XXI. *Stability of Fluid Motion (continued from the May and June numbers).—Rectilineal Motion of Viscous Fluid between two Parallel Planes**. *By* Sir W. THOMSON, *LL.D., F.R.S.*

27. SINCE the communication of the first of this series of articles to the Royal Society of Edinburgh in April, and its publication in the Philosophical Magazine in May and June, the stability or instability of the steady motion of a viscous fluid has been proposed as subject for the Adams Prize of the University of Cambridge for 1888 †. The present communication (§§ 27–40) solves the simpler of the two cases specially referred to by the Examiners in their announcement, and prepares the way for the investigation of the less simple by a preliminary laying down, in §§ 27–29, and equations (7) to (12) below, of the fundamental equations of motion of a viscous fluid kept moving by gravity between two infinite plane boundaries inclined to the horizon at any angle I, and given with any motion deviating infinitely little from the determinate steady motion which would be the unique and essentially stable solution if the viscosity were sufficiently large. It seems probable, almost certain indeed, that analysis similar to that of §§ 38 and 39 will demonstrate that the steady motion is stable for any viscosity, however small; and that the practical unsteadiness pointed out by Stokes forty-four years ago, and so admirably investigated experimentally five or six years ago by Osborne Reynolds, is to be explained by limits of stability becoming narrower and narrower the smaller is the viscosity.

Let OX be chosen in one of the bounding planes, parallel to the direction of the rectilineal motion; and OY perpendicular to the two planes. Let the x-, y-, z-, component velocities, and the pressure, at (x, y, z, t), be denoted by $\mathrm{U}+u, v, w$, and p respectively; U denoting a function of (y, t). Then, calling the density of the fluid unity, and the viscosity μ, we have, as the equations of motion ‡,

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad . \quad . \quad . \quad . \quad . \quad (1);$$

* Communicated by the Author, having been read before the Royal Society of Edinburgh, July 18, 1887.

† See Phil. Mag. July 1887, p. 142.

‡ Stokes's Collected Papers, vol. i. p. 93.

$$\left.\begin{aligned}\frac{d}{dt}(\mathrm{U}+u)+(\mathrm{U}+u)\frac{du}{dx}+v\frac{d}{dy}(\mathrm{U}+u)+w\frac{dw}{dz}&=\mu\nabla^2(\mathrm{U}+u)-\frac{dp}{dx}+g\sin\mathrm{I},\\ \frac{dv}{dt}+(\mathrm{U}+u)\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}&=\mu\nabla^2 v-\frac{dp}{dy}-g\cos\mathrm{I},\\ \frac{dw}{dt}+(\mathrm{U}+u)\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}&=\mu\nabla^2 w-\frac{dp}{dz},\end{aligned}\right\}\quad(2);$$

where ∇^2 denotes the "Laplacian" $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$.

28. If we have $u=0$, $v=0$, $w=0$; $p=\mathrm{C}-g\cos\mathrm{I}\,y$; the four equations are satisfied identically; except the first of (2), which becomes

$$\frac{d\mathrm{U}}{dt}=\mu\frac{d^2\mathrm{U}}{dy^2}+g\sin\mathrm{I}\quad.\quad.\quad.\quad.\quad.\quad(3).$$

This is reduced to

$$\frac{dv}{dt}=\mu\frac{d^2v}{dy^2}\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(4),$$

if we put

$$\mathrm{U}=v+\tfrac{1}{2}g\sin\mathrm{I}/\mu\,.\,(b^2-y^2)\quad.\quad.\quad.\quad(5).$$

For terminal conditions (the bounding planes supposed to be $y=0$ and $y=b$), we may have

$$\left.\begin{aligned}v&=\mathrm{F}(t)\ \text{ when } y=0\\ v&=\mathfrak{F}(t)\ \quad,,\quad y=b\end{aligned}\right\}\quad.\quad.\quad.\quad.\quad(6),$$

where F and $\mathfrak{F}$ denote arbitrary functions. These equations (4) and (6) show (what was found forty-two years ago by Stokes) that the diffusion of velocity in parallel layers, *provided it is exactly in parallel layers*, through a viscous fluid, follows Fourier's law of the "linear" diffusion of heat through a homogeneous solid. Now, towards answering the highly important and interesting question which Stokes raised,—Is this laminar motion unstable in some cases?—go back to (1) and (2), and in them suppose u, v, w to be each infinitely small: (1) is unchanged; (2), with U eliminated by (5), become

