

CURVATURE COMPUTATIONS FOR A TWO-COMPONENT CAMASSA-HOLM EQUATION WITH VORTICITY

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ABSTRACT. In the present paper, a two-component Camassa-Holm (2CH) system with vorticity is studied as a geodesic flow on a suitable Lie group. The paper aims at presenting various details of the geometric formalism and a major result is the computation of the sectional curvature K of the underlying configuration manifold. As a further result, we show that there are directions for which K is strictly positive and bounded away from zero.

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1. INTRODUCTION

As a mathematical model for two-dimensional shallow water waves with constant vorticity, the following generalized two-component Camassa-Holm (2CH) system has attracted a considerable amount of interest recently:

$$(1) \quad \begin{cases} m_t &= \alpha u_x - au_x m - um_x - \kappa \rho \rho_x, \\ \rho_t &= -u \rho_x - (a-1)u_x \rho, \\ \alpha_t &= 0. \end{cases}$$

Here $a \in \mathbb{R} \setminus \{1\}$, α is a constant, $\kappa > 0$ and $m = Au$ with A denoting the Fourier multiplication operator $A = (1 - \partial_x^2)^s$ for $s \geq 1$. The functions u and ρ depend on time t and a spatial variable $x \in \mathbb{S} \simeq \mathbb{R}/\mathbb{Z}$. A derivation of the system (1)

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with $s = 1$ by means of formal asymptotic methods applied to the full-governing equations for two-dimensional water waves with constant vorticity is the subject of the paper [11]. A special case of the system (1) is a one-parameter family of evolution equations obtained for $\alpha = 0$, $s = 1$ and $\rho \equiv 0$ and it is also called the b -equation [9, 7, 19] (here with the parameter $a \neq 1$). To further special cases of this family are the Camassa-Holm (CH) equation ($a = 2$)

$$(2) \quad u_t - u_{txx} = 2u_x u_{xx} - 3uu_x + uu_{xxx},$$

cf. [3], and the Degasperis-Procesi (DP) equation ($a = 3$)

$$u_t - u_{txx} = 3u_x u_{xx} - 4uu_x + uu_{xxx},$$

cf. [8]. Apart from the fact that both the CH equation and the DP equation are of hydrodynamical relevance, see e.g. [9, 22], they share some interesting mathematical properties: the b -equation is integrable only if $b \in \{2, 3\}$ in the sense that for $b \in \{2, 3\}$ there exists a bi-Hamiltonian formulation and a Lax pair representation [3, 7]. Moreover, both the CH and the DP equation allow for finite-time solutions that can be interpreted as breaking waves [4] or shock waves [15] as well as for global solutions [18] and peculiar traveling wave solutions [27, 28]. The 2CH equation without vorticity ($\alpha = 0$, $a = 2$ and $s = 1$ in (1)) has been the subject of [5, 14] where the authors proved local-in-time well-posedness by applying Kato's semigroup approach, discussed blow-up and established its integrable structure. In addition, the hydrodynamical relevance of the 2CH system without vorticity as a two-component extension of the CH equation is presented.

A remarkable property of the b -family equations is that they allow for a geometric reformulation on the diffeomorphism group of the circle. The group of all smooth and orientation-preserving diffeomorphisms $\mathbb{S} \rightarrow \mathbb{S}$, denoted as $\text{Diff}^\infty(\mathbb{S})$, is a Fréchet Lie group that can be equipped with an affine connection ∇ so that the b -equation is the geodesic equation on $\text{Diff}^\infty(\mathbb{S})$ with respect to the connection ∇ , see, e.g. [13, 24]. Furthermore, the geodesic flow is the minimizer of a length functional if and only if $b = 2$; in this case the resulting equation, precisely the CH equation, is a metric Euler equation with respect to the H^1 metric. The DP equation belongs to the class of non-metric Euler equations [17, 24]. An analogous geometric framework has been established for the 2CH equation without vorticity in [12, 20] where the authors showed that it can be recast as a geodesic flow on the semidirect product $\text{Diff}^\infty(\mathbb{S}) \ltimes C^\infty(\mathbb{S})$ equipped with the H^1 metric for the first component plus the L_2 metric for the second component.

The geometric theory for evolution equations arising in hydrodynamics is not only a technical game: There are various applications of the geometric picture concerning the features of solutions to report on. In [13], the authors make use of the geometric reformulation to establish the well-posedness of the periodic b -equation on a scale of Sobolev spaces that are used as a Banach space approximation for $C^\infty(\mathbb{S})$. In [6], finite time solutions to the CH equation are related to a breakdown of the geodesic flow. In particular, computations of the sectional curvature have been performed as the sign of the sectional curvature of the underlying configuration manifold has implications for the stability of its geodesics. The curvature tensor

for the CH equation on the diffeomorphism group of \mathbb{S} has been computed in [30] where the author also shows that the non-normalized sectional curvature S_{CH} is positive on an infinite-dimensional subspace containing the vector $\cos kx$ and that the normalized sectional curvature is bounded away from zero on this subspace. Moreover,

$$S_{\text{CH}}(u, v) = \langle \Gamma_{\text{CH}}(u, v), \Gamma_{\text{CH}}(u, v) \rangle_{H^1} - \langle \Gamma_{\text{CH}}(u, u), \Gamma_{\text{CH}}(v, v) \rangle_{H^1}$$

with Γ_{CH} denoting the Christoffel operator for the CH equation. Similarly, it could be shown in [12] that the non-normalized sectional curvature $S_{2\text{CH}}$ for the 2CH equation without vorticity is strictly positive in directions spanned by vectors of the type $(\cos kx, \cos lx)$, that the normalized sectional curvature is bounded away from zero in directions spanned by vectors of the type $(0, \cos kx)$ and that

$$S_{2\text{CH}}(u, v) = \langle \Gamma_{2\text{CH}}(u, v), \Gamma_{2\text{CH}}(u, v) \rangle_{H^1 \oplus L_2} - \langle \Gamma_{2\text{CH}}(u, u), \Gamma_{2\text{CH}}(v, v) \rangle_{H^1 \oplus L_2},$$

with $\Gamma_{2\text{CH}}$ denoting the Christoffel operator for the 2CH equation. We refer the reader to [2, 23] for further curvature computations for related equations of hydrodynamical relevance.

In the present paper, we first focus in detail on the geometric picture for Eq. (1) with the fixed parameters $a = 2$ and $\kappa, s = 1$. The fact that (1) represents geodesic motion on the Lie group $(\text{Diff}^\infty(\mathbb{S}) \otimes C^\infty(\mathbb{S})) \times \mathbb{R}$ with a suitable right-invariant metric $\langle \cdot, \cdot \rangle_{\mathbb{A}}$ induced by an operator \mathbb{A} has been established in [11] where the authors compute the adjoint of the adjoint action on the Lie algebra with respect to $\langle \cdot, \cdot \rangle_{\mathbb{A}}$ in order to identify Eq. (1) with the geodesic equation $U_t = -\text{ad}_U^* U$, where $U = (u, \rho, \alpha)$. However, the various analogies and their consequences when comparing the geometric picture for Eq. (1) with the geometric picture for the rigid body motion pioneered by Arnold [1] in 1966 have not been worked out to the best of the author's knowledge. Section 2 of the present work has the goal to provide some further aspects of the geometric theory for the 2CH equation (1). In Section 3, we present the following main theorem on the sectional curvature associated with Eq. (1). It clearly gets in line with the above mentioned results on the sectional curvature for the CH equation and the 2CH equation with $\alpha = 0$.

Theorem 1. *Let R denote the curvature tensor associated with the 2CH equation (1) on $(\text{Diff}^\infty(\mathbb{S}) \otimes C^\infty(\mathbb{S})) \times \mathbb{R}$ and denote by $S(u, v) = \langle R(u, v)v, v \rangle_{\mathbb{A}}$ the non-normalized sectional curvature at the identity. Then*

$$(3) \quad S(u, v) = \langle \Gamma(u, v), \Gamma(u, v) \rangle_{\mathbb{A}} - \langle \Gamma(u, u), \Gamma(v, v) \rangle_{\mathbb{A}}.$$

Moreover, $S(u, v) > 0$ for all vectors of the form

$$u = \begin{pmatrix} \cos k_1 x \\ \cos k_2 x \\ \alpha \end{pmatrix}, \quad v = \begin{pmatrix} \cos l_1 x \\ \cos l_2 x \\ 1 \end{pmatrix}, \quad k_1, k_2, l_1, l_2 \in 2\pi\mathbb{N}, \quad \alpha \geq 6 \max\{k_1^2, l_1^2\},$$

and the normalized sectional curvature

$$K(u, v) = \frac{S(u, v)}{\langle u, u \rangle_{\mathbb{A}} \langle v, v \rangle_{\mathbb{A}} - \langle u, v \rangle_{\mathbb{A}}^2}$$

is bounded away from zero for fixed values $k_1 \neq k_2, l_1 \neq l_2$ and as $\alpha \rightarrow \infty$.

