +\frac{k^2e^{-ikr}}{r}\left(\frac{T}{N}\frac{x^2+y^2}{r^2} + \frac{T}{4\pi-M}\frac{x}{r}\right). \quad . \quad (128)$$

Next to the sphere the case of greatest interest is that of a flat circular disk (radius$=$R). The volume of the obstacle then vanishes, but the effect remains finite in certain cases notwithstanding. Thus, if the axis of the disk be parallel to x, that is to the direction of primary propagation, we have (21), (25),

$$\frac{\mathrm{T}}{\mathrm{N}}=\frac{4\mathrm{R}^3}{3\pi}, \qquad \frac{\mathrm{T}}{4\pi-\mathrm{M}}=0. \quad . \quad . \quad . \quad . \quad (129)$$

In spite of its thinness, the plate being a perfect conductor disturbs the electric field in its neighbourhood; but the magnetic disturbance vanishes, the zero permeability having no effect upon the magnetic flow parallel to its face. If the axis of the disk be parallel to y {see (24)},

$$\frac{\mathrm{T}}{\mathrm{N}}=\frac{4\mathrm{R}^3}{3\pi}, \qquad \frac{\mathrm{T}}{4\pi-\mathrm{M}}=\frac{2\mathrm{R}^3}{3\pi}; \quad . \quad . \quad . \quad (130)$$

and if the axis be parallel to z,

$$\frac{\mathrm{T}}{\mathrm{N}}=0, \qquad \frac{\mathrm{T}}{4\pi-\mathrm{M}}=0, \quad . \quad . \quad . \quad . \quad (131)$$

so that in this case the obstacle produces no effect at all.

Circular Aperture in Conducting Screen.

The problem proposed is the incidence of plane waves ($h_0=e^{ikx}$) upon an infinitely thin screen at $x=0$ endowed with perfect electric conductivity and perforated by a circular aperture. In the absence of a perforation there would of course be no waves upon the negative side, and upon the positive side the effect of the screen would merely be to superpose the reflected waves denoted by $h_0=-e^{-ikx}$. We wish to calculate the influence of a small circular aperture of radius R.

In accordance with the general principle the condition of things is determined by what happens in the neighbourhood of the aperture, and this is substantially the same as if the wave-length were infinite. The problem is then expressible by means of a common potential. The magnetic force at a distance from the aperture on the positive side is altogether $8\pi\mathrm{V}$, and on the negative side zero; while the condition to be satisfied upon the faces of the screen is that the force be entirely tangential. The general character of the flow is indicated in fig. 1.

The problem here proposed is closely connected with those which we have already considered where no infinite screen was present, but a flat finite obstacle, which may be imagined to coincide with the proposed aperture. The primary magnetic field being $\beta=4\pi V$, and the disk of radius R being of infinite permeability, the potential at a distance great

Fig. 1.

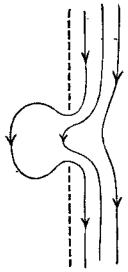


Fig. 2.

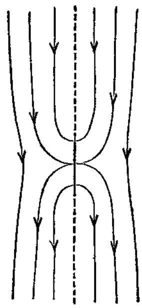
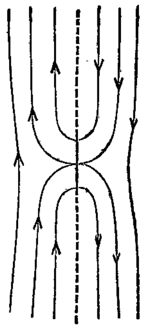


Fig. 3.



compared with R (but small compared with λ) is by (27) (28)

$$\psi=-4\pi V \frac{4R^3}{3\pi}\frac{y}{r^3}. \quad . \quad . \quad . \quad . \quad (132)$$

By the symmetry the part of the plane $x=0$ external to the disk is not crossed by the lines of flow, and thus it will make no difference in the conditions if this area be filled up by a screen of zero permeability. On the other hand, the part of the plane $x=0$ represented by the disk is met normally by the lines of flow. This state of things is indicated in fig. 2.

The introduction of the lamina of zero permeability effects the isolation of the positive and negative sides. We may therefore now reverse the flow upon the negative side, giving the state of things indicated in fig. 3. But the plate of infinite permeability then loses its influence and may be removed, so as to re-establish a communication between the positive and negative sides through an aperture. The passage from the present state of things to that of fig. 1 is effected by superposition upon the whole field of $\beta=4\pi V$, so as to destroy the field at a distance from the aperture upon the negative side and upon the positive side to double it.

As regards the solution of the proposed problem we have then on the positive side

$$\psi=8\pi Vy-4\pi V\frac{4R^3}{3\pi}\frac{y}{r^3}, \quad . \quad . \quad . \quad (133)$$

and on the negative side

$$\psi=4\pi V\frac{4R^3}{3\pi}\frac{y}{r^3}. \quad . \quad . \quad . \quad . \quad . \quad (134)$$

Thus on the negative side at a distance great in comparison with the wave-length we get, as in (99), (111), (112),

$$f, g, h = -\frac{4R^3}{3\pi}\frac{k^2e^{-ikr}}{r}\left(\frac{z}{r}, 0, -\frac{x}{r}\right), \quad . \quad . \quad . \quad . \quad . \quad (135)$$

$$\frac{\alpha, \beta, \gamma}{4\pi V} = -\frac{4R^3}{3\pi}\frac{k^2e^{-ikr}}{r}\left(-\frac{xy}{r^2}, \frac{x^2+z^2}{r^2}, -\frac{zy}{r^2}\right). \quad . \quad (136)$$

On the positive side these values are to be reversed, and addition made of

$$h_0 = e^{ikx} - e^{-ikx}, \qquad \beta_0 = 4\pi V(e^{ikx} + e^{-ikx}), \quad . \quad (137)$$

representing the plane waves incident and reflected.

The solution for h in (135) may be compared with that obtained (27), (28) in a former paper *, where, however, the primary waves were supposed to travel in the positive, instead of, as here, in the negative direction. It had at first been supposed that the solution for ϕ there given might be applied directly to h, which satisfies the condition (imposed upon ϕ) of vanishing upon the faces of the screen. If this were admitted, as also $g=0$ throughout, the value of h would follow by (95). The argument was, however, felt to be insufficient on account of the discontinuities which occur at the *edge* of the aperture, and the value now obtained, though of the same form, is doubly as great.

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VI. *Thermal Transpiration and Radiometer Motion.*

To the Editors of the Philosophical Magazine.

GENTLEMEN,

IN the Phil. Mag. for Feb. 1897, Prof Osborne Reynolds, in commenting upon my paper on Thermal Transpiration and Radiometer Motion, remarks freely on errors into which I have fallen therein. I fancy that most readers of my paper will recognize that the particular errors mentioned by Prof. Reynolds are rather the result of his own misinterpretation than of my blundering; but still, as he has taken six pages of the Phil. Mag. in which to lay these errors to my charge, I should like to point out briefly how the errors are his own.

First:—On page 143 Prof. Reynolds writes:—“while Mr. Sutherland expressly excludes the action of these walls

* “On the Passage of Waves through Apertures in Plane Screens, and Allied Problems,” Phil. Mag. vol. xliii. p. 264 (1897).