$$\frac{du}{dt}+[v+\tfrac{1}{2}c(b^2-y^2)]\frac{du}{dx}+v\left(\frac{dv}{dy}-cy\right)=\mu\nabla^2u-\frac{dp}{dx}\quad.\quad.\quad(7),$$

$$\frac{dv}{dt}+[v+\tfrac{1}{2}c(b^2-y^2)]\frac{dv}{dx}=\mu\nabla^2v-\frac{dp}{dy}\quad.\quad.\quad(8),$$

$$\frac{dw}{dt}+[v+\tfrac{1}{2}c(b^2-y^2)]\frac{dw}{dx}=\mu\nabla^2w-\frac{dp}{dz}\quad.\quad.\quad(9);$$

where

$$c=g\sin\mathrm{I}/\mu.\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(10)$$

and, for brevity, p now denotes, instead of as before the pressure, the pressure $+g\cos \mathrm{I}\,y$.

We still suppose v to be a function of y and t determined by (4) and (6). Thus (1) and (7), (8), (9) are four equations which, with proper initial and boundary conditions, determine the four unknown quantities u, v, w, p; in terms of x, y, z, t.

29. It is convenient to eliminate u and w; by taking $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ of (7), (8), (9), and adding. Thus we find, in virtue of (1),

$$2\left(\frac{dv}{dy}-cy\right)\frac{dv}{dx}=-\nabla^2 p \quad . \quad . \quad . \quad . \quad (11).$$

This and (8) are two equations for the determination of v and p. Eliminating p between them, we find

$$\frac{d\nabla^2 v}{dt}-\left(\frac{d^2 v}{dy^2}-c\right)\frac{dv}{dx}+\left[v-\tfrac{1}{2}c(b^2-y^2)\right]\frac{d\nabla^2 v}{dx}=\mu\nabla^4 v \quad . \quad . \quad (12),$$

a single equation which, with proper initial and boundary conditions, determines the one unknown, v. When v is thus found, (8), (7), (9) determine p, u, and w.

30. An interesting and practically important case is presented by supposing one or both of the bounding planes to be kept oscillating in its own plane; that is, F and $\mathfrak{F}$ of (6) to be periodic functions of t. For example, take

$$\mathrm{F}=a\cos\omega t,\ \mathfrak{F}=0 \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The corresponding periodic solution of (4) is

$$v=a\frac{\epsilon^{(b-y)\sqrt{\frac{\omega}{2\mu}}}-\epsilon^{-(b-y)\sqrt{\frac{\omega}{2\mu}}}}{\epsilon^{b\sqrt{\frac{\omega}{2\mu}}}-\epsilon^{-b\sqrt{\frac{\omega}{2\mu}}}}\cos\left(\omega t-y\sqrt{\frac{\omega}{2\mu}}\right) \quad . \quad . \quad (14).$$

In connexion with this case there is no particular interest in supposing a current to be maintained by gravity; and we shall therefore take $c=0$, which reduces (7), (8), (9), (11), (12), to

$$\frac{du}{dt}+v\frac{du}{dx}+\frac{dv}{dy}v=\mu\nabla^2 u-\frac{dp}{dx} \quad . \quad . \quad . \quad (15),$$

$$\frac{dv}{dt}+v\frac{dv}{dx}=\mu\nabla^2 v-\frac{dp}{dy} \quad . \quad . \quad . \quad (16),$$

$$\frac{dw}{dt}+v\frac{dw}{dx}=\mu\nabla^2 w-\frac{dp}{dz} \quad . \quad . \quad . \quad (17),$$

$$2\frac{dv}{dy}\frac{dv}{dx}=-\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad (18),$$

$$\frac{d\nabla^2 v}{dt}-\frac{d^2 v}{dy^2}\frac{dv}{dx}+v\frac{d\nabla^2 v}{dx}=\mu\nabla^4 v \quad . \quad . \quad . \quad . \quad . \quad (19);$$

in all of which v is the function of (y, t) expressed by (14).

These equations (15) ... (19) are of course satisfied by $u=0$, $v=0$, $w=0$, $p=0$. The question of stability is, Does every possible solution of them come to this in time? It seems to me probable that it does; but I cannot, at present at all events, enter on the investigation. The case of $b=\infty$ is specially important and interesting.