2. THE GEOMETRIC FORMALISM

In 1966, Arnold [1] showed that the motion of a rigid body rotating around its center of mass is in fact geodesic motion on the group $G = SO(3)$. The configuration of the body at time t is given by a rotation matrix $R(t)$ which maps the position of a particle in body coordinates to its spatial position. The quantities $\omega = \dot{R}R^{-1}$ and $\Omega = R^{-1}\dot{R}$ are elements of $\mathfrak{so}(3)$, the Lie algebra \mathfrak{g} of $SO(3)$, and correspond to the angular velocity in spatial or body coordinates respectively. A moment of inertia tensor \mathbb{I} maps the body velocity to its momentum $\Pi = \mathbb{I}\Omega$; the spatial momentum is denoted by $\pi = R\Omega$. As $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$ are canonically identified with \mathbb{R}^3 , the Adjoint and Co-Adjoint actions $\text{Ad}_R, \text{Ad}_R^*$ are maps $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ and they link the velocity and the momentum in the spatial and the body frame of reference. Euler's equations for the rigid body motion particularly imply that the spatial momentum is a conserved quantity.

Ebin and Marsden [10] proved in 1970 that Arnold's formalism can also be applied to the motion of an ideal fluid for which the configuration space is the group of all volume-preserving diffeomorphisms of the fluid domain. A major difference to the rigid body motion is that the Riemannian metric on the diffeomorphism group is right-invariant whereas the geodesic equation for the rigid body is induced by a left-invariant metric. Details of this geometric approach have been elaborated in detail for the Camassa-Holm equation (2) in [25, 29] where the authors show that Eq. (2) with periodic boundary conditions is equivalent to a geodesic equation on the group $\text{Diff}^\infty(\mathbb{S})$ of all smooth and orientation-preserving diffeomorphisms on \mathbb{S} . In [12, 20] the authors showed that the 2CH equation without vorticity ($\alpha = 0$) allows for a geometric reformulation on the semidirect product group $\text{Diff}^\infty(\mathbb{S}) \ltimes C^\infty(\mathbb{S})$. We refer the reader to Appendix A of the paper [12] where the analogy of Arnold's approach to the geometric picture for CH and 2CH without vorticity is explained.

In this section, we present in detail the geometric picture for Eq. (1) with the fixed parameters $a = 2$ and $\kappa, s = 1$.

2.1. The Lie group. Consider the Fréchet Lie group

$$C^\infty G := (\text{Diff}^\infty(\mathbb{S}) \ltimes C^\infty(\mathbb{S})) \times \mathbb{R}$$

where $\text{Diff}^\infty(\mathbb{S})$ denotes the group of smooth and orientation-preserving diffeomorphisms of $\mathbb{S} := \mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ and \ltimes denotes a semidirect product. Writing \circ for the composition of functions, the group product on $C^\infty G$ is given by

$$(\varphi_1, f_1, s_1) * (\varphi_2, f_2, s_2) = (\varphi_1 \circ \varphi_2, f_2 + f_1 \circ \varphi_2, s_1 + s_2),$$

for $(\varphi_1, f_1, s_1), (\varphi_2, f_2, s_2) \in \text{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \times \mathbb{R}$. The neutral element on $C^\infty G$ is $(\text{id}, 0, 0)$ and one easily checks that $(\varphi, f, s) \in C^\infty G$ has the inverse

$$(\varphi, f, s)^{-1} = (\varphi^{-1}, -f \circ \varphi^{-1}, -s).$$

Let R_g and L_g denote right and left translation on $C^\infty G$ and write $I_g h = L_g R_{g^{-1}} h$ for the inner automorphism. We observe that

$$I_{(\varphi_1, f_1, s_1)}(\varphi_2, f_2, s_2) = (\varphi_1 \circ \varphi_2 \circ \varphi_1^{-1}, (f_2 - f_1) \circ \varphi_1^{-1} + f_1 \circ (\varphi_2 \circ \varphi_1^{-1}), s_2).$$

Writing T_g for the tangent map at $g \in C^\infty G$, we have that

$$\begin{aligned} \text{Ad}_{(\varphi_1, f_1, s_1)}(u_2, \rho_2, \alpha_2) &= [T_{(\text{id}, 0, 0)} I_{(\varphi_1, f_1, s_1)}] (u_2, \rho_2, \alpha_2) \\ &= ((u_2 \varphi_{1x}) \circ \varphi_1^{-1}, (\rho_2 + f_{1x} u_2) \circ \varphi_1^{-1}, \alpha_2) \end{aligned}$$

and

$$\begin{aligned} \text{ad}_{(u_1, \rho_1, \alpha_1)}(u_2, \rho_2, \alpha_2) &= T_{(\text{id}, 0, 0)} [\text{Ad}_{(\cdot)}(u_2, \rho_2, \alpha_2)] (u_1, \rho_1, \alpha_1) \\ &= (u_{1x} u_2 - u_{2x} u_1, \rho_{1x} u_2 - \rho_{2x} u_1, 0) \\ &= [(u_1, \rho_1, \alpha_1), (u_2, \rho_2, \alpha_2)]; \end{aligned}$$

here, $[\cdot, \cdot]$ denotes the Lie bracket on the Lie algebra

$$C^\infty \mathfrak{g} := T_{(\text{id}, 0, 0)} C^\infty G \simeq C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \times \mathbb{R}.$$

For the following considerations, it will be also important to note the trivialization

$$TC^\infty G \simeq (\text{Diff}^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \times \mathbb{R}) \times (C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \times \mathbb{R}).$$

For a smooth path $g(t) = (\varphi, f, s)(t)$ in $C^\infty G$, the associated Eulerian velocity $(u, \rho, \alpha)(t) = T_{g(t)} R_{g^{-1}(t)} g'(t) \in C^\infty \mathfrak{g}$ is given by

$$(4) \quad (u, \rho, \alpha)(t) = (\varphi' \circ \varphi^{-1}, f' \circ \varphi^{-1}, s')(t)$$

and

$$U_0(t) := T_{g(t)} L_{g^{-1}(t)} g'(t) = \left(\frac{\varphi_t}{\varphi_x}, f_t - f_x \frac{\varphi_t}{\varphi_x}, s' \right)$$

so that, for $U = (u_1, u_2, u_3) \in C^\infty \mathfrak{g}$,

$$\text{Ad}_{(\varphi, f, s)} U = (\text{Ad}_\varphi u_1, (u_2 + f_x u_1) \circ \varphi^{-1}, u_3),$$

where $\text{Ad}_\varphi u_1 = (u_1 \varphi_x) \circ \varphi^{-1}$ is the Adjoint action with respect to $\text{Diff}^\infty(\mathbb{S})$.