31. The present communication is confined to the much simpler case in which the two bounding planes are kept moving relatively with constant velocity; including as sub-case, the two planes held at rest, and the fluid caused by gravity to move between them. But we shall first take the much simpler sub-case, in which there is relative motion of the two planes, and no gravity. This is the very simplest of all cases of the general question of the Stability or Instability of the Motion of a Viscous Fluid. It is the second of the two cases prescribed by the Examiners for the Adams Prize of 1888. I have ascertained, and I now give (§§ 32 ... 39 below) the proof, that in this sub-case the steady motion is wholly stable, however small or however great be the viscosity; and this without limitation to two-dimensional motion of the admissible disturbances.

32. In our present sub-case, let βb be the relative velocity of the two planes; so that in (6) we may take $\mathrm{F}=0, \mathfrak{G}=\beta b$; and the corresponding steady solution of (4) is

$$v = \beta y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Thus equation (19) becomes reduced to

$$\left.\begin{array}{l} \dfrac{d\sigma}{dt}+\beta y\dfrac{d\sigma}{dx}=\mu\nabla^2\sigma, \\[2ex] \text{where} \qquad \sigma=\nabla^2 v \end{array}\right\} \quad . \quad . \quad . \quad . \quad (21);$$

and (18), (15), (16), (17) become

$$2\beta\frac{dv}{dx}=-\nabla^2 p \quad . \quad . \quad . \quad . \quad . \quad . \quad (22),$$

$$\frac{du}{dt}+\beta y\frac{du}{dx}+\beta v=\mu\nabla^2 u-\frac{dp}{dx} \quad . \quad . \quad . \quad . \quad (23),$$

$$\frac{dv}{dt}+\beta y\frac{dv}{dx} \qquad =\mu\nabla^2 v-\frac{dp}{dy} \quad . \quad . \quad . \quad . \quad (24),$$

$$\frac{dw}{dt}+\beta y\frac{dw}{dx} \qquad =\mu\nabla^2 w-\frac{dp}{dz} \quad . \quad . \quad . \quad . \quad (25).$$

It may be remarked that equations (22) ... (25) imply (1), and that any four of the five determines the four quantities u, v, w, p. It will still be convenient occasionally to use (1).

We proceed to find the complete solution of the problem before us, consisting of expressions for u, v, w, p satisfying (22)...(25) for all values of x, y, z, t; and the following initial and boundary conditions :—

$$\left.\begin{array}{r}\text{when } t=0: \; u, v, w \text{ to be arbitrary functions} \\ \text{of } x, y, z, \text{ subject only to (1)}\end{array}\right\} \quad . \quad (26);$$

$$\left.\begin{array}{l}u=0,\; v=0,\; w=0, \text{ for } y=0 \text{ and all values of } x, z, t \\ u=0,\; v=0,\; w=0, \text{ for } y=b \quad\quad ,, \quad\quad ,,\end{array}\right\} \; (27).$$

33. First let us find a particular solution $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p}$, which shall satisfy the initial conditions (26), irrespectively of the boundary conditions (27), except as follows :—

$$\left.\begin{array}{l}\mathbf{v}=0, \text{ when } t=0 \text{ and } y=0 \\ \mathbf{v}=0, \text{ when } t=0 \text{ and } y=b\end{array}\right\} \quad . \quad . \quad . \quad (28).$$

Next, find another particular solution, $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \mathfrak{p}$, satisfying the following initial and boundary equations :—

$$\mathfrak{u}=0,\; \mathfrak{v}=0,\; \mathfrak{w}=0, \text{ when } t=0 \quad . \quad . \quad . \quad (29);$$

$$\left.\begin{array}{r}\mathfrak{u}+\mathbf{u}=0,\; \mathfrak{v}+\mathbf{v}=0,\; \mathfrak{w}+\mathbf{w}=0, \text{ when } y=0 \\ \text{and when } y=b\end{array}\right\} \quad . \quad . \quad (30).$$

The required complete solution will then be

$$u=\mathfrak{u}+\mathbf{u}, \quad v=\mathfrak{v}+\mathbf{v}, \quad w=\mathfrak{w}+\mathbf{w} \quad . \quad . \quad . \quad (31).$$

34. To find $\mathbf{u}, \mathbf{v}, \mathbf{w}$, remark that, if μ were zero, the complete integral of (21) would be

$$\sigma = \text{arb. func. } (x-\beta yt) \; ;$$

and take therefore as a trial for a type-solution with μ not zero,

$$\sigma = \mathrm{T}\epsilon^{\iota[mx+(n-m\beta t)y+qz]} \quad . \quad . \quad . \quad . \quad (32);$$

where T is a function of t, and ι denotes $\sqrt{-1}$. Substituting accordingly in (21), we find

$$\frac{d\mathrm{T}}{dt} = -\mu[m^2+(n-m\beta t)^2+q^2]\mathrm{T} \quad . \quad . \quad . \quad (33);$$

whence, by integration,

$$\mathrm{T}=\mathrm{C}\epsilon^{-\mu t\left[m^2+n^2+q^2-nm\beta t+\frac{m^2}{3}\beta^2t^2\right]} \quad . \quad . \quad . \quad (34).$$

By the second of (21), and (32), we find

$$\mathbf{v} = -\mathrm{T}\frac{\epsilon^{\iota[mx+(n-m\beta t)y+qz]}}{m^2+(n-m\beta t)^2+q^2} \quad . \quad . \quad . \quad . \quad (35);$$

whence, by (22),

$$p=-2\beta m\iota\mathrm{T}\frac{\epsilon^{\iota[mx+(n-m\beta t)y+qz]}}{[m^2+(n-m\beta t)^2+q^2]^2} \quad . \quad . \quad . \quad (36).$$

Using this in (25), and putting

$$w=\mathrm{W}\epsilon^{\iota[mx+(n-m\beta t)y+qz]} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

we find

$$\frac{d\mathrm{W}}{dt}=-\mu[m^2+(n-m\beta t)^2+q^2]\mathrm{W}-\frac{2\beta mq\mathrm{T}}{[m^2+(n-m\beta t)^2+q^2]} \quad . \quad (38),$$

which, integrated, gives W.

Having thus found v and w, we find u by (1), as follows:—

$$u=-\frac{(n-m\beta t)v+qw}{m} \quad . \quad . \quad . \quad . \quad . \quad (39).$$

35. Realizing, by adding type-solutions for $\pm\iota$ and $\pm n$, with proper values of C, we arrive at a complete real type-solution with, for v, the following—in which K denotes an arbitrary constant:

$$v=\tfrac{1}{2}\mathrm{K}\left\{\frac{\epsilon^{-\mu t[m^2+n^2+q^2-nm\beta t+\frac{1}{3}m^2\beta^2t^2]}}{m^2+(n-m\beta t)^2+q^2}\,\frac{\cos}{\sin}\,[mx+(n-m\beta t)y+qz]\right.$$
$$\left.-\frac{\epsilon^{-\mu t[m^2+n^2+q^2+nm\beta t+\frac{1}{3}m^2\beta^2t^2]}}{m^2+(n+m\beta t)^2+q^2}\,\frac{\cos}{\sin}\,[mx-(n+m\beta t)y+qz]\right\} \quad (40).$$

This gives, when $t=0$,

$$v=\frac{\mp\mathrm{K}}{m^2+n^2+q^2}\sin ny\,\frac{\sin}{\cos}\,(mx+qz) \quad . \quad . \quad . \quad (41),$$

which fulfils (28) if we make

$$n=i\pi y/b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42);$$

and allows us, by proper summation for all values of i from 1 to ∞, and summation or integration with reference to m and q, with properly determined values of K, after the manner of Fourier, to give any arbitrarily assigned value to $v_{t=0}$ for every value of x, y, z,

$$\left.\begin{array}{llll}\text{from} & x=-\infty & \text{to} & x=+\infty, \\ \;\;,, & y=0 & \;\;,, & y=b, \\ \;\;,, & z=-\infty & \;\;,, & z=+\infty.\end{array}\right\} \quad . \quad . \quad . \quad (43).$$

The same summation and integration applied to (40) gives $\mathbf{v}$ for all values of t, x, y, z; and then by (38), (37), (39) we find corresponding determinate values of w and u.