2.2. The right-invariant metric. We define an inner product on $C^\infty \mathfrak{g}$ by setting

$$(5) \quad \langle U, V \rangle_{(\text{id}, 0, 0)} := \int_{\mathbb{S}} u_1 v_1 dx + \int_{\mathbb{S}} u_{1x} v_{1x} dx + \int_{\mathbb{S}} u_2 v_2 dx - \frac{1}{2} \int_{\mathbb{S}} (u_1 v_3 + u_3 v_1) dx + \frac{1}{2} u_3 v_3,$$

where $U = (u_1, u_2, u_3), V = (v_1, v_2, v_3) \in C^\infty \mathfrak{g}$. It is shown in [16] that $\langle \cdot, \cdot \rangle_{(\text{id}, 0, 0)}$ is indeed positive definite. With the inertia operator $\mathbb{A}: C^\infty \mathfrak{g} \rightarrow (C^\infty \mathfrak{g})^*$ given by

$$\mathbb{A}U := \left(Au_1 - \frac{1}{2} u_3, u_2, \frac{1}{2} \left(u_3 - \int_{\mathbb{S}} u_1 dx \right) \right),$$

where $A = 1 - \partial_x^2$, we observe that

$$\langle U, V \rangle_{\mathbb{A}} := \int_{\mathbb{S}} (\mathbb{A}U) \cdot V dx = \langle U, V \rangle_{(\text{id}, 0, 0)}$$

and that the associated quadratic form is equivalent to the Hilbert norm $\|u_1\|_{H^1}^2 + \|u_2\|_{L^2}^2 + |u_3|^2$. We define a right-invariant metric on $C^\infty G$ by setting

$$\langle U, V \rangle_{(\varphi, f, s)} = \langle TR_{(\varphi, f, s)^{-1}} U, TR_{(\varphi, f, s)^{-1}} V \rangle_{\mathbb{A}}$$

for all $U, V \in T_{(\varphi, f, s)} C^\infty G \simeq C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \times \mathbb{R}$. It is well-known that the right-invariant metric for the 2CH equation without vorticity depends smoothly on (φ, f) , cf. [12], and by the definition of $\langle \cdot, \cdot \rangle_{(\varphi, f, s)}$ and the fact that $C^\infty G$ is a Lie

group, it is immediately clear that $\langle \cdot, \cdot \rangle_{(\varphi, f, s)}$ depends smoothly on (φ, f, s) so that $(C^\infty G, \langle \cdot, \cdot \rangle_{\mathbb{A}})$ is indeed a (weak) Riemannian manifold.

The operator \mathbb{A} maps the Eulerian velocity (u, ρ, α) to the momentum

$$\mu := \mathbb{A}(u, \rho, \alpha) = \left(Au - \frac{1}{2}\alpha, \rho, \frac{1}{2} \left(\alpha - \int_{\mathbb{S}} u \, dx \right) \right).$$

Using the L_2 -pairing to identify the regular part of $(C^\infty \mathfrak{g})^*$ with $C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \times \mathbb{R}$ and that $\text{Ad}_\varphi^* m = (m \circ \varphi) \varphi_x^2$, $m = Au$, we observe that

$$\begin{aligned} \langle \mu, \text{Ad}_{(\varphi, f, s)} V \rangle &= \int_{\mathbb{S}} \left(Au - \frac{\alpha}{2} \right) \text{Ad}_\varphi v_1 \, dx + \int_{\mathbb{S}} \rho [(f_x v_1 + v_2) \circ \varphi^{-1}] \, dx + \\ &\quad + \int_{\mathbb{S}} \left(\frac{\alpha}{2} - \frac{1}{2} \int_{\mathbb{S}} u \, dx \right) v_3 \, dx \\ &= \int_{\mathbb{S}} \left\{ \left[(m \circ \varphi) - \frac{\alpha}{2} \right] \varphi_x^2 + (\rho \circ \varphi) f_x \varphi_x \right\} v_1 \, dx + \int_{\mathbb{S}} (\rho \circ \varphi) \varphi_x v_2 \, dx \\ &\quad + \int_{\mathbb{S}} \left(\frac{\alpha}{2} - \frac{1}{2} \int_{\mathbb{S}} u \, dx \right) v_3 \, dx \end{aligned}$$

so that $\mu_0 := \text{Ad}_{(\varphi, f, s)}^* \mu$ is given by

$$\mu_0 = \left(\left[(m \circ \varphi) - \frac{\alpha}{2} \right] \varphi_x^2 + (\rho \circ \varphi) f_x \varphi_x, (\rho \circ \varphi) \varphi_x, \frac{1}{2} \left(\alpha - \int_{\mathbb{S}} u \, dx \right) \right).$$

We also show that, in analogy to the rigid body motion, we now obtain a conservation law for the 2CH equation (1).

Proposition 2. *The quantity μ_0 corresponding to the body momentum of the rigid body motion is a conserved quantity for the 2CH equation (1), i.e.*

$$\frac{d}{dt} \mu_0 = 0.$$

Proof. A simple calculation shows that

$$\begin{aligned} &\frac{d}{dt} \left[\left(m \circ \varphi - \frac{\alpha}{2} \right) \varphi_x^2 + (\rho \circ \varphi) f_x \varphi_x \right] \\ &= [(m_t + 2u_x m + u m_x - \alpha u_x + \rho \rho_x) \circ \varphi] \varphi_x^2 + [(\rho_t + u \rho_x + u_x \rho) \circ \varphi] f_x \varphi_x \\ &= 0. \end{aligned}$$

That the second component of μ_0 is conserved follows from Lemma 6.1 in [11] and the time derivative of the third component of μ_0 is zero as

$$u_t = -\partial_x \left[A^{-1} \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2 - \alpha u \right) + \frac{1}{2} u^2 \right],$$

cf. Eq. (6). □

The adjoint of the operator $\text{ad}: C^\infty \mathfrak{g} \times C^\infty \mathfrak{g} \rightarrow C^\infty \mathfrak{g}$ has been computed in [11, 16] and writing $\text{ad}_{(u_1, \rho_1, \alpha_1)}^*(u_2, \rho_2, \alpha_2) = (\tilde{u}, \tilde{\rho}, \tilde{\alpha})$, one has

$$\begin{aligned} \tilde{u} &= A^{-1} (2u_{1x} A u_2 + u_1 A u_{2x} - \alpha_2 u_{1x} + \rho_{1x} \rho_2) + \int_{\mathbb{S}} (u_{1x} A u_2 + \rho_{1x} \rho_2) \, dx, \\ \tilde{\rho} &= (u_1 \rho_2)_x, \end{aligned}$$

$$\tilde{\alpha} = 2 \int_{\mathbb{S}} (u_{1x} A u_2 + \rho_{1x} \rho_2) dx.$$

2.3. The geodesic spray. To obtain the weak formulation of (1), we apply the operator A^{-1} to the first equation and add the term uu_x so that

$$\begin{aligned} u_t + uu_x &= A^{-1}(\alpha u_x - 2u_x m - um_x - \rho \rho_x + A(uu_x)) \\ &= -A^{-1} \partial_x \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2 - \alpha u \right); \end{aligned}$$

observe that the terms including third order derivatives of u cancel out on the right hand side. We may thus rewrite (1) as

$$(6) \quad \begin{cases} u_t + uu_x &= -A^{-1} \partial_x \left(u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2 - \alpha u \right), \\ \rho_t + u \rho_x &= -\rho u_x, \\ \alpha_t &= 0. \end{cases}$$

Writing $U = (u_1, u_2, u_3), V = (v_1, v_2, v_3) \in C^\infty \mathfrak{g}$ and $\tilde{U} = (u_1, u_2)$ and $\tilde{V} = (v_1, v_2)$, we introduce the bilinear operator

$$(7) \quad \begin{aligned} \Gamma(U, V) &:= \begin{pmatrix} -A^{-1} \partial_x \left(u_1 v_1 + \frac{1}{2} u_{1x} v_{1x} + \frac{1}{2} u_2 v_2 - \frac{1}{2} u_3 v_1 - \frac{1}{2} u_1 v_3 \right) \\ -\frac{1}{2} u_2 v_{1x} - \frac{1}{2} u_{1x} v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \Gamma^0(\tilde{U}, \tilde{V}) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} A^{-1} \partial_x (u_3 v_1 + u_1 v_3) \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

where

$$(8) \quad \Gamma^0(\tilde{U}, \tilde{V}) := \begin{pmatrix} -A^{-1} \partial_x \left(u_1 v_1 + \frac{1}{2} u_{1x} v_{1x} + \frac{1}{2} u_2 v_2 \right) \\ -\frac{1}{2} u_2 v_{1x} - \frac{1}{2} u_{1x} v_2 \end{pmatrix}$$

denotes the Christoffel operator for the 2CH equation without vorticity [12]. Again, the map $\Gamma: C^\infty \mathfrak{g} \times C^\infty \mathfrak{g} \rightarrow C^\infty \mathfrak{g}$ can be extended to a right-invariant bilinear operator $\Gamma_{(\varphi, f, s)}: T_{(\varphi, f, s)} C^\infty G \times T_{(\varphi, f, s)} C^\infty G \rightarrow T_{(\varphi, f, s)} C^\infty G$ by setting

$$\Gamma_{(\varphi, f, s)} = TTR_{(\varphi, f, s)} \circ \Gamma_{(\text{id}, 0, 0)} \circ TR_{(\varphi, f, s)}^{-1}.$$

The second order vector field $TC^\infty G \rightarrow TTC^\infty G$, $(g, U) \mapsto (g, U, U, \Gamma_g(U, U))$ is called the geodesic spray for the 2CH equation (1).