36. To give now an arbitrary initial value, $\mathbf{w}_0$, to the

z-component of velocity, for every value of x, y, z, add to the solution (u, v, w), which we have now found, a particular solution (u', v', w') fulfilling the following conditions:—

$$\left.\begin{array}{l} v'=0 \text{ for all values of } t,\ x,\ y,\ z\,; \\ w'=\mathrm{w}_0-w_0 \text{ for } t=0, \text{ and all values of } x,\ y,\ z \end{array}\right\} \quad . \quad (44),$$

and to be found from (25) and (1), by remarking that $v'=0$ makes, by (22), $p'=0$, and therefore (23) and (25) become

$$\frac{du'}{dt}+\beta y\frac{du'}{dx}=\mu\nabla^2 u' \quad . \quad . \quad . \quad . \quad . \quad (45),$$

$$\frac{dw'}{dt}+\beta y\frac{dw'}{dx}=\mu\nabla^2 w' \quad . \quad . \quad . \quad . \quad . \quad (46).$$

Solving (46); just as we solved (21), by (32), (33), (34); and then realizing and summing to satisfy the arbitrary initial condition, as we did for v in (40), (41), (42), we achieve the determination of w'; and by (1) we determine the corresponding u', *ipso facto* satisfying (45). Lastly, putting together our two solutions, we find

$$\mathrm{u}=u+u', \quad \mathrm{v}=\mathrm{v}, \quad \mathrm{w}=w+w' \quad . \quad . \quad . \quad (47)$$

as a solution of (26) without (27), in answer to the first requisition of § 33. It remains to find $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$, in answer to the second requisition of § 33.

37. This we shall do by first finding a real (simple harmonic) periodic solution of (21), (22), (23), (25), fulfilling the condition

$$\left.\begin{array}{l} \left.\begin{array}{l} u=\mathrm{A}\cos\omega t+\mathrm{B}\sin\omega t \\ v=\mathrm{C}\cos\omega t+\mathrm{D}\sin\omega t \\ w=\mathrm{E}\cos\omega t+\mathrm{F}\sin\omega t \end{array}\right\} \text{ when } y=0 \\ \left.\begin{array}{l} u=\mathfrak{A}\cos\omega t+\mathfrak{B}\sin\omega t \\ v=\mathfrak{C}\cos\omega t+\mathfrak{D}\sin\omega t \\ w=\mathfrak{E}\cos\omega t+\mathfrak{F}\sin\omega t \end{array}\right\} \text{ when } y=b \end{array}\right\} \quad . \quad . \quad (48),$$

where A, B, C, D, E, F, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$ are twelve arbitrary functions of (x, z). Then, by taking $\int_0^\infty d\omega f(\omega)$ of each of these after the manner of Fourier, we solve the problem of determining the motion produced throughout the fluid, by giving to every point of each of its approximately plane boundaries an infinitesimal displacement of which each of the three components is an arbitrary function of x, z, t. Lastly, by taking these functions each $=0$ from $t=-\infty$ to $t=0$, and each equal to minus the value of $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ for every point of each boundary, we find the $\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$ of § 33. The solution of our problem of § 32 is then completed by equations (31). To do all this is a mere routine after an imaginary type solution is provided as follows.

38. To satisfy (21) assume

$$v=\epsilon^{\iota(\omega t+mx+qz)}\mathcal{V}$$
$$=\epsilon^{\iota(\omega t+mx+qz)}\{H\epsilon^{y\sqrt{(m^2+q^2)}}+K\epsilon^{-y\sqrt{(m^2+q^2)}}+Lf(y)+MF(y)\}\ . \quad (49),$$

where H, K, L, M are arbitrary constants and f, F any two particular solutions of

$$\iota(\omega+m\beta y)\sigma=\mu\left[\frac{d^2\sigma}{dy^2}-(m^2+q^2)\sigma\right] \quad . \quad . \quad (50).$$