2.4. The geodesic equation. Introducing Lagrangian variables $(\varphi, f, s)(t)$ for Eq. (1) by setting

$$\varphi' = u \circ \varphi, \quad f' = \rho \circ \varphi, \quad s' = \alpha$$

it follows that (6) is equivalent to the geodesic equation

$$(9) \quad (\varphi, f, s)''(t) = \Gamma_{(\varphi, f, s)(t)}((\varphi, f, s)'(t), (\varphi, f, s)'(t)).$$

We finally give a rigorous proof of the fact that the geodesics $(\varphi, f, s)(t)$ are in fact length-minimizing with respect to the metric $\langle \cdot, \cdot \rangle_{\mathbb{A}}$.

Proposition 3. *Let $\gamma(t): [0, T] \rightarrow C^\infty G$ denote the shortest path on $C^\infty G$ between fixed endpoints with respect to the metric $\langle \cdot, \cdot \rangle_{\mathbb{A}}$. Then $(u, \rho, \alpha)(t) = T_{\gamma(t)} R_{\gamma^{-1}(t)} \gamma'(t)$ is a solution to the 2CH equation (1), i.e. (1) is the Euler-Lagrange equation for the action functional*

$$\mathfrak{a}(\gamma) = \frac{1}{2} \int_0^T \langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)} dt.$$

Proof. Assume that γ is a critical point in the space of paths for the functional \mathfrak{a} . Then

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathfrak{a}(\gamma + \varepsilon\eta) = 0$$

for every path $\eta: [0, T] \rightarrow C^\infty G$ with endpoints at zero and such that $\gamma + \varepsilon\eta$ is a small variation of γ on $C^\infty G$. As

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathfrak{a}(\gamma + \varepsilon\eta) &= \int_0^T \int_{\mathbb{S}} (\gamma'_1 \circ \gamma_1^{-1}) \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} [(\gamma'_1 + \varepsilon\eta'_1) \circ (\gamma_1 + \varepsilon\eta_1)^{-1}] dt dx \\ &\quad + \int_0^T \int_{\mathbb{S}} (\gamma'_1 \circ \gamma_1^{-1})_x \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} [(\gamma'_1 + \varepsilon\eta'_1)_x \circ (\gamma_1 + \varepsilon\eta_1)^{-1}]_x dt dx \\ &\quad + \int_0^T \int_{\mathbb{S}} (\gamma'_2 \circ \gamma_1^{-1}) \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} [(\gamma'_2 + \varepsilon\eta'_2) \circ (\gamma_1 + \varepsilon\eta_1)^{-1}] dt dx \\ &\quad - \frac{1}{2} \int_0^T \int_{\mathbb{S}} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \{(\gamma'_3 + \varepsilon\eta'_3) \cdot [(\gamma'_1 + \varepsilon\eta'_1) \circ (\gamma_1 + \varepsilon\eta_1)^{-1}]\} dt dx \\ &\quad + \frac{1}{4} \int_0^T \int_{\mathbb{S}} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} (\gamma'_3 + \varepsilon\eta'_3)^2 dt dx \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5, \end{aligned}$$

where the prime indicates differentiation with respect to time, we can invoke a result presented in [21] to conclude that

$$\mathcal{I}_1 + \mathcal{I}_2 = - \int_0^T \int_{\mathbb{S}} (\eta_1 \circ \gamma_1^{-1}) [u_t + 3uu_x - u_{txx} - 2u_x u_{xx} - uu_{xxx}] dt dx$$

with $u = \gamma'_1 \circ \gamma_1^{-1}$. Differentiating the equation $\gamma_1 \circ \gamma_1^{-1} = \text{id}$ with respect to t and x yields expressions for the derivatives of γ_1^{-1} that help us to conclude that

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} [(\gamma'_2 + \varepsilon\eta'_2) \circ (\gamma_1 + \varepsilon\eta_1)^{-1}] &= \eta'_2 \circ \gamma_1^{-1} - [(\partial_x \gamma'_2) \circ \gamma_1^{-1}] \frac{\eta_1 \circ \gamma_1^{-1}}{(\partial_x \gamma_1) \circ \gamma_1^{-1}} \\ &= \partial_t (\eta_2 \circ \gamma_1^{-1}) + (\gamma'_1 \circ \gamma_1^{-1}) \partial_x (\eta_2 \circ \gamma_1^{-1}) - (\eta_1 \circ \gamma_1^{-1}) \partial_x (\gamma'_2 \circ \gamma_1^{-1}). \end{aligned}$$

Writing $\rho = \gamma'_2 \circ \gamma_1^{-1}$, integration by parts and the boundary conditions for η now show that

$$\mathcal{I}_3 = - \int_0^T \int_{\mathbb{S}} (\eta_2 \circ \gamma_1^{-1}) [\rho_t + (\rho u)_x] dt dx - \int_0^T \int_{\mathbb{S}} (\eta_1 \circ \gamma_1^{-1}) \rho \rho_x dt dx.$$

Similar calculations yield that

$$\mathcal{I}_4 = - \frac{1}{2} \int_0^T \int_{\mathbb{S}} \{ \eta'_3 u + \gamma'_3 [\partial_t (\eta_1 \circ \gamma_1^{-1}) + (\gamma'_1 \circ \gamma_1^{-1}) \partial_x (\eta_1 \circ \gamma_1^{-1})]$$

$$- (\eta_1 \circ \gamma_1^{-1}) \partial_x (\gamma_1' \circ \gamma_1^{-1}) \} dt dx.$$

As $\eta_3(t) \in \mathbb{R}$ for any $t \in [0, T]$, the first term in \mathcal{I}_4 vanishes due to the fact that

$$\begin{aligned} \int_0^T \int_{\mathbb{S}} \eta_3' u dt dx &= - \int_0^T \int_{\mathbb{S}} \eta_3 u_t dt dx \\ &= - \int_0^T \eta_3 \left(\int_{\mathbb{S}} u_t dx \right) dt \\ &= \int_0^T \eta_3 \left(\int_{\mathbb{S}} \partial_x [A^{-1}(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 - \alpha u) + \frac{1}{2}u^2] dx \right) dt \\ &= 0. \end{aligned}$$

With $\gamma_3' = \alpha$, the remaining term can be rewritten as

$$\mathcal{I}_4 = \int_0^T \int_{\mathbb{S}} (\eta_1 \circ \gamma_1^{-1}) (\frac{1}{2}\alpha_t + \alpha u_x) dt dx.$$

We finally observe that

$$\mathcal{I}_5 = \frac{1}{2} \int_0^T \int_{\mathbb{S}} \gamma_3' \eta_3' dt dx = -\frac{1}{2} \int_0^T \int_{\mathbb{S}} \alpha_t \eta_3 dt dx.$$

Hence the critical point of the length functional \mathbf{a} is obtained from the equation

$$\begin{aligned} &\int_0^T \int_{\mathbb{S}} (\eta_1 \circ \gamma_1^{-1}) [u_t + 3uu_x - u_{txx} - 2u_x u_{xx} - uu_{xxx} - \alpha u_x + \rho \rho_x - \frac{1}{2}\alpha_t] dt dx \\ &+ \int_0^T \int_{\mathbb{S}} (\eta_2 \circ \gamma_1^{-1}) [\rho_t + (\rho u)_x] dt dx + \frac{1}{2} \int_0^T \int_{\mathbb{S}} \alpha_t \eta_3 dt dx = 0. \end{aligned}$$

Since we can choose η arbitrarily, we immediately obtain the system (1) from the above identity. \square

2.5. The affine connection. The geodesic flow $(\varphi, f, s)(t)$ is not only the minimizer of the length functional on $(C^\infty G, \langle \cdot, \cdot \rangle_{\mathbb{A}})$, it is also the geodesic flow corresponding to the affine connection

$$(10) \quad (\nabla_X Y)(\varphi, f, s) := DY(\varphi, f, s) \cdot X(\varphi, f, s) - \Gamma_{(\varphi, f, s)}(X, Y)$$

where X and Y are smooth vector fields on $C^\infty G$. It follows immediately from the definition (10) that ∇ is a Riemannian covariant derivative as defined in [29], i.e.,

- (i) ∇ is \mathbb{R} -bilinear,
- (ii) $X(\varphi, f, s) = 0$ implies that $(\nabla_X Y)(\varphi, f, s) = 0$,
- (iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ for $f \in C^\infty(C^\infty G)$ and
- (iv) $\nabla_X Y - \nabla_Y X = [X, Y]$

with the Lie bracket given locally by

$$[X, Y](g) = DY(g) \cdot X(g) - DX(g) \cdot Y(g), \quad g \in C^\infty G.$$

Proposition 4. *The Riemannian covariant derivative $(X, Y) \xrightarrow{\nabla} \nabla_X Y$ defined in (10) is compatible with the metric (5).*

Proof. Let $X(\varphi, f, s)$, $Y(\varphi, f, s)$ and $Z(\varphi, f, s)$ be smooth vector fields on $C^\infty G$ and let $u = X(\varphi, f, s) \circ \varphi^{-1}$, $v = Y(\varphi, f, s) \circ \varphi^{-1}$ and $w = Z(\varphi, f, s)$. Let $\gamma(\varepsilon) \subset C^\infty G$ be a smooth path such that $\gamma(0) = (\varphi, f, s)$ and $\gamma'(0) = X(\varphi, f, s)$. Then

$$\begin{aligned}
(X \langle Y, Z \rangle_{\mathbb{A}})(\varphi, f, s) &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \left\langle \left(\begin{array}{c} Y_1(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} \\ Y_2(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} \\ Y_3(\gamma(\varepsilon)) \end{array} \right), \left(\begin{array}{c} Z_1(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} \\ Z_2(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} \\ Z_3(\gamma(\varepsilon)) \end{array} \right) \right\rangle_{\mathbb{A}} \\
&= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \left\{ \left\langle \tilde{Y}(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1}, \tilde{Z}(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} \right\rangle_{H^1 \oplus L_2} \right. \\
&\quad \left. - \frac{1}{2} \int_{\mathbb{S}} [Y_1(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} Z_3(\gamma(\varepsilon)) + Z_1(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} Y_3(\gamma(\varepsilon))] dx \right. \\
&\quad \left. + \frac{1}{2} Y_3(\gamma(\varepsilon)) Z_3(\gamma(\varepsilon)) \right\} \\
&= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \left\langle \tilde{Y}(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1}, \tilde{Z}(\gamma(\varepsilon)) \circ \gamma_1(\varepsilon)^{-1} \right\rangle_{H^1 \oplus L_2} \\
&\quad - \frac{1}{2} \int_{\mathbb{S}} [w_3(DY_1 \cdot X) \circ \varphi^{-1} + v_1 DZ_3 \cdot X - u_1 v_{1x} w_3] dx \\
&\quad - \frac{1}{2} \int_{\mathbb{S}} [v_3(DZ_1 \cdot X) \circ \varphi^{-1} + w_1 DY_3 \cdot X - u_1 v_3 w_{1x}] dx \\
&\quad + \frac{1}{2} (w_3 DY_3 \cdot X + v_3 DZ_3 \cdot X).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\langle \nabla_X Y, Z \rangle_{(\varphi, f, s)} &= \left\langle \left(\begin{array}{c} (D\tilde{Y} \cdot X) \circ \varphi^{-1} - \Gamma^0(\tilde{u}, \tilde{v}) \\ DY_3 \cdot X \end{array} \right) - \frac{1}{2} \left(\begin{array}{c} A^{-1} \partial_x (u_3 v_1 + u_1 v_3) \\ 0 \\ 0 \end{array} \right), w \right\rangle_{\mathbb{A}} \\
&= \left\langle (D\tilde{Y} \cdot X) \circ \varphi^{-1} - \Gamma^0(\tilde{u}, \tilde{v}), \tilde{w} \right\rangle_{H^1 \oplus L_2} \\
&\quad - \frac{1}{2} \int_{\mathbb{S}} [w_1 DY_3 \cdot X + w_3 (DY_1 \cdot X) \circ \varphi^{-1} - w_{1x} (u_3 v_1 + u_1 v_3)] dx + \frac{1}{2} w_3 DY_3 \cdot X
\end{aligned}$$

where we have used that integrals of the type

$$\int_{\mathbb{S}} \Gamma^0(\tilde{u}, \tilde{v}) w_3 dx = \int_{\mathbb{S}} A^{-1} (u_1 v_1 + \frac{1}{2} u_{1x} v_{1x} + \frac{1}{2} u_2 v_2) w_{3x} dx = 0$$

vanish. Clearly,

$$\begin{aligned}
\langle \nabla_X Z, Y \rangle_{(\varphi, f, s)} &= \left\langle (D\tilde{Z} \cdot X) \circ \varphi^{-1} - \Gamma^0(\tilde{u}, \tilde{w}), \tilde{v} \right\rangle_{H^1 \oplus L_2} \\
&\quad - \frac{1}{2} \int_{\mathbb{S}} [v_1 DZ_3 \cdot X + v_3 (DZ_1 \cdot X) \circ \varphi^{-1} - v_{1x} (u_3 w_1 + u_1 w_3)] dx + \frac{1}{2} v_3 DZ_3 \cdot X.
\end{aligned}$$

Applying [12, Prop. 3.1] completes the proof of the proposition. \square

2.6. Summary and conclusions. In the following tabular, we summarize some unifying features of the approach pioneered by V.I. Arnold by comparing the geometric quantities for the rigid body motion with the corresponding quantities that have been presented in this section for the 2CH equation with vorticity.

	Rigid body	2CH with vorticity
configuration space	$SO(3)$	$C^\infty G = (\text{Diff}^\infty(\mathbb{S}) \otimes C^\infty(\mathbb{S})) \times \mathbb{R}$
Lie algebra	$\mathfrak{so}(3)$	$C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S}) \times \mathbb{R}$
material velocity	$\dot{R}(t)$	$(\varphi, f, s)'(t)$
spatial velocity	$\omega = \dot{R}R^{-1}$	$(u, \rho, \alpha) = (\varphi' \circ \varphi^{-1}, f' \circ \varphi^{-1}, s')$
body velocity	$\Omega = R^{-1}\dot{R}$	$U_0 = \left(\frac{\varphi_t}{\varphi_x}, f_t - f_x \frac{\varphi_t}{\varphi_x}, s' \right)$
inertia operator	\mathbb{I}	\mathbb{A}
spatial momentum	$\pi = R\mathbb{I}$	$\mu = \left(Au - \frac{\alpha}{2}, \rho, \frac{\alpha}{2} - \frac{1}{2} \int_{\mathbb{S}} u \, dx \right)$
body momentum	$\Pi = \mathbb{I}\Omega$	$\mu_0 = \left(\begin{array}{c} [(m \circ \varphi) - \frac{\alpha}{2}] \varphi_x^2 + (\rho \circ \varphi) f_x \varphi_x \\ (\rho \circ \varphi) \varphi_x \\ \frac{\alpha}{2} - \frac{1}{2} \int_{\mathbb{S}} u \, dx \end{array} \right)$
spatial velocity (Ad)	$\omega = \text{Ad}_R \Omega$	$(u, \rho, \alpha) = \text{Ad}_{(\varphi, f, s)} U_0$
body momentum (Ad*)	$\Pi = \text{Ad}_R^* \pi$	$\mu_0 = \text{Ad}_{(\varphi, f, s)}^* \mu$
momentum conservation	$\pi = \text{const.}$	$\mu_0 = \text{const.}$
Lie bracket (ad)	$[A, B] = AB - BA$	$[(u_1, \rho_1, \alpha_1), (u_2, \rho_2, \alpha_2)] = \begin{pmatrix} u_{1x}u_2 - u_{2x}u_1 \\ \rho_{1x}u_2 - \rho_{2x}u_1 \\ 0 \end{pmatrix}$
ad*	$\text{ad}_A^* B = [B, A]$	$\text{ad}_{(u_1, \rho_1, \alpha_1)}^* (u_2, \rho_2, \alpha_2) = \begin{pmatrix} A^{-1}(2u_{1x}Au_2 + u_1Au_{2x} - \alpha_2u_{1x} + \rho_{1x}\rho_2) \\ + \int_{\mathbb{S}} (u_{1x}Au_2 + \rho_{1x}\rho_2) \, dx \\ (u_1\rho_2)_x \\ 2 \int_{\mathbb{S}} (u_{1x}Au_2 + \rho_{1x}\rho_2) \, dx \end{pmatrix}$