This equation, if we put

$$m\beta/\mu=\gamma,\text{ and } m^2+q^2+\iota\omega/\mu=\lambda \quad . \quad . \quad . \quad (51),$$

becomes

$$\frac{d^2\sigma}{dy^2}=(\lambda+\iota\gamma y)\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (52);$$

which, integrated in ascending powers of $(\lambda+\iota\gamma y)$, gives two particular solutions, which we may conveniently take for our f and F, as follows :—

$$\left.\begin{aligned} f(y)&=1-\frac{\gamma^{-2}(\lambda+\iota\gamma y)^3}{3\,.\,2}+\frac{\gamma^{-4}(\lambda+\iota\gamma y)^6}{6\,.\,5\,.\,3\,.\,2}-\frac{\gamma^{-6}(\lambda+\iota\gamma y)^9}{9\,.\,8\,.\,6\,.\,5\,.\,3\,.\,2}+\&c.\\ F(y)&=\lambda+\iota\gamma y-\frac{\gamma^{-2}(\lambda+\iota\gamma y)^4}{4\,.\,3}+\frac{\gamma^{-4}(\lambda+\iota\gamma y)^7}{7\,.\,6\,.\,4\,.\,3}-\frac{\gamma^{-6}(\lambda+\iota\gamma y)^{10}}{10\,.\,9\,.\,7\,.\,6\,.\,4\,.\,3}+\&c.\end{aligned}\right\}(53).$$

39. *These series are essentially convergent for all values of y.* Hence in (49) we have a solution continuous from $y=0$ to $y=b$; and by its four arbitrary constants we can give any prescribed values to $\mathcal{V}$, and $\frac{d\mathcal{V}}{dy}$, for $y=0$ and $y=b$. This done, find p determinately by (24); and then integrate (25) for w in *an essentially convergent series* of ascending powers of $\lambda+\iota\gamma y$, which is easily worked out, but need not be written down at present, except in abstract as follows :—

$$w=\mathcal{W}\epsilon^{\iota(\omega t+mx+qz)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54):$$

where

$$\left.\begin{aligned}\mathcal{W}=H\mathfrak{F}_1(\lambda+\iota\gamma y)+K\mathfrak{F}_2(\lambda+\iota\gamma y)+L\mathfrak{F}_3(\lambda+\iota\gamma y)\\ +M\mathfrak{F}_4(\lambda+\iota\gamma y)+P\epsilon^{y\sqrt{(m^2+q^2)}}+Q\epsilon^{-y\sqrt{(m^2+q^2)}}\end{aligned}\right\}(55).$$

Here P and Q are the two fresh constants, due to the integration for w. By these we can give to $\mathcal{W}$ any prescribed values for $y=0$ and $y=b$. Lastly, by (1), with (49), we have

$$\left.\begin{aligned} u &= \mathscr{U}\epsilon^{\iota(\omega t+mx+qz)} \\ \text{where} \quad \mathscr{U} &= -\left(\frac{1}{m\iota}\frac{d\mathscr{V}}{dy}+\frac{q}{m}\mathscr{W}\right) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (56).$$

Our six arbitrary constants, H, K, L, M, P, Q, clearly allow us to give any prescribed values to each of $\mathscr{U}$, $\mathscr{V}$, $\mathscr{W}$, for $y=0$ and for $y=b$. Thus the completion of the realized problem with real data of arbitrary functions, as described in § 37, becomes a mere affair of routine.

40. Now remark that the (**u**, **v**, **w**) solution of § 34 comes essentially to nothing, asymptotically as time advances, as we see by (33), (34), and (38). Hence the ($\mathfrak{u}$, $\mathfrak{v}$, $\mathfrak{w}$) of § 37, which rise gradually from zero at $t=0$, comes asymptotically to zero again as t increases to ∞. We conclude that the steady motion is stable.

[To be continued.]

XXII. *On Evaporation and Dissociation.*—Part VI. (*continued*). *On the Continuous Change from the Gaseous to the Liquid State at all Temperatures.* *By* William Ramsay, *Ph.D.*, *and* Sydney Young, *D.Sc.**

[Plates III.-V.]

THE following pages give a further proof of the correctness of the relation $p=bt-a$, where $v=$constant, applicable both to gases and liquids. The data for methyl alcohol apply solely to the gaseous state, for the very high pressures which its vapour exerts precluded measurements at temperatures above its critical point. With ethyl alcohol the determinations of the compressibility of the liquid are more complete than with ether; the experimental observations in the neighbourhood of the critical volume are, however, not very numerous, for the highest temperature for which an isothermal was constructed is 246°, the critical temperature being 243°·1. The values of a and b at volumes near the critical are consequently somewhat uncertain. The data for the gaseous condition are, however, pretty full. We have also a considerable number of data for acetic acid (Trans. Chem. Soc. 1886, p. 790). Here the temperature at which the highest isothermal was measured was the highest conveniently attainable by our method, viz. 280°. But as the critical temperature

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