The geometric theory is not only aesthetically appealing but also helps to understand some important features of the solutions to the 2CH equation:

The authors of [11] showed that the geodesic spray $((\varphi, f, s), U, U, \Gamma_{(\varphi, f, s)}(U, U))$ is smooth as a map $TH^sG \rightarrow TTH^sG$, for $s > 5/2$, where H^sG denotes the group $(\text{Diff}^s(\mathbb{S}) \otimes H^{s-1}(\mathbb{S})) \times \mathbb{R}$ and $\text{Diff}^s(\mathbb{S})$ is the group of all orientation-preserving H^s diffeomorphisms $\mathbb{S} \rightarrow \mathbb{S}$. The groups H^sG are only topological groups (but not Lie groups), instead they are Banach manifolds (and not Fréchet manifolds) so that the Picard-Lindelöf Theorem can be applied to conclude the existence of a local-in-time solution $(\varphi, f, s)(t)$ to the geodesic equation (9) for any pair of initial values $(u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$ and $\alpha \in \mathbb{R}$. A Hilbert approximation of $C^\infty G$ by the groups H^sG then shows that Eq. (9) also possesses a unique non-extendable solution with smooth dependence on the initial data in the smooth category. As $C^\infty G$ is a Lie group, composition and inversion are smooth maps so that the relation (4) immediately implies that Eq. (1) possesses a unique maximal solution $(u, \rho)(t) \in C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$, $t \in J$, for any initial datum $(u_0, \rho_0) \in C^\infty(\mathbb{S}) \times C^\infty(\mathbb{S})$ and any $\alpha \in \mathbb{R}$.

By Theorem 6.5 of [11], the solution $(u, \rho)(t)$ exists for all $t \geq 0$ provided $\|u_x(t)\|_\infty$ is bounded on any bounded subinterval of J .

3. THE SECTIONAL CURVATURE

In this section, we present some curvature computations providing a proof of Theorem 1.

We begin with the term $\langle \Gamma(u, v), \Gamma(u, v) \rangle_{\mathbb{A}}$ on the right hand side of (3) which we intend to rewrite as $\langle \Gamma^0(\tilde{u}, \tilde{v}), \Gamma^0(\tilde{u}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}}$ plus additional terms; again Γ^0 denotes the spray for the 2CH equation without vorticity, cf. Eq. (8), $\tilde{u} = (u_1, u_2)$, $\tilde{v} = (v_1, v_2)$ and $\tilde{\mathbb{A}} = \text{diag}(A, 1)$. By (7) and the definition of the metric (5), we have that

$$\langle \Gamma(u, v), \Gamma(u, v) \rangle_{\mathbb{A}} = \langle \Gamma^0(\tilde{u}, \tilde{v}), \Gamma^0(\tilde{u}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}} + \left\langle \Gamma^0(\tilde{u}, \tilde{v}), \begin{pmatrix} A^{-1} \partial_x (u_3 v_1 + u_1 v_3) \\ 0 \end{pmatrix} \right\rangle_{\tilde{\mathbb{A}}}$$

$$\begin{aligned}
& + \frac{1}{4} \langle A^{-1} \partial_x (u_3 v_1 + u_1 v_3), A^{-1} \partial_x (u_3 v_1 + u_1 v_3) \rangle_A \\
& = \langle \Gamma^0(\tilde{u}, \tilde{v}), \Gamma^0(\tilde{u}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}} \\
& \quad - \int_{\mathbb{S}} \partial_x (u_1 v_1 + \frac{1}{2} u_{1x} v_{1x} + \frac{1}{2} u_2 v_2) A^{-1} \partial_x (u_3 v_1 + u_1 v_3) dx \\
(11) \quad & + \frac{1}{4} \int_{\mathbb{S}} \partial_x (u_3 v_1 + u_1 v_3) A^{-1} \partial_x (u_3 v_1 + u_1 v_3) dx.
\end{aligned}$$

The second term on the right hand side of (3) is computed similarly and we find that

$$\begin{aligned}
\langle \Gamma(u, u), \Gamma(v, v) \rangle_{\mathbb{A}} & = \langle \Gamma^0(\tilde{u}, \tilde{u}), \Gamma^0(\tilde{v}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}} \\
& \quad - \int_{\mathbb{S}} \partial_x (u_1^2 + \frac{1}{2} u_{1x}^2 + \frac{1}{2} u_2^2) A^{-1} \partial_x (v_1 v_3) dx \\
& \quad - \int_{\mathbb{S}} \partial_x (v_1^2 + \frac{1}{2} v_{1x}^2 + \frac{1}{2} v_2^2) A^{-1} \partial_x (u_1 u_3) dx \\
(12) \quad & + \int_{\mathbb{S}} \partial_x (u_1 u_3) A^{-1} \partial_x (v_1 v_3) dx.
\end{aligned}$$

Let $u, v, w \in T_p C^\infty G$ be three tangent vectors at a point $p \in C^\infty G$. The curvature tensor R for $(C^\infty G, \langle \cdot, \cdot \rangle_{\mathbb{A}})$ is given locally by

$$(13) \quad R_p(u, v)w = D_1 \Gamma_p(w, u)v - D_1 \Gamma_p(w, v)u + \Gamma_p(\Gamma_p(w, v), u) - \Gamma_p(\Gamma_p(w, u), v),$$

cf. [26], where D_1 denotes differentiation with respect to p :

$$D_1 \Gamma_p(w, u)v = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \Gamma_{p+\epsilon v}(w, u).$$

We apply Eq. (13) at $p = \text{id}$ and with $w = v$ in order to rewrite $S(u, v) = \langle R(u, v)v, u \rangle_{\mathbb{A}}$ as a sum of terms involving Γ^0 and first and second components of u and v plus additional terms involving the third components. Therefore, we make use of the identities

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} u_1 \circ (\text{id} + \epsilon v_1)^{-1} = -u_{1x} v_1$$

and

$$D_1 \Gamma^0(\tilde{w}, \tilde{u})\tilde{v} = -\Gamma^0(\tilde{w}_x v_1, \tilde{u}) - \Gamma^0(\tilde{u}_x v_1, \tilde{w}) + \Gamma^0(\tilde{w}, \tilde{u})_x v_1,$$

see the proof of [12, Prop. 5.1], to infer that

$$\begin{aligned}
D_1 \Gamma(v, u)v & = \begin{pmatrix} -\Gamma^0(\tilde{v}_x v_1, \tilde{u}) - \Gamma^0(\tilde{u}_x v_1, \tilde{v}) + \Gamma^0(\tilde{v}, \tilde{u})_x v_1 \\ 0 \end{pmatrix} \\
& + \begin{pmatrix} -\frac{1}{2} A^{-1} \partial_x (u_{1x} v_1 v_3 + u_3 v_1 v_{1x}) \\ 0 \\ 0 \end{pmatrix} \\
& + \begin{pmatrix} \frac{1}{2} v_1 A^{-1} \partial_x^2 (u_1 v_3 + u_3 v_1) \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

and similarly that

$$D_1\Gamma(v, v)u = \begin{pmatrix} -2\Gamma^0(\tilde{v}_x u_1, \tilde{v}) + \Gamma^0(\tilde{v}, \tilde{v})_x u_1 \\ 0 \\ -A^{-1}\partial_x(u_1 v_{1x} v_3) + u_1 A^{-1}\partial_x^2(v_1 v_3) \\ 0 \\ 0 \end{pmatrix}.$$

Using once more the definition (7), we also find that

$$\Gamma(\Gamma(v, v), u) = \begin{pmatrix} \Gamma^0(\Gamma^0(\tilde{v}, \tilde{v}), \tilde{u}) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}A^{-1}\partial_x(u_3\Gamma^0(\tilde{v}, \tilde{v})_1) \\ 0 \\ 0 \end{pmatrix} \\ + \begin{pmatrix} -A^{-1}\partial_x(u_1 A^{-1}\partial_x(v_1 v_3) + \frac{1}{2}u_1 A^{-1}\partial_x^2(v_1 v_3) - \frac{1}{2}u_3 A^{-1}\partial_x(v_1 v_3)) \\ -\frac{1}{2}u_2 A^{-1}\partial_x^2(v_1 v_3) \\ 0 \end{pmatrix}$$

and similarly that

$$\Gamma(\Gamma(v, u), v) = \begin{pmatrix} \Gamma^0(\Gamma^0(\tilde{v}, \tilde{u}), \tilde{v}) \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}A^{-1}\partial_x(v_3\Gamma^0(\tilde{v}, \tilde{u})_1) \\ 0 \\ 0 \end{pmatrix} \\ + \begin{pmatrix} -\frac{1}{2}A^{-1}\partial_x(v_1 A^{-1}\partial_x(u_1 v_3 + u_3 v_1) + \frac{1}{2}v_{1x} A^{-1}\partial_x^2(u_1 v_3 + u_3 v_1)) \\ -\frac{1}{2}v_3 A^{-1}\partial_x(u_1 v_3 + u_3 v_1) \\ -\frac{1}{4}v_2 A^{-1}\partial_x^2(u_1 v_3 + u_3 v_1) \\ 0 \end{pmatrix}.$$

As shown in the proof of [12, Prop. 5.1]

$$\langle \Gamma^0(\tilde{u}, \tilde{v}), \Gamma^0(\tilde{u}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}} - \langle \Gamma^0(\tilde{u}, \tilde{u}), \Gamma^0(\tilde{v}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}} = \langle \Gamma^0(\tilde{v}, \tilde{u})_x v_1, \tilde{u} \rangle_{\tilde{\mathbb{A}}} - \langle \Gamma^0(\tilde{v}, \tilde{v})_x u_1, \tilde{u} \rangle_{\tilde{\mathbb{A}}} \\ - \langle \Gamma^0(\tilde{v}_x v_1, \tilde{u}), \tilde{u} \rangle_{\tilde{\mathbb{A}}} - \langle \Gamma^0(\tilde{v}, \tilde{u}_x v_1), \tilde{u} \rangle_{\tilde{\mathbb{A}}} + 2 \langle \Gamma^0(\tilde{v}_x u_1, \tilde{v}), \tilde{u} \rangle_{\tilde{\mathbb{A}}} \\ + \langle \Gamma^0(\Gamma^0(\tilde{v}, \tilde{v}), \tilde{u}), \tilde{u} \rangle_{\tilde{\mathbb{A}}} - \langle \Gamma^0(\Gamma^0(\tilde{v}, \tilde{u}), \tilde{v}), \tilde{u} \rangle_{\tilde{\mathbb{A}}}$$

so that, using once more the definition (5), we obtain that

$$(14) \quad S(u, v) = \langle \Gamma^0(\tilde{u}, \tilde{v}), \Gamma^0(\tilde{u}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}} - \langle \Gamma^0(\tilde{u}, \tilde{u}), \Gamma^0(\tilde{v}, \tilde{v}) \rangle_{\tilde{\mathbb{A}}} + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$$

where

$$\mathcal{J}_1 = -\frac{1}{2} \int_{\mathbb{S}} (u_1 v_3 + u_3 v_1) \Gamma^0(\tilde{v}, \tilde{u})_{1x} dx + \frac{1}{2} \int_{\mathbb{S}} (Au_1) v_1 A^{-1} \partial_x^2 (u_1 v_3 + u_3 v_1) dx \\ - \frac{1}{4} \int_{\mathbb{S}} u_3 v_1 A^{-1} \partial_x^2 (u_1 v_3 + u_3 v_1) dx + \frac{1}{4} \int_{\mathbb{S}} u_2 v_2 A^{-1} \partial_x^2 (u_1 v_3 + u_3 v_1) dx \\ - \frac{1}{2} \int_{\mathbb{S}} u_{1x} [v_1 A^{-1} \partial_x (u_1 v_3 + u_3 v_1) + \frac{1}{2} v_{1x} A^{-1} \partial_x^2 (u_1 v_3 + u_3 v_1) \\ - \frac{1}{2} v_3 A^{-1} \partial_x (u_1 v_3 + u_3 v_1)] dx \\ + \frac{1}{2} \int_{\mathbb{S}} u_{1x} (u_{1x} v_1 v_3 + v_1 v_{1x} u_3) dx,$$

$$\begin{aligned}
\mathcal{J}_2 &= \int_{\mathbb{S}} u_1 u_3 \Gamma^0(\tilde{v}, \tilde{v})_{1x} dx + \frac{1}{2} \int_{\mathbb{S}} u_1 u_3 A^{-1} \partial_x^2 (v_1 v_3) dx - \frac{1}{2} \int_{\mathbb{S}} u_{1x} u_3 A^{-1} \partial_x (v_1 v_3) dx, \\
\mathcal{J}_3 &= \int_{\mathbb{S}} u_{1x} [u_1 A^{-1} \partial_x (v_1 v_3) + \frac{1}{2} u_{1x} A^{-1} \partial_x^2 (v_1 v_3)] dx - \frac{1}{2} \int_{\mathbb{S}} u_2^2 A^{-1} \partial_x^2 (v_1 v_3) dx \\
&\quad - \int_{\mathbb{S}} (A u_1) u_1 A^{-1} \partial_x^2 (v_1 v_3) dx - \int_{\mathbb{S}} u_1 u_{1x} v_{1x} v_3 dx.
\end{aligned}$$

Now the proof of formula (3) is completed by verifying that all the terms including third components on the right hand sides of (11) and (12) are equal to the third component terms $\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3$ on the right hand side of (14). Using that $u_{3x} = v_{3x} = 0$, that $\partial_x^2 A^{-1} = A^{-1} \partial_x^2 = -1 + A^{-1}$ and integration by parts, some tedious computations which are omitted for the convenience of the reader show that indeed

$$\begin{aligned}
\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 &= - \int_{\mathbb{S}} \partial_x (u_1 v_1 + \frac{1}{2} u_{1x} v_{1x} + \frac{1}{2} u_2 v_2) A^{-1} \partial_x (u_3 v_1 + u_1 v_3) dx \\
&\quad + \frac{1}{4} \int_{\mathbb{S}} \partial_x (u_3 v_1 + u_1 v_3) A^{-1} \partial_x (u_3 v_1 + u_1 v_3) dx \\
&\quad + \int_{\mathbb{S}} \partial_x (u_1^2 + \frac{1}{2} u_{1x}^2 + \frac{1}{2} u_2^2) A^{-1} \partial_x (v_1 v_3) dx \\
&\quad + \int_{\mathbb{S}} \partial_x (v_1^2 + \frac{1}{2} v_{1x}^2 + \frac{1}{2} v_2^2) A^{-1} \partial_x (u_1 u_3) dx \\
&\quad - \int_{\mathbb{S}} \partial_x (u_1 u_3) A^{-1} \partial_x (v_1 v_3) dx.
\end{aligned}$$

We now let

$$u = \begin{pmatrix} \cos k_1 x \\ \cos k_2 x \\ \alpha \end{pmatrix}, \quad v = \begin{pmatrix} \cos l_1 x \\ \cos l_2 x \\ \beta \end{pmatrix},$$

for $k_1, k_2, l_1, l_2 \in 2\pi\mathbb{N}$, and apply the identities

$$\begin{aligned}
\cos \xi_1 \cos \xi_2 &= \frac{1}{2} (\cos(\xi_1 - \xi_2) + \cos(\xi_1 + \xi_2)), \\
\sin \xi_1 \sin \xi_2 &= \frac{1}{2} (\cos(\xi_1 - \xi_2) - \cos(\xi_1 + \xi_2)), \\
\sin \xi_1 \cos \xi_2 &= \frac{1}{2} (\sin(\xi_1 - \xi_2) + \sin(\xi_1 + \xi_2)),
\end{aligned}$$

and

$$\begin{aligned}
A^{-1} \cos \xi x &= \frac{1}{1 + \xi^2} \cos \xi x, \quad \xi \in \mathbb{R}, \\
\int_0^1 \cos(\xi_1 x) \cos(\xi_2 x) dx &= \frac{1}{2} (\delta_{\xi_1, \xi_2} + \delta_{\xi_1, -\xi_2}), \quad \xi_1, \xi_2 \in 2\pi\mathbb{Z}, \\
\int_0^1 \sin(\xi_1 x) \sin(\xi_2 x) dx &= \frac{1}{2} (\delta_{\xi_1, \xi_2} - \delta_{\xi_1, -\xi_2}), \quad \xi_1, \xi_2 \in 2\pi\mathbb{Z}, \\
\int_0^1 \cos(\xi_1 x) \sin(\xi_2 x) dx &= 0, \quad \xi_1, \xi_2 \in 2\pi\mathbb{Z},
\end{aligned}$$

to observe that, by the definition of Γ and $\langle \cdot, \cdot \rangle_{\mathbb{A}}$,

$$S(u, v) = \langle \Gamma^0(\tilde{u}, \tilde{v}), \Gamma^0(\tilde{u}, \tilde{v}) \rangle_{\mathbb{A}} - \langle \Gamma^0(\tilde{u}, \tilde{u}), \Gamma^0(\tilde{v}, \tilde{v}) \rangle_{\mathbb{A}}$$

$$\begin{aligned}
& + \frac{1}{4} \int_{\mathbb{S}} \partial_x (\alpha \cos l_1 x + \beta \cos k_1 x) A^{-1} \partial_x (\alpha \cos l_1 x + \beta \cos k_1 x) dx \\
& - \int_{\mathbb{S}} \partial_x (\alpha \cos k_1 x) A^{-1} \partial_x (\beta \cos l_1 x) dx \\
& - \int_{\mathbb{S}} \partial_x (\cos k_1 x \cos l_1 x + \frac{1}{2} k_1 l_1 \sin k_1 x \sin l_1 x + \frac{1}{2} \cos k_2 x \cos l_2 x) \times \\
& \quad \times A^{-1} \partial_x (\alpha \cos l_1 x + \beta \cos k_1 x) dx \\
& + \int_{\mathbb{S}} \partial_x (\cos^2 k_1 x + \frac{1}{2} k_1^2 \sin^2 k_1 x + \frac{1}{2} \cos^2 k_2 x) A^{-1} \partial_x (\beta \cos l_1 x) dx \\
& + \int_{\mathbb{S}} \partial_x (\cos^2 l_1 x + \frac{1}{2} l_1^2 \sin^2 l_1 x + \frac{1}{2} \cos^2 l_2 x) A^{-1} \partial_x (\alpha \cos k_1 x) dx.
\end{aligned}$$

By [12, Prop. 5.1] the sum of the first two terms equals the (non-normalized) sectional curvature $S_{2\text{CH}}(\tilde{u}, \tilde{v})$ for the 2CH equation without vorticity and

$$S_{2\text{CH}} \left(\begin{pmatrix} \cos k_1 x \\ \cos k_2 x \end{pmatrix}, \begin{pmatrix} \cos l_1 x \\ \cos l_2 x \end{pmatrix} \right) \geq \frac{k_1^2 l_1^2}{16} \left[\frac{1}{k_1^2 l_1^2} - \frac{1}{k_1 l_1} + \frac{1}{4} - \frac{1}{2k_1^2 l_1^2} - \frac{2}{k_1 l_1} \right] > 0.$$

We may thus write

$$S(u, v) = S_{2\text{CH}}(\tilde{u}, \tilde{v}) + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4 + \mathcal{K}_5$$

and conclude that

$$\begin{aligned}
\mathcal{K}_1 &= \frac{1}{4} \left(\frac{1}{2} \alpha^2 + \frac{1}{2} \beta^2 + \alpha \beta (\delta_{k_1, l_1} + \delta_{k_1, -l_1}) \right) \\
& - \frac{1}{4} \left(\frac{\alpha^2}{2(1+l_1^2)} + \frac{\beta^2}{2(1+k_1^2)} + \frac{1}{2} \left(\frac{\alpha \beta}{1+l_1^2} + \frac{\alpha \beta}{1+k_1^2} \right) (\delta_{k_1, l_1} + \delta_{k_1, -l_1}) \right), \\
\mathcal{K}_2 &= \frac{1}{2} \alpha \beta \left(-\delta_{k_1, l_1} - \delta_{k_1, -l_1} + \frac{1}{1+l_1^2} (\delta_{k_1, l_1} + \delta_{k_1, -l_1}) \right), \\
\mathcal{K}_3 &= -\frac{\alpha}{2} \frac{l_1^2}{1+l_1^2} \left[\frac{1}{2} (\delta_{k_1+l_1, l_1} + \delta_{k_1+l_1, -l_1} + \delta_{k_1-l_1, l_1} + \delta_{k_1-l_1, -l_1}) \right. \\
& + \frac{1}{4} k_1 l_1 (\delta_{k_1-l_1, l_1} + \delta_{k_1-l_1, -l_1} - \delta_{k_1+l_1, l_1} - \delta_{k_1+l_1, -l_1}) \\
& + \left. \frac{1}{4} (\delta_{k_2+l_2, l_1} + \delta_{k_2+l_2, -l_1} + \delta_{k_2-l_2, l_1} + \delta_{k_2-l_2, -l_1}) \right] \\
& - \frac{\beta}{2} \frac{k_1^2}{1+k_1^2} \left[\frac{1}{2} (\delta_{k_1+l_1, k_1} + \delta_{k_1+l_1, -k_1} + \delta_{k_1-l_1, k_1} + \delta_{k_1-l_1, -k_1}) \right. \\
& + \frac{1}{4} k_1 l_1 (\delta_{k_1-l_1, k_1} + \delta_{k_1-l_1, -k_1} - \delta_{k_1+l_1, k_1} - \delta_{k_1+l_1, -k_1}) \\
& + \left. \frac{1}{4} (\delta_{k_2+l_2, k_1} + \delta_{k_2+l_2, -k_1} + \delta_{k_2-l_2, k_1} + \delta_{k_2-l_2, -k_1}) \right], \\
\mathcal{K}_4 &= \frac{\beta}{4} \frac{l_1^2}{1+l_1^2} \left((1-k_1^2) (\delta_{2k_1, l_1} + \delta_{2k_1, -l_1}) + \delta_{2k_2, l_1} + \delta_{2k_2, -l_1} \right), \\
\mathcal{K}_5 &= \frac{\alpha}{4} \frac{k_1^2}{1+k_1^2} \left((1-l_1^2) (\delta_{2l_1, k_1} + \delta_{2l_1, -k_1}) + \delta_{2l_2, k_1} + \delta_{2l_2, -k_1} \right).
\end{aligned}$$

We let $\alpha > 0$ and $\beta = 1$ and recall that $k_1, l_1, k_2, l_2 \in \{2\pi, 4\pi, \dots\}$ so that at most one Kronecker delta within each pair $\delta_{\xi_1, \xi_2} + \delta_{\xi_1, -\xi_2}$ gives a nonzero contribution. Then a lower estimate for the sectional curvature is given by

$$\begin{aligned} S(u, v) &\geq \frac{1}{8}(\alpha^2 + 1) - \frac{\alpha^2 + 1}{8(1 + 4\pi^2)} - \frac{\alpha}{4(1 + 4\pi^2)} - \frac{\alpha}{2} - \frac{\alpha + 1}{2} \left(\frac{3}{2} + \frac{1}{4}k_1l_1 \right) \\ &\quad - \frac{1}{4} \frac{k_1^2 l_1^2}{1 + l_1^2} - \frac{\alpha}{4} \frac{k_1^2 l_1^2}{1 + k_1^2} \\ &\geq \frac{1}{10}(\alpha^2 - 5\alpha M^2 - 5M^2), \end{aligned}$$

where $M = \max\{k_1, l_1\}$. The right hand side of the above inequality is positive for

$$\alpha \geq 6M^2 > \frac{5}{2}M^2 + \sqrt{\frac{25}{4}M^4 + 5M^2}.$$

As

$$\begin{aligned} \langle u, u \rangle_{\mathbb{A}} &= 1 + \frac{1}{2}(k_1^2 + \alpha^2), \\ \langle v, v \rangle_{\mathbb{A}} &= 1 + \frac{1}{2}(l_1^2 + 1) \text{ and} \\ \langle u, v \rangle_{\mathbb{A}} &= \frac{\alpha}{2}, \end{aligned}$$

it is clear that

$$K(u, v) \geq \frac{\frac{1}{10}(\alpha^2 - 5\alpha M^2 - 5M^2)}{(1 + \frac{1}{2}(k_1^2 + \alpha^2))(1 + \frac{1}{2}(l_1^2 + 1)) - \frac{\alpha^2}{4}} \rightarrow \frac{\frac{1}{10}}{\frac{1}{2}(1 + \frac{1}{2}(l_1^2 + 1)) - \frac{1}{4}} > 0$$

as $\alpha \rightarrow \infty$. Thus the proof of Theorem 1 is completed. \square